Trefftz and Collocation Methods

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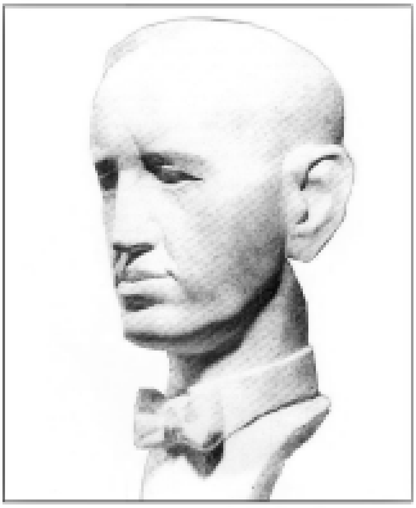
To our friends,

Wen-Jang Huang and Mong-Na Lo Huang.

Contents



Erich Trefftz
21/02/1888 – 21/01/1937

The bust is on display in the Willers building of the Technical University of Dresden
(Photo courtesy of Professor Andrezej P. Zieliñski)

Preface

This book covers a class of numerical methods that are generally referred to as *Collocation Methods*. Different from the finite element and the finite difference methods, the discretization and approximation of the collocation method is based on a set of unstructured points in space. This *meshless* feature is attractive because it eliminates the bookkeeping requirements of the element-based methods, particularly, if the basis functions used satisfy the governing equation; the collocation is conducted only on the boundary. The boundary collocation methods are also known as Trefftz methods. The main advantages of these methods include the flexible representation of the irregular and deforming geometry, ease of data input and preprocessing, high accuracy of the numerical solution, and the efficient computation.

This book contains an introduction, an appendix, and eleven chapters in which several types of collocation methods are discussed. These include the radial basis function method, the Trefftz method, and the coupled collocation and finite element method. Governing equations investigated include Laplace, Poisson, Helmholtz, and bi-harmonic equations. Regular boundary value problems, boundary value problems with singularity, and eigenvalue problems are also examined. Rigorous mathematical proofs are contained in these chapters, and numerical experiments are also provided to support the algorithms and to verify the theory. A tutorial on the applications of these methods is provided in the introduction and a historic review of boundary methods in the appendix.

This book is an extension of the leading author's earlier books on combined methods based on the theoretical analysis of finite element method (FEM). However, this book has several distinct features, which are addressed as follows:

1. In this book, the boundary collocation method, which is a form of Trefftz method (TM) [1], is presented, and referred to as the collocation Trefftz method (CTM). The boundary approximation method (BAM) as discussed in [2], which involves numerical integration, is also classified as a CTM. New analysis of exponential convergence and excellent numerical results are demonstrated. The CTM is shown to be the most accurate numerical method not only for the global solutions, but also for the leading coefficient of the singularity expansion, which is important for problems like fracture mechanics.

2. This book also covers the original TM, the hybrid TM, the direct TM and the indirect TM. There was a special journal issue published in 1995 celebrating 70 years of Trefftz method [3], [4]. Besides, the first and the second International Workshops of Trefftz methods held in Cracow, Poland, 1996, organized by A.P. Zielinski, and in Sintra, Portugal, 1999, organized by J.A.T. Treitas and J.P.M. Almeida, and the invited talks are published in Computer Assisted Mechanics and Engineering Sciences (CAMES) in Vol. 4 (1977) and Vol. 8 (2001). Although a number of papers on TMs were collected, only a few involved analysis. The analysis of TMs lags behind that of the FEM and the boundary element method (BEM). For the TM, there is a significant gap between the excellent computation and the theoretical analysis to support the results. This book presents a systematic analysis for the CTM, the hybrid TM, the indirect TM, and the direct TM to bridge the gap.
3. It also demonstrates the advantages of the CTM over other TMs. The CTM is the simplest algorithm because the collocation equations can be assembled in a straightforward way. For solving Motz's problem, the CTM provides the most accurate solutions not only in the global H^1 sense, but also in its leading singular term. More importantly, the condition number of the stiffness matrix from the CTM is significantly smaller than that of the other TMs. It should be mentioned that the application of CTM is limited to those PDEs whose particular solutions or local particular solutions can be found explicitly. Particular solutions are used in this book in a wide sense, to satisfy the homogeneous or the non-homogeneous elliptic equation with partial or no boundary conditions.
4. More topics are explored in this book, such as the biharmonic equation, the Helmholtz equation, and eigenvalue problems by means of particular solutions of elliptic equations. The combinations of the collocation Trefftz method with high order FEM are also discussed, as compared to the linear and bilinear FEMs reported earlier [5], [6].
5. Particular solutions are essential to the Trefftz methods. We provide the series expansion solutions for the Laplace equation on a polygon, particularly those involving mild singularity.
6. The collocation method (CM) on the entire domain, in contrast to boundary collocation, is studied. The CM can be interpreted as the least squares method with numerical integration. The analysis can be conducted by means of the FEM approach, and optimal weights for different collocation equations resulting from the PDE and different boundary conditions can be found theoretically.
7. The radial basis functions (RBF) are a new approximation tool for smooth functions. In this book the RBF has been developed to solve the elliptic equation with singularities. Moreover, the convergence of Kansa's method is proved with error estimates in H^1 norm.

8. To enhance the education value, a historical review of the boundary methods is provided as an appendix.

This book is organized as follows. The introduction reviews the fundamentals of the collocation and the Trefftz methods from several viewpoints. The remainder of the book is divided into three parts, Part I: The Collocation Trefftz Method; Part II: The Collocation Methods; and Part III: Advanced Topics. Part I is mainly concerned with the algorithms, the error estimates, and stability analysis of both the Trefftz method and the collocation Trefftz method. Several popular examples of PDEs with singularities, including Poisson's equation (Motz's and its variants), and the biharmonic equations with crack singularities are examined. Part II gives a unified framework of combinations of collocation methods with other numerical methods. Part III introduces advanced topics for the collocation Trefftz method.

To appeal to both applied mathematicians and engineers, we have carefully selected only the important analyses and the significant numerical experiments. The mathematics retained is necessary to provide a deeper insight into the numerical algorithms proposed. For easy reading of the book, each chapter can be treated as an independent unit; hence, readers can directly refer to the chapters that interest them. We hope that through this book we can bring the engineering and applied mathematics community a step closer to recognizing the power of the collocation and Trefftz methods.

The Authors, 2008

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