

Trefftz and Collocation Methods

WIT*PRESS*

WIT Press publishes leading books in Science and Technology.

Visit our website for the current list of titles.

www.witpress.com

WIT*eLibrary*

Home of the Transactions of the Wessex Institute, the WIT electronic-library provides the international scientific community with immediate and permanent access to individual papers presented at WIT conferences. Visit the eLibrary at

<http://library.witpress.com>

Trefftz and Collocation Methods

Z.-C. Li

*National Sun Yat-sen University, Taiwan and
National Center for Theoretical Science, Taiwan*

T.-T. Lu

*National Sun Yat-sen University, Taiwan and
National Center for Theoretical Science, Taiwan*

H.-Y. Hu

Tunghai University, Taiwan

A. H.-D. Cheng

University of Mississippi, USA

WITPRESS Southampton, Boston



Z.-C. Li

*National Sun Yat-sen University, Taiwan and
National Center for Theoretical Science, Taiwan*

T.-T. Lu

*National Sun Yat-sen University, Taiwan and
National Center for Theoretical Science, Taiwan*

H.-Y. Hu

Tunghai University, Taiwan

A. H.-D. Cheng

University of Mississippi, USA

Published by

WIT Press

Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK

Tel: 44 (0) 238 029 3223; Fax: 44 (0) 238 029 2853

E-Mail: witpress@witpress.com

<http://www.witpress.com>

For USA, Canada and Mexico

WIT Press

25 Bridge Street, Billerica, MA 01821, USA

Tel: 978 667 5841; Fax: 978 667 7582

E-Mail: infousa@witpress.com

<http://www.witpress.com>

British Library Cataloguing-in-Publication Data

A Catalogue record for this book is available from the British Library

ISBN: 978-1-84564-153-5

Library of Congress Catalog Card Number: 2007922336

No responsibility is assumed by the Publisher, the Editors and Authors for any injury and/or damage to persons or property as a matter of products liability, negligence or otherwise, or from any use or operation of any methods, products, instructions or ideas contained in the material herein. The Publisher does not necessarily endorse the ideas held, or views expressed by the Editors or Authors of the material contained in its publications.

© WIT Press 2008

Printed in Great Britain by Cambridge Printing.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the Publisher.

To our friends,

Wen-Jang Huang and Mong-Na Lo Huang.

Contents

The bust image of Erich Trefftz (1888–1937)	xiii
Preface	xv
Acknowledgements	xix
Tutorial introduction	1
I.1 Algorithms of CM, TM, and CTM	1
I.1.1 Algorithms of CM	1
I.1.2 Viewpoint of boundary approximation methods	4
I.1.3 Viewpoint of least squares methods	8
I.1.4 Complete systems of solutions	10
I.2 Coupling techniques	13
I.2.1 Six combinations	14
I.2.2 The original TM	16
I.2.3 The direct TM	16
I.2.4 Trefftz–Herrera approaches for coupling problems	18
I.3 Boundary element methods	21
I.3.1 Green theorem	21
I.3.2 Discrete approximation	22
I.3.3 Natural BEM	24
I.4 Other kinds of boundary methods	25
I.5 Comparisons	28
Part I Collocation Trefftz method	31
1 Basic algorithms and theory	37
1.1 Notations and preliminaries	38
1.2 Approximation problems	40

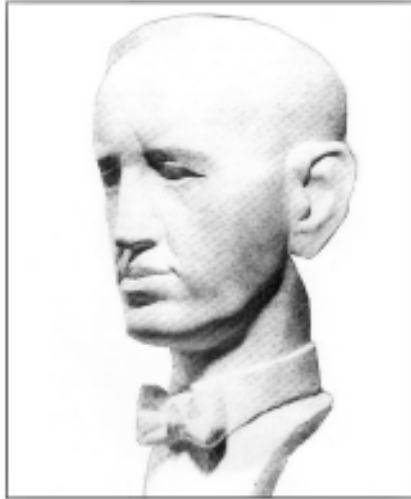
1.3	Error estimates	42
1.4	Debye–Huckel equation	46
1.5	Stability analysis	52
2	Motz’s problem and its variants	59
2.1	Introduction	59
2.2	Basic algorithms of CTM	61
2.3	Error bounds and integration approximation	70
2.4	Variants of Motz’s problem	75
2.5	Concluding remarks	81
3	Coupling techniques	83
3.1	Introduction	84
3.2	Description of generalized TMs	85
3.3	Penalty TMs	88
3.4	Simplified hybrid TMs	93
3.5	Penalty plus hybrid TMs	95
3.6	Lagrange multiplier TM	101
3.7	Effective condition number	105
3.8	Numerical experiments	109
4	Biharmonic equations with singularities	117
4.1	Introduction	117
4.2	The Green formulas of $\Delta^2 u$	118
4.2.1	On rectangular domains	118
4.2.2	Corner effects on polygons	120
4.2.3	Boundary conditions for biharmonic equations on polygons	124
4.3	The collocation Trefftz methods	127
4.3.1	Three singularity models	127
4.3.2	Description of the method	129
4.3.3	The collocation Trefftz method with natural corners	130
4.3.4	Formulas of partial derivatives	131
4.4	Error bounds	132
	Part II Collocation methods	135
5	Collocation methods	139
5.1	Introduction	139
5.2	Description of collocation methods	140
5.3	Error analysis	144

5.4	Robin boundary conditions	151
5.5	Inverse inequalities	155
5.6	Final remarks	160
6	Combinations of collocation and finite element methods	163
6.1	Introduction	163
6.2	Combinations of FEMs	164
6.3	Linear algebraic equations of combination of FEM and CM	168
6.4	Uniformly V_h^0 -elliptic inequality	173
6.5	Uniformly V_h^0 -elliptic inequality involving integration approximation	179
6.6	Final remarks	185
7	Radial basis function collocation methods	187
7.1	Introduction	187
7.2	Radial basis functions	189
7.3	Description of radial basis function collocation methods	190
7.3.1	RBFCM for different boundary conditions	190
7.3.2	Combination of FEM and RBFCM	197
7.4	Inverse estimates for radial basis functions	202
7.5	Numerical experiments	206
7.5.1	Different boundary conditions	206
7.5.2	Adding method of singular functions	208
7.5.3	Subtracting method of singular functions	210
7.6	Comparisons and conclusions	212
	Part III Advanced topics	217
8	Combinations with high-order FEMs	221
8.1	Introduction	221
8.2	Combinations of TM and Lagrange FEMs	222
8.3	Global superconvergence	228
8.4	Adini's elements	233
9	Eigenvalue problems	237
9.1	Introduction	237
9.2	New numerical algorithms for eigenvalue problems	239
9.2.1	The Trefftz methods for eqn. (9.1.6)	239
9.2.2	Heuristic ideas of degeneracy of eqn. (9.1.6)	241
9.2.3	New iteration algorithms	243
9.2.4	Solution of non-linear equations	243
9.3	Error bounds of eigenvalues	246
9.4	Error bounds of eigenfunctions	251

9.5	Computational models and numerical experiments	254
9.5.1	A basic sample of eigenvalue problems and particular solutions	254
9.5.2	Description of detailed algorithms for TM	259
9.5.3	Investigation of behavior for function $f(k)$ as $k^2 \rightarrow \lambda_1$	261
9.5.4	Minimal eigenvalue and corresponding eigenfunction	263
9.6	Eigenvalues for the singularity problem	268
9.7	Summaries and discussions	271
10	The Helmholtz equation	273
10.1	Introduction	273
10.2	The Trefftz method	276
10.3	Error analysis	277
10.3.1	Preliminary lemmas	278
10.3.2	Error bounds	281
10.3.3	Exponential rates of convergence	284
10.3.4	Estimates on bounds of constant $K_{m,n}$	287
10.4	Summaries and discussions	289
11	Explicit harmonic solutions of Laplace's equation	291
11.1	Introduction	291
11.2	Harmonic functions	292
11.2.1	General cases	294
11.2.2	Formulas for special Θ	298
11.3	Harmonic solutions involving Neumann conditions	300
11.3.1	The case of N–D type	300
11.3.2	The case of D–N type	302
11.3.3	The case of N–N type	303
11.4	Extensions and analysis on singularity	305
11.4.1	Particular solutions for Poisson's equations	305
11.4.2	Extensions to not smooth functions of g_D and g_N	307
11.4.3	Regularity and singularity of the solutions of $\Theta =$ $\frac{\pi}{2}, \frac{3\pi}{2}, \pi, 2\pi$	308
11.4.3.1	For the case of $\Theta = \frac{\pi}{2}$	309
11.4.3.2	For the case of $\Theta = \pi$	310
11.4.3.3	For the case of $\Theta = \frac{3\pi}{2}$	311
11.4.3.4	For the case of $\Theta = 2\pi$	312
11.5	New models of singularities for Laplace's equation	314
11.5.1	Two models	314
11.5.2	Trefftz methods	317
11.6	Concluding remarks	319
Appendix	Historic review of boundary methods	323
A.1	Potential theory	324
A.1.1	Euler	326
A.1.2	Lagrange	326

A.1.3	Laplace	327
A.1.4	Fourier	327
A.1.5	Poisson	328
A.1.6	Hamilton	329
A.2	Existence and uniqueness	329
A.2.1	Dirichlet	331
A.2.2	Neumann	332
A.2.3	Kellogg	332
A.3	Reduction in dimensions and Green's formula	333
A.3.1	Gauss	334
A.3.2	Green	335
A.3.3	Ostrogradski	336
A.3.4	Stokes	336
A.4	Integral equations	337
A.4.1	Cauchy	339
A.4.2	Hadamard	340
A.4.3	Fredholm	341
A.5	Extended Green's formula	342
A.5.1	Helmholtz	345
A.5.2	Betti	346
A.5.3	Kelvin	346
A.5.4	Rayleigh	347
A.5.5	Volterra	348
A.5.6	Somigliana	349
A.5.7	Kolosov	349
A.6	Pre-electronic computer era	349
A.6.1	Ritz	354
A.6.2	Von Kármán	354
A.6.3	Trefftz	355
A.6.4	Muskhelishvili	356
A.7	Electronic computer era	356
A.7.1	Kupradze	362
A.7.2	Jaswon	363
A.8	Boundary integral equation and boundary element methods	363
A.8.1	Rizzo	366
A.8.2	Cruse	366
A.8.3	Brebbia	367

References	369
Glossary of symbols	393
Index	397



Erich Trefftz
21/02/1888 – 21/01/1937

The bust is on display in the Willers building of the Technical University of Dresden
(Photo courtesy of Professor Andrezej P. Zieliński)

Preface

This book covers a class of numerical methods that are generally referred to as *Collocation Methods*. Different from the finite element and the finite difference methods, the discretization and approximation of the collocation method is based on a set of unstructured points in space. This *meshless* feature is attractive because it eliminates the bookkeeping requirements of the element-based methods, particularly, if the basis functions used satisfy the governing equation; the collocation is conducted only on the boundary. The boundary collocation methods are also known as Trefftz methods. The main advantages of these methods include the flexible representation of the irregular and deforming geometry, ease of data input and preprocessing, high accuracy of the numerical solution, and the efficient computation.

This book contains an introduction, an appendix, and eleven chapters in which several types of collocation methods are discussed. These include the radial basis function method, the Trefftz method, and the coupled collocation and finite element method. Governing equations investigated include Laplace, Poisson, Helmholtz, and bi-harmonic equations. Regular boundary value problems, boundary value problems with singularity, and eigenvalue problems are also examined. Rigorous mathematical proofs are contained in these chapters, and numerical experiments are also provided to support the algorithms and to verify the theory. A tutorial on the applications of these methods is provided in the introduction and a historic review of boundary methods in the appendix.

This book is an extension of the leading author's earlier books on combined methods based on the theoretical analysis of finite element method (FEM). However, this book has several distinct features, which are addressed as follows:

1. In this book, the boundary collocation method, which is a form of Trefftz method (TM) [1], is presented, and referred to as the collocation Trefftz method (CTM). The boundary approximation method (BAM) as discussed in [2], which involves numerical integration, is also classified as a CTM. New analysis of exponential convergence and excellent numerical results are demonstrated. The CTM is shown to be the most accurate numerical method not only for the global solutions, but also for the leading coefficient of the singularity expansion, which is important for problems like fracture mechanics.

2. This book also covers the original TM, the hybrid TM, the direct TM and the indirect TM. There was a special journal issue published in 1995 celebrating 70 years of Trefftz method [3], [4]. Besides, the first and the second International Workshops of Trefftz methods held in Cracow, Poland, 1996, organized by A.P. Zielinski, and in Sintra, Portugal, 1999, organized by J.A.T. Treitas and J.P.M. Almeida, and the invited talks are published in Computer Assisted Mechanics and Engineering Sciences (CAMES) in Vol. 4 (1977) and Vol. 8 (2001). Although a number of papers on TMs were collected, only a few involved analysis. The analysis of TMs lags behind that of the FEM and the boundary element method (BEM). For the TM, there is a significant gap between the excellent computation and the theoretical analysis to support the results. This book presents a systematic analysis for the CTM, the hybrid TM, the indirect TM, and the direct TM to bridge the gap.
3. It also demonstrates the advantages of the CTM over other TMs. The CTM is the simplest algorithm because the collocation equations can be assembled in a straightforward way. For solving Motz's problem, the CTM provides the most accurate solutions not only in the global H^1 sense, but also in its leading singular term. More importantly, the condition number of the stiffness matrix from the CTM is significantly smaller than that of the other TMs. It should be mentioned that the application of CTM is limited to those PDEs whose particular solutions or local particular solutions can be found explicitly. Particular solutions are used in this book in a wide sense, to satisfy the homogeneous or the non-homogeneous elliptic equation with partial or no boundary conditions.
4. More topics are explored in this book, such as the biharmonic equation, the Helmholtz equation, and eigenvalue problems by means of particular solutions of elliptic equations. The combinations of the collocation Trefftz method with high order FEM are also discussed, as compared to the linear and bilinear FEMs reported earlier [5], [6].
5. Particular solutions are essential to the Trefftz methods. We provide the series expansion solutions for the Laplace equation on a polygon, particularly those involving mild singularity.
6. The collocation method (CM) on the entire domain, in contrast to boundary collocation, is studied. The CM can be interpreted as the least squares method with numerical integration. The analysis can be conducted by means of the FEM approach, and optimal weights for different collocation equations resulting from the PDE and different boundary conditions can be found theoretically.
7. The radial basis functions (RBF) are a new approximation tool for smooth functions. In this book the RBF has been developed to solve the elliptic equation with singularities. Moreover, the convergence of Kansa's method is proved with error estimates in H^1 norm.

8. To enhance the education value, a historical review of the boundary methods is provided as an appendix.

This book is organized as follows. The introduction reviews the fundamentals of the collocation and the Trefftz methods from several viewpoints. The remainder of the book is divided into three parts, Part I: The Collocation Trefftz Method; Part II: The Collocation Methods; and Part III: Advanced Topics. Part I is mainly concerned with the algorithms, the error estimates, and stability analysis of both the Trefftz method and the collocation Trefftz method. Several popular examples of PDEs with singularities, including Poisson's equation (Motz's and its variants), and the biharmonic equations with crack singularities are examined. Part II gives a unified framework of combinations of collocation methods with other numerical methods. Part III introduces advanced topics for the collocation Trefftz method.

To appeal to both applied mathematicians and engineers, we have carefully selected only the important analyses and the significant numerical experiments. The mathematics retained is necessary to provide a deeper insight into the numerical algorithms proposed. For easy reading of the book, each chapter can be treated as an independent unit; hence, readers can directly refer to the chapters that interest them. We hope that through this book we can bring the engineering and applied mathematics community a step closer to recognizing the power of the collocation and Trefftz methods.

The Authors, 2008

References

- [1] Trefftz, E., Konvergenz und fehlerabschätzung beim Ritz'schen verfahren, *Math. Ann.*, **100**, pp. 503–521, 1928.
- [2] Li, Z.C., Mathon, R. & Sermer, P., Boundary methods for solving elliptic problem with singularities and interfaces, *SIAM J. Numer. Anal.*, **24**, pp. 487–498, 1987.
- [3] Kamiya, N. & Kita, E., Trefftz method 70 years, *Adv. Eng. Softw.*, **24** (1-3), pp. 1, 1995.
- [4] Kita, E. & Kamiya, N., Trefftz method, An overview, *Adv. Eng. Softw.*, **24** (1-3), pp. 3-12, 1995.
- [5] Li, Z.C., *Numerical Methods for Elliptic Equations with Singularities, Boundary Methods and Nonconforming Combinations*, World Scientific, Singapore, 1990.
- [6] Li, Z.C., *Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities*, Kluwer Academic Publishers, Boston, London, 1998.

Acknowledgements

We are obliged to Professors C.S. Huang, H.T. Huang, I. Herrera, G.G. Georgiou, C. Xenohontos, J. Huang, T. Lü, S. Wang, L. Ling, and R.C.D. Chen for their collaboration or valuable suggestions on the research works related to this book, and to the graduate students Y.L. Chen, H.S. Tsai, C.H. Huang, S.Y. Tsai, W.L. Jhou, S.L. Huang, S.J. Jian and J.R. Wang for reading the manuscript and giving their comments and suggestions. We also are indebted to S.Y. Wang, W.L. Jhou and S.L. Huang for typing the manuscript.

We are grateful to Professor C.A. Brebbia and the editorial staff at the WIT Press for their encouragement and assistance in publishing this book. We also wish to thank Professors C.C. Chang, J.C. Yao, N.C. Wong and C.K. Law at National Sun Yat-sen University, Taiwan, for their support during the writing process.

This book and the related works were supported in part by the National Science Council of Taiwan, and National Center for Theoretical Sciences, Mathematics Division, Taiwan.