Trends in Applications of Pure Mathematics to Mechanics

A collection of papers presented at a conference at the University of Lecce, Italy, in May 1975

Edited by
Gaetano Fichera
Università di Roma

Pitman Publishing
London San Francisco Melbourne

W. Nowacki

Coupled Fields in Elasticity

1 Introduction

In the 19th century the mechanics of a solid deformable body consisted mainly of the theory of elasticity, treated as a branch of the mathematical physics.

Parallel with the theory of elasticity its technological applications were being developed within the frame of the strength of materials, the theory of plates and structural mechanics.

During the after-war period new branches of the mechanics of deformable solids started developing, namely, the theory of plasticity, visco-elasticity, and rheology. Simultaneously one could observe a revival of interest in the theory of elasticity itself. The nonlinear theory was successfully developing. Within the linear theory of elasticity the problems of cracks, important in the physics of fracture, played the main role.

Simultaneously we observe a rapid development of the coupled theory of elastic bodies. By this name we understand an interrelation of two or more branches of phenomenological physics, so far being developed separately. As a typical example we may mention thermoelasticity. Here the classical theory of elasticity and the theory of heat conduction in solid bodies are coupled into one synthesized branch. We investigate the effect of temperature deviation on solid deformation and the effect of change of deformation on variation of temperature.

Investigations in the coupled fields have been stimulated by the development of technology, progress in aviation and machine constructions, and principally by the development of chemical engineering (especially nuclear engineering). Elements of constructions are more and more exposed to elevated temperature, higher pressure; they work in conditions of radiation, diffusion and in strong magnetic fields.

The investigation of coupled fields in elasticity is connected with a revision of the thermodynamical fundamentals. The thermodynamics of irreversible processes has to be used.

In this review we shall consider only some coupled fields, namely those in which the system is hyperbolic with respect to some of the unknown functions, and parabolic with respect to the remaining ones. We shall consider only fields with energy dissipation. As an illustration we have selected the three domains: thermoelasticity, thermodiffusion in solids, and magnetoelasticity. The main role, in our discussion, will be played by the field of displacements, disturbed by the remaining fields: temperature, concentration, and electromagnetic.

We shall consider the linear theory of elasticity, and we confine ourselves to elastic, homogeneous and isotropic bodies.

2 Thermoelasticity

It is a known fact that different thermodynamical assumptions are applied in the theory of elasticity in the case of statics and elastodynamics. Moreover other assumptions lie in the basis of the theory of thermal stresses.

In elastostatics we assume that during a slow increase of loading, and the resulting deformation, exchange of heat with the environment is complete. It is assumed that there is a constant temperature T_0 over the entire solid, called the temperature of the natural state. The displacement vector $\mathbf{u}(\mathbf{x})$ satisfies the differential equation

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = 0 \tag{2.1}$$

where

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{2.2}$$

Here $\mathbf{X}$ denotes the body forces vector, while μ and λ are Lamé's material constants for the isothermal state. Inequalities (2.2) result from the assumption that the deformation energy constitutes a positively defined quadratic form.

On the other hand, it is assumed in classical elastodynamics that the heat exchange due to heat conduction is very slow, and that there are no heat sources within the solid. This assumption corresponds to the conditions of an adiabatic process.

Now the displacement vector $\mathbf{u}(\mathbf{x}, t)$ satisfies the equation of motion

$$\square_2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = 0 \tag{2.3}$$

where

$$\square_2 = \mu \nabla^2 - \rho\, \partial_t^2, \qquad \partial_t^2 = \frac{\partial^2}{\partial t^2}$$

Here ρ denotes the density while $\square_2$ is d'Alembert's operator. The Lamé constants μ, λ in equation (2.3) refer to the adiabatic state. Also here the same inequalities hold

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{2.4}$$

A different type of assumptions is assumed in the theory of thermal stresses. In this case a heat source may exist, but the effect of the change of deformation on the deviation of temperature is neglected.

The total strain ε_{ij} consists of two parts, namely thermal distortion ε^0_{ij}, and elastic deformation ε'_{ij}. In the result we obtain

$$\varepsilon_{ij} = \alpha_t \theta \delta_{ij} + 2\mu' \sigma_{ij} + \lambda' \delta_{ij} \sigma_{kk} \tag{2.5}$$

where

$$2\mu' = \frac{1}{2\mu}, \qquad \lambda' = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}.$$

Here $\theta = T - T_0$ denotes the increment of temperature; $T(\mathbf{x}, t)$ is the absolute temperature, and α_t denotes the coefficient of linear thermal expansion. The relation $\varepsilon^0_{ij} = \alpha_t \theta \delta_{ij}$ describes a known physical phenomenon, namely the proportionality of the thermal distortion to the temperature deviation. σ_{ij} denotes the stress tensor. Solving equations (2.5) with respect to stresses we obtain the Duhamel–Neumann equations:

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda \varepsilon_{kk} - \gamma \theta)\delta_{ij}, \qquad \gamma = (3\lambda + 2\mu)\,\alpha_t. \tag{2.6}$$

Substituting the relations (2.6) into the equations of motion (for $\mathbf{X} = 0$)

$$\sigma_{ji,j} - \rho \ddot{u}_i = 0 \tag{2.7$'$}$$

and making use of the definition of the total deformation $\varepsilon_{ij} = u_{(i,j)}$ we arrive at the equation of the theory of elasticity with the thermal term

$$\Box_2 \mathbf{u} + (\lambda + \mu)\,\text{grad div}\,\mathbf{u} = \gamma\,\text{grad}\,\theta. \tag{2.7$''$}$$

Now we have to determine the field of temperature. From the equation of thermal equilibrium

$$\text{div}\,\mathbf{q} + c_\varepsilon \frac{\partial \theta}{\partial t} = -W \tag{2.8}$$

and the Fourier law of thermal conduction

$$\mathbf{q} = -k\,\text{grad}\,\theta, \qquad k > 0 \tag{2.9}$$

we obtain the equation of thermal conduction

$$k\nabla^2 \theta - c_\varepsilon \dot{\theta} = -W. \tag{2.10}$$

Here $\mathbf{q}$ denotes the vector of heat flux, W is the heat amount generated in a unit of volume and unit of time, c_ε denotes the specific heat at constant deformation. Here the equation of heat conduction has been derived without taking into account the body deformation. This is why it is not possible to determine the change of temperature generated by the deformation from equations (2.7″) and (2.10).

In the cases considered we have obtained systems of differential equations for different thermodynamic assumptions. The theory of coupled thermoelasticity attempts to derive a unified system of differential equations for all possible thermodynamic processes.

The coupling between the deformation and temperature fields was first postulated by Duhamel [1], the founder of the theory of thermal stresses, who introduced the dilatation term into the equation of thermal conductivity:

$$D\theta - \eta \operatorname{div} \dot{\mathbf{u}} = -W, \qquad D = k\nabla^2 - c_\varepsilon \partial_t. \tag{2.11}$$

This equation, however, was not well justified on thermodynamic grounds. An attempt to justify thermodynamically equation (2.11) was undertaken by Voigt [2] and Jeffreys [3]. Only recently, however, in 1956, Biot [4] gave a satisfactory justification of the thermal conduction equation on the basis of the thermodynamics of irreversible processes.

As the point of departure of the discussion we take the principle of energy, the entropy balance, and the Clausius–Duhem inequality [5]:

$$\dot{U} = \sigma_{ij}\dot{\varepsilon}_{ij} - q_{i,i} + W \tag{2.12}$$

$$\dot{S} + \left(\frac{q_i}{T}\right)_{,i} - \frac{W}{T} = \Theta \tag{2.13}$$

$$\Theta = -\frac{q_i T_{,i}}{T^2} > 0. \tag{2.14}$$

Here U is the internal energy, S denotes the entropy while Θ is a source of entropy. The Clausius–Duhem inequality satisfies the postulates of the thermodynamics of irreversible processes. Inequality (2.14) is satisfied by the Fourier law of heat conduction

$$q_i = -kT_{,i}, \qquad k > 0. \tag{2.15}$$

Introducing the Helmholtz free energy $F = U - ST$ and eliminating the quantity $q_{i,i}$ from equations (2.12) and (2.13), we obtain

$$\dot{F} = \sigma_{ij}\dot{\varepsilon}_{ij} - S\dot{T}. \tag{2.16}$$

The free energy F is a function of state. Comparing the terms in the total differential of F we obtain

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}}, \qquad S = -\frac{\partial F}{\partial T}. \tag{2.17}$$

Expanding the free energy F into the Taylor series in the neighbourhood of the natural state, for an isotropic body, we obtain the following representation

$$F = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{kk}^2 - \gamma\theta\varepsilon_{kk} - \frac{n}{2}\theta^2. \tag{2.18}$$

In the expansion we have retained only the quadratic terms, thus obtaining the linear relations between the stresses σ_{ij}, strains ε_{ij} and the deviation of temperature θ.

From equation (2.17) we obtain the constitutive equations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\theta)\delta_{ij}$$
$$S = \gamma\varepsilon_{kk} + \frac{c_\varepsilon}{T_0}\theta. \tag{2.19}$$

Combining equations (2.13) and (2.15) we arrive at the non-linear heat conduction equation. The equation can be linearized by the assumption that the temperature field differs only slightly from a prescribed uniform temperature T_0, i.e. $|\theta/T_0| \ll 1$, and that the rate of change of the temperature and the temperature gradient are small. Thus we obtain [4]:

$$k\nabla^2\theta - c_\varepsilon\dot{\theta} - \eta \operatorname{div} \dot{\mathbf{u}} = -W, \qquad \eta = \gamma T_0. \tag{2.20}$$

Equation (2.20), together with the displacement equation

$$\Box_2\mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \gamma \operatorname{grad} \theta \tag{2.21}$$

constitute a complete system of thermoelasticity equations. The equations are coupled. The deformation and temperature are caused by body forces, heat sources, prescribed boundary conditions (loadings, displacements and the temperature over the surface A bounding the body), and the initial conditions.

Thermoelasticity describes a wide range of phenomena. It constitutes a generalization of the classical theory of elasticity and thermal conduction. Thermoelasticity is of fundamental importance in cases for which the investigation of elastic dissipation is the main purpose. The significance of thermoelasticity consists primarily in its recognition of the generalized nature of the associated phenomena. Despite its mathematical complexity, thermoelasticity permits a deeper insight into the mechanism of deformation processes connected with thermal effects in elastic solids.

There are a number of particular cases of the thermoelasticity equations (2.20) and (2.21). If the causes generating the displacement and the change of temperature vary very slowly in time, then the inertia term $\rho\ddot{u}_i$ in equation (2.21) can be neglected. The coupled equations of thermoelasticity can be uncoupled in the case of a stationary process. The heat conduction equation becomes Poisson's equation, while the displacement equation is of elliptic type. The case of classical elastodynamics, considered previously, becomes also a particular case of the thermoelasticity equations. Namely, in the adiabatic case we have $\dot{S} = 0$, and making use of equation $(2.19)_2$ we obtain

$$\theta = -\beta \operatorname{div} \mathbf{u}, \qquad \beta = \gamma T_0/c_\varepsilon. \tag{2.22}$$

Substituting (2.22) into equation (2.21) we obtain the displacement equation of classical elastodynamics (2.3).

In many problems of practical importance a simplifying assumption is made, facilitating the solution of the boundary value problem. Thus the uncoupled theory is obtained and the term η div $\dot{\mathbf{u}}$ in the heat conduction equation can be neglected.

Let us return to the equations of thermoelasticity (2.20) and (2.21) and decompose the displacement vector and the body force vector into the potential and rotational parts:

$$\begin{aligned} \mathbf{u} &= \operatorname{grad} \Phi + \operatorname{curl} \mathbf{\Psi}, \qquad \operatorname{div} \mathbf{\Psi} = 0 \\ \mathbf{X} &= \operatorname{grad} \vartheta + \operatorname{curl} \boldsymbol{\chi}, \qquad \operatorname{div} \boldsymbol{\chi} = 0 \end{aligned} \tag{2.23}$$

Substituting then relations (2.23) into equations (2.20) and (2.21) we transform the equations of thermoelasticity to the following form [6]:

$$\Box_1 \Phi = \gamma\theta - \vartheta \tag{2.24}$$

$$\Box_2 \mathbf{\Psi} = -\boldsymbol{\chi} \tag{2.25}$$

$$D\theta - \eta \nabla^2 \dot{\Phi} = -W \tag{2.26}$$

where the following symbols have been used

$$\Box_1 = (\lambda + 2\mu)\nabla^2 - \rho\partial_t^2, \qquad D = k\nabla^2 - c_\varepsilon \partial_t.$$

Equations (2.24) and (2.26) are coupled. Eliminating the function θ we obtain the longitudinal wave equation

$$(\Box_1 D - \gamma\eta\partial_t \nabla^2)\Phi = -\gamma W - D\vartheta. \tag{2.27'}$$

Equation (2.25) describes the shear wave. Eliminating the function Φ from equations (2.24) and (2.26) we obtain the equation

$$(\Box_1 D - \gamma\eta\partial_t \nabla^2)\theta = -\Box_1 W - \eta\partial_t \nabla^2 \vartheta \tag{2.27''}$$

which has the same form as equation (2.27′). The completeness of these solutions has been established by Sternberg [7].

One can prove, considering the propagation of a plane wave and causes and effects varying harmonically in time, that the longitudinal wave is attenuated and dispersed. The same effect refers to the thermal wave. The shear wave, propagating in an infinite space, is purely elastic and is neither attenuated nor dispersed.

It should be noted that the solutions obtained within the framework of thermoelasticity differ quantitatively only slightly from those of the classical theory of elasticity or the theory of thermal stresses. Although the coupling between the deformation and the temperature field is weak, qualitative differences are fundamental. Neglecting the term η div $\dot{\mathbf{u}}$ in equation (2.20) we obtain the following system of waves (for $\mathbf{X} = 0$)

$$\left.\begin{aligned} \Box_1 D\Phi &= \gamma W \\ \Box_2 \mathbf{\Psi} &= 0 \\ D\theta &= -W \end{aligned}\right\}. \tag{2.28}$$

It is apparent from equation (2.28) that the longitudinal wave consists of two waves, namely an elastic wave Φ_1, which is neither attenuated nor dispersed, and a diffusive wave Φ_2, where

$$\Box_1\Phi_1 = 0, \qquad D\Phi_2 = 0, \qquad \Phi = \Phi_1 + \Phi_2$$

To illustrate the procedure we consider the function Φ for a longitudinal spherical wave. Let a concentrated instant heat source of intensity W_0 act in an infinite space at the origin of the co-ordinate system. The function Φ is given by the formula [8]:

$$\Phi(\zeta, \tau) = \frac{mW_0}{4\pi\zeta}[F_1(\zeta, \tau) + E_2(\zeta, \tau)H(\tau - \zeta)] \tag{2.29}$$

Here

$$F_1 = \operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}}\right) - \tfrac{1}{2}e^{-\tau}\left[e^{\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} + \sqrt{\tau}\right) + e^{-\zeta}\operatorname{erfc}\left(\frac{\zeta}{2\sqrt{\tau}} - \sqrt{\tau}\right)\right] \tag{2.30}$$

$$F_2 = e^{\tau-\zeta} - 1, \qquad \zeta = \frac{r}{\varkappa\sigma}, \qquad \tau = \frac{t}{\varkappa\sigma^2}, \qquad \sigma = \frac{1}{c_1}, \qquad \varkappa = k/c_\varepsilon,$$
$$r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$$

F_1 is a purely elastic wave, while F_2 denotes a diffusive wave.

Podstrichač [9], Kaliski [10] and Rüdiger [11] have considered an analogue of the Cauchy–Kovalewski–Somigliana solution of the isothermal theory. The displacement vector $\mathbf{u}$ and the temperature field θ can be represented by the vector function $\boldsymbol{\varphi}$ and scalar function ψ:

$$\mathbf{u} = -\Omega\boldsymbol{\varphi} + \operatorname{grad}\operatorname{div}\Gamma\boldsymbol{\varphi} - \gamma\operatorname{grad}\psi \tag{2.31}$$

$$\theta = -\eta\partial_t\operatorname{div}\Box_2\boldsymbol{\varphi} - \Box_1\psi \tag{2.32}$$

where

$$\Omega = \Box_1 D - \gamma\eta\partial_t\nabla^2, \qquad \Gamma = (\lambda + \mu)D - \gamma\eta\,\partial_t.$$

Substituting the above representation into the equations of thermoelasticity we obtain the following set of wave equations:

$$\Box_2(\Box_1 D - \gamma\eta\partial_t\nabla^2)\boldsymbol{\varphi} = \mathbf{X} \tag{2.33}$$

$$(\Box_1 D - \gamma\eta\partial_t\nabla^2)\psi = W. \tag{2.34}$$

Here the vector $\boldsymbol{\varphi}$ plays a role analogous to that of Galerkin's vector in elastostatics. The form of equations (2.33) and (2.34) is particularly suitable for the determination of fundamental solutions in an infinite elastic space.

At present thermoelasticity constitutes a fully developed field theory. Numerous methods of solving the differential equations have been derived and the fundamental energy theorem and variational theorems have been deduced.

A number of theorems of classical elastodynamics have been generalized on thermoelasticity.

The following theorems have been generalized: the reciprocity theorem, variational theorems, the Somigliana, Helmholtz, Kirchhoff, Weber, and Volterra theorems. The radiation conditions have been given. Here we discuss only Helmholtz's formula [12] and the reciprocity theorem [13].

Consider an internal region B^+ bounded by a surface A. Let B^- denote the complement of B^+ with respect to all of three-dimensional space. We shall study the propagation of longitudinal waves inside the region B^+. We are interested in an expression for the value of the functions Φ and θ at the point $\boldsymbol{\zeta}$ in terms of distribution of Φ, $\partial\Phi/\partial n$, θ, $\partial\theta/\partial n$ on the closed surface A.

For the special case of causes harmonically varying in time we obtain the following formulae:

$$\Phi^*(\zeta)=\int_A\left(F^*\frac{\partial\Phi^*}{\partial n}-\Phi^*\frac{\partial F^*}{\partial n}\right)\mathrm{d}A+m\int_A\left(G^*\frac{\partial\theta^*}{\partial n}-\theta^*\frac{\partial G^*}{\partial n}\right)\mathrm{d}A \tag{2.35}$$

$$\theta^*(\zeta)=\int_A\left(S^*\frac{\partial\theta^*}{\partial n}-\theta^*\frac{\partial S^*}{\partial n}\right)\mathrm{d}A+q\varepsilon\sigma^2\int_A\left(G^*\frac{\partial\Phi^*}{\partial n}-\Phi^*\frac{\partial G^*}{\partial n}\right)\mathrm{d}A. \tag{2.36}$$

Here Φ^*, θ^*,... are the amplitudes of the functions Φ, θ,.... We have introduced the notation

$$G^*=\frac{1}{4\pi R(k_2^2-k_1^2)}\left(\mathrm{e}^{ik_1R}-\mathrm{e}^{ik_2R}\right)$$

$$S^*=(\nabla^2+\sigma^2)G^*,\qquad F^*=(\nabla^2+q_0)G^*$$

The quantities k_1, k_2 are roots of the equation

$$k^4-k^2(\sigma^2+q(1+\varepsilon))+q\sigma^2=0 \tag{2.37}$$

where

$$q=\frac{i\omega}{\varkappa},\qquad \sigma=\omega/c_1,\qquad q_0=q(1+\varepsilon),\qquad \varepsilon=\eta m\varkappa,\qquad c_1=\left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}.$$

The roots k_1, k_2 are complex

$$k_\beta=a_\beta+ib_\beta,\qquad a_\beta>0,\qquad b_\beta>0,\qquad \beta=1,2.$$

G^* is here the Green function of the wave equation (2.27′). Equation (2.35), obtained within the framework of coupled thermoelasticity, is similar to the Helmholtz formula in classical elastodynamics. Equation (2.36) may be regarded as a generalization of Green's theorem in the theory of heat conduction to the problems in coupled thermoelasticity.

The reciprocity theorem has the following form [13]

$$\eta\varkappa\left\{\int_V X_i\odot u_i'\,\mathrm{d}V+\int_A p_i\odot u_i'\,\mathrm{d}A\right\}+\gamma k\int_A\theta*\frac{\partial\theta'}{\partial n}\mathrm{d}A-\gamma\int_V W*\theta'\,\mathrm{d}V=$$

$$\eta\varkappa\left\{\int_V X_i'\odot u_i\,\mathrm{d}V+\int_A p_i'\odot u_i\,\mathrm{d}A\right\}+\gamma k\int_A\theta'*\frac{\partial\theta}{\partial n}\mathrm{d}A-\gamma\int_V W'*\theta\,\mathrm{d}V. \tag{2.38}$$

Here we have introduced the following notation

$$X_i \odot u_i' = \int_0^t X_i(\mathbf{x}, t-\tau)\frac{\partial u_i'(\mathbf{x}, \tau)}{\partial \tau}\,\mathrm{d}\tau,$$

$$W * \theta' = \int_0^t W(\mathbf{x}, t-\tau)\theta'(\mathbf{x}, \tau)\,\mathrm{d}\tau, \ldots$$

and so on.

All the causes and effects of both the systems occur in equation (2.38). The reciprocity theorem includes a number of particular cases, for example Graffi's theorem [14] for elastodynamics and Maysel's theorem [15] for the theory of thermal stresses.

The proof of the uniqueness of the thermoelasticity equations has been given by Weiner [16] under the assumption that the elasticity tensor c_{ijkl} is positive semi-definite. Knops & Payne [17] have proved the same theorem disregarding that assumption and have shown that the solution depends continuously on the initial date. Brun [18] has shown that the elasticity tensor c_{ijkl} does not need to be positive semi-definite. Existence, uniqueness and asymptotic stability theorems for the relevant mixed problem have been derived by Dafermos [19]. Under the assumption of periodic time-dependence Kupradze and his co-workers [20] have proved the existence theorem for internal and external problems.

Compared with classical elastodynamics only few solutions of coupled thermoelasticity have been obtained in the closed form: only in the case of the simplest types of initial and boundary value problems, first of all for the causes and effects varying periodically with time.

Let us return to the classical heat conduction equation. The equation is of parabolic type, implying infinite speed of thermal wave propagation.

Since the idea of a thermal disturbance propagating with infinite speed is unacceptable, modifications of the Fourier law have recently had to be devised to provide for a finite signal time. Cattaneo [21] proposes to replace Fourier's law by the extended law:

$$(1+\tau\partial_t)\mathbf{q} = -k \operatorname{grad} \theta. \tag{2.39}$$

Here τ is a positive quantity of time dimension, small on the time scale. The substitution of (2.39) into the equation of heat flow balance leads to the 'telegraphic' heat conduction equation

$$k\nabla^2\theta - c_\varepsilon\dot{\theta} - \tau c_\varepsilon\ddot{\theta} = 0. \tag{2.40}$$

This equation is hyperbolic. The wave velocity v is defined by the equation

$$v^2 = k/\tau c_\varepsilon. \tag{2.41}$$

The other modifications of Fourier's law have been given by Gurtin & Pipkin [22].

Below we present the coupled equations of thermoelasticity with the modified Fourier's law taken into account (2.39). These equations form a system of four hyperbolic equations [23].

$$\square_2 \mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} = \gamma \text{ grad } \theta \tag{2.42}$$

$$k\nabla^2\theta - \partial_t(1+\tau\partial_t)(c_e\theta + \eta \text{ div } \mathbf{u}) = 0. \tag{2.43}$$

3 Diffusion and Thermo-diffusion in Solids

Let us consider the diffusion of a substance into a solid. We deal with a two-component model in which a mobile component co-exists with the immobile one. In particular, a gas diffusing into a solid body may be described by such a model (also diffusion in stable solutions.) A crystal lattice of the immobile component can be assumed as a reference system for the diffusion flow.

As in the theory of thermal stresses, also here we shall introduce the initial deformation ε^0_{ij} due to diffusion into an elastic solid. Here we have

$$\varepsilon^0_{ij} = \alpha_c c \delta_{ij} \tag{3.1}$$

where $c(\mathbf{x}, t)$ denotes the concentration, while α_c is the coefficient of diffusive expansion. The relations between the stresses and deformations are here analogous to the Duhamel–Neumann equations

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma_0 c)\delta_{ij}, \qquad \gamma_c = (3\lambda + 2\mu)\alpha_c. \tag{3.2}$$

Substitution of the above relations into the equations of motion leads to the displacement equations of the theory of elasticity

$$\square_2 \mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} + \mathbf{X} = \gamma_c \text{ grad } c. \tag{3.3}$$

The concentration variation in time and space is described by the following equations

$$\frac{\partial c}{\partial t} = -\text{div } \boldsymbol{\eta} + \sigma \qquad \boldsymbol{\eta} = -D_c \text{ grad } c. \tag{3.4}$$

Here $\boldsymbol{\eta}(\mathbf{x}, t)$ denotes the vector of the flow of diffusing mass, σ is the amount of mass generated in a unit of time and unit of volume. D_c is the diffusion coefficient.

Equation $(3.4)_1$ represents the balance of the flow of mass, the relation $(3.4)_2$ is the phenomenological Fick's law. The elimination of the vector $\boldsymbol{\eta}$ from equations (3.4) leads to the diffusion equation

$$(D_c\nabla^2 - \partial_t)c = -\sigma. \tag{3.5}$$

Equations (3.3) and (3.5) constitute a complete system of uncoupled equations of diffusion in solid bodies, valid for the isothermal state. These equations enable us to represent the effect of diffusion on a solid deformation, but on the other hand they do not give an answer if we wish to estimate the influence of the mechanical quantities (ultrasonic vibrations, say) on the process of diffusion.

An analogous system of equations can be obtained for the problem of humidity change in a porous solid body, in problems of shrinkage and bulging.

Let us consider the diffusion phenomenon combined with the process of heating (or cooling) of a solid body and its deformation. In order to combine the three fields, concentration, temperature and deformation, we depart from the equations of the balance of energy and entropy, assuming that the procedure is thermodynamically irreversible

$$\dot{F} = \sigma_{ij}\dot{\varepsilon}_{ij} - S\dot{T} + \vartheta\dot{c} \tag{3.6}$$

$$\dot{S} + \left(\frac{q_i}{T}\right)_{,i} - \left(\frac{\vartheta\eta_i}{T}\right)_{,i} = \Omega, \tag{3.7}$$

$$\Omega = -\frac{1}{T}\left[\frac{q_i T_{,i}}{T^2} + \eta_i T\left(\frac{\vartheta}{T}\right)_{,i}\right] > 0. \tag{3.8}$$

In these equations ϑ denotes the chemical potential. Here we have made a simplifying assumption, namely that the temperature of both the components is the same.

The following equations result from the Clausius–Duhem inequalities

$$q_i = -\frac{L_{qq}}{T^2} T_{,i} - L_{q\eta}\left(\frac{\vartheta}{T}\right)_{,i} \tag{3.9}$$

$$\eta_i = -\frac{L_{\eta q}}{T} T_{,i} - L_{\eta\eta}\left(\frac{\vartheta}{T}\right)_{,i} \tag{3.10}$$

where

$$L_{qq} > 0, \qquad L_{\eta\eta} > 0, \qquad L_{qq}L_{\eta\eta} - L_{q\eta}^2 > 0, \qquad L_{\eta q} = L_{q\eta}$$

The above equations generalize Fourier's law and Fick's law to the problems of thermodiffusion. From the expression for free energy one obtains the following constitutive equations

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma_\theta\theta - \gamma_c c)\delta_{ij} \tag{3.11}$$

$$S = -\frac{\partial F}{\partial T} = \gamma_\theta\varepsilon_{kk} + n\theta + dc \tag{3.12}$$

$$\vartheta = \frac{\partial F}{\partial c} = \gamma_\vartheta\varepsilon_{kk} - d\theta + ac. \tag{3.13}$$

Equations (3.11) constitute here a generalization of the Duhamel–Neumann equations. Combining the equations of motion with the entropy balance, and the flow equation we arrive at the system of three coupled differential equations, where the following functions enter: the displacement $\mathbf{u}$, temperature θ and the chemical potential ϑ. These equations were derived by Podstrichač [24], and take the following form:

$$\Box_2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \operatorname{grad} (\gamma_\theta \theta + \gamma_\vartheta \vartheta), \tag{3.14}$$

$$D_1 \theta - d\dot{\vartheta} - \gamma_\theta \operatorname{div} \dot{\mathbf{u}} = -W_1 \tag{3.15}$$

$$D_2 \vartheta - d\dot{\theta} - \gamma_\vartheta \operatorname{div} \dot{\mathbf{u}} = -W_2. \tag{3.16}$$

Here we have introduced the following symbols

$$D_\alpha = k_\alpha \nabla^2 - c_\alpha \partial_t, \qquad \alpha = 1, 2.$$

Equation (3.14) is the equation of the theory of elasticity, equation (3.15) is the heat conduction equation, and finally equation (3.16) is the diffusion equation. The equations are coupled, the uncoupling occurring for a stationary state.

The system of thermodiffusion equations includes a number of particular cases. If the diffusion process is isothermal, then $\theta = 0$; and the remaining equations (3.14) and (3.16) do not contain the thermal terms. In the absence of diffusion, $\vartheta = 0$, we obtain the equations of thermoelasticity (3.14) and (3.15) without the diffusion terms. If we neglect the dilatation terms in equations (3.15) and (3.16) then we obtain the system of parabolic equations:

$$D_1 \theta - d\dot{\vartheta} = -W_1 \tag{3.17}$$

$$D_2 \vartheta - d\dot{\theta} = -W_2. \tag{3.18}$$

The solutions of this system of equations, namely the functions θ and ϑ, can be substituted into equation (3.14) as the known functions.

The coupled equations of thermodiffusion (3.14) ÷ (3.16) can be reduced to simple wave equations. Similarly to thermoelasticity the longitudinal waves are attenuated and dispersed while the transversal wave is independent of thermal and diffusion effects and propagates with the speed $v = (\mu/\rho)^{1/2}$.

The theory of thermodiffusion in solids, in the form derived by Podstrichač is in the course of development. A number of general theorems have already been obtained (variational theorem and the reciprocity theorem with the resulting conclusions [25]).

An important contribution in thermodiffusion is due to Fichera [26] who investigated the problem of uniqueness in existence, and estimated the solution of the dynamical problem. The coupled thermoelasticity constitutes a particular case.

The system of thermodiffusion equations (3.14) + (3.16) will be extended considerably in the case of diffusion of the multicomponent systems.

4 Magneto-elasticity and Magneto-thermoelasticity

In the last two decades a new domain has been developed in which the investigations concern the interactions between the strain and electromagnetic fields. This new discipline is called magneto-elasticity. Its development was stimulated by possible applications to the problems of geophysics, certain topics in optics, acoustics, investigations on the damping of acoustic waves in magnetic fields, etc.

If an elastic solid in a strong magnetic static field is set into motion by external mechanical forces, then in addition to a deformation an electromagnetic field is generated. The two fields interact and are coupled. In the equations of electrodynamics of slowly moving bodies strain rate effects appear while the derivatives of the Maxwell electromagnetic stresses, the Lorentz forces enter the equations of motion.

In magneto-elasticity the motion is regarded as adiabatic, as in classical elastodynamics. It is assumed that the heat exchange between parts of the solid is slow.

In the sequel we present a complete system of magneto-elasticity equations for an isotropic solid with a finite electric conductivity. The first system of equations constitutes the systems of equations of electrodynamics of slowly moving bodies [27–29]

$$\operatorname{curl}\mathbf{E} = -\frac{\mu_0}{c}\frac{\partial \mathbf{h}}{\partial t} \tag{4.1}$$

$$\operatorname{curl}\mathbf{h} = \frac{4\pi}{c}\mathbf{j} \tag{4.2}$$

$$\operatorname{div}\mathbf{h} = 0 \tag{4.3}$$

$$\operatorname{div}\mathbf{E} = 0 \tag{4.4}$$

where

$$\begin{aligned} \mathbf{j} &= \lambda_0\left[\mathbf{E} + \frac{\mu_0}{c}\left(\frac{\partial \mathbf{u}}{\partial t}\times\mathbf{H}\right)\right] \\ D &= \varepsilon_0\left[\mathbf{E} + \frac{1}{c}\,\frac{\mu_0\varepsilon_0 - 1}{\varepsilon_0}\left(\frac{\partial \mathbf{u}}{\partial t}\times\mathbf{H}\right)\right]. \end{aligned} \tag{4.5}$$

Here the vectors $\mathbf{h}$ and $\mathbf{E}$ denote the perturbations of the magnetic and electric fields respectively, $\mathbf{j}$ is the electric current density vector, $\mathbf{H}$ denotes the initial constant magnetic field, $\mathbf{D}$ is the electric induction vector, $\mathbf{B}$ is the magnetic induction vector, $\mathbf{u}$ denotes the displacement vector, while c is the speed of light. μ_0 and ε_0 denote the magnetic and electric permeability, respectively, λ_0 is the electric conductivity.

The equations of motion constitute the second group of equation

$$\sigma_{ji,j} + T_{ji,j} + X_i = \rho\ddot{u}_i \tag{4.6}$$

where T_{ij} denotes the Maxwell electro-magnetic stress tensor. The quantities $f_i = T_{ji,j}$ are the Lorentz forces. The Maxwell tensor is related to the vector $\mathbf{h}$ in the following manner:

$$T_{ij} = \frac{\mu_0}{4\pi}\left(H_i h_j + H_j h_i - \delta_{ij}H_k h_k\right), \qquad i, j = 1, 2, 3. \tag{4.7}$$

Since the values of the quantities μ_0 and ε_0 are close to one, the term $\varepsilon_0\mu_0 - 1$ can be neglected. Moreover, the displacement current $(\varepsilon_0/c)(\partial \mathbf{E}/\partial t)$ can also be neglected, since its influence on the electro-magnetic quantities is very small. Eliminating the vector $\mathbf{E}$ and $\mathbf{j}$, and the stresses and strains, and expressing the tensor T_{ij} by the components of the vector $\mathbf{h}$ we arrive at the system of equations

$$\nabla^2\mathbf{h} - \beta\dot{\mathbf{h}} = -\beta \text{ curl } (\dot{\mathbf{u}} \times \mathbf{H}), \qquad \beta = \frac{4\pi\lambda_0\mu_0}{c^2} \tag{4.8}$$

$$\Box_2\mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} + \frac{\mu_0}{4\pi} \text{ curl } (\mathbf{h} \times \mathbf{H}) + \mathbf{X} = 0. \tag{4.9}$$

Equations (4.8) and (4.9) constitute a complete system of the fundamental differential equations of magneto-elasticity. They represent quasistatic electric fields and dynamic mechanical behaviour.

Consider now the mechanical, electro-magnetic and thermal couplings. Equations of electro-magnetics (4.1)–(4.4) equations of motion (4.5) and the tensor T_{ij} remain unchanged. Hooke's law relations are now replaced by the Duhamel–Neumann equations (2.6). The complete system of magneto-thermoelastic equations assumes the form [30]:

$$\nabla^2\mathbf{h} - \beta\dot{\mathbf{h}} = -\beta \text{ curl } (\dot{\mathbf{u}} \times \mathbf{H}) \tag{4.10}$$

$$\Box_2\mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} + \frac{\mu_0}{4\pi} \text{ curl } (\mathbf{h} \times \mathbf{H}) + \mathbf{X} = \gamma \text{ grad } \theta \tag{4.11}$$

$$k\nabla^2\theta - c_\varepsilon\dot{\theta} - \eta \text{ div } \dot{\mathbf{u}} = -W. \tag{4.12}$$

The last equation is a well known heat conduction equation. Magneto-elasticity and magneto-thermoelasticity are little developed domains. Even the particular case of a solid ideally conducting electricity [$\beta = \infty$, $\mathbf{h} = \text{curl } (\mathbf{u} \times \mathbf{H})$] is not well analysed. This case leads to the following system of equations:

$$\Box_2\mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} + \frac{\mu_0}{4\pi} [\text{curl curl } (\mathbf{u} \times \mathbf{H}) \times \mathbf{H}] = \gamma \text{ grad } \theta \tag{4.13}$$

$$k\nabla^2\theta - c_\varepsilon\dot{\theta} - \eta \text{ div } \dot{\mathbf{u}} = -W. \tag{4.14}$$

So far only few one- and two-dimensional problems have been solved: the propagation of plane and cylindrical waves. In these cases dispersion and attenuation of waves is observed.

The interrelation of three coupled fields—displacement, temperature and magneto-electric field—distinctly occurs in the fundamental energy theorem expressing both the thermodynamic laws [30]:

$$\frac{d}{dt}(\mathcal{K}+\mathcal{W}+\mathcal{P}+\mathcal{L})+\chi_\theta+\chi_h=\int_V X_i v_i \, dV+\int_A p_i v_i \, dA$$
$$+\frac{k}{T_0}\int_A \theta\frac{\partial\theta}{\partial n}\, dA+\frac{\mu_0}{4\pi\beta}\int_A h_i\frac{\partial h_i}{\partial n}\, dA$$
$$-\frac{\mu_0 H_3}{4\pi}\int_A v_3 h_j n_j \, dA, \qquad H=(0,0,H_3). \quad (4.15)$$

Here $\mathcal{K}$ and $\mathcal{W}$ denote kinetic energy and the strain-energy, respectively:

$$\mathcal{K}=\frac{\rho}{2}\int_V v_i v_i \, dV, \qquad \mathcal{W}=\int_V\left(\mu\varepsilon_{ij}\varepsilon_{ij}+\frac{\lambda}{2}\varepsilon_{kk}\varepsilon_{nn}\right)dV. \quad (4.16)$$

$\mathcal{P}$ is the heat potential, while $\mathcal{L}$ denotes the electrodynamic potential

$$\mathcal{P}=\frac{c_\varepsilon}{2T_0}\int_V \theta^2\, dV, \qquad \mathcal{L}=\frac{\mu_0}{4\pi}\int_V h_i^2\, dV. \quad (4.17)$$

Finally, χ_θ and χ_h denote the thermal and magnetic dissipation functions respectively:

$$\chi_\theta=\frac{k}{T_0^2}\int_V(\theta_{,i})^2\, dV, \qquad \chi_h=\frac{\mu_0}{4\pi\beta}\int_V(h_{i,j})^2\, dV. \quad (4.18)$$

The first expression on the left-hand side of the equation denotes the time derivative of the kinetic energy, the strain-energy, the thermal and electrodynamical potentials. The two remaining terms determine the thermal and magnetic dissipation respectively. On the right-hand side of the equation the first two terms denote the power of external forces, the remaining terms denote the non-mechanical power. All the causes generating the motion of the body occur on the right-hand side of equation.

The above discussion can easily be generalized on more complicated interactions of electromagnetic and elastic fields in elastic dielectrics, piezoelectrics, magnetizable elastic solids, etc. [31–33].

The above presentation of coupled fields can be enlarged on bodies with microstructure, the Cosserat media, micromorphic media, etc.

Below we cite the system of differential equations of thermoelasticity for a micropolar media [34]

$$(\mu+\alpha)\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\varphi}+\mathbf{X}=\rho\ddot{\mathbf{u}}+\gamma\,\mathrm{grad}\,\theta \quad (4.19)$$

$$((\gamma+\varepsilon)\nabla^2-4\alpha)\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}+2\alpha\,\mathrm{rot}\,\mathbf{u}+\mathbf{Y}=\mathcal{J}\ddot{\boldsymbol{\varphi}} \quad (4.20)$$

$$k\,\nabla^2\theta-c_\varepsilon\dot{\theta}-\eta\,\mathrm{div}\,\dot{\mathbf{u}}=-W. \quad (4.21)$$

The motion of a solid body in the micropolar theory of elasticity is described by the displacement vector $\mathbf{u}$, and independent of it by the rotation vector $\boldsymbol{\varphi}$.

The quantities μ, λ, α, β, γ and ε entering equations (4.19) and (4.20) denote the material constants.

As I have mentioned already the coupling of various fields has become a main trend of development. It concerns not only the disturbance of the elastic field but also the coupling of the general mechanics of continuous media with other fields. A number of papers show this trend, among them two recently published by Collet & Maugin [35,36]. Their general theory of continuous media with interactions comprises as particular cases magneto-hydrodynamics and electro-hydrodynamics, as well as the theory of elastic dielectrics and magnetizable solids.

The mechanicians are first of all interested in the influence of fields on the state of stresses and strains. They investigate the conditions for which the non-mechanical interactions lead to significant, sometimes dominating, stresses. Such is the case with elevated temperatures, strong magnetic fields, strong radiation, etc.

The coupled field equations of the theory of elasticity lead to more complicated types of differential equation, and frequently to new types of mathematical physics equations. Here I see the further field of collaboration between mathematicians and mechanicians.

References

1 Duhamel, J. H. C. Second mémoire sur les phénomenes thermomécaniques. *J. de l'Ecole. Poltychn.* **15,** 1–15, 1837.

2 Voigt, W. *Lehrbuch der Kristallphysik.* Teubner, 1910.

3 Jeffreys, H. The thermodynamics of an elastic solid. *Proc. Camb. Phil. Soc.* **26,** 101–6, 1930.

4 Biot, M. A. Thermoelasticity and irreversible thermodynamics. *J. appl. Phys.* **27,** 240–53, 1956.

5 Groot, S. R. de. *Thermodynamics of Irreversible Processes.* Amsterdam, 1952.

6 Zorski, H. Singular solutions for the thermoelastic media. *Bull. Acad. Pol. Sci., Sér. Sci. Tech.* **6,** 331–9, 1958.

7 Sternberg, E. On the integration of the equation of motion in the classical theory of elasticity. *Arch. Ratl. Mech. Anal.* **6,** 37–50, 1960.

8 Nowacki, W. Some dynamic problems in thermoelasticity. *Arch. Mech. Stos.* **11,** 259–83, 1959.

9 Podstrichač, Y. S. General solution of the non-steady thermoelastic problem. *Pryklad. Mekh.* **6,** 215–19, 1960. [*in Ukranian*].

10 Kaliski, S. Some boundary value problems of the dynamical theory of elasticity. *Biul.* Wojskova Akademia Techniczna, Warszawa, 1957. [In Polish].

11 Rüdiger, D. Bemerkung zur Integration der thermo-elastischen Grundgleichungen. *Österr. Ing. Arch.* **18,** 1–2, 1964.

12 Nowacki, W. Quelques théorems de la thermoélasticité. (In Revue Roumain des Sciences Techniques.) *Acad. Rep. Soc. Roum.* **11,** 1173–83, 1966.

13 Ionescu-Cazimir, V. Problem of linear coupled thermoelasticity—I. *Bull. Acad. Pol. Sci., Sér. Sci. Tech.* **12,** 473–83, 1964.

14 Graffi, D. Sui teoremi di reciprocita nei fenomeni non stazionari. *Atti Acad. Sci. Bologna* **10,** Ser. 11, 33–44, 1963.

15 Maysel, V. M. *Temperature Problems of the Theory of Elasticity*. Kiev, 1951. [In Russian].

16 Weiner, J. H. A uniqueness theorem for the coupled thermoelastic problem. *Q. appl. Math.* **15,** 102–5, 1957.

17 Knops, R. J. & Payne, L. E. On uniqueness and continuous dependence in dynamical problems of linear thermoelasticity. *Int. J. Sols Structs* **6,** 1173, 1970.

18 Brun, L. Sur l'unicité et thermoelasticité dynamique et diverses expressions analogues à la formule de Clapeyron. *Compt. rend. Acad. Sci. Paris* **261,** 2584–7, 1965.

19 Dafermos, C. M. On the existence and asymptotic stability of solutions to the equations of linear thermoelasticity. *Arch. Ratl. Mech. Anal.* **29,** 241–71, 1968.

20 Kupradse, V. D., Gegelia, T. G., Bachelischvili, M. O. & Burchuladse, T. W. *Three-dimensional Problems of the Mathematical Theory of Elasticity*. Tbilisi, 1968. [In Russian].

21 Cattaneo, C. Sulla conduzione del calore. *Atti sem. Mat. Fis. Univ. Modena*, vol. 3, 1948.

22 Gurtin, M. E. & Pipkin, A. C. A general theory of heat conduction with finite wave speeds. *Arch. Ratl. Mech. Anal.* **31,** 113–26, 1968.

23 Kaliski, S. Wave equation of heat conduction. *Bull. Acad. Pol. Sci., Ser. Sci. Tech.* **13,** 211–9, 1965.

24 Podstrichač, Y. S. Differential equations of the problem of thermodiffusion in isotropic deformable solids. *Dokl. Akad. Nauk USSR* **2,** 169–71, 1961. [In Ukrainian].

25 Nowacki, W. Dynamic problems of thermodiffusion in elastic solids. *Proc. Vibr. Probl.* **15,** 105–28, 1974.

26 Fichera, G. Uniqueness, existence and estimate of the solution in the dynamical problem of thermodiffusion in an elastic solid. *Arch. Mech.* **26,** 903–20, 1974.

27 Kaliski, S. & Petykiewicz, J. Equations of motion coupled with the field of temperature in the magnetic field, involving mechanical and electro-magnetic relaxation for anisotropic bodies. *Proc. Vibr. Probl.* 1–12, 1959.

28 Kaliski, S. & Nowacki, W. Thermal excitations in coupled fields. In *Progress in Thermoelasticity* (VIIIth European Mechanics Colloquium). Państwowe Wydawnictwo Naukowe, Warsaw, 1969.

29 Dunkin, J. W. & Eringen, A. C. On the propagation of waves in an electromagnetic elastic solid. *Int. Jl. Engng Sci.* 461–95, 1963.

30 Nowacki, W. Problem of linear coupled magneto-thermoelasticity—I. *Bull. Acad. Pol. Sci., Sér. Sci. Tech.* **13,** 235–40, 1965.

31 Toupin, R. A. A dynamical theory of elastic dielectrics. *Int. Jl. Engng Sci.* **1,** 101–26, 1963.

32 Hutter, K. & Pao, Vih-Hsing. A dynamic theory of magnetizable elastic solids with thermal and electrical conduction. *J. Elasticity* **4,** 89–114, 1974.

33 Ven, A. A. F. van de. Interactions of electromagnetic and elastic fields in solids (Doctoral thesis). Valkenswaard, Netherlands, 1975.

34 Nowacki, W. *Theory of Micropolar Elasticity*. J. Springer, Vienna, 1970.

35 Collet, B. & Maugin, G. Sur l'électrodynamique des milieux continus avec interactions. *Compt. rend. Acad. Sci. Paris* (B) **279,** 379–82, 1974.

36 Maugin, G. & Collet, B. Thermodynamic des milieux continus électromagnétique avec interactions. *Compt. rend. Acad. Sci. Paris* (B) **279,** 439–42, 1974.

Professor W. Nowacki,
University of Warsaw, Department of Mathematics,
Institute of Mechanics,
P.K.i N., p. 9,
Warsaw,
Poland.