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TRIANGLE CENTERS AS FUNCTIONS

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ABSTRACT. We consider a kind of problem that appears to be new to Euclidean geometry, since it depends on an understanding of a *point* as a *function* rather than a position in a two-dimensional plane. Certain special points we call *centers*, including the centroid, incenter, circumcenter, and orthocenter. For example, the centroid, as a function of the class of triangles with sidelengths in the ratio $a_1 : a_2 : a_3$, is given by the formula $1/a_1 : 1/a_2 : 1/a_3$. The kind of problem introduced here leads to functional equations whose solutions are centers.

1. Introduction. A triangle $\Delta A_1A_2A_3$ with respective sidelengths a_1, a_2, a_3 and angles $\alpha_1, \alpha_2, \alpha_3$ (as in Figure 1) is often studied by means of homogeneous coordinates, as introduced by Möbius [**6**]; for a historical account, see Boyer [**1**]. In many discussions of triangles, homogeneous *barycentric* coordinates are preferred, but here we shall use homogeneous *trilinear* coordinates instead. The main reason for this choice is that our results depend on a formula for the distance between two points, and this formula (4a) is much shorter in trilinears than in barycentrics. Another reason is that a single reference (Carr [**2**]) gives many useful formulas in terms of trilinears, whereas no comparable reference seems to exist for barycentric formulas. Typical representations in trilinears, written as $x_1 : x_2 : x_3$ and defined in Section 2, are the following:

$$\begin{aligned} &\text{centroid} && x_1 : x_2 : x_3 = 1/a_1 : 1/a_2 : 1/a_3 \\ &\text{circumcenter} && x_1 : x_2 : x_3 = a_1(a_2^2 + a_3^2 - a_1^2) : a_2(a_3^2 + a_1^2 - a_2^2) : \\ & && a_3(a_1^2 + a_2^2 - a_3^2) = \cos\alpha_1 : \cos\alpha_2 : \cos\alpha_3 \\ &\text{circumcircle} && a_1/x_1 + a_2/x_2 + a_3/x_3 = 0 \\ &\text{Euler line} && x_1 \sin 2\alpha_1 \sin(\alpha_2 - \alpha_3) + x_2 \sin 2\alpha_2 \sin(\alpha_3 - \alpha_1) \\ & && \quad + x_3 \sin 2\alpha_3 \sin(\alpha_1 - \alpha_2) = 0 \end{aligned}$$

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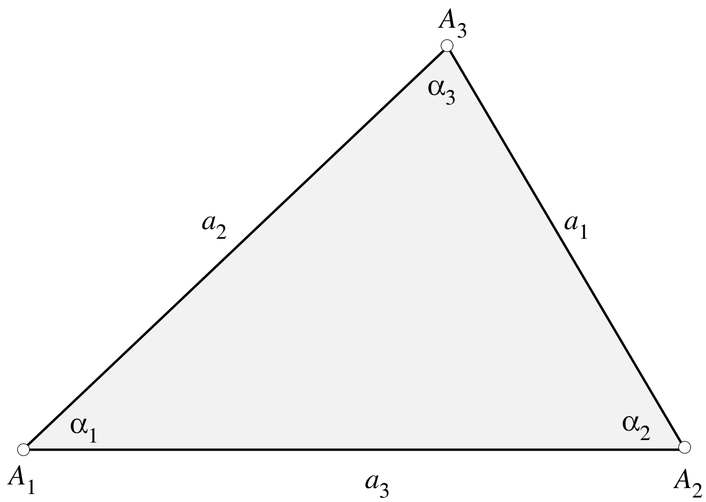

FIGURE 1. Triangle $A_1A_2A_3$ with sidelengths a_1, a_2, a_3 and angles $\alpha_1, \alpha_2, \alpha_3$.

Collinearity, perspectivity, and other relations are easily proved using trilinears by showing that appropriate determinants vanish for all a_1, a_2, a_3. This algebraic approach suggests that (a_1, a_2, a_3), (or, more precisely, the *similarity class*, $a_1 : a_2 : a_3$, of triangles with given sidelength-ratios $a_1 : a_2$ and $a_2 : a_3$), be treated as a *variable* and that geometric objects be treated as functions of $a_1 : a_2 : a_3$. For example, we *define* centroid as the function whose domain depends on the set of all triangles (the set $\mathbf{T}$ in Section 2) and whose value at any given $\Delta A_1A_2A_3$ is the point $1/a_1 : 1/a_2 : 1/a_3$. Now, there would seem to be little point in regarding centroid and other geometric objects as functions, *unless*

(1) among such objects, there exist relationships that can be understood only in terms of a functional meaning of point.

It is the purpose of this paper to present such relationships, as typified by the following problem:

Problem $\mathbf{X_1X_2X_3}$. *For any center* $\mathbf{X}$ *of a* (*variable*) *triangle* $A_1B_1C_1$*, let* X, X_1, X_2, X_3 *be the values of* $\mathbf{X}$ *in the triangles* $\Delta A_1A_2A_3$*,* ΔXA_2A_3*,* ΔA_1XA_3*,* ΔA_1A_2X*, respectively. For what* *choices of* $\mathbf{X}$ *is* $\Delta X_1X_2X_3$ *perspective with* $\Delta A_1A_2A_3$ *in the sense that the lines* A_1X_1, A_2X_2, A_3X_3 *concur in a point*?

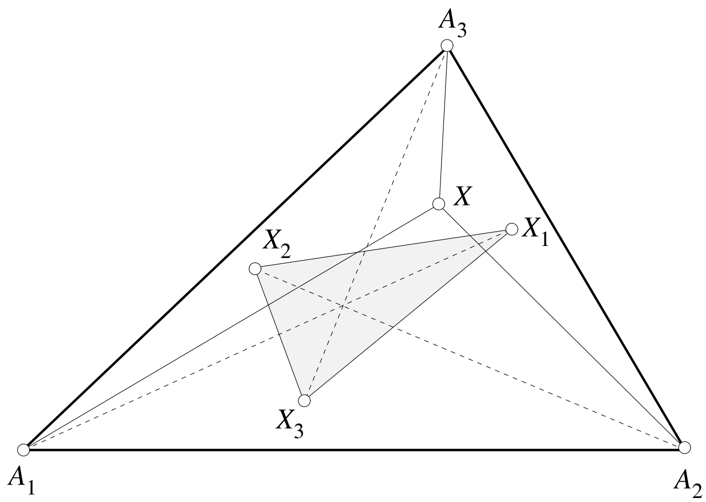

FIGURE 2. Solutions $\mathbf{X}$ of Problem $X_1X_2X_3$ include centroid, circumcenter, and orthocenter, but not *all* the centers on the Euler line (See Section 5).

To see the significance of Problem $X_1X_2X_3$, or more properly, the significance of a large class of problems which it represents, consider the fact that the geometry of special points in the plane of a triangle $\Delta A_1A_2A_3$ consists largely of theorems of the form "special point X has property $\mathcal{P}$." For example, the centroid $\mathbf{G}$ of $\Delta A_1A_2A_3$ has the following easily proved property: the centroids of the three triangles $\mathbf{G}A_2A_3$, $A_1\mathbf{G}A_3$, $A_1A_2\mathbf{G}$ form a triangle that is perspective with $\Delta A_1A_2A_3$. Now, what *other* "points" have this property?—or better yet: what are necessary and sufficient conditions for a "point" $\mathbf{X}$ to have this property? In order to answer this question, one must understand the notion of "point" not as a single location but, instead, as a set of rules or a formula that specifies a location. That is to say, a "point" must be understood as a *function*.

The meaning of Problem $X_1X_2X_3$ may be further clarified by an attempt to state it *without* regarding $\mathbf{X}$ as a function, like this:

Theorem $\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$. *Let $\mathfrak{X}$ be a point inside a triangle $\mathcal{A} = A_1A_2A_3$. Let $\mathcal{B}_i$ be the triangle obtained by replacing vertex A_i by $\mathfrak{X}$ but keeping the other two vertices fixed. Let $\mathfrak{X}_i$ be similarly placed in $\mathcal{B}_i$ as $\mathfrak{X}$ is in $\mathcal{A}$; that is, $\mathfrak{X}_i$ subdivides $\mathcal{B}_i$ into three triangles, the ratios of whose areas are respectively proportional to the ratios of areas of triangles into which $\mathfrak{X}$ subdivides $\mathcal{A}$. Then $\Delta\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$ is perspective with $\mathcal{A}$, and the center of perspective is $\mathfrak{X}$.*

To see that Theorem $\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$ (which is easy to prove) is quite different from Problem $X_1X_2X_3$, consider any point inside $\mathcal{A}$ that does not solve Problem $X_1X_2X_3$, such as the Fermat point. The Fermat point of $\mathcal{B}_i$ subdivides $\mathcal{B}_i$ into triangles whose areas are *not* proportional to those into which the Fermat point of $\mathcal{A}$ subdivides $\mathcal{A}$. Thus, the point X of Problem $X_1X_2X_3$ differs from the point $\mathfrak{X}$ of Theorem $\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$.

For another example, consider the incenter, which *does* happen to solve Problem $X_1X_2X_3$, but alas: $\Delta X_1X_2X_3 \neq \Delta\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}_3$, and also, the two centers of perspective differ.

2. Definitions: Point and Center. Following [**3**] and [**4**], we represent the set of all triangles as

$$\mathbf{T} = \{(a_1, a_2, a_3) : 0 < a_1 < a_2+a_3, 0 < a_2 < a_3+a_1, 0 < a_3 < a_1+a_2\}$$

and let

$$\sqrt{\mathcal{P}} = (1/4)\sqrt{(a_1+a_2+a_3)(a_2+a_3-a_1)(a_3+a_1-a_2)(a_1+a_2-a_3)},$$

this being the area of the triangle $A_1A_2A_3$ having sidelengths a_1, a_2, a_3. Let $(\mathbf{R}, +, \cdot)$ be the ring of polynomial functions in $a_1, a_2, a_3, \sqrt{\mathcal{P}}$ over the real number field, and let $(\mathbf{F}, +, \cdot)$ be the quotient field of $(\mathbf{R}, +, \cdot)$. A *point*, P, is an equivalence class of ordered triples (f_1, f_2, f_3) of functions f_i in $\mathbf{F}$, at least one of which is not the zero function, where two such triples (f_1, f_2, f_3) and (g_1, g_2, g_3) are *equivalent* if the following two conditions hold:

$$g_i = 0 \quad \text{iff } f_i = 0 \quad \text{for } i = 1, 2, 3; \quad \text{and } f_1/g_1 = g_2/g_2 = f_3/g_3$$

on all of $\mathbf{T}$ except the zero-set of $g_1g_2g_3$. Note that P has infinitely many representatives (f, g, h) in $\mathbf{R}^3$. For any such (f, g, h) in $\mathbf{F}^3$, we write P with colons instead of commas, like this:

$$P = f(a_1, a_2, a_3) : g(a_1, a_2, a_3) : h(a_1, a_2, a_3).$$

Thus, for example, $\sin\alpha_1 : \sin\alpha_2 : \sin\alpha_3$ and $a_1 : a_2 : a_3$ are identical, whereas $(\sin\alpha_1, \sin\alpha_2, \sin\alpha_3)$ and (a_1, a_2, a_3) are distinct, in the same way that $3/6 = 1/2$ even though $(3, 6) \neq (1, 2)$.)

The expression on the right-hand side of the above equation will be called *trilinears* for P. This is short for *homogeneous trilinear coordinates*, namely, any triple x_1, x_2, x_3 of numbers proportional to the directed distances from P to the sides A_2A_3, A_3A_1, A_1A_2, respectively, of the refernece triangle $\Delta A_1A_2A_3$. The *actual trilinear distances* are kx_1, kx_2, kx_3, where $k = 2\sqrt{\mathcal{P}}/(a_1x_1 + a_2x_2 + a_3x_3)$.

A point $\hat{f}(x_1, x_2, x_3) : \hat{g}(x_1, x_2, x_3) : \hat{h}(x_1, x_2, x_3)$ is a *center* if there exists a function $f(x_1, x_2, x_3)$ in $\mathbf{R}$ such that the following conditions hold:

(F1) $\hat{f}(x_1, x_2, x_3) : \hat{g}(x_1, x_2, x_3) : \hat{h}(x_1, x_2, x_3) = f(x_1, x_2, x_3) : f(x_2, x_3, x_1) : f(x_3, x_1, x_2)$;

(F2) $f(x_1, x_3, x_2) = f(x_1, x_2, x_3)$;

(F3) $f(x_1, x_2, x_3)$ is homogeneous in x_1, x_2, x_3; that is, $f(tx_1, tx_2, tx_3) = t^n f(x_1, x_2, x_3)$ for some nonnegative integer n and all $t > 0$.

The fact that this definition of "center" is more algebraic than geometric calls for some explanation. We shall see that each of the three algebraic properties (substitution, symmetry, homogeneity) matches a geometric property that is shared by familiar examples of triangle centers.

First, consider the geometric meaning of cyclic substitution. Long ago, we learned how to "keep doing the same thing in different places" in order to construct the centroid, incenter, or circumcenter. For the circumcenter, for example, we first draw the perpendicular bisector of side A_2A_3, and then repeat, but this time for the side A_3A_1, and then once again, for the side A_1A_2. Such "cyclic repetition" occurs in the construction of most of the named special points in the plane of a triangle.

The symmetry of $f(a_1, a_2, a_3)$ in a_2 and a_3 corresponds to interchanging the roles of A_2 and A_3 (or a_2 and a_3; or α_2 and α_3) when carrying out a construction relative to the vertices in the order A_1, A_2, A_3. For example, when drawing the perpendicular bisector of the segment from A_2 to A_3, we get the same thing if we go from A_3 to A_2.

Finally, homogeneity ensures that similar triangles have similarly situated centers. For, if a triangle $T = B_1B_2B_3$ is similar to the reference triangle $A_1A_2A_3$, then, for some $t > 0$, the sidelengths of T are ta_1, ta_2, ta_3. Suppose X is a center, and let f be a center function for X. Then the value of X in $A_1A_2A_3$ is

$$f(x_1, x_2, x_3) : f(x_2, x_3, x_1) : f(x_3, x_1, x_2),$$

and the value of X in T is

$$f(tx_1, tx_2, tx_3) : f(tx_2, tx_3, tx_1) : f(tx_3, tx_1, tx_2),$$

which by homogeneity equals $f(x_1, x_2, x_3) : f(x_2, x_3, x_1) : f(x_3, x_1, x_2)$. That is, the *ratios* of distances from X to the sidelines remain unchanged.

3. Derived triangles and coordinate transformations. Any three points $P_i = f_i(a_1, a_2, a_3) : g_i(a_1, a_2, a_3) : h_i(a_1, a_2, a_3)$, $i = 1, 2, 3$, determine a triangle with vertices P_1, P_2, P_3. The triangle can be represented as a matrix:

$$(2) \qquad M = \begin{pmatrix} f_1(a_1, a_2, a_3) & g_1(a_1, a_2, a_3) & h_1(a_1, a_2, a_3) \\ f_2(a_1, a_2, a_3) & g_2(a_1, a_2, a_3) & h_2(a_1, a_2, a_3) \\ f_3(a_1, a_2, a_3) & g_3(a_1, a_2, a_3) & h_3(a_1, a_2, a_3) \end{pmatrix}.$$

Let F_i, G_i, H_i denote the functions satisfying

$$(3) \quad M^{-1} = \frac{1}{|M|} \begin{pmatrix} F_1(a_1, a_2, a_3) & G_1(a_1, a_2, a_3) & H_1(a_1, a_2, a_3) \\ F_2(a_1, a_2, a_3) & G_2(a_1, a_2, a_3) & H_2(a_1, a_2, a_3) \\ F_3(a_1, a_2, a_3) & G_3(a_1, a_2, a_3) & H_3(a_1, a_2, a_3) \end{pmatrix}.$$

It will be convenient to refer to a matrix such as M as a *matrix-triangle*. Note that every triangle is represented by many matrix-triangles, since the rows, as trilinears, are determined only up to multiplication by an element of $\mathbf{F}$.

Theorem 1. *Suppose M and M^{-1} are as in* (2) *and* (3)*, and x_1, x_2, x_3 are actual trilinear distances with respect to $\Delta A_1A_2A_3$ for* *a point X. Let x_1', x_2', x_3' be actual trilinear distances with respect to M for X. Then*

$$(x_1, x_2, x_3) = (x_1', x_2', x_3')DM/|M|, \tag{4}$$

where

$$D = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix},$$

where

$$\begin{aligned} D_1 &= \sqrt{F_1^2+F_2^2+F_3^2-2F_2F_3\cos\alpha_1-2F_3F_1\cos\alpha_2-2F_1F_2\cos\alpha_3} \\ D_2 &= \sqrt{G_1^2+G_2^2+G_3^2-2G_2G_3\cos\alpha_1-2G_3G_1\cos\alpha_2-2G_1G_2\cos\alpha_3} \\ D_3 &= \sqrt{H_1^2+H_2^2+H_3^2-2H_2H_3\cos\alpha_1-2H_3H_1\cos\alpha_2-2H_1H_2\cos\alpha_3}. \end{aligned}$$

Proof. We use the notation $t_1 : t_2 : t_3$ for a variable point in trilinears. An equation for A_2A_3 is then

$$\begin{vmatrix} g_2 & h_2 \\ g_3 & h_3 \end{vmatrix} t_1 + \begin{vmatrix} h_2 & f_2 \\ h_3 & f_3 \end{vmatrix} t_2 + \begin{vmatrix} f_2 & g_2 \\ f_3 & g_3 \end{vmatrix} t_3 = 0,$$

which is $F_1t_1 + F_2t_2 + F_3t_3 = 0$. The directed distance from X to this line is

$$x_1' = \frac{x_1F_1 + x_2F_2 + x_3F_3}{D_1} \tag{4a}$$

(e.g., Carr [**2**, Article 4624]) and similarly for x_2' and x_3'. Consequently,

$$\begin{aligned} (x_1', x_2', x_3') &= (x_1, x_2, x_3) \begin{pmatrix} F_1 & G_1 & H_1 \\ F_2 & G_2 & H_2 \\ F_3 & G_3 & H_3 \end{pmatrix} \begin{pmatrix} 1/D_1 & 0 & 0 \\ 0 & 1/D_2 & 0 \\ 0 & 0 & 1/D_3 \end{pmatrix} \\ &= (x_1, x_2, x_3)|M|M^{-1}D^{-1}, \end{aligned}$$

and (4) follows. $\square$

Corollary 1. *Let M be a matrix triangle as in* (2), *and let $x_1 : x_2 : x_3$ be coordinates* (*not necessarily actual trilinear distances*) *with respect to $\Delta A_1A_2A_3$ for a point X. Let $x'_1 : x'_2 : x'_3$ be coordinates with respect to the triangle M for X. Then*

$$x_1 : x_2 : x_3 = (x'_1 : x'_2 : x'_3)DM. \tag{4'}$$

Before continuing with our primary concern, expressed in (1), we note that Corollary 1 yields useful special cases that are difficult to find in the literature. These involve five much studied triangles: *medial* (the vertices are the points where the medians of $\Delta A_1A_2A_3$ meet the sides of $\Delta A_1A_2A_3$), *orthic* (where the altitudes meet the sides), *anticomplementary* (the triangle whose medial is $\Delta A_1A_2A_3$), the *tangential* formed by the lines tangent to the circumcircle of $\Delta A_1A_2A_3$ at A_1, A_2, A_3, and the *tritangent* (the vertices are the excenters of $\Delta A_1A_2A_3$); its orthic is $\Delta A_1A_2A_3$).

Example 1.1. If $x_1 : x_2 : x_3$ are trilinears for a point P with respect to $\Delta A_1A_2A_3$ and $x'_1 : x'_2 : x'_3$ are trilinears for P with respect to the *medial* triangle M given by

$$M = \begin{pmatrix} 0 & 1/a_2 & 1/a_3 \\ 1/a_1 & 0 & 1/a_3 \\ 0 & 1/a_2 & 1/a_3 \end{pmatrix},$$

then (4′) leads to

$$x'_1 : x'_2 : x'_3 = (-a_1x_1+a_2x_2+a_3x_3)/a_1 : (-a_2x_2+a_3x_3+a_1x_1)/a_2 : (-a_3x_3+a_1x_1+a_2x_2)/a_3,$$

and

$$x_1 : x_2 : x_3 = (a_2x'_2+a_3x'_3)/a_1 : (a_3(x'_3+a_1x'_1)/a_2 : (a_1x'_1 + a_2x'_2)/a_3.$$

Example 1.2. If $x_1 : x_2 : x_3$ are trilinears for a point P with respect to $\Delta A_1A_2A_3$ and $x'_1 : x'_2 : x'_3$ are trilinears for P with resepct to the *orthic* triangle M given by

$$M = \begin{pmatrix} 0 & \sec\alpha_2 & \sec\alpha_3 \\ \sec\alpha_1 & 0 & \sec\alpha_3 \\ \sec\alpha_1 & \sec\alpha_2 & 0 \end{pmatrix},$$

then

$$\begin{aligned} x_1' : x_2' : x_3' = &-x_1 \cos\alpha_1 + x_2 \cos\alpha_2 + x_3 \cos\alpha_3 : \\ &- x_2 \cos\alpha_2 + x_3 \cos\alpha_3 + x_1 \cos\alpha_1 : \\ &- x_3 \cos\alpha_3 + x_1 \cos\alpha_1 + x_2 \cos\alpha_2, \end{aligned}$$

and

$$x_1 : x_2 : x_3 = (x_2' + x_3') \sec\alpha_1 : (x_3' + x_1') \sec\alpha_2 : (x_1' + x_2') \sec\alpha_3.$$

Example 1.3. If $x_1 : x_2 : x_3$ are trilinears for a point P with respect to $\Delta A_1A_2A_3$ and $x_1' : x_2' : x_3'$ are trilinears for P with respect to the *anticomplementary* triangle M given by

$$M = \begin{pmatrix} -1/a_1 & 1/a_2 & 1/a_3 \\ 1/a_1 & -1/a_2 & 1/a_3 \\ 1/a_1 & 1/a_2 & -1/a_3 \end{pmatrix},$$

then

$$x_1' : x_2' : x_3' = (a_2x_2 + a_3x_3)/a_1 : (a_3x_3 + a_1x_1)/a_2 : (a_1x_1 + a_2x_2)/a_3,$$

and

$$\begin{aligned} x_1 : x_2 : x_3 = (-a_1x_1' + a_2x_2' + a_3x_3')/a_1 : (-a_2x_2' + a_3x_3' + a_1x_1')/a_2 : \\ (-a_3x_3' + a_1x_1' + a_2x_2')/a_3. \end{aligned}$$

Example 1.4. if $x_1 : x_2 : x_3$ are trilinears for a point P with respect to $\Delta A_1A_2A_3$ and $x_1' : x_2' : x_3'$ are trilinears for P with respect to the *tangential* triangle M given by

$$M = \begin{pmatrix} -a_1 & a_2 & a_3 \\ a_1 & -a_2 & a_3 \\ a_1 & a_2 & -a_3 \end{pmatrix},$$

then

$$x_1' : x_2' : x_3' = (a_3x_2+a_2x_3)/a_1 : (a_1x_3+a_3x_1)/a_2 : (a_2x_1+a_1x_2)/a_3,$$

and

$$x_1 : x_2 : x_3 = a_1(-a_1^2x_1'+a_2^2x_2'+a_3^2x_3') : a_2(-a_2^2x_2'+a_3^2x_3'+a_1^2x_1') : \\ a_3(-a_3^2x_3' + a_1^2x_1^2x_1' + a_2^2x_2').$$

Example 1.5. If $x_1 : x_2 : x_3$ are trilinears for a point P with respect to $\Delta A_1A_2A_3$ and $x_1' : x_2' : x_3'$ are trilinears for P with respect to the *tritangent* triangle M given by

$$M = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

then

$$x_1' : x_2' : x_3' = (x_2+x_3)\csc\alpha_1/2 : (x_3+x_1)\csc\alpha_2/2 : (x_1+x_2)\csc\alpha_3/2,$$

and

$$\begin{aligned} x_1 : x_2 : x_3 = &-x_1'\sin\alpha_1/2 + x_2'\sin\alpha_2/2 + x_3'\sin\alpha_3/2 : \\ &- x_2'\sin\alpha_2/2 + x_3'\sin\alpha_3/2 + x_1'\sin\alpha_1/2 : \\ &- x_3'\sin\alpha_3/2 + x_1'\sin\alpha_1/2 + x_2'\sin\alpha_2/2. \end{aligned}$$

4. Two-triangle problems. The designation *two-triangle problem* means any problem whose statement necessarily refers to two but not more than two triangles, with respect to which points, as functions, are specified. More generally, we speak of *n-triangle problems*. The solution of such a problem will often entail $n-1$ applications of formula (4).

Examples 1.1–1.5 are two-triangle problems. We shall consider here a somewhat different two-triangle problem:

Problem A_1XX_1. For any center $\mathbf{X}$ of a variable triangle $A_1A_2A_3$, let X be the value of $\mathbf{X}$ in $\Delta A_1A_2A_3$, and let X_1 be its value in ΔXA_2A_3. For what choices of $\mathbf{X}$ are A_1, X, X_1 collinear?

Theorem 4. *Let* $\mathbf{X} = x : y : z = f(a_1,a_2,a_3) : f(a_2,a_3,a_1) : f(a_3,a_1,a_2)$ *be a center. Let* X *be its value in* $\Delta A_1A_2A_3$ *and* X_1 *its value in* ΔXA_2A_3. *Let*

$$r_1 = |A_2A_3|, \qquad r_2 = |A_3X|, \qquad r_3 = |XA_2|$$

(*in* ΔXA_2A_3*, the side opposite vertex* A_3 *is* XA_2),

$$X_1 = x_1 : y_1 : z_1$$

(*with respect to* $\Delta A_1A_2A_3$),

$$X_1 = \hat{x}_1 : \hat{y}_1 : \hat{z}_1 = f(r_1,r_2,r_3) : f(r_2,r_3,r_1) : f(r_3,r_1,r_2)$$

(*with respect to* ΔXA_2A_3). *Then*

$$x_1 : y_1 : z_1 = x\hat{x}_1 : y\hat{x}_1 + s_3\hat{y}_1 : z\hat{x}_1 + s_2\hat{z}_1 \tag{5}$$

where

$$s_1 = \sqrt{y^2+z^2+2yz\cos\alpha_1}, \qquad s_2 = \sqrt{z^2+x^2+2zx\cos\alpha_2}, \qquad s_3 = \sqrt{x^2+y^2+2xy\cos\alpha_3} \tag{6}$$

are the sidelengths of the pedal triangle of X *with respect to* $\Delta A_1A_2A_3$. (*See Figure* 3.)

Proof. Equation (5) results directly from (4) using the matrix for ΔXA_2A_3 given by

$$M = \begin{pmatrix} x & y & z \\ 0 & 2\sqrt{\mathcal{P}}/a_2 & 0 \\ 0 & 0 & 2\sqrt{\mathcal{P}}/a_3 \end{pmatrix}, \qquad \text{where } \sqrt{\mathcal{P}} = \text{area of } \Delta A_1A_2A_3. \quad \square$$

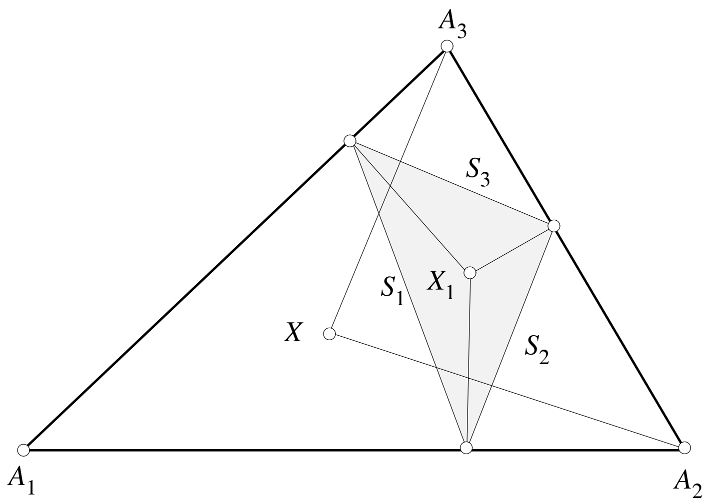

FIGURE 3. Triangles $A_1A_2A_3$, XA_2A_3 and the pedal triangle of X_1.

Corollary 4.1. X *solves Problem* A_1XX_1 *if and only if* f *solves the functional equation*

$$s_2 f(a_2, a_3, a_1) f(r_3, r_1, r_2) = s_3 f(a_3, a_1, a_2) f(r_2, r_3, r_1). \tag{7}$$

Proof. Collinearity of the points A_1, X, X_1 is equivalent (e.g., Carr [**2**, Article 4615]) to $yz_1 = zy_1$, and (7) now follows by substituting from (5). $\square$

Equation (7) is far from simple, since (e.g., Carr [**2**, Article 4602])

$$r_2^2 = \frac{a_1a_2a_3}{4\mathcal{P}}\Bigg(a_1(kf(a_1, a_2, a_3))^2 \cos\alpha_1 + a_2(kf(a_2, a_3, a_1))^2 \cos\alpha_2 + a_3\left(kf(a_3, a_1, a_2) - \frac{2\sqrt{\mathcal{P}}}{a_3}\right)^2 \cos\alpha_3\Bigg)$$

where

$$k = \frac{2\sqrt{\mathcal{P}}}{a_1 f(a_1, a_2, a_3) + a_2 f(a_2, a_3, a_1) + a_3 f(a_3, a_1, a_2)},$$

and similarly for r_3. Nevertheless, (7) is easily sampled by computer. Among the well-known triangle centers, only those two which are obvious solutions—centroid and orthocenter—have been found to satisfy (7). Are there others?

5. A four-triangle problem.

Problem $\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3$. For any center $\mathbf{X}$ of a (variable) triangle $A_1A_2A_3$, let X, X_1, X_2, X_3 be the values of $\mathbf{X}$ in the triangles $\Delta A_1A_2A_3$, ΔXA_2A_3, ΔA_1XA_3, ΔA_1A_2X, respectively. For what choices of $\mathbf{X}$ is $\Delta X_1X_2X_3$ perspective with $\Delta A_1A_2A_3$?

Theorem 5. *In the notation of Problem $X_1X_2X_3$, let $\mathbf{X} = x : y : z = f(a_1, a_2, a_3) : f(a_2, a_3, a_1) : f(a_3, a_1, a_2)$,*

$$\begin{aligned} X_1 &= x_1 : y_1 : z_1 \ (\textit{with respect to } \Delta A_1A_2A_3) \\ &= \hat{x}_1 : \hat{y}_1 : \hat{z}_1 \ (\textit{with respect to } \Delta XA_2A_3) \\ X_2 &= x_2 : y_2 : z_2 \ (\textit{with respect to } \Delta A_1A_2A_3) \\ &= \hat{x}_2 : \hat{y}_2 : \hat{z}_2 \ (\textit{with respect to } \Delta A_1XA_2) \\ X_3 &= x_3 : y_3 : z_3 \ (\textit{with respect to } \Delta A_1A_2A_3) \\ &= \hat{x}_3 : \hat{y}_3 : \hat{z}_3 \ (\textit{with respect to } \Delta XA_1A_2), \end{aligned}$$

and let s_1, s_2, s_3 be as in (6). *Then $\mathbf{X}$ solves Problem $X_1X_2X_3$ if and only if*

(8)
$$\begin{aligned} s_1x\hat{x}_1(z\hat{y}_2\hat{y}_3 - y\hat{z}_2\hat{z}_3) + s_2y\hat{y}_2(x\hat{z}_3\hat{z}_1 - z\hat{x}_3\hat{x}_1) + s_3z\hat{z}_3(y\hat{x}_1\hat{x}_2 - x\hat{y}_1\hat{y}_2) \\ + s_2s_3(y\hat{x}_2\hat{z}_3\hat{z}_1 - z\hat{x}_3\hat{y}_1\hat{y}_2) + s_3s_1(z\hat{y}_3\hat{x}_1\hat{x}_2 - x\hat{y}_1\hat{z}_2\hat{z}_3) \\ + s_1s_2(x\hat{z}_1\hat{y}_2\hat{y}_3 - y\hat{z}_2\hat{x}_3\hat{x}_1) + s_1s_2s_3(\hat{x}_2\hat{y}_3\hat{z}_1 - \hat{x}_3\hat{y}_1\hat{z}_2) = 0. \end{aligned}$$

Proof. Just as in the proof of Theorem 4, we find

$$x_1 : y_1 : z_1 = x\hat{x}_1 : y\hat{x}_1 + s_3\hat{y}_1 : z\hat{x}_1 + s_2\hat{z}_1 \tag{5}$$

$$x_2 : y_2 : z_2 = x\hat{y}_2 + s_3\hat{x}_2 : y\hat{y}_2 : z\hat{y}_2 + s_1\hat{z}_2 \tag{9}$$

$$x_3 : y_3 : z_3 = x\hat{z}_3 + s_2\hat{x}_3 : y\hat{z}_3 + s_1\hat{y}_3 : z\hat{z}_3 \tag{10}$$

Triangles $\Delta X_1X_2X_3$ and $\Delta A_1A_2A_3$ are perspective if and only if the lines A_1X_1, A_2X_2, A_3X_3 concur, and this is equivalent to

$$x_2y_3z_1 = x_3y_1z_2. \tag{11}$$

Equation (8) now follows by substituting from (5), (9), (10) into (11) and simplifying. $\square$

Corollary 5.1. *Suppose $h \in \mathbf{F}$ and h is a function of a_1 only (so that we write $h(a_1)$). Then the center $\mathbf{X} = h(a_1) : h(a_2) : h(a_3)$ solves Problem $X_1X_2X_3$. (In particular, solutions include centroid, incenter, and symmedian point, for which $h(a_1) = 1/a_1, 1, a_1$, respectively.)*

Proof. In the notation of Theorem 5,

$$\begin{aligned}
\hat{x}_1 : \hat{y}_1 : \hat{z}_1 &= h(a_1) : h(s_3) : h(s_2)\\
\hat{x}_2 : \hat{y}_2 : \hat{z}_2 &= h(s_3) : h(a_2) : h(s_1)\\
\hat{x}_3 : \hat{y}_3 : \hat{z}_3 &= h(s_2) : h(s_1) : h(a_3),
\end{aligned}$$

where s_1, s_2, s_3 are the distances from X to A_1, A_2, A_3, respectively. Substitution of these into (8) leads quickly to zero for each expression between parentheses in (8), so that $\mathbf{X}$ is a solution, by Theorem 5. $\square$

Corollary 5.2. *The following centers are solutions of Problem $X_1X_2X_3$: circumcenter, orthocenter, and center of the nine-point circle.*

Proof. Substitute into the left-hand side of (8). Simplification is best carried out by computer. $\square$

6. Thinlines. In order to account more fully for solutions to Problem $X_1X_2X_3$, it will be expedient to define lines and thinlines as follows. A *line* is the set of points (as defined in Section 2) $x_1 : x_2 : x_3$ satisfying

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix} = 0$$

for all (a_1, a_2, a_3) in $\mathbf{T}$, for some pair of points

$$(12) \qquad P = x_{11} : x_{12} : x_{13} \quad \text{and} \quad Q = x_{21} : x_{22} : x_{23}.$$

Equivalently, a line $\mathbf{L}$ consists of all points of the form

$$(13) \quad s(a_1, a_2, a_3)x_{11} + t(a_1, a_2, a_3)x_{21} : \\ s(a_1, a_2, a_3)x_{12} + t(a_1, a_2, a_3)x_{21} : s(a_1, a_2, a_3)x_{13} + t(a_1, a_2, a_3)x_{31}$$

where $s(a_1, a_2, a_3)$ and $t(a_1, a_2, a_3)$ are in $\mathbf{F}$ and are not both zero. By a *thinline*, we mean the set of points of the form (13) in the special case that $s(a_1, a_2, a_3)$ and $t(a_1, a_2, a_3)$ range only through all pairs of real numbers s and t satisfying $st \neq 0$. Note that this definition depends on the six particular functions x_{11}, x_{12}, x_{13} and x_{21}, x_{22}, x_{23}, so that appropriate notation is

$$\mathbf{L}(x_{11}, x_{12}, x_{13}; x_{21}, x_{22}, x_{23}).$$

Any thinline through P and Q we call a *PQ-thinline*. In accord with Theorem 6 below, for any P and Q, there are infinitely many PQ-thinlines. To distinguish one from another, for given functions as in (12) we call $\mathbf{L}(x_{11}, x_{12}, x_{13}; x_{21}, x_{22}, x_{23})$ *the thinline based on the functions* x_{11}, x_{12}, x_{13} *and* x_{21}, x_{22}, x_{23}). If P and Q have the form

$$f(a_1, a_2, a_3) : f(a_2, a_3, a_1) : f(a_3, a_1, a_2)$$

and

$$g(a_1, a_2, a_3) : g(a_2, a_3, a_1) : g(a_3, a_1, a_2)$$

then we speak of the thinline based on $(f(a_1, a_2, a_3), f(a_2, a_3, a_1), f(a_3, a_1, a_2))$ and $(g(a_1, a_2, a_3), g(a_2, a_3, a_1), g(a_3, a_1, a_2))$ as the *thinline based on f and g*, and we write $\mathbf{L}(f, g)$. In particular, this shorter notation applies to all pairs P, Q of centers.

Theorem 6. *Distinct PQ-thinlines* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *have only two points in common.*

Proof. Let (x_{i1}, x_{i2}, x_{i3}), for $i = 1, 2, 3, 4$, be functions for which $\mathbf{L}_1 = \mathbf{L}(x_{11}, x_{12}, x_{13}; x_{21}, x_{22}, x_{23})$ and $\mathbf{L}_2 = \mathbf{L}(x_{31}, x_{32}, x_{33}; x_{41}, x_{42}, x_{43})$. There exist functions $d = d(a_1, a_2, a_3)$ and $e = e(a_1, a_2, a_3)$ in $\mathbf{F}$ such that

$$x_{31} = dx_{11}, \qquad x_{32} = dx_{12}, \qquad x_{33} = dx_{13} \tag{14a}$$

and

$$x_{41} = ex_{21}, \qquad x_{42} = ex_{22}, \qquad x_{43} = ex_{23}. \tag{14b}$$

Clearly both P and Q lie on both $\mathbf{L}_1$ and $\mathbf{L}_2$. Suppose R is yet another point common to $\mathbf{L}_1$ and $\mathbf{L}_2$. Then

$$\begin{aligned} R &= sx_{11} + tx_{21} : sx_{12} + tx_{22} : sx_{13} + tx_{23} \\ &= udx_{11} + vex_{21} : udx_{12} + vex_{22} : udx_{13} + vex_{23} \end{aligned}$$

for some numbers s, t, u, v satisfying $stuv \neq 0$. But then

$$(sx_{11} + tx_{21})(udx_{12} + vex_{22}) = (sx_{12} + tx_{22})(udx_{11} + vex_{21}),$$

which leaves $(sve - tud)(x_{22}x_{11} - x_{12}x_{21}) = 0$. Likewise, $(sve - tud)(x_{23}x_{11} - x_{13}x_{21}) = 0$. These imply $sve = tud$, since if not then $x_{22}x_{11} = x_{12}x_{21}$ and $x_{23}x_{11} = x_{13}x_{21}$, contrary to the distinctness of $x_{11} : x_{12} : x_{13}$ and $x_{21} : x_{22} : x_{23}$. Thus $x_{41} = kdx_{21}$, $x_{42} = kdx_{22}$, $x_{43} = kdx_{23}$, where k is the real number tu/sv. These with (14a) imply $\mathbf{L}_2 = \mathbf{L}_1$, contrary to the hypothesis. Therefore, no such R exists. □

Corollary 5.3. *Every point on the thinline* $\mathbf{L}(\cos\alpha_1, \cos\alpha_2 \cos\alpha_3)$ *solves Problem* $X_1X_2X_3$.

Proof. Substitute into the left-hand side of (8) and simplify via a computer algebra system. □

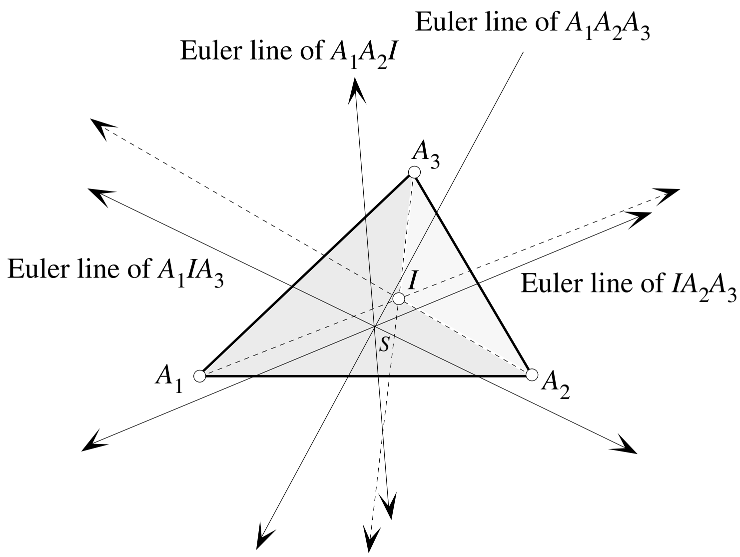

FIGURE 4. Four Euler lines concur in the Schiffler point, S.

Conjecture. *The only points on the Euler line that solve Problem $X_1X_2X_3$ are the points*

$$\mathbf{E}(s,t) = s\cos\alpha_1 + t\cos\alpha_2\cos\alpha_3 : s\cos\alpha_2 + t\cos\alpha_3\cos\alpha_1 : \\ s\cos\alpha_3 + t\cos\alpha_1\cos\alpha_2.$$

Concerning this conjecture we note that the thinline $\mathbf{E}(s,t)$ includes the circumcenter, orthocenter, centroid, center of the nine-point circle, the de Longchamps point, and a point on the line at infinity, as these correspond respectively to $(s,t) = (1,0), (0,1), (1,1), (1,2), (1,-1), (1, -2)$. (In case coordinates for the de Longchamps point A have not appeared earlier in the literature, we note that Example 1.3 applies, since A is the orthocenter of the anticomplementary triangle.)

7. Conclusion. Problem $X_1X_2X_3$ typifies a wide range of problems involving derived triangles of a variable reference triangle. These problems are new in the sense that their statements and solutions depend on a functional meaning of triangle center and on the notion of thinlines. Many problems of the traditional sort generalize naturally to the new sort of problem. We conclude with an example:

Schiffler Problem (original). *Let I be the incenter of a triangle $A_1A_2A_3$. The Euler lines of the triangles $A_1A_2A_3$, IA_2A_3, A_2IA_3, A_1A_2I concur in a point.* (*For a solution, see* [**7**]).

Schiffler Problem (new). *For any centers, $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, let X, X_1, X_2, X_3 be the values of $\mathbf{X}$ relative to the triangles $A_1A_2A_3$, XA_2A_3, A_1XA_3, A_1A_2X, and similarly for Y, Y_1, Y_2, Y_3 and Z, Z_1, Z_2, Z_3. For what choices of $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ do the three lines Y_iZ_i* (*or these together with YZ*) *concur?*

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