Triangle Contact Representations and Duality*

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Abstract. A contact representation by triangles of a graph is a set of triangles in the plane such that two triangles intersect on at most one point, each triangle represents a vertex of the graph and two triangles intersects if and only if their corresponding vertices are adjacent. de Fraysseix, Ossona de Mendez and Rosenstiehl proved that every planar graph admits a contact representation by triangles. We strengthen this in terms of a simultaneous contact representation by triangles of a planar map and of its dual.

A primal-dual contact representation by triangles of a planar map is a contact representation by triangles of the primal and a contact representation by triangles of the dual such that for every edge uv, bordering faces f and g, the intersection between the triangles corresponding to u and v is the same point as the intersection between the triangles corresponding to f and g. We prove that every 3-connected planar map admits a primal-dual contact representation by triangles. Moreover, the interiors of the triangles form a tiling of the triangle corresponding to the outer face and each contact point is a node of exactly three triangles. Then we show that these representations are in one-to-one correspondence with generalized Schnyder woods defined by Felsner for 3-connected planar maps.

1 Introduction

A *contact system* is a set of curves (closed or not) in the plane such that two curves cannot cross but may intersect tangentially. A *contact point* of a contact system is a point that is in the intersection of at least two curves. A *contact representation* of a graph $G = (V, E)$ is a contact system $\mathcal{C} = \{c(v) : v \in V\}$, such that two curves intersect if and only if their corresponding vertices are adjacent.

The Circle Packing Theorem of Koebe [14] states that every planar graph admits a contact representation by circles.

Theorem 1 (Koebe [14]). *Every planar graph admits a contact representation by circles.*

Theorem 1 implies that every planar graph has a contact representation by convex polygons, and de Fraysseix et al. [8] strengthened this by showing that every planar graph admits a contact representation by triangles. A contact representation by triangles is *strict* if each contact point is a node of exactly one triangle. de Fraysseix et al. [8] proved the following:

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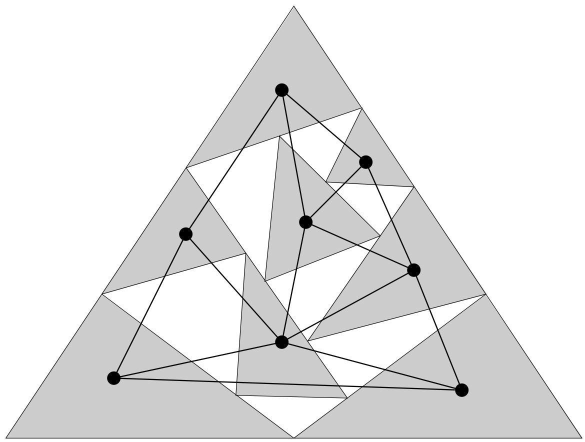

Fig. 1. A strict tiling primal-dual contact representation by triangles

Theorem 2 (de Fraysseix et al. [8]). *Every planar graph admits a strict contact representation by triangles.*

Moreover, de Fraysseix et al. [8] proved that strict contact representations by triangles of a planar triangulation are in one-to-one correspondence with its Schnyder woods defined by Schnyder [17].

Andre'ev [1] strengthen Theorem 1 in terms of a simultaneous contact representation of a planar map and of its dual. The *dual* of a planar map $G = (V, E)$ is noted $G^* = (V^*, E^*)$. A *primal-dual contact representation* $(\mathcal{V}, \mathcal{F})$ of a planar map G is two contact systems $\mathcal{V} = \{c(v) : v \in V\}$ and $\mathcal{F} = \{c(f) : f \in V^*\}$, such that $\mathcal{V}$ is a contact representation of G, and $\mathcal{F}$ is a contact representation of G^*, and for every edge uv, bordering faces f and g, the intersection between $c(u)$ and $c(v)$ is the same point as the intersection between $c(f)$ and $c(g)$. A *contact point* of a primal-dual contact representation is a contact point of V or a contact point of F. Andre'ev [1] proved the following:

Theorem 3 (Andre'ev [1]). *Every 3-connected planar map admits a primal-dual contact representation by circles.*

Our main result is an analogous strengthening of Theorem 2. We say that a primal-dual contact representation by triangles is *tiling* if the triangles corresponding to vertices and those corresponding to bounded faces form a tiling of the triangle corresponding to the outer face (see Figure 1). We say that a primal-dual contact representation by triangles is *strict* if each contact point is a node of exactly three triangles corresponding to vertices or faces (see Figure 1). We prove the following :

Theorem 4. *Every 3-connected planar map admits a strict tiling primal-dual contact representation by triangles.*

In [12], Gansner et al. study representation of graphs by triangles where two vertices are adjacent if and only if their corresponding triangles are intersecting on a side (touching representation by triangles). Theorem 4 shows that for 3-connected planar graphs, the incidence graph between vertices and faces admits a touching representation by triangles.

The tools needed to prove Theorem 4 are introduced in section 2. In section 2.1, we present a result of de Fraysseix et al. [10] concerning the stretchability of a contact system of arcs. In section 2.2, we define (generalized) Schnyder woods and present related results obtained by Felsner [4]. In Section 3, we define a contact system of arc, based on a Schnyder wood, and show that this system of arc is stretchable. When stretched, this system gives the strict tiling primal-dual contact representation by triangles. In Section 4, we show that strict tiling primal-dual contact representations by triangles of a planar map are in one-to-one correspondence with its Schnyder woods. In Section 5, we define the class of planar maps admitting a Schnyder wood and thus a strict tiling primal-dual contact representation by triangles. In Section 6, we discuss possible improvements of Theorem 4.

2 Tools

2.1 Stretchability

An *arc* is a non-closed curve. An *internal point* of an arc is a point of the arc distinct from its extremities. A contact system of arcs is *strict* if each contact points is internal to at most one arc. A contact system of arcs is *stretchable* if there exists a homeomorphism which transforms it into a contact system whose arcs are straight line segments. An *extremal point* of a contact system of arcs is a point on the outer-boundary of the system and which is internal to no arc.

We define in Section 3 a contact system of arcs such that when stretched it gives a strict tiling primal-dual contact representation by triangles. To prove that our contact system of arcs is stretchable, we need the following theorem of de Fraysseix et al. [10].

Theorem 5 (de Fraysseix et al. [10]). *A strict contact system of arcs is stretchable if and only if each subsystem of cardinality at least two has at least three extremal points.*

2.2 Schnyder Woods

The contact system of arcs defined in Section 3 is constructed from a Schnyder wood.

Schnyder woods where introduced by Schnyder [17] and then generalized by Felsner [4]. Here we use the definition from [4] except if explicitly mentioned. We refer to *classic Schnyder woods* defined by Schnyder [17] or *generalized Schnyder woods* defined by Felsner [4] when there is a discussion comparing both.

Given a planar map G. Let x_0, x_1, x_2 be three distinct vertices occurring in clockwise order on the outer face of G. The *suspension* G^σ is obtained by attaching a half-edge that reaches into the outer face to each of these special vertices. A *Schnyder wood* rooted at x_0, x_1, x_2 is an orientation and coloring of the edges of G^σ with the colors 0, 1, 2 satisfying the following rules (see Figures 2 and 3):

- Every edge e is oriented in one direction or in two opposite directions. We will respectively say that e is uni- or bi-directed. The directions of edges are colored such that if e is bi-directed the two directions have distinct colors.
- The half-edge at x_i is directed outwards and colored i.
- Every vertex v has out-degree one in each color. The edges $e_0(v)$, $e_1(v)$, $e_2(v)$ leaving v in colors 0, 1, 2, respectively, occur in clockwise order. Each edge entering v in color i enters v in the clockwise sector from $e_{i+1}(v)$ to $e_{i-1}(v)$ (where $i+1$ and $i-1$ are understood modulo 3).
- There is no interior face the boundary of which is a directed monochromatic cycle.

The difference with the original definition of Schnyder [17] it that edges can be oriented in two opposite directions.

A Schnyder wood of G^σ defines a labelling of the angles of G^σ where every angle in the clockwise sector from $e_{i+1}(v)$ to $e_{i-1}(v)$ is labeled i.

A *Schnyder angle labellings* of G^σ is a labeling of the angles of G^σ with the labels 0, 1, 2 satisfying the following rules (see Figures 2 and 3):

- The two angles at the half-edge of the special vertex x_i have labels $i+1$ and $i-1$ in clockwise order.
- Rule of vertices: The labels of the angles at each vertex form, in clockwise order, a nonempty interval of 0's, a nonempty interval of 1's and a nonempty interval of 2's.
- Rule of faces: The labels of the angles at each interior face form, in clockwise order, a nonempty interval of 0's, a nonempty interval of 1's and a nonempty interval of 2's. At the outer face the same is true in counterclockwise order.

Felsner [5] proved the following correspondence:

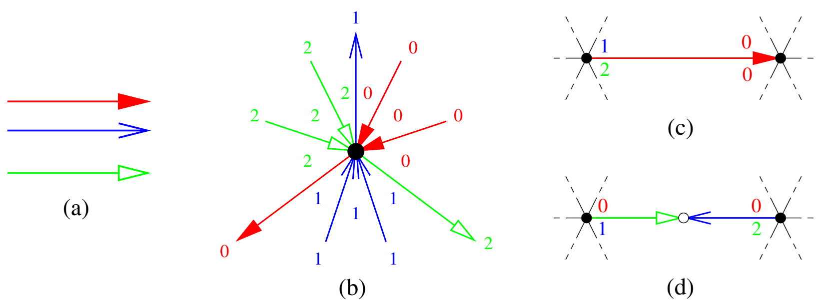

Fig. 2. (a) Edge colored respectively with color 0, 1, and 2. We use distinct arrow types to distinguish those colors. (b) Rules for Schnyder woods and angle labellings. (c) Example of angle labelling around an uni-directed egde colored with color 0. (d) Example of angle labelling around a bi-directed edge colored with colors 2 and 1.

Theorem 6 (Felsner [5]). *Schnyder woods of G^σ are in one-to-one correspondence with Schnyder angle labellings.*

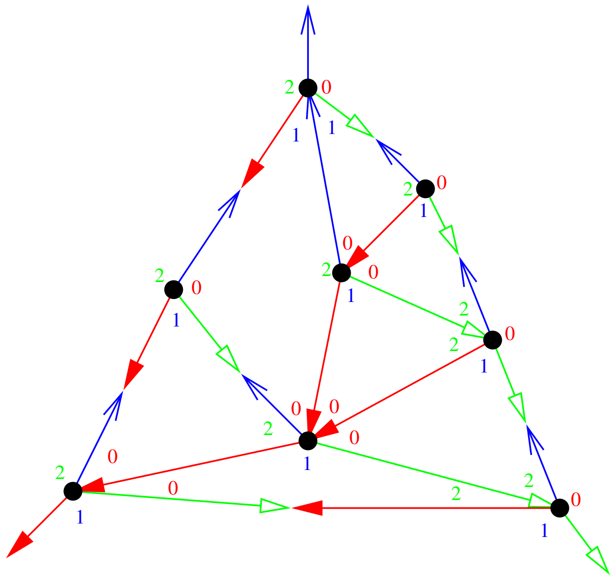

Fig. 3. A Schnyder wood with its corresponding angle labeling

3 Mixing Tools

Given a planar map G and a Schnyder wood of G rooted at x_0, x_1, x_2 we construct a contact system of arcs $\mathcal{A}$ *corresponding* to the Schnyder wood by the following method (see Figure 4):

Each vertex v is represented by three arcs $a_0(v), a_1(v), a_2(v)$, where the arc $a_i(v)$ is colored i and represent the interval of angles labeled i of v. It may be the case that $a_i(u) = a_i(v)$ for some values of i, u and v. For every edge e of G, we choose a point on its interior that we note $p(e)$. There is also such a point on the half-edge leaving x_i, for $i \in \{0, 1, 2\}$. The points $p(e)$ are the contact points of the contact system of arcs.

Actually the arcs of $\mathcal{A}$ are completely defined by the following subarcs : For each angle labeled i at a vertex v in-between the edges e and e', there is a subarc of $a_i(v)$ going from $p(e)$ to $p(e')$ along e and e'. Each contact point $p(e)$ is the end of 4 such subarcs. The Schnyder labelling implies that the three colors are represented at $p(e)$ and so the two subarcs with the same color are merged and form a longer arc.

One can easily see that this defines a contact system (there is no crossing arcs) of arcs (there is no closed curve) whose contact points are the points $p(e)$. It is also clear that the arcs satisfy the following rules:

- For every edge $e = vw$ uni-directed from v to w in color i: The arcs $a_{i+1}(v)$ and $a_{i-1}(v)$ end at $p(e)$ and the arc $a_i(w)$ goes through $p(e)$.
- For every edge $e = vw$ bi-directed, leaving v in color i and leaving w in color j: Let k be such that $\{i, j, k\} = \{0, 1, 2\}$. The arcs $a_j(v)$ and $a_i(w)$ ends at $p(e)$, and the arcs $a_k(v)$ and $a_k(w)$ are equal and go through $p(e)$.

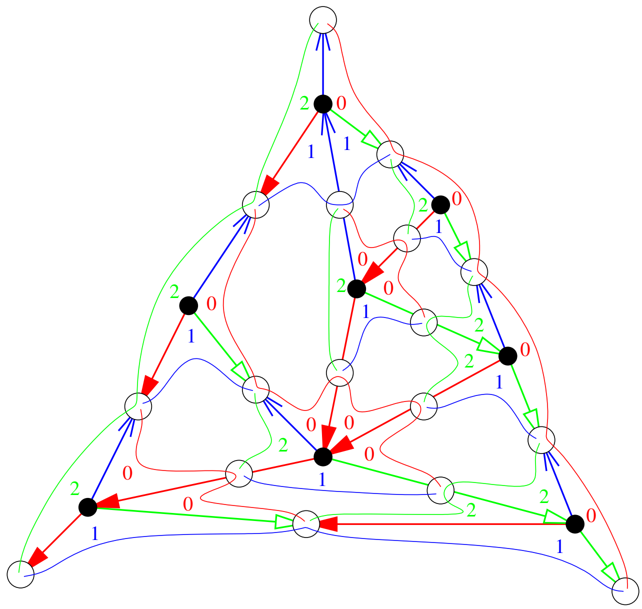

Fig. 4. A Schnyder wood with its corresponding angle labeling and contact system of arcs

The following lemma will be used to transform the contact system of arcs into a strict tiling primal-dual contact representation by triangles.

Lemma 1. *The contact system of arcs corresponding to a Schnyder wood is stretchable.*

Proof. Let G be a planar map, given with a Schnyder wood rooted at x_0, x_1, x_2. Let $\mathcal{A}$ be the contact system of arcs corresponding to the Schnyder wood as defined before. By definition of $\mathcal{A}$, every point $p(e)$, corresponding to an edge e uni- or bi-directed, is interior to one arc and is the end of two other arcs, so the contact system of arcs $\mathcal{A}$ is strict. By Theorem 5, we have to prove that each subsystem of $\mathcal{A}$, of cardinality at least two, has at least three extremal points. Let $\mathcal{B}$ be a subsystem of arcs of cardinality at least two. We have to prove that $\mathcal{B}$ has at least three extremal points.

The rest of this technical proof is omitted due to lack of space.

4 One-to-One Correspondence

De Fraysseix et al. [8] already proved that strict contact representations by triangles of a planar triangulation are in one-to-one correspondence with its Schnyder woods defined by Schnyder [17]. In this section, we are going to prove a similar result for primal-dual contact representations.

De Fraysseix et al. [9] proved that classic Schnyder woods of a planar triangulation are in one-to-one correspondence with orientation of the edges of the graph where each interior vertex has out-degree 3. This shows that it is possible to retrieve the coloring of the edges of a classic Schnyder wood from the orientation of all the edges of this Schnyder wood.

For generalized Schnyder woods (with some edges bi-directed) such a property is not true: it is not always possible to retrieve the coloring of the edges of a generalized Schnyder wood from the orientation of the edges (see for example the graph of Figure 8 in [6]). But Felsner proved that a Schnyder wood of a planar map uniquely defines a Schnyder wood of the dual and when both the orientation of the edges of the primal and the dual are given, then the coloring of the Schnyder wood can be retrieved. We will use this to obtain the one-to-one correspondence with strict tiling primal-dual contact representations by triangles. To this purpose, we need to introduce some formalism from [6].

The *suspension dual* $G^{\sigma *}$ is obtained from the dual G^* by the following: The dual-vertex corresponding to the unbounded face is replaced by a triangle with vertices y_0, y_1, y_2. More precisely, let X_i be the set of edges on the boundary of the outer face of G between vertices x_j and x_k, with $\{i,j,k\} = \{0,1,2\}$. Let Y_i be the set of dual edges to the edges in X_i, i.e. $Y_0 \cup Y_1 \cup Y_2$ is the set of edges containing the vertex f_∞ of G^* which corresponds to the unbounded face of G. Exchange f_∞ by y_i at all the edges of Y_i, add three edges y_0y_1, y_1y_2, y_2y_0, and finally add a half-edge at each y_i inside the face $y_0y_1y_2$. The resulting graph is the suspension dual $G^{\sigma *}$. Felsner [5,6] proved that Schnyder woods of G^σ are in one-to-one correspondence with Schnyder woods of $G^{\sigma *}$.

The *completion* of a plane suspension G^σ and its dual $G^{\sigma *}$ is obtain by the following: Superimpose G^σ and $G^{\sigma *}$ so that exactly the primal dual pairs of edges cross (the half-edge at x_i cross the dual edge y_jy_k, for $\{i,j,k\} = \{0,1,2\}$). The common subdivision of each crossing pair of edges is a new edge-vertex. Add a new vertex v_∞ which is the second endpoint of the six half-edges reaching into the unbounded face. The resulting graph is the completion $\widetilde{G^\sigma}$.

A *s-orientation* of $\widetilde{G^\sigma}$ is an orientation of the edges of $\widetilde{G^\sigma}$ satisfying the following out-degrees :

- $d^+(v) = 3$ for all primal- and dual-vertices v
- $d^+(e) = 1$ for all edge-vertices e.
- $d^+(v_\infty) = 0$ for the special vertex v_∞.

Felsner [6] proved the following:

Theorem 7 (Felsner [6]). *Schnyder woods of G^σ are in one-to-one correspondence with s-orientations of $\widetilde{G^\sigma}$.*

We are now able to prove the following correspondence:

Theorem 8. *The non-isomorphic strict tiling primal-dual contact representations by triangles of a planar map are in one-to-one correspondence with its Schnyder woods.*

Proof. Given a strict tiling primal-dual contact representation by triangles $(\mathcal{V}, \mathcal{F})$ of a graph G, one can associate a corresponding suspension G^σ, its suspension dual $G^{\sigma *}$, the completion $\widetilde{G^\sigma}$ and a s-orientation of the completion. The three vertices x_0, x_1, x_2 that define the suspension G^σ are, in clockwise order, the three triangles of $\mathcal{V}$ that share a node with the triangle corresponding to the outer face. We modify our contact system by exchanging the triangle $c(f_\infty)$, representing the outer face f_∞, by three triangles $c(y_0), c(y_1), c(y_2)$ each one representing y_0, y_1, y_2 of the suspension dual. Each $c(y_i)$ share a side with $c(f_\infty)$ and two $c(y_i)$ have parallel and intersecting sides. The interiors of the triangles of this new system still form a tiling of a triangle $c(v_\infty)$ representing the vertex v_∞ of the completion. The edge-vertices of the completion corresponds to the nodes of the triangles of the new system.

The s-orientation of $\widetilde{G^\sigma}$ is obtained by the following. For a primal- or dual-vertex v, represented by a triangle $c(v)$, all edges ve of $\widetilde{G^\sigma}$ are directed from v to e if e corresponds to a node of $c(v)$ and from e to v otherwise. For the special vertex v_∞, all its incident edges are directed towards itself. Clearly, for every primal- or dual-vertex v, we have $d^+(v) = 3$ as $c(v)$ is a triangle and for v_∞ we have $d^+(v_\infty) = 0$. As the primal-dual contact representation $(\mathcal{V}, \mathcal{F})$ is strict, i.e. each contact point is a node of exactly three triangles, we have $d^+(e) = 1$ for every edge-vertex that is a contact point of $(\mathcal{V}, \mathcal{F})$. For edge-vertices between special vertices x_i, y_j and v_∞ one can check that the out-degree constraint is also satisfied.

One can remark that two non-isomorphic triangle contact systems representing the same planar map G define two distinct orientations of $\widetilde{G^\sigma}$ and thus two different Schnyder woods of G^σ by Theorem 7.

Conversely, let G be a planar map, given with a Schnyder wood rooted at x_0, x_1, x_2 and the corresponding s-orientation of $\widetilde{G^\sigma}$. Let $\mathcal{A}$ be the contact system of arcs corresponding to the Schnyder wood as defined in Section 3. For each vertex $v \in V$, we note $c(v)$ the closed curve that is the union, for $i \in \{0, 1, 2\}$, of the part of the arc $a_i(v)$ between the contact point with $a_{i-1}(v)$ and $a_{i+1}(v)$. The set of curves $\mathcal{V} = (c(v))_{v \in V}$ is a contact representation of G by closed curves. For each interior face F, the labels of its angles form a nonempty interval of 0's, a nonempty interval of 1's and a nonempty interval of 2's by Theorem 6. By definition of the arcs, each interval of i's corresponds to only one arc, noted $a_i(f)$. We note $c(f)$ the closed curve that is the union, for $i \in \{0, 1, 2\}$, of the part of the arc $a_i(f)$ between the contact point with $a_{i-1}(f)$ and $a_{i+1}(f)$. For the outer face f_∞, the curve $c(f_\infty)$ is the union, for $i \in \{0, 1, 2\}$, of $a_{i+1}(x_i)$. The set of curves $\mathcal{F} = (c(f))_{f \in V^*}$ is a contact representation of G^* by closed curves.

By Lemma 1, the contact system of arcs $\mathcal{A}$ is stretchable. For each $v \in V \cup V^*$, the closed curves $c(v)$ is the union of three part of arcs of $\mathcal{A}$, so when stretched it becomes a triangle. Thus, we obtain a primal-dual contact representation by triangles $(\mathcal{V}, \mathcal{F})$ of G. By definition of $(\mathcal{V}, \mathcal{F})$ the interiors of the triangles form a tiling of the triangle corresponding to the outer face. Thus, the primal-dual contact representation by triangles $(\mathcal{V}, \mathcal{F})$ is tiling. By definition of $\mathcal{A}$, every contact point, corresponding to an uni- or bi-directed edge, is interior to one arc and is the extremity of two arcs. So each contact point of $(\mathcal{V}, \mathcal{F})$ is a node of exactly three triangles. Thus, the primal-dual contact representation by triangles $(\mathcal{V}, \mathcal{F})$ is strict. The strict tiling primal-dual contact representation by triangles $(\mathcal{V}, \mathcal{F})$ corresponds to the s-orientation of $\widetilde{G^\sigma}$ and thus to the Schnyder wood by Theorem 7.

5 Internally 3-Connected Planar Maps

A planar map G is *internally 3-connected* if there exists three vertices on the outer face such that the graph obtain from G by adding a vertex adjacent to the three vertices is 3-connected. Miller [16] proved the following (see also [4] for existence of Schnyder woods for 3-connected planar maps and [3] were the following result is stated in this form):

Theorem 9 (Miller [16]). *A planar map admits a Schnyder wood if and only if it is internally 3-connected.*

As a corollary of Theorems 8 and 9, we obtain the following:

Corollary 1. *A planar map admits a strict tiling primal-dual contact representation by triangles if and only if it is internally 3-connected.*

A 3-connected planar map is obviously internally 3-connected, so we obtain Theorem 4 as a consequence of Corollary 1.

6 Particular Types of Triangles

The construction given by de Fraysseix et al. [8] to obtain a strict contact representation by triangles of a planar triangulation can be slightly modified to give a strict tiling primal-dual contact representation by triangles (the three triangles corresponding to the outer face have to be modified to obtain the tiling property). In de Fraysseix et al.'s construction all the triangles have a horizontal side at their bottom and moreover it is possible to require that all the triangles are right (with the right angle on the left extremity of the horizontal side). This leads us to propose the following conjecture.

Conjecture 1. Every 3-connected planar map admits a strict tiling primal-dual contact representation by right triangles where all triangles have a horizontal and a vertical side and where the right angle is bottom-left for primal vertices and the outer face and top-right otherwise.

One may wonder if further requirements can be asked. Is it possible to obtain primal-dual contact representation by homothetic triangles ? The 4-connected planar triangulation of Figure 5 has a unique contact representation by homothetic triangles (for a fixed size of the external triangles). The central face corresponds to an empty triangle and thus this graph has no primal-dual contact representation by homothetic triangles. Moreover if one add a vertex in the central face adjacent to all the vertices of this face, then, there is no contact representation by homothetic triangles. In this case, the planar triangulation that is obtain is not 4-connected anymore. This leads Kratochvil [15] (see also [2]) to conjecture that every 4-connected planar triangulation admits a contact representation by homothetic triangles. Actually this conjecture holds by an application of the following theorem of O. Schramm [18] that is a generalization of Theorem 1 (in the sense that circle are replaced by convex bodies).

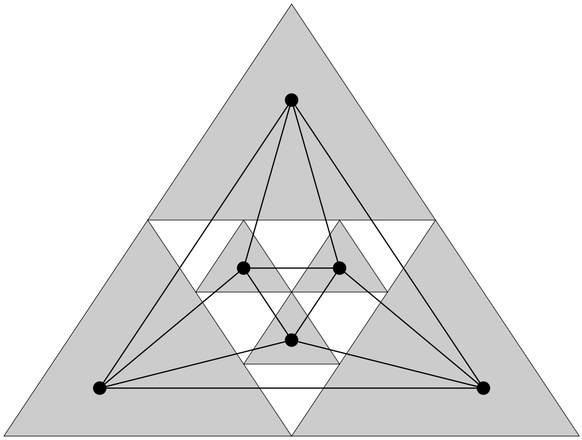

Fig. 5. A contact representation by homothetic triangles

Theorem 10 (Convex Packing Theorem). *Let T be a planar triangulation with outerface abc. Let C be a simple closed curve in the plane, and let $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ be three arcs composing $\mathcal{C}$, which are determined by three distinct points of $\mathcal{C}$. For each vertex $v \in V(T) \setminus \{a, b, c\}$, let there be a prototype $\mathcal{P}_v$, which is a convex set in the plane containing more than one point. Then there is a contact system in the plane $\mathcal{Q} = \{\mathcal{Q}_v : v \in V(T)\}$, where $\mathcal{Q}_a = \mathcal{P}_a$, $\mathcal{Q}_b = \mathcal{P}_b$, $\mathcal{Q}_c = \mathcal{P}_c$ and each $\mathcal{Q}_v$ (for $v \in V(T) \setminus \{a, b, c\}$ is either a point or (positively) homothetic to $\mathcal{P}_v$, and such that T is a subgraph of the graph induced by $\mathcal{Q}$.*

This theorem makes an intersecting link between Theorem 1 and Theorem 2.

Theorem 11. *Every 4-connected planar triangulation T admits a contact representation by homothetic triangles.*

Proof. Indeed, in the Convex Packing Theorem if we let the prototypes be homothetic triangles and the curves $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$ be segments with appropriate slopes (in such a way that those segment can be the sides of homothetic triangles added in the outer-region), we obtain a contact system of homothetic triangles $\mathcal{Q}$, where the triangles may be reduced to a point, and that induces a graph $G \supseteq T$. Thus, to prove the theorem we just have to show that (a) none of the triangles are reduced to a point and (b) that $E(G) = E(T)$.

(a) If there was a vertex v such that its triangle $\mathcal{Q}_v$ is reduced to a point p then by taking a sufficiently small circle C around p we intersect at most three non-degenerated triangles. Since a path P from x to y in H clearly corresponds to a curve in $\cup_{z \in P} \mathcal{Q}_z$ from $\mathcal{Q}_x$ to $\mathcal{Q}_y$, the triangles intersecting C correspond to a set of vertices separating v to some $u \in \{a, b, c\}$ in G, contradicting the 4-connectedness of G and T.
(b) Since none of the triangles is degenerated, the contact points are either the intersection of two or three triangles. In those cases the contact point respectively correspond to one or three edges of H. Then, since these contact points are respectively

the nodes of at least one, or exactly three triangles, and according to the position of the segments $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$, we have that $|E(G)| \leq 3 + 3(n-3) = 3n-6 = |E(T)|$. Thus $E(G) = E(T)$ and we are done.

It is still an open question to know whether these representations by homothetic triangles are unique for a given 4-connected triangulation. These representations being not strict (three triangles can meet at one point, see Figure 5) we can not always derive a unique Schnyder wood as in [8]. However, we can define a set of Schnyder woods corresponding to the representation as follow. All the triangles of the representation are homothetic to a triangle with nodes colored $0, 1, 2$ in clockwise order. The out-going arc of color i of a vertex v corresponds to the contact point with the node i of its corresponding triangle. For the particular case where three triangles meet in one point we have to choose arbitrarily the clockwise or anti-clockwise cycle. This set of Schnyder woods can be embedded on an orthogonal drawing where edge-points are coplanar (by allowing a degenerate patterns for each point that is the intersection of three triangles, see Felsner and Zickfeld [7]). Another interesting conjecture concerning contact system of triangles is the following.

Conjecture 2. Every planar graph has a contact representation by equilateral (not necessarily homothetic) triangles.

Concerning intersection systems (not contact systems) of triangles, M. Kaufmann *et al.* [13] proved that Max-tolerance graphs are exactly those graphs that have an intersection representation by homothetic triangles. Then K. Lehmann conjectured that every planar graph has such a representation. We can derive from Theorem 11 that her conjecture holds.

Theorem 12. *A graph G is planar if and only if it has an intersection representation by homothetic triangles where no three triangles intersect.*

It is interesting to notice that this theorem implies a result of Gansner et al. [11], that planar graphs have a representation by touching hexagons. Indeed consider an intersection model of G by homothetic triangles where no three triangles intersect, and where two intersecting triangles intersect in more than one point (inflate the triangles if necessary). Then remove, for each pair of intersecting triangles $T(u)$ and $T(v)$ (where $T(u)$ has a node strictly inside $T(v)$), the triangle $T(u) \cap T(v)$ from $T(u)$. However, Gansner et al.'s construction also provides, for triangulations, a model of touching polygons (with at most six sides) that form a tilling.

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