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Françoise Tisseur, Ion Zaballa

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Tisseur, Françoise and Zaballa, Ion

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Manchester Institute for Mathematical Sciences

School of Mathematics

The University of Manchester

TRIANGULARIZING QUADRATIC MATRIX POLYNOMIALS[*]

FRANÇOISE TISSEUR[†] AND ION ZABALLA[‡]

Abstract. We show that any regular quadratic matrix polynomial can be reduced to an upper triangular quadratic matrix polynomial over the complex numbers preserving the finite and infinite elementary divisors. We characterize the real quadratic matrix polynomials that are triangularizable over the real numbers and show that those that are not triangularizable are quasi-triangularizable with diagonal blocks of sizes 1×1 and 2×2. We also derive complex and real Schur-like theorems for linearizations of quadratic matrix polynomials with nonsingular leading coefficients. In particular, we show that for any monic linearization $\lambda I + A$ of an $n \times n$ quadratic matrix polynomial there exists a nonsingular matrix defined in terms of n orthonormal vectors that transforms A to a companion linearization of a (quasi-)triangular quadratic matrix polynomial. This provides the foundation for designing numerical algorithms for the reduction of quadratic matrix polynomials to upper (quasi-)triangular form.

1. Introduction. There is no analogue of the generalized Schur decomposition for quadratic matrix polynomials in the sense that $Q(\lambda) = A_2 \lambda^2 + A_1 \lambda + A_0$ cannot, in general, be reduced to triangular form $T(\lambda) = T_2 \lambda^2 + T_1 \lambda + T_0$ by unitary or strict equivalence transformations (i.e., with constant matrices P and R such that $PQ(\lambda)R$ is triangular). Hence for eigenvalue and frequency response computations, it is common practice to convert $Q(\lambda)$ to a linear polynomial $L(\lambda) = A\lambda + B$ of twice the dimension, a process called linearization, and then triangularize $L(\lambda)$ in place of $Q(\lambda)$ (see the survey paper [18]). However, a reliable process for reducing $Q(\lambda)$ to triangular form while preserving its eigenstructure would have undeniable benefits: the eigenvalues of $Q(\lambda)$ could be directly obtained from the diagonal elements of $T(\lambda)$, and, assuming that the inverse of $Q(\omega)$ can be expressed in terms of the inverse of $T(\omega)$, solving for x linear systems of the form $Q(\omega)x = b$ for many values of the parameter ω would reduce to solving many triangular systems with the same dimension as Q, one for each new value of ω.

The first main contribution of this paper is to establish that, theoretically, the triangularization process is always possible. It turns out that if unimodular transformations are used in place of strict equivalence transformations, any monic quadratic matrix polynomial is equivalent to a triangular monic quadratic matrix polynomial over the complex numbers. In other words, there exist matrix polynomials $U(\lambda)$ and $V(\lambda)$ with nonzero constant determinant such that

$$U(\lambda)Q(\lambda)V(\lambda) = T(\lambda),$$

where $T(\lambda)$ is a monic triangular quadratic and $Q(\lambda)$ and $T(\lambda)$ have the same eigen-structure. We say that $Q(\lambda)$ is *triangularizable*. This apparently little-known fact appears in the proof of a theorem on the inverse problem for linearization in the classic reference by Gohberg, Lancaster, and Rodman [9, Thm. 1.7]. Using the notion of strong equivalence so as to preserve the elementary divisors at infinity, we prove in section 3 that *any* (not necessarily monic) quadratic matrix polynomial that is regular (i.e., $\det Q(\lambda) \not\equiv 0$) is triangularizable over the complex numbers. The proof is constructive and uses unimodular transformations.

Like for real square constant matrices, not all real quadratic matrix polynomials are triangularizable over the reals. We characterize those that are triangularizable and show in section 4 that, as in the constant case, the real quadratic matrix polynomials that are not triangularizable are quasi-triangularizable with 2×2 and 1×1 diagonal blocks. In this case, a characterization of the elementary divisors of the 2×2 diagonal blocks as well as an expression for the minimum number of 2×2 diagonal blocks are provided. Unlike for the real Schur form of $A \in \mathbb{R}^{n \times n}$, the presence of 2×2 diagonal blocks is not linked only to the existence of nonreal eigenvalues. We show that in addition, a real eigenvalue, λ_0, say, must exist with a geometric multiplicity, $\dim \operatorname{Ker} Q(\lambda_0)$, greater than $n - n_c$, where n is the size of $Q(\lambda)$ and $2n_c$ is the sum of the algebraic multiplicities of its nonreal eigenvalues (Theorem 4.1). This is not a generic property and indeed the majority of real quadratic eigenvalue problems in the NLEVP collection [3] are triangularizable over the reals. The proofs of the results in section 4 are constructive, but long and technical. They are provided in Appendix A.

It is numerically not practical to use unimodular transformations to reduce $Q(\lambda)$ to (quasi-)triangular form. Instead we consider structure preserving similarities [8] acting on a linearization $\lambda A + B$ of $Q(\lambda)$ and producing a pencil from which the triangular quadratic $T(\lambda)$ is extracted. This leads to the second main contribution of this paper.

The classical complex Schur form theorem for a single $A \in \mathbb{C}^{n \times n}$ can be stated in terms of subspaces spanned by orthonormal vectors that decompose $\mathbb{C}^n$ as a direct sum (Theorem 5.1). The fact that every complex quadratic matrix polynomial is triangularizable implies the existence of two-dimensional Krylov subspaces, generated by n vectors (the generating vectors) that form an orthogonal system, decomposing $\mathbb{C}^{2n}$ as a direct sum (Theorem 5.2). In matrix terms, for any linearization $A \in \mathbb{C}^{2n \times 2n}$ of $Q(\lambda)$, there is a $2n \times n$ matrix U with orthonormal columns (corresponding to the generating vectors) such that $S = [U \ AU]$ is nonsingular and $S^{-1}AS = \begin{bmatrix} 0 & -T_0 \\ I & -T_1 \end{bmatrix}$ is a companion linearization of a triangular monic quadratic matrix polynomial $T(\lambda) = I\lambda^2 + T_1\lambda + T_0$ equivalent to $Q(\lambda)$. When $Q(\lambda)$ is linear instead of quadratic, the above result reduces to the classical Schur form. Thus, $T(\lambda)$ can be said to be a Schur-like form of $Q(\lambda)$.

The real Schur forms of any $A \in \mathbb{R}^{n \times n}$ are the quasi-triangular matrices (with 2×2 and 1×1 diagonal blocks) that are unitary similar to A and that have the minimum number of 2×2 blocks (see Remark 5.6). This fact can be expressed in terms of the existence of one- and two-dimensional subspaces decomposing $\mathbb{R}^n$. This result can be generalized to real quadratic matrix polynomials (Theorem 5.5) and its matrix version establishes that for any real quadratic matrix polynomial $Q(\lambda)$ and any linearization $A \in \mathbb{R}^{2n \times 2n}$ of $Q(\lambda)$, there is a $2n \times n$ matrix U, with orthonormal columns, such that $S = [U \ AU]$ is nonsingular and $S^{-1}AS = \begin{bmatrix} 0 & -T_0 \\ I & -T_1 \end{bmatrix}$ is a companion linearization of a quasi-triangular monic quadratic matrix polynomial $T(\lambda) = I\lambda^2 + T_1\lambda + T_0$ equivalent

to $Q(\lambda)$ and with the minimum possible number of 2×2 diagonal blocks. Again such a $T(\lambda)$ can be said to be a real Schur-like form of $Q(\lambda)$.

The results in section 5 show that to triangularize $Q(\lambda)$ it suffices to construct a set of n orthogonal generating vectors. We hope these results will lead to the development of new classes of algorithms for quadratic matrix polynomials. A first step in that direction is presented in [14], where it is shown that the generating vectors can be obtained from the Schur vectors of any monic linearization of $Q(\lambda)$. Although the approach in [14] is unsuitable for eigenvalue computation, it yields an efficient algorithm for solving linear systems of the form $Q(\omega)x = b$ for many values of ω. Finally, we note that some (but not all) of the results presented in this paper extend to matrix polynomials of arbitrary degree [17].

2. Preliminaries. The eigenstructure of a matrix polynomial comprises the eigenvalues and their partial multiplicities or, equivalently, the elementary divisors (or invariant factors) of the matrix polynomial, including those at infinity. Two matrix polynomials with the same finite elementary divisors are said to be *equivalent*. The equivalence of matrix polynomials can be characterized by the action of the unimodular group as follows. As mentioned in the introduction, a matrix polynomial is called *unimodular* if it is square and its determinant is a nonzero constant polynomial. These matrices form a noncommutative group (the group of units of the ring of square matrix polynomials of fixed size) and two matrix polynomials $A(\lambda)$ and $B(\lambda)$ are equivalent if and only if there are unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$. If in addition $A(\lambda)$ and $B(\lambda)$ have the same elementary divisors at infinity, they are called *strongly equivalent*. We say that a regular quadratic matrix polynomial $Q(\lambda)$ with nonsingular leading coefficient A_2 is *triangularizable* if it is equivalent to a triangular monic quadratic matrix polynomial. When $\det A_2 = 0$, $Q(\lambda)$ is *triangularizable* if it is *strongly* equivalent to a triangular quadratic matrix polynomial whose diagonal entries are monic.

Any $m \times n$ matrix polynomial $A(\lambda)$ with coefficients in an arbitrary field $\mathbb{F}$ is equivalent to a diagonal matrix polynomial (of different degree, in general) called the *Smith form* of $A(\lambda)$. That is to say, there are unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ such that

$$U(\lambda)A(\lambda)V(\lambda) = D(\lambda) = \begin{bmatrix} \mathrm{diag}\big(\alpha_1(\lambda), \ldots, \alpha_r(\lambda)\big) & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \mathrm{rank}\, A(\lambda)$ and $\alpha_1(\lambda)| \cdots |\alpha_r(\lambda)$ are monic polynomials. Here, $|$ stands for divisibility, so that $\alpha_j(\lambda)$ is divisible by $\alpha_{j-1}(\lambda)$. These polynomials are the *invariant factors* of $A(\lambda)$ and are uniquely determined by $A(\lambda)$.

In what follows, $\mathbb{F}$ denotes either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. Since $Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ is regular, $\mathrm{rank}\, Q(\lambda) = n$ and its Smith form is $\mathrm{diag}\big(\alpha_1(\lambda), \ldots, \alpha_n(\lambda)\big)$ with $\alpha_1(\lambda)| \cdots |\alpha_n(\lambda)$. The invariant factors of $Q(\lambda)$ can be decomposed into irreducible factors over $\mathbb{F}$ as follows [7, Chap. VI, sect. 3]:

$$
\begin{array}{rcl}
\alpha_n(\lambda) & = & \phi_1(\lambda)^{m_{11}} \cdots \phi_s(\lambda)^{m_{s1}}, \\
\alpha_{n-1}(\lambda) & = & \phi_1(\lambda)^{m_{12}} \cdots \phi_s(\lambda)^{m_{s2}}, \\
\vdots & & \vdots \\
\alpha_1(\lambda) & = & \phi_1(\lambda)^{m_{1n}} \cdots \phi_s(\lambda)^{m_{sn}},
\end{array}
$$
(2.1)

where $\phi_i(\lambda)$, $i = 1\colon s$, are distinct monic polynomials irreducible over $\mathbb{F}[\lambda]$, and

(2.2) $$m_{i1} \geq m_{i2} \geq \cdots \geq m_{in} \geq 0, \quad i = 1\colon s.$$

The factors $\phi_i(\lambda)^{m_{ij}}$ with $m_{ij} > 0$ are the *elementary divisors* of $Q(\lambda)$. Notice that when $\mathbb{F} = \mathbb{C}$, $\phi_i(\lambda) = \lambda - \lambda_i$, and when $\mathbb{F} = \mathbb{R}$, $\phi_i(\lambda) = \lambda - \lambda_i$, if $\lambda_i \in \mathbb{R}$ and $\phi_i(\lambda) = \lambda^2 - 2\mathrm{Re}(\lambda_i)\lambda + |\lambda_i|^2$ for the complex eigenvalues $\lambda_i, \bar{\lambda}_i$. The elementary divisors associated with real eigenvalues will be called *real elementary divisors*. We will use in sections 3 and 4 the procedure in (2.1) to construct the invariant factors of a matrix polynomial from its elementary divisors.

The elementary divisors at infinity (or infinite elementary divisors) of a regular polynomial matrix $Q(\lambda)$ can be introduced in a number of ways. They are defined in [20] as the zeros at $\lambda = 0$ in the Smith–McMillan form of $\lambda Q(\lambda^{-1})$, and the so-called pole-zero structure at infinity of $Q(\lambda)$ is defined in [10] to be the poles and zeros at $\lambda = 0$ in the Smith–McMillan form of $Q(\lambda^{-1})$. These concepts can also be defined using the local ring of proper rational functions (see, for example, [6], [19]). Nowadays a commonly accepted definition is that the infinite elementary divisors of $Q(\lambda)$ are the elementary divisors of rev $Q(\lambda)$ at 0, where

$$\mathrm{rev}\, Q(\lambda) = \lambda^2 Q\left(\lambda^{-1}\right) = A_0\lambda^2 + A_1\lambda + A_2$$

is the reversal of $Q(\lambda)$. This definition has its foundation in the fact that these are the elementary divisors at $(1,0)$ of the homogenization $Q_h(\lambda, \mu) = A_2\lambda^2 + A_1\lambda\mu + A_0\mu^2$ of $Q(\lambda)$. It follows from the definition that $Q(\lambda)$ has infinite elementary divisors if and only if its leading coefficient is singular.

In this paper the following methodology will be used. Results will be first proved assuming that $\det A_2 \neq 0$, and when $\det A_2 = 0$, a conformal transformation of the form

(2.3) $$f(\lambda) = a + \frac{1}{\lambda - b}$$

will be applied to $Q(\lambda)$ that reduces the study to the former case. Notice that for $a = b = 0$, rev $Q(\lambda) = \lambda^2 Q(f(\lambda))$. It may happen, however, that $\det A_0 = 0$ and rev $Q(\lambda)$ may still have infinite elementary divisors. In this case, $a \in \mathbb{F}$ can be chosen such that $\det Q(a) \neq 0$. We can write

$$Q(\lambda) = \widetilde{A}_2(\lambda - a)^2 + \widetilde{A}_1(\lambda - a) + \widetilde{A}_0$$

so that $\det \widetilde{A}_0 \neq 0$. When $Q(\lambda)$ is written in this form the notation $Q_a(\lambda)$ will be used to emphasize the role of a. With such an a and any $b \in \mathbb{F}$, define $f(\lambda)$ as in (2.3) and

$$\mathrm{rev}_b\, Q_a(\lambda) = (\lambda - b)^2 Q(f(\lambda)) = \widetilde{A}_0(\lambda - b)^2 + \widetilde{A}_1(\lambda - b) + \widetilde{A}_2.$$

So, rev$_b$ $Q_a(\lambda)$ has no infinite elementary divisors. Although the simple choice $b = 0$ is always at hand, other choices may produce coefficient matrices in rev$_b$ $Q_a(\lambda)$ with better numerical properties.

We need to know how the finite and infinite elementary divisors of $Q(\lambda)$ are related to those of rev$_b$ $Q_a(\lambda)$ for any choice of a and b. Results about this issue are scattered in the literature (see, for example, [2, sect. 4.2], [9, Thm. 7.3], [11], [22], and [4, Lem. 10]).

The following result is a simple consequence of [15, Thm. 4.1] or [21, Thm. 3.4] and can also be derived from [11, Thm. 5.3].

PROPOSITION 2.1. *For a regular* $Q(\lambda) = \sum_{j=0}^{\ell} A_j\lambda^j$ *with* $A_j \in \mathbb{F}^{n \times n}$, $j = 0 : \ell$, *and for* $a, b \in \mathbb{F}$ *with* a *such that* $Q(a) \neq 0$, *let* rev$_b$ $Q_a(\lambda) = (\lambda - b)^\ell Q(f(\lambda))$, *where* $f(\lambda)$ *is as in (2.3). Then the following holds:*

   (i) *If $\phi(\lambda) \neq (\lambda - a)$ is an irreducible polynomial and $\phi(\lambda)^{d_1}$, ..., $\phi(\lambda)^{d_n}$ with $d_1 \geq \cdots \geq d_n \geq 0$ are the elementary divisors of $Q(\lambda)$ with respect to $\phi(\lambda)$, then $\phi_b(\lambda)^{d_1}$, ..., $\phi_b(\lambda)^{d_n}$ are the elementary divisors of $\mathrm{rev}_b\, Q_a(\lambda)$ with respect to $\phi_b(\lambda)$, where $\phi_b(\lambda)$ is $(\lambda - b)^d \phi(f(\lambda))$ normalized so as to be monic and $d = \deg(\phi(\lambda))$.*
   (ii) *If $(\lambda - a)^{d_1}, \ldots, (\lambda - a)^{d_n}$ with $d_1 \geq \cdots \geq d_n \geq 0$ are the elementary divisors of $Q(\lambda)$ with respect to $(\lambda - a)$, then $\mu^{d_1}, \ldots, \mu^{d_n}$ are the infinite elementary divisors of $\mathrm{rev}_b\, Q_a(\lambda)$.*
   (iii) *If $\mu^{d_1}, \ldots, \mu^{d_n}$, with $d_1 \geq \cdots \geq d_n \geq 0$ are the infinite elementary divisors of $Q(\lambda)$, then $(\lambda - b)^{d_1}, \ldots, (\lambda - b)^{d_n}$ are the elementary divisors of $\mathrm{rev}_b\, Q_a(\lambda)$ with respect to $(\lambda - b)$.*

The following lemma allows us to reduce the case of quadratic matrix polynomials with singular leading coefficient to the case where the leading coefficient is nonsingular.

LEMMA 2.2. *Let $Q(\lambda) = A_2 \lambda^2 + A_1 \lambda + A_0 \in \mathbb{F}[\lambda]^{n \times n}$ with $\det A_2 = 0$ (but $A_2 \neq 0$) and let $a, b \in \mathbb{F}$ with a such that $\det Q(a) \neq 0$. If $\mathrm{rev}_b\, Q_a(\lambda)$ is equivalent, over $\mathbb{F}$, to a (block-)triangular quadratic matrix polynomial, then $Q(\lambda)$ is strongly equivalent, over $\mathbb{F}$, to a (block-)triangular quadratic matrix polynomial.*

Proof. Assume that $R(\lambda) = \mathrm{rev}_b\, Q_a(\lambda) = (\lambda - b)^2 Q(f(\lambda))$ is equivalent to a (block-)triangular quadratic $T(\lambda)$ with $f(\lambda)$ as in (2.3). Then $R(\lambda)$ and $T(\lambda)$ have the same finite elementary divisors and no elementary divisors at infinity. Define $\widetilde{f}(\lambda) = b + \frac{1}{\lambda - a}$. By Proposition 2.1, $(\lambda - a)^2 R(\widetilde{f}(\lambda))$ and $(\lambda - a)^2 T(\widetilde{f}(\lambda))$ have the same finite and infinite elementary divisors provided that $R(b) \neq 0$ and $T(b) \neq 0$. But $R(b) = A_2 \neq 0$ and $T(b)$ and $R(b)$ have the same rank because $T(\lambda) = U(\lambda) R(\lambda) V(\lambda)$ for some unimodular matrices $U(\lambda)$ and $V(\lambda)$. So $T(b) \neq 0$ as well. Finally,

$$(\lambda - a)^2 R(\widetilde{f}(\lambda)) = (\lambda - a)^2 ((\widetilde{f}(\lambda) - b)^2 Q(f(\widetilde{f}(\lambda)))) = Q(\lambda).$$

Hence $Q(\lambda)$ is strongly equivalent to $(\lambda - a)^2 T(\widetilde{f}(\lambda))$, which is triangular or block-triangular accordingly as $T(\lambda)$ is triangular or block-triangular, respectively. □

We will need the following result, which is a particular case of [12, Thm. 5.2] and [9, Thm. 1.7]. It is also a consequence of Rosenbrock's theorem on pole placement (see [1, Thm. 1.1], [16]). Here $\deg(\cdot)$ denotes "the degree of."

LEMMA 2.3. *Let $\mathbb{F}$ be an arbitrary field and let $\alpha_1(\lambda) | \alpha_2(\lambda) | \cdots | \alpha_n(\lambda)$ be monic polynomials with coefficients in $\mathbb{F}$. There exists a quadratic $Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ with nonsingular leading coefficient and $\alpha_1(\lambda) | \alpha_2(\lambda) | \cdots | \alpha_n(\lambda)$ as invariant factors if and only if $\sum_{i=1}^{n} \deg(\alpha_i(\lambda)) = 2n$.*

We will have to deal with block-triangular matrices whose diagonal blocks are quadratic and the off-diagonal blocks of possible higher degree. The following lemma can be used to reduce their degree.

LEMMA 2.4. *Let $T(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be upper block-triangular with ith diagonal block $T_i(\lambda) \in \mathbb{F}[\lambda]^{n_i \times n_i}$, $i = 1 : s$. If all the $T_i(\lambda)$'s are quadratic with nonsingular leading coefficient matrices, then $T(\lambda)$ is equivalent to a quadratic upper block-triangular matrix polynomial with the same diagonal blocks as $T(\lambda)$ and off-diagonal blocks of degree less than 2.*

Proof. A procedure to obtain such a quadratic upper block-triangular matrix is as follows. Let $j = 2$ and $i = j - 1$.

   Step 1. Compute the euclidean division [10, Thm. 6.3-15] of $T_{ij}(\lambda) \in \mathbb{F}^{n_i \times n_j}$ by $T_j(\lambda)$: $T_{ij}(\lambda) = Q_{ij}(\lambda) T_j(\lambda) + R_{ij}(\lambda)$ with $R_{ij}(\lambda) = 0$ or $\deg R_{ij}(\lambda) < 2$.
   Step 2. Add the jth block-row multiplied by $-Q_{ij}(\lambda)$ to the ith block-row so as to replace $T_{ij}(\lambda)$ by $R_{ij}(\lambda)$. (All blocks $T_{ik}(\lambda)$, for $k \geq j$, are modified.)

Step 3. If $i > 1$, then $i = i - 1$; else $j = j + 1$, $i = j - 1$. If $j \leq s$ go to Step 1, else stop. □

3. Triangularizable quadratic matrix polynomials. As mentioned in the introduction, the proof of [9, Thm. 1.7] shows that any complex matrix polynomial of degree ℓ with nonsingular leading coefficient is equivalent to a monic triangular matrix polynomial of the same degree. We recall the procedure for $\ell = 2$.

Step 1. Compute the invariant factors $\alpha_1(\lambda)|\alpha_2(\lambda)|\cdots|\alpha_n(\lambda)$ of the given matrix polynomial and let $D(\lambda) = \mathrm{diag}(\alpha_1(\lambda), \ldots, \alpha_n(\lambda))$.

Step 2. If $\deg(\alpha_1) = 2$, then $\deg(\alpha_i) = 2$, $i = 2\colon n$. Then $D(\lambda)$ is a monic triangular quadratic matrix polynomial equivalent to $Q(\lambda)$ and the construction is done. Otherwise, go to Step 3.

Step 3. If $\ell_1 = \deg(\alpha_1) < 2$, then $\ell_n = \deg(\alpha_n) > 2$ and there is a monic polynomial $s(\lambda)$ of degree $\ell_n - 2$ such that $\alpha_{j-1}(\lambda)|\alpha_1(\lambda)s(\lambda)|\alpha_j$ for some index j, $1 < j \leq n$. The $s(\lambda)$ is the product of $\alpha_{j-1}(\lambda)/\alpha_1(\lambda)$ and some of the factors in the prime factorization of $\alpha_j(\lambda)/\alpha_{j-1}(\lambda)$.

Step 4. Perform the following elementary transformations on $D(\lambda)$:
- (i) Add to column n the first column multiplied by $s(\lambda)$. Then add to row n the first row multiplied by $-\alpha_n(\lambda)/\big(\alpha_1(\lambda)s(\lambda)\big)$ and permute columns 1 and n.
- (ii) Successively interchange rows k and $k + 1$ for $k = 1, 2, \ldots, j - 2$ so that rows 1, 2, …, $j - 2$, $j - 1$ of the new matrix are rows 2, 3, …, $j - 1$ and 1, respectively, of the former one.
- (iii) Permute columns 1 to $j - 1$ in the same way as the rows in (ii).

The resulting matrix polynomial has the form

$$T_1(\lambda) = \begin{bmatrix} \alpha_2(\lambda) & & & & & & \\ & \ddots & & & & & \\ & & \alpha_{j-1}(\lambda) & & & & \\ & & & \alpha_1(\lambda)s(\lambda) & \cdots & \cdots & \alpha_1(\lambda) \\ & & & & \alpha_j(\lambda) & & \vdots \\ & & & & & \ddots & \\ & & & & & \alpha_{n-1}(\lambda) & \vdots \\ & & & & & & -\alpha_n(\lambda)/s(\lambda) \end{bmatrix}.$$

Step 5. Let $D_1(\lambda) = \mathrm{diag}(\alpha_2(\lambda), \ldots, \alpha_{j-1}(\lambda), \alpha_1(\lambda)s(\lambda), \ldots, \alpha_{n-1}(\lambda))$. If $D_1(\lambda)$ is diagonal of degree 2, the construction is done; otherwise use $D_1(\lambda)$ to choose the new index j and polynomials $s(\lambda)$ as in Step 3 and perform the elementary transformations described in Step 4 on the whole matrix $T_1(\lambda)$. Repeat these steps until all diagonal entries have degree 2.

Example 3.1. We consider the quadratic matrix polynomial

(3.1)
$$Q(\lambda) = \begin{bmatrix} \lambda^2 + 1 & 0 & 0 \\ \lambda^2 & \lambda - 1 & \lambda^2 - \lambda \\ \lambda & \lambda^2 - \lambda & 1 - \lambda \end{bmatrix}$$

with nonmonic, nonsingular leading coefficient. The invariant polynomials of $Q(\lambda)$ are $\alpha_1(\lambda) = 1$, $\alpha_2(\lambda) = (\lambda - 1)(\lambda^2 + 1)$, and $\alpha_3(\lambda) = (\lambda - 1)(\lambda^2 + 1)$. In Step 3 of the procedure, $\ell_1 = 0$, $\ell_2 = \ell_3 = 3$. We look for a degree 1 polynomial $s(\lambda)$ such

that $\alpha_1(\lambda)|\alpha_1(\lambda)s(\lambda)|\alpha_2(\lambda)$ (i.e., $j = 2$). We can take, for example, $s(\lambda) = \lambda - 1$.
(Another possibility for $s(\lambda)$ would be $s(\lambda) = \lambda - i$.) Applying Step 4 to the Smith
form $D(\lambda) = \text{diag}(1, (\lambda - 1)(\lambda^2 + 1), (\lambda - 1)(\lambda^2 + 1))$ leads to

$$T_1(\lambda) = \begin{bmatrix} \lambda - 1 & 0 & 1 \\ 0 & (\lambda - 1)(\lambda^2 + 1) & 0 \\ 0 & 0 & -(\lambda^2 + 1) \end{bmatrix}.$$

For Step 5, let $D_1(\lambda) = \text{diag}(\lambda - 1, (\lambda - 1)(\lambda^2 + 1))$. We find that $j = 2$. Choosing,
for instance, $s(\lambda) = \lambda - i$, we obtain

$$T(\lambda) = \begin{bmatrix} (\lambda - 1)(\lambda - i) & \lambda - 1 & 1 \\ 0 & -(\lambda - 1)(\lambda + i) & -(\lambda + i) \\ 0 & 0 & -(\lambda^2 + 1) \end{bmatrix}.$$

We only have to multiply rows 2 and 3 by -1 to produce a monic quadratic triangular
matrix equivalent to $Q(\lambda)$.

For a given $Q(\lambda)$, the above construction produces different triangular matrices
depending on the choice of the polynomial $s(\lambda)$ and index j. A natural question is
how much freedom one may expect to have in choosing the diagonal elements in the
triangularization process. An answer to this question is provided by the following
result.

THEOREM 3.2 (see [13]). *Let $\alpha_1(\lambda)|\cdots|\alpha_n(\lambda)$ and $\delta_1(\lambda),\ldots,\delta_n(\lambda)$ be monic poly-
nomials with coefficients in an arbitrary field $\mathbb{F}$. Then there exists a triangular matrix
polynomial in $\mathbb{F}[\lambda]^{n\times n}$ with diagonal entries $\delta_1(\lambda),\ldots,\delta_n(\lambda)$ and $\alpha_1(\lambda),\ldots,\alpha_n(\lambda)$ as
invariant factors if and only if $\prod_{j=1}^n \alpha_j(\lambda) = \prod_{j=1}^n \delta_j(\lambda)$ and*

$$\prod_{j=1}^k \alpha_j(\lambda)\Big|\gcd\left\{\prod_{j=1}^k \delta_{i_j}(\lambda), 1 \leq i_1 < \cdots < i_k \leq n\right\}, \quad 1 \leq k \leq n - 1.$$

In Example 3.1, the condition $\alpha_1(\lambda)\alpha_2(\lambda)|\gcd(\delta_1(\lambda)\delta_2(\lambda), \delta_1(\lambda)\delta_3(\lambda), \delta_2(\lambda)\delta_3(\lambda))$
imposes strong restrictions on the choice of the diagonal elements. Namely, one of
them must be $\lambda^2 + 1$, another $(\lambda - 1)(\lambda - i)$, and the third $(\lambda - 1)(\lambda + i)$. Provided that
these are the three diagonal elements, they may be placed in any diagonal position.

The Gohberg–Lancaster–Rodman triangularization procedure cannot be applied
when $Q(\lambda)$ has a singular leading coefficient. We next show that any regular quadratic
matrix polynomial is strongly equivalent over $\mathbb{C}[\lambda]$ to a triangular quadratic matrix
polynomial with monic diagonal polynomials and that real quadratics with finite
eigenvalues all real are triangularizable over $\mathbb{R}[\lambda]$.

THEOREM 3.3. *Let $Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a regular quadratic matrix polynomial.
If $\det Q(\lambda)$ admits a prime factorization in linear factors over $\mathbb{F}[\lambda]$, then $Q(\lambda)$ is
triangularizable.*

Proof. We have seen that when $Q(\lambda)$ has a nonsingular leading coefficient it is
always triangularizable over $\mathbb{C}[\lambda]$. If $\mathbb{F} = \mathbb{R}$ and $\det Q(\lambda)$ has a prime factorization
in linear factors over $\mathbb{R}$, then the Gohberg–Lancaster–Rodman procedure produces a
real triangular quadratic matrix polynomial.

Now if the leading coefficient of $Q(\lambda)$ is singular, we choose $a \in \mathbb{F}$ such that
$\det Q(a) \neq 0$ and $b \in \mathbb{F}$ arbitrary so that $R(\lambda) = \text{rev}_b Q_a(\lambda)$ has no infinite el-
ementary divisors. Then by Proposition 2.1 $\det R(\lambda)$ admits a prime factorization
in linear factors. Hence $R(\lambda)$ is triangularizable and, by Lemma 2.2, $Q(\lambda)$ is also
triangularizable.   ☐

The following result is a direct consequence of Theorem 3.3.

COROLLARY 3.4. *Let $Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be a quadratic matrix polynomial with nonsingular leading coefficient. If $\det Q(\lambda)$ admits a prime factorization in linear factors over $\mathbb{F}[\lambda]$, then $Q(\lambda)$ is equivalent to an upper triangular quadratic matrix polynomial whose strictly upper triangular part is linear in λ at most.*

Proof. By Theorem 3.3, $Q(\lambda)$ is equivalent to a triangular quadratic matrix polynomial $T(\lambda) = \lambda^2 T_2 + \lambda T_1 + T_0$. We can observe that T_2 is nonsingular because $Q(\lambda)$ has nonsingular leading coefficient. Hence $Q(\lambda)$ is equivalent to the monic triangular quadratic $T_2^{-1}T(\lambda) = \lambda^2 I + \lambda T_2^{-1}T_1 + T_2^{-1}T_0$. □

We have shown in Example 3.1 how to reduce a quadratic matrix polynomial with nonsingular leading coefficient to triangular form. Such quadratics have no elementary divisors at infinity. In the following example we consider the opposite situation: the given matrix only has infinite elementary divisors, i.e., it is unimodular.

Example 3.5. Consider the following quadratic matrix polynomial:

$$Q(\lambda) = \begin{bmatrix} \lambda^2+1 & \lambda & \lambda^2-\lambda \\ \lambda^2+\lambda & \lambda+1 & \lambda^2 \\ \lambda^2 & \lambda & \lambda^2-\lambda+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\lambda + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det Q(\lambda) = 1$, $Q(\lambda)$ is unimodular. As $\det A_0 \neq 0$ we can take $a = b = 0$ so that $\text{rev}_b\, Q_a(\lambda) = \text{rev}\, Q(\lambda)$. The invariant factors of this matrix are $\alpha_1(\lambda) = 1$, $\alpha_2(\lambda) = \lambda^2$, and $\alpha_3(\lambda) = \lambda^4$. This means that the exponents of the elementary divisors of $Q(\lambda)$ at infinity are 2 and 4. We now use the Gohberg–Lancaster–Rodman procedure to construct a triangular quadratic matrix polynomial $\text{rev}\, T(\lambda) = \lambda^2 I_3 + e_1 e_3^T$ with $\alpha_j(\lambda)$, $j = 1{:}\,3$ as invariant factors. Here e_j denotes the jth column of the 3×3 identity matrix. Then, its reversal

$$T(\lambda) = \begin{bmatrix} 1 & 0 & \lambda^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is triangular and strongly equivalent to $Q(\lambda)$.

We mentioned that as a consequence of Theorem 3.2 the only possible diagonal elements of any triangular matrix polynomial equivalent to $Q(\lambda)$ in (3.1) are $\lambda^2 + 1$, $(\lambda - i)(\lambda - 1)$, and $(\lambda + i)(\lambda - 1)$. Hence $Q(\lambda)$ cannot be triangularizable over $\mathbb{R}[\lambda]$. In fact, the reduction process may fail for matrices over arbitrary fields due to the possible existence of irreducible polynomials of any degree. The real quadratic matrix polynomials that are triangularizable over $\mathbb{R}[\lambda]$ are characterized in the following theorem, for which we need additional notation.

If $Q(\lambda)$ has real coefficients, its elementary divisors are powers of linear or quadratic polynomials. That is to say, with the notation of (2.1), $\phi_i(\lambda) = \lambda^2 + a_i\lambda + b_i$ or $\phi_i(\lambda) = (\lambda - \lambda_i)$. Thus we can write the list of the elementary divisors of $Q(\lambda)$ explicitly as follows:

$$
\begin{aligned}
(\lambda^2 + a_i\lambda + b_i)^{n_{ij}}, & \quad i = 1{:}\,s,\ j = 1{:}\,t_i, \\
(\lambda - \lambda_i)^{m_{ij}}, & \quad i = 1{:}\,r,\ j = 1{:}\,p_i, \\
\mu^{m_{0j}}, & \quad j = 1{:}\,p_0,
\end{aligned}
$$

(3.2)

where $\lambda^2 + a_i\lambda + b_i$ is irreducible over $\mathbb{R}[\lambda]$, and the $\mu^{m_{0j}}$, $j = 1{:}\,p_0$, are the elementary divisors at infinity. We assume that the partial multiplicities m_{ij}, n_{ij} are ordered as in (2.2). Denote

(3.3)
$$n_c = \sum_{i=1}^{s} \sum_{j=1}^{t_i} n_{ij}, \qquad p = \max_{0 \le i \le r} p_i,$$

that is, n_c is the sum of the exponents of the elementary divisors that are powers of irreducible polynomials of degree 2 and p is the largest chain of elementary divisors corresponding to the same real eigenvalue or to the eigenvalue at infinity. In other words, this eigenvalue has geometric multiplicity p and there is no other real or infinite eigenvalue with larger geometric multiplicity. We are now ready to state the theorem.

THEOREM 3.6. *A quadratic matrix polynomial $Q(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ is triangularizable over $\mathbb{R}[\lambda]$ if and only if $p \le n - n_c$.*

Proof. Assume first that the leading coefficient of $Q(\lambda)$ is nonsingular. This means that $p_0 = 0$ and $p = \max_{1 \le i \le r} p_i$.

($\Rightarrow$) Suppose that $Q(\lambda)$ is upper triangular and that $q_{ii}(\lambda) = \lambda^2 + a_i \lambda + b_i$ is irreducible and $q_{jj}(\lambda) = (\lambda - \lambda_{j_1})(\lambda - \lambda_{j_2})$ with $\lambda_{j_1}, \lambda_{j_2} \in \mathbb{R}$ and these two real numbers may be equal. Thus $q_{ii}(\lambda)$ and $q_{jj}(\lambda)$ are coprime polynomials and if $i < j$ there exist polynomials $x_{ij}(\lambda)$ and $y_{ij}(\lambda)$ such that

$$x_{ij}(\lambda) q_{ii}(\lambda) + y_{ij}(\lambda) q_{jj}(\lambda) = q_{ij}(\lambda).$$

Replace the jth column of $Q(\lambda)$ by the jth column minus the ith column times $x_{ij}(\lambda)$ and the ith row by the ith row minus the jth row multiplied by $y_{ij}(\lambda)$. Then we obtain a matrix equivalent to $Q(\lambda)$, which is upper triangular and the entry in position (i, j) is zero.

Similarly, if $j < i$ there exist polynomials $x_{ji}(\lambda)$ and $y_{ji}(\lambda)$ such that

$$x_{ji}(\lambda) q_{ii}(\lambda) + y_{ji}(\lambda) q_{jj}(\lambda) = q_{ji}(\lambda).$$

If we replace the ith column of $Q(\lambda)$ by the ith column minus the jth column times $y_{ji}(\lambda)$ and the jth row by the jth row minus the ith row multiplied by $x_{ji}(\lambda)$, then we get an upper triangular matrix equivalent to $Q(\lambda)$ with zero in position (j, i).

After a finite number of such elementary transformations (starting from the bottom of the matrix in order to preserve the already zeroed elements), $Q(\lambda)$ is transformed to an equivalent upper triangular matrix with the property that $q_{ij}(\lambda) = 0$ if either $q_{ii}(\lambda) = \lambda^2 + a_i \lambda + b_i$ and $q_{jj}(\lambda) = (\lambda - \lambda_{j_1})(\lambda - \lambda_{j_2})$ or $q_{ii}(\lambda) = (\lambda - \lambda_{i_1})(\lambda - \lambda_{i_2})$ and $q_{jj}(\lambda) = \lambda^2 + a_j \lambda + b_j$. Now, by permuting rows and columns we transform that matrix to block diagonal form,

$$T(\lambda) = \begin{matrix} {} \\ n_c \\ n - n_c \end{matrix} \overset{\begin{matrix} n_c & \quad n - n_c \end{matrix}}{\begin{bmatrix} T_c(\lambda) & 0 \\ 0 & T_r(\lambda) \end{bmatrix}},$$

where $T_c(\lambda)$ and $T_r(\lambda)$ are upper triangular and such that the elementary divisors of $T_c(\lambda)$ are of the form $(\lambda^2 + a_i \lambda + b_i)^{n_{ij}}$ with $a_i^2 - 4b_i < 0$ and the elementary divisors of $T_r(\lambda)$ are of the form $(\lambda - \lambda_i)^{m_{ij}}$ with $\lambda_i \in \mathbb{R}$. Since $T_r(\lambda)$ is $(n - n_c) \times (n - n_c)$, the largest chain of elementary divisors corresponding to the same real eigenvalue must not be bigger that $n - n_c$, i.e., $p \le n - n_c$.

($\Leftarrow$) Conversely, take all elementary divisors of $Q(\lambda)$ of the form $(\lambda^2 + a_i \lambda + b_i)^{n_{ij}}$ with $\lambda^2 + a_i \lambda + b_i$ irreducible over $\mathbb{R}[\lambda]$ and construct the upper triangular quadratic matrix polynomials

$$T_{ij}(\lambda) = (\lambda^2 + a_i \lambda + b_i) I_{n_{ij}} + N_{n_{ij}}, \quad 1 \le j \le t_i, \; 1 \le i \le s,$$

whose elementary divisors are $(\lambda^2 + a_i\lambda + b_i)^{n_{ij}}$. Here $N_{n_{ij}}$ is an $n_{ij} \times n_{ij}$ Jordan block with eigenvalue zero. Using the notation $\bigoplus_{j=1}^{s} F_j(\lambda)$ to denote the direct sum of the matrix polynomials $F_1(\lambda), \ldots, F_s(\lambda)$ we have that $T_c(\lambda) = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{t_i} T_{ij}(\lambda)$ is an upper triangular quadratic matrix polynomial with elementary divisors $(\lambda^2 + a_i\lambda + b_i)^{n_{ij}}$, $1 \leq j \leq t_i$, $1 \leq i \leq s$ (see [7, Chap. VI, Thm. 5]).

Consider now the elementary divisors of $Q(\lambda)$ of the form $(\lambda - \lambda_i)^{m_{ij}}$ with $\lambda_i \in \mathbb{R}$. First,

$$\sum_{i=1}^{r}\sum_{j=1}^{p_i} m_{ij} = 2(n - n_c) \geq 2p.$$

Put $n_r = n - n_c$ and consider the process of constructing the invariant factors of a polynomial matrix out of its elementary divisors as described in section 2. Assuming that $m_{i1} \geq m_{i2} \geq \cdots \geq m_{ip_i}$ and bearing in mind that p is the largest chain associated with any real eigenvalue and $p \leq n_r$, we have that

(3.4)
$$
\begin{aligned}
\gamma_{n_r}(\lambda) &= (\lambda - \lambda_1)^{m_{11}} \cdots (\lambda - \lambda_r)^{m_{r1}}, \\
&\vdots \\
\gamma_{n_r-p+1}(\lambda) &= (\lambda - \lambda_1)^{m_{1p}} \cdots (\lambda - \lambda_r)^{m_{rp}}, \\
\gamma_{n_r-p}(\lambda) &= 1, \\
&\vdots \\
\gamma_1(\lambda) &= 1,
\end{aligned}
$$

where for notational simplicity we may have included exponents $m_{ij} = 0$. Since $\sum_{i=1}^{n_r} \deg(\gamma_i(\lambda)) = 2n_r$, Lemma 2.3 guarantees that $\gamma_1(\lambda) | \cdots | \gamma_{n_r}(\lambda)$ are the invariant factors of an $n_r \times n_r$ quadratic matrix polynomial with nonsingular leading coefficient and $(\lambda - \lambda_i)^{m_{ij}}$ as elementary divisors. Taking into account that $\gamma_{n_r}(\lambda)$ factorizes into linear factors over $\mathbb{R}[\lambda]$, by Theorem 3.3 there exists an $n_r \times n_r$ upper triangular quadratic matrix polynomial $T_r(\lambda)$ whose elementary divisors are the $(\lambda - \lambda_i)^{m_{ij}}$. Then $T(\lambda) = T_c(\lambda) \oplus T_r(\lambda)$ has $(\lambda^2 + a_i\lambda + b_i)^{n_{ij}}$ and $(\lambda - \lambda_i)^{m_{ij}}$ as elementary divisors and is quadratic. Therefore $Q(\lambda)$ is triangularizable.

Finally, if the leading matrix coefficient of $Q(\lambda)$ is singular (i.e., $p_0 > 0$), and $Q(\lambda)$ is triangular, then there is $a, b \in \mathbb{R}$ such that $\det Q(a) \neq 0$, and $\mathrm{rev}_b\, Q_a(\lambda)$ has nonsingular leading coefficient and it is triangular. According to Proposition 2.1 its elementary divisors are

$$
\left((\lambda - b)^2 + \frac{\tilde{a}_i}{\tilde{b}_i}(\lambda - b) + \frac{1}{\tilde{b}_i}\right)^{n_{ij}}, \quad i = 1 : s,\ j = 1 : t_i,
$$

$$
\left(\lambda - b - \frac{1}{\lambda_i - a}\right)^{m_{ij}}, \qquad i = 1 : r,\ j = 1 : p_i
$$

$$
(\lambda - b)^{m_{0j}}, \qquad\qquad j = 1 : p_0,
$$

with $\tilde{a}_i = 2a + a_i$, $\tilde{b}_i = a^2 + a_i a + b_i$, $\left(\left(\frac{\tilde{a}_i}{\tilde{b}_i}\right)^2 - 4\frac{1}{\tilde{b}_i}\right) < 0$ and $0 \neq \lambda_i - a \in \mathbb{R}$. Therefore, $p \leq n - n_c$. Conversely, if $p \leq n - n_c$, then $\mathrm{rev}_b Q_a(\lambda)$ is triangularizable and, by Lemma 2.2, so is $Q(\lambda)$. $\square$

Theorem 3.6 says that if the real eigenvalues and eigenvalues at infinity of a real quadratic matrix polynomial have geometric multiplicity less than or equal to $n - n_c$, then the quadratic is triangularizable over the real numbers.

Example 3.7. As already stated, the matrix $Q(\lambda)$ of Example 3.1 is not triangularizable over $\mathbb{R}[\lambda]$. Let us check that the condition of Theorem 3.6 is not satisfied. The elementary divisors of $Q(\lambda)$ are $\lambda^2 + 1$, $\lambda^2 + 1$, $\lambda - 1$, and $\lambda - 1$ so that $n_c = 2$ and $p = 2$. Hence $n - n_c = 1 < 2 = p$.

Example 3.8. As an example of a quadratic matrix polynomial with structure at infinity consider

$$Q(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \lambda + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

with elementary divisors $\lambda^2 + 1$, $\lambda^2 + 1$, μ, and μ. Then $n - n_c = 1 < 2 = p$ and $Q(\lambda)$ is not triangularizable over $\mathbb{R}[\lambda]$.

We remark that the property $p \leq n - n_c$ is generic for the set of real quadratic matrix polynomials. This means that, unlike for square constant matrices, "most" real quadratic matrix polynomials are triangularizable over the reals.

4. Quasi-triangularizable real quadratic matrix polynomials. A natural question, in analogy with the triangularization problem for real square matrices, is whether any real quadratic matrix polynomial is equivalent to a real quasi-triangular quadratic matrix polynomial with 2×2 and 1×1 diagonal blocks. We show now that the answer is in the affirmative.

THEOREM 4.1. *Let $Q(\lambda)$ be an $n \times n$ regular quadratic matrix polynomial and, with the notation (3.2)–(3.3), let $\rho = \max\{0, p + n_c - n\}$. When $\rho > 0$ let $\lambda_1 \in \mathbb{R}$ or infinity be an eigenvalue with geometric multiplicity p. Then $Q(\lambda)$ is strongly equivalent to a block-triangular quadratic matrix polynomial*

$$
(4.1) \qquad T(\lambda) = \begin{matrix} {\scriptstyle 2\rho} \\ {\scriptstyle n-2\rho} \end{matrix} \overset{\displaystyle \overset{2\rho \quad\;\; n-2\rho}{}}{\begin{bmatrix} T_1(\lambda) & T_3(\lambda) \\ 0 & T_2(\lambda) \end{bmatrix}},
$$

where $T_2(\lambda)$ is triangular and if $\rho > 0$, then

  (i) *$T_1(\lambda)$ is quasi-triangular with ρ 2×2 diagonal blocks having the property that either all the ρ blocks have elementary divisors $\lambda - \lambda_1$, $\lambda - \lambda_1$, $\lambda^2 + a_i\lambda + b_i$ or all the ρ blocks have elementary divisors μ, μ, $\lambda^2 + a_i\lambda + b_i$, where the polynomials $\lambda^2 + a_i\lambda + b_i$, $i = 1:\rho$, are irreducible over $\mathbb{R}[\lambda]$ and not necessarily distinct;*
 (ii) *$\lambda - \lambda_1$ (2ρ times) are the only real elementary divisors of $T_1(\lambda)$ or μ (2ρ times) are the only infinite elementary divisors of $T_1(\lambda)$;*
(iii) *the real elementary divisors and the elementary divisors at infinity of $Q(\lambda)$ are those of $T_1(\lambda)$ and $T_2(\lambda)$ together with possible repetitions.*

Proof. See Appendix A.1. □

Example 4.2. Let $Q(\lambda)$ be the quadratic of Example 3.8. Its reversal has λ, λ, $\lambda^2 + 1$, and $\lambda^2 + 1$ as elementary divisors and is equivalent to the quasi-triangular quadratic rev $T(\lambda) = \begin{bmatrix} \lambda^2 & -\lambda \\ \lambda & \lambda^2 \end{bmatrix} \oplus [\lambda^2 + 1]$. Hence $T(\lambda) = \begin{bmatrix} 1 & -\lambda \\ \lambda & 1 \end{bmatrix} \oplus [\lambda^2 + 1]$ is a quasi-triangular quadratic matrix polynomial strongly equivalent to $Q(\lambda)$ with the properties of Theorem 4.1.

The next result is a direct consequence of Theorem 4.1 and its proof is similar to that of Corollary 3.4.

COROLLARY 4.3. *A quadratic matrix polynomial $Q(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ with nonsingular leading coefficient is equivalent to a block-triangular quadratic matrix polynomial $T(\lambda)$ as in Theorem 4.1 with off-diagonal elements linear in λ at most.*

Remark 4.4. The 2×2 diagonal blocks of $T(\lambda)$ in Theorem 4.1 appear in the upper left part of the matrix polynomial. We show that $T(\lambda)$ can be modified so that the 2×2 diagonal blocks appear in any desired diagonal position.

1. With real elementary transformations we bring each 2×2 diagonal block of $T(\lambda)$ to the equivalent block $\mathrm{diag}((\lambda - \lambda_1), (\lambda - \lambda_1)(\lambda^2 + a_i\lambda + b_i\lambda))$ so that $T(\lambda)$ is equivalent to a real (not quadratic) triangular matrix $R(\lambda)$, say.
2. Theorem 3.2 guarantees the existence of a triangular matrix, $\widetilde{R}(\lambda)$, equivalent to $R(\lambda)$, with diagonal elements those of $R(\lambda)$ in any desired order. In particular we can arrange the diagonal elements as we wish but keeping together the blocks $\mathrm{diag}((\lambda - \lambda_1), (\lambda - \lambda_1)(\lambda^2 + a_i\lambda + b_i))$.
3. Reversing the transformations of item 1, we recover a real monic block-triangular matrix (perhaps not quadratic) with the 2×2 and 1×1 diagonal blocks in the desired order.
4. For quadratics with nonsingular leading coefficient, the 2×2 diagonal blocks can be constructed so that their nondiagonal entries are of degree less than 2. If the other off-diagonal elements have degree more than 1, then Lemma 2.4 Can be used to reduce their degree below 2.

The nonnegative integer $\rho = \max\{0, p + n_c - n\}$ appearing in Theorem 4.1 is an invariant for the equivalence of quadratic matrix polynomials. It turns out that for a given $n \times n$ quadratic matrix polynomial $Q(\lambda)$, this is the minimum number of 2×2 diagonal blocks that a quasi-triangular quadratic matrix polynomial $T(\lambda)$ equivalent to $Q(\lambda)$ may have. This property plays an important role in the next section. For this purpose we say that a monic quasi-triangular quadratic $T(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ with σ 2×2 diagonal blocks is *irreducible* if any monic quasi-triangular quadratic matrix polynomial equivalent to $T(\lambda)$ has at least σ 2×2 diagonal blocks.

THEOREM 4.5. *Let $Q(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ be quadratic with nonsingular leading coefficient and, with the notation (3.2)–(3.3), let $\rho = \max\{0, p + n_c - n\}$. A monic quasi-triangular quadratic matrix polynomial $T(\lambda)$ equivalent to $Q(\lambda)$ is irreducible if and only if the number of its 2×2 diagonal blocks is ρ.*

Proof. See Appendix A.2. ☐

We conclude this section by pointing out that according to Theorem 4.5 and Lemma A.5, the proof of Theorem 4.1 in Appendix A provides a procedure to construct an irreducible block-triangular quadratic matrix polynomial equivalent to $Q(\lambda)$.

5. A Schur-like theorem for quadratic matrix polynomials.

5.1. The complex case. The theorem of Schur for complex matrices states that for any $A \in \mathbb{C}^{n \times n}$ there is a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that $U^*AU = T$ is a triangular matrix. Schur's theorem can be rewritten as follows.

THEOREM 5.1. *Let $A \in \mathbb{C}^{n \times n}$. There are subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of $\mathbb{C}^n$ satisfying the following:*

(i) $\mathbb{C}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$;
(ii) *the subspaces $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_k$, $k = 1 : n$, are A-invariant;*
(iii) $\mathcal{V}_k = \langle u_k \rangle$, $k = 1 : n$, *where $u_1, \ldots, u_n$ form an orthonormal system of vectors of $\mathbb{C}^n$.*

We investigate in this section how the matrix version ($U^*AU = T$) and the subspaces version (Theorem 5.1) of Schur's theorem extend to any linearizations of quadratic matrix polynomials with nonsingular leading coefficient. Recall that a pencil $\lambda I - A$ is a monic linearization of $Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ if $A \in \mathbb{F}^{2n \times 2n}$ and $\lambda I - A$ has the same elementary divisors as $Q(\lambda)$. For example, the left companion matrix

$$(5.1) \qquad A = \begin{bmatrix} 0 & -A_0 A_2^{-1} \\ I_n & -A_1 A_2^{-1} \end{bmatrix} =: C_L(Q)$$

defines a linearization of $Q(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$.

We start with a Schur-like theorem for quadratic matrix polynomials along the lines of Theorem 5.1.

THEOREM 5.2. *Let $Q(\lambda) \in \mathbb{C}[\lambda]^{n \times n}$ be quadratic with nonsingular leading coefficient and let $\lambda I - A$ be any $2n \times 2n$ monic linearization of $Q(\lambda)$. There are subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of $\mathbb{C}^{2n}$ satisfying the following*:
  (i) $\mathbb{C}^{2n} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$;
  (ii) *the subspaces $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_k$, $k = 1:n$, are A-invariant*;
  (iii) $\dim \mathcal{V}_k = 2$ *and $\mathcal{V}_k = \langle u_k, Au_k \rangle$, $k = 1:n$, where the generating vectors $u_1, \ldots, u_n$ are linearly independent and can be chosen to form an orthonormal system of vectors of $\mathbb{C}^{2n}$.*

Proof. By Corollary 3.4, $Q(\lambda)$ is equivalent to a monic upper triangular quadratic matrix polynomial $T(\lambda) = I\lambda^2 + T_1\lambda + T_0$. Using the permutation matrix,

$$(5.2) \qquad P = \begin{bmatrix} e_1 & e_{n+1} & e_2 & e_{n+2} & \cdots & e_n & e_{2n} \end{bmatrix},$$

where e_i is the ith column of the $2n \times 2n$ identity matrix, we have

$$P^T C_L(T) P = P^T \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix} P =: A_T = (A_{ij})_{1 \le i,j \le n},$$

where, with the notation $T_1 = (t_{ij}^{(1)})_{1 \le i,j \le n}$ and $T_0 = (t_{ij}^{(0)})_{1 \le i,j \le n}$,

$$(5.3) \quad A_{ii} = \begin{bmatrix} 0 & -t_{ii}^{(0)} \\ 1 & -t_{ii}^{(1)} \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} 0 & -t_{ij}^{(0)} \\ 0 & -t_{ij}^{(1)} \end{bmatrix} \text{ for } i < j, \quad A_{ij} = 0_{2 \times 2} \text{ for } i > j.$$

Since $T(\lambda)$ is equivalent to $Q(\lambda)$, $\lambda I - A_T$ is also a monic linearization of $Q(\lambda)$, and hence there is a nonsingular matrix $S \in \mathbb{C}^{2n \times 2n}$ such that $S^{-1}AS = A_T$. The quasi-triangular structure of A_T reveals the existence of invariant subspaces with respect to A. Indeed let $s_i = Se_i$ and $\mathcal{V}_k = \langle s_{2k-1}, s_{2k} \rangle$, $k = 1, \ldots, n$. Then it is plain that $\mathbb{C}^{2n} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$ and since $AS = SA_T$ we have that $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$ is A-invariant. Now, by (5.3) and since $AS = SA_T$,

$$s_{2k} = Se_{2k} = SA_T e_{2k-1} = ASe_{2k-1} = As_{2k-1}.$$

Hence there are linearly independent vectors $s_1, \ldots, s_{2n-1}$ such that $\{s_1, As_1, \ldots, s_{2k-1}, As_{2k-1}\}$ is a basis of $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$. Note that the $\mathcal{V}_k$, $k = 1:n$, are Krylov subspaces of dimension 2.

Let P be the permutation in (5.2) and $X \in \mathbb{C}^{2n \times n}$ be such that $Xe_k = s_{2k-1}$. Then $SP^T = [X \ AX]$ and it follows from $AS = SA_T$ that

$$(5.4) \qquad A[X \ AX] = [X \ AX] \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix}.$$

Now let $X = UR$ be a QR factorization of X. Then (5.4) becomes

$$A[U \ AU] = [U \ AU] \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}^{-1} = [U \ AU] \begin{bmatrix} 0 & -\widetilde{T}_0 \\ I_n & -\widetilde{T}_1 \end{bmatrix},$$

where $\widetilde{T}_0$ and $\widetilde{T}_1$ are still upper triangular. If we let $\widetilde{S} = [U\ AU]P$, then $A\widetilde{S} = \widetilde{S}A_{\widetilde{T}}$, where $A_{\widetilde{T}}$ has the same block structure as A_T. As above, we have that $\mathcal{V}_k = \langle \widetilde{s}_{2k-1}, \widetilde{s}_{2k}\rangle = \langle \widetilde{s}_{2k-1}, A\widetilde{s}_{2k-1}\rangle = \langle u_k, Au_k\rangle$, showing that the generating vectors of the Krylov subspaces $\mathcal{V}_k$ can be taken to form an orthonormal system.   □

When A is the left companion matrix in (5.1), the similarity $[X\ AX]$ in (5.4) is called a *left companion structure preserving similarity* [8, Def. 2.1].

The next result is a matrix version of Theorem 5.2.

THEOREM 5.3. *For any* $2n \times 2n$ *monic linearization* $\lambda I - A$ *of a quadratic* $Q(\lambda) \in \mathbb{C}[\lambda]^{n \times n}$ *with nonsingular leading coefficient, there exists* $U \in \mathbb{C}^{2n \times n}$ *with orthonormal columns such that* $[U\ AU]$ *is nonsingular and*

$$[U\ AU]^{-1}A[U\ AU] = \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix},$$

where $T(\lambda) = I_n\lambda^2 + T_1\lambda + T_0$ *is upper triangular and equivalent to* $Q(\lambda)$.

Proof. Let $U = [u_1\ \dots\ u_n]$ whose columns are the n orthonormal generating vectors defined in Theorem 5.2(iii). By (i) of this same theorem, $S = [U\ AU]$ is nonsingular and if we let $B = S^{-1}AS$, then

$$AS = SB \iff A[U\ AU] = [U\ AU] \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

which implies that $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, that is, B is in left companion form. Theorem 5.2(ii) implies that B_{12} and B_{22} are upper triangular, so if we let $B_{12} = -T_0$ and $B_{22} = -T_1$, then $Q(\lambda)$ is equivalent to the triangular quadratic $T(\lambda) = \lambda^2 I + \lambda T_1 + T_0$.   □

Theorems 5.2 and 5.3 show that to triangularize a quadratic matrix polynomial, it suffices to construct a set of n orthonormal generating vectors.

5.2. The real case. The extension of a Schur theorem from linear to quadratic matrix polynomials is more involved in the real case. Recall that for $A \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ such that $U^T A U = T$, where the real Schur form T is a quasi-triangular matrix with 1×1 and 2×2 diagonal blocks corresponding, respectively, to the real and the nonreal complex conjugate eigenvalues of A. We aim to write this decomposition as a theorem that, properly generalized, reflects the fact that any real quadratic matrix polynomial can be reduced to quasi-triangular form.

THEOREM 5.4. *Let* $A \in \mathbb{R}^{n \times n}$ *and let* ρ *be the number of pairs of nonreal complex conjugate eigenvalues of* A. *Then there are subspaces* $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{n-\rho}$ *of* $\mathbb{R}^n$ *satisfying the following:*
  (i) $\mathbb{R}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_{n-\rho}$;
  (ii) *the subspaces* $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_k$, $k = 1\colon n - \rho$, *are* A-*invariant*;
  (iii) $\mathcal{V}_k = \langle u_{2k-1}, u_{2k}\rangle$, $k = 1\colon \rho$, *and* $\mathcal{V}_{\rho+k} = \langle u_{2\rho+k}\rangle$, $k = 1\colon n - 2\rho$, *where* $u_1, \dots, u_n$ *form an orthonormal basis of* $\mathbb{R}^n$.

Proof. There is an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ such that

$$(5.5) \qquad U^T A U = \begin{array}{c} \\ 2\rho \\ n-2\rho \end{array} \overset{\displaystyle \begin{array}{cc} 2\rho & n-2\rho \end{array}}{\begin{bmatrix} T_1 & T_3 \\ 0 & T_2 \end{bmatrix}},$$

where T_1 is quasi-triangular with ρ 2×2 diagonal blocks containing the pairs of nonreal eigenvalues and T_2 is upper triangular. Properties (i)–(iii) follow by defining $\mathcal{V}_k = \langle u_{2k-1}, u_{2k}\rangle$ for $k = 1\colon \rho$ and $\mathcal{V}_{\rho+k} = \langle u_{2\rho+k}\rangle$ for $k = 1\colon n - 2\rho$.   □

The next theorem can be thought of as a Schur-like theorem for real quadratic matrix polynomials (compared with Theorems 5.2 and 5.4).

THEOREM 5.5. *Let $Q(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ be quadratic with nonsingular leading co-efficient and let $\lambda I - A$ be any $2n \times 2n$ monic linearization of $Q(\lambda)$. Let $\rho = \max\{0, p + n_c - n\}$, where p and n_c are defined in (3.3). Then there are subspaces $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_{n-\rho}$ of $\mathbb{R}^{2n}$ satisfying the following:*

(i) $\mathbb{R}^{2n} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_{n-\rho}$;

(ii) *the subspaces $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$, $k = 1: n - \rho$, are A-invariant;*

(iii) $\dim \mathcal{V}_k = 4$ *with* $\mathcal{V}_k = \langle u_{2k-1}, Au_{2k-1}, u_{2k}, Au_{2k} \rangle$ *for* $k = 1: \rho$ *and* $\dim \mathcal{V}_{\rho+k} = 2$ *with* $\mathcal{V}_{\rho+k} = \langle u_{2\rho+k}, Au_{2\rho+k} \rangle$ *for* $k = 1: n - 2\rho$, *where the generating vectors* $u_1, \ldots, u_n$ *are linearly independent and can be chosen to form an orthonormal system of vectors of $\mathbb{R}^{2n}$.*

Proof. By Theorem 4.1 and Corollary 4.3, $Q(\lambda)$ is equivalent to an irreducible quasi-triangular monic quadratic matrix polynomial $T(\lambda) = I\lambda^2 + T_1\lambda + T_0$ with ρ 2×2 blocks on the diagonal as in (4.1). If we permute the left companion form of $T(\lambda)$ with the permutation P in (5.2) we find that the resulting matrix

$$A_T := P^T \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix} P = \begin{matrix} \\ 4\rho \\ 2\sigma \end{matrix} \overset{\displaystyle 4\rho \quad 2\sigma}{\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}}$$

is upper block-triangular with ρ 4×4 diagonal blocks on the diagonal of A_1 and $\sigma := n - 2\rho$ 2×2 blocks on the diagonal of A_2. The ith 4×4 diagonal block of A_1 with $1 \le i \le \rho$ and the jth 2×2 diagonal block of A_2 with $1 \le j \le \sigma$ are given by, respectively,

$$(A_1)_{ii} = \begin{bmatrix} 0 & -t^{(0)}_{2i-1,2i-1} & 0 & -t^{(0)}_{2i-1,2i} \\ 1 & -t^{(1)}_{2i-1,2i-1} & 0 & -t^{(1)}_{2i-1,2i} \\ 0 & -t^{(0)}_{2i,2i-1} & 0 & -t^{(0)}_{2i,2i} \\ 0 & -t^{(1)}_{2i,2i-1} & 1 & -t^{(1)}_{2i,2i} \end{bmatrix}, \quad (A_2)_{jj} = \begin{bmatrix} 0 & -t^{(0)}_{2\rho+j,2\rho+j} \\ 1 & -t^{(1)}_{2\rho+j,2\rho+j} \end{bmatrix},$$

where $T_1 = \left(t^{(1)}_{ij}\right)_{1 \le i,j \le n}$ and $T_0 = \left(t^{(0)}_{ij}\right)_{1 \le i,j \le n}$ as in the proof of Theorem 5.2. It is easy to check that $A_T e_{2k-1} = e_{2k}$, $k = 1: n$. Since $T(\lambda)$ is equivalent to $Q(\lambda)$, $\lambda I - A_T$ is also a monic linearization of $Q(\lambda)$, and hence there is a nonsingular matrix $S \in \mathbb{C}^{2n \times 2n}$ such that $S^{-1}AS = A_T$. Let $s_k = Se_k$ and $\mathcal{V}_k = \langle s_{4k-3}, s_{4k-2}, s_{4k-1}, s_{4k} \rangle$ for $k = 1: \rho$ and $\mathcal{V}_{\rho+k} = \langle s_{4\rho+2k-1}, s_{4\rho+2k} \rangle$ for $k = 1: n - 2\rho$. Then it is plain that $\mathbb{R}^{2n} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_{n-\rho}$ and since $AS = SA_T$ we have that $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$ is A-invariant. Also, since $A_T e_{2k-1} = e_{2k}$ and $AS = SA_T$, $s_{2k} = SA_T e_{2k-1} = As_{2k-1}$ for $k = 1: n$. Putting $u_k = s_{2k-1}$ property (iii) follows at once. The proof that the generating vectors can be taken orthogonal is analogous to that for the complex case. $\quad\blacksquare$

Remark 5.6. Notice that in Theorem 5.4, ρ is the minimum number for which conditions (i)–(iii) hold true for a given matrix A. In fact, if $\mathbb{R}^n$ admits a decomposition as a direct sum of $n - \rho'$ subspaces satisfying properties (i)–(iii) and $\rho' < \rho$, then A has at most ρ' pairs of nonreal complex conjugate eigenvalues and so $\rho' \ge \rho$, a contradiction.

The nonnegative integer $\rho = \max\{0, p + n_c - n\}$ in Theorem 5.5 has a similar meaning: it is the minimum number for which there are subspaces $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_{n-\rho}$ of $\mathbb{R}^{2n}$ satisfying (i)–(iii) of that theorem. To see this, let $A = C_L(Q)$ be the left companion linearization of $Q(\lambda)$ and assume that $\mathbb{R}^{2n} = \mathcal{V}'_1 \oplus \mathcal{V}'_2 \oplus \cdots \oplus \mathcal{V}'_{n-\rho'}$ is another decomposition of $\mathbb{R}^{2n}$ as a direct sum of subspaces satisfying properties (i)–(iii) with $\rho' < \rho$. Let $X' = [u'_1 \ u'_2 \ \ldots \ u'_n]$, where the u'_j, $j = 1: n$, generate $\mathcal{V}'_1, \mathcal{V}'_2, \ldots, \mathcal{V}'_{n-\rho'}$. Then properties (i)–(iii) in Theorem 5.5 imply that $[X' \ C_L(Q)X']$ is nonsingular and that

$$[X' \ C_L(Q)X']^{-1}C_L(Q)[X' \ C_L(Q)X'] = \begin{bmatrix} 0 & -T_0' \\ I_n & -T_1' \end{bmatrix}$$

with T_0' and T_1' such that $T'(\lambda) = I_n\lambda^2 + T_1'\lambda + T_0'$ is upper quasi-triangular with 2×2 and 1×1 diagonal blocks on the diagonal, the number of 2×2 diagonal blocks being $\rho' < \rho$. But this is impossible because $Q(\lambda)$ and $T'(\lambda)$ are equivalent and by Theorem 4.5 the minimum number of 2×2 diagonal blocks in any monic block-triangular quadratic matrix polynomial $T(\lambda)$ equivalent to $Q(\lambda)$ is ρ.

We now give an analogue of Theorem 5.3 for real quadratic matrix polynomials, which follows from Theorem 5.5 and the above remark.

THEOREM 5.7. *For any monic linearization $\lambda I - A \in \mathbb{R}[\lambda]^{2n \times 2n}$ of a quadratic $Q(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ with nonsingular leading coefficient, there exists $U \in \mathbb{R}^{2n \times n}$ with orthonormal columns such that $[U \ AU]$ is nonsingular and*

$$[U \ AU]^{-1}A[U \ AU] = \begin{bmatrix} 0 & -T_0 \\ I_n & -T_1 \end{bmatrix},$$

where $T(\lambda) = I_n\lambda^2 + T_1\lambda + T_0$ is equivalent to $Q(\lambda)$ and upper quasi-triangular with $\rho = \max\{0, p + n_c - n\}$ 2×2 diagonal blocks, where p and n_c are defined in (3.3).

Appendix A. This appendix deals with the proofs of Theorems 4.1 and 4.5. In order to do so we need some technical lemmas.

LEMMA A.1. *Let an $n \times n$ real quadratic matrix polynomial with nonsingular leading coefficient have the elementary divisors list in (3.2) with*

(A.1) $$p_1 \geq p_2 \geq \cdots \geq p_r, \quad t_1 \geq t_2 \geq \cdots \geq t_s.$$

Let p and n_c be defined as in (3.3). If $n > 2$ and $p > n - n_c$, then
 (i) $n_c > 0$, $p \geq 2$, and $p_i \leq p - 2$, $i = 2, \ldots r$;
 (ii) $m_{1p} = m_{1(p-1)} = 1$, and if $p = n$ and $n_{1t_1} \geq 2$, then $m_{1(p-2)} = 1$ as well;
 (iii) $\max\{p_2, t_1 - 1, t_2\} \leq \begin{cases} n - 3 \ \text{if } n_{1t_1} \geq 2, \\ n - 2 \ \text{otherwise.} \end{cases}$

Proof.

(i) Recall that the largest chain of elementary divisors corresponding to the same real eigenvalue cannot exceed the size of the matrix polynomial, i.e., $p \leq n$. But $p > n - n_c$ so $n_c > n - p \geq 0$. Also, the assumption $p > n - n_c$ implies that

$$2p - 2 \geq 2n - 2n_c = \sum_{i=1}^{r} \sum_{j=1}^{p_i} m_{ij}.$$

Bearing in mind that $p_1 = p$ we get

(A.2) $$p - 2 = 2p - 2 - p \geq \sum_{i=1}^{r} \sum_{j=1}^{p_i} m_{ij} - p = \sum_{j=1}^{p}(m_{1j} - 1) + \sum_{i=2}^{r} \sum_{j=1}^{p_i} m_{ij} =: \Phi.$$

Taking into account that $m_{ij} > 0$ we conclude that $\Phi \geq 0$, that is, $p \geq 2$, and $\Phi \geq \sum_{i=2}^{r} \sum_{j=1}^{p_i} m_{ij} \geq p_i$, $i = 2: r$. Thus property (i) follows at once.

(ii) It also follows from (A.2) that

$$p - 2 \geq \Phi \geq \sum_{j=1}^{p}(m_{1j} - 1) \geq \#\{j : m_{1j} \geq 2\},$$

where # stands for "number of elements of." That is $p-2$ is greater than or equal to the number of elementary divisors that are power of $(\lambda - \lambda_1)$ with exponents at least 2. Since the total number of such elementary divisors is p, we conclude that at least two elementary divisors are equal to $\lambda - \lambda_1$, that is, $m_{1p} = m_{1(p-1)} = 1$.

If $p = n$ and $n_{1t_1} \geq 2$, there are at least three elementary divisors equal to $\lambda - \lambda_1$ because if there are only two, then $m_{11} + \cdots + m_{1p} \geq 2 + 2(n-2)$ and so

$$2n \geq 2n_c + \sum_{j=1}^{p} m_{1j} \geq 2n_{1t_1} + 2(n-2) + 2 \geq 4 + 2n - 4 + 2 = 2n + 2,$$

which is a contradiction.

(iii) By (i), $p_2 \leq n - 2$. If $n_{1t_1} \geq 2$ and $p = n$, then we show that $p_2 \neq n - 2$. Indeed, if $p_2 = n - 2$, then $2n \geq 2n_c + p + p_2 = 2n_c + 2n - 2$ so that $n_c = 1$ and this implies that $n_{1t_1} = 1$, which is a contradiction. If $n_{1t_1} \geq 2$ and $p \leq n - 1$, then by (i), $p_2 \leq p - 2 \leq n - 3$. So $p_2 \leq n - 3$ if $n_{1t_1} \geq 2$ and $p_2 \leq n - 2$ otherwise.

Next we show that $t_1 \leq n - 1$. If $t_1 = n$, then since by (i) $p \geq 2$ we have that

$$2n_c \geq \sum_{i=1}^{t_1} 2n_{1i} \geq 2t_1 = 2n \geq 2n_c + p \geq 2n_c + 2,$$

which is impossible. Hence $t_1 - 1 \leq n - 2$. Now if $n_{1t_1} \geq 2$ and $t_1 = n - 1$, then since $n_{11} \geq \cdots \geq n_{1t_1} \geq 2$, we have that $n_c \geq 2t_1 = 2n - 2$. But $2n \geq 2n_c \geq 4n - 4$, that is, $n \leq 2$ contradicting $n > 2$. Thus when $n_{1t_1} \geq 2$, $t_1 - 1 \leq n - 3$.

Finally, $t_2 \leq t_1 \leq n - 1$ and $t_2 = t_1 = n - 1$ leads to a contradiction because, as $p \geq 2$ and $2n \geq 2n_c + p$, we find that $2n - 2 \geq 2n_c \geq 2(t_1 + t_2) = 4t_1 = 4n - 4$, and so $n \leq 1$ and we are assuming $n > 2$. Thus, if $t_1 = t_2$, then $t_1 = t_2 \leq n - 2$. Now suppose that $n_{1t_1} \geq 2$. If $t_2 = n - 2$, then $t_1 = t_2 = n - 2$ and this leads to a contradiction again. Indeed, since $n_{1j} \geq n_{1t_1} \geq 2$ and $n_{2j} \geq 1$ $j = 1, \ldots, t_2$ we have that $n_c \geq 2(n-2) + (n-2) = 3n - 6$. But $2n > 2n_c$. Then $2n > 6n - 12$ and so $3 > n$ contradicting that $n > 2$. Hence when $n_{1t_1} \geq 2$, $t_2 \leq n - 3$ and this completes the proof. □

LEMMA A.2. *Let $Q(\lambda) \in \mathbb{R}[\lambda]^{n \times n}$ be quadratic with nonsingular leading coefficient and with the elementary divisors in (3.2), where $p_1 \geq p_2 \geq \cdots \geq p_r$ and $t_1 \geq t_2 \geq \cdots \geq t_s$. Assume that $p > n - n_c$, where p and n_c are defined in (3.3). Then $Q(\lambda)$ is equivalent to a real monic quadratic matrix polynomial of the form*

$$\begin{array}{cc} & \begin{array}{cc} 2 & \quad n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \begin{bmatrix} Q_1(\lambda) & X(\lambda) \\ 0 & Q_2(\lambda) \end{bmatrix}, \end{array}$$

where

(i) $Q_1(\lambda)$ *has elementary divisors* $\lambda - \lambda_1, \lambda - \lambda_1$ *and* $\lambda^2 + a_1\lambda + b_1$,
(ii) $Q_2(\lambda)$ *has elementary divisors*

$$(A.3) \qquad \begin{aligned} (\lambda - \lambda_1)^{m_{1j}}, & \qquad j = 1: p - 2, \\ (\lambda - \lambda_i)^{m_{ij}}, & \qquad j = 1: p_i, \ i = 2: r, \\ (\lambda^2 + a_1\lambda + b_1)^{n_{1j}}, & \qquad j = 1: t_1 - 1 \ \text{if} \ t_1 > 1, \\ (\lambda^2 + a_1\lambda + b_1)^{n_{1t_1} - 1} & \qquad \text{if} \ n_{1t_1} > 1, \\ (\lambda^2 + a_i\lambda + b_i)^{n_{ij}}, & \qquad j = 1: t_i, \ i = 2: s. \end{aligned}$$

Proof. The case $n = 1$ does not arise because if $n = 1$, then either $p = 0$ or $n_c = 0$, which cannot happen when $p > n - n_c$. Indeed, $n \geq p$ and $n \geq n_c$. Thus $p > n - n_c \geq 0$ and so $n_c > n - p \geq 0$.

If $n = 2$, then it follows from $p > 0$ and $n_c > 0$ that $p = 2$ and $n_c = 1$ so that the elementary divisors of $Q(\lambda)$ are $\lambda - \lambda_1$, $\lambda - \lambda_1$ and $\lambda^2 + a_1\lambda + b_1$. Hence, if A_2 is the nonsingular leading coefficient of $Q(\lambda)$, then $Q_1(\lambda) = A_2^{-1}Q(\lambda)$ is a matrix with the desired properties.

From now on we assume that $n > 2$. By Lemma A.1(ii), $Q(\lambda)$ has at least two elementary divisors equal to $\lambda - \lambda_1$. Let us define

(A.4) $$S_1(\lambda) = \text{diag}(\lambda - \lambda_1, (\lambda - \lambda_1)(\lambda^2 + a_1\lambda + b_1)).$$

We now split the study into three possible cases according as (a) $n_{1t_1} = 1$, (b) $n_{1t_1} \geq 2$ and $p = n$, and (c) $n_{1t_1} \geq 2$ and $p < n$.

(a) $\underline{n_{1t_1} = 1}$. In this case, we remove $\lambda - \lambda_1$ twice and $\lambda^2 + a_1\lambda + b_1$ from the list of elementary divisors of $Q(\lambda)$ so as to end up with the list of powers of prime polynomials displayed in (A.3). The sum of the degrees of all these polynomials is $2n-4$. We now show that there is an $(n-2) \times (n-2)$ real quadratic matrix polynomial with these polynomials as elementary divisors. In order to apply Lemma 2.3 we use the procedure outlined in (3.4) to construct an invariant factors chain of length $n - 2$. For the procedure to go through, the maximal length of any chain of elementary divisors must be at most $n - 2$. In other words we must show that

(A.5) $$\max\{p - 2, p_2, \ldots, p_r, t_1 - 1, t_2, \ldots, t_s\} \leq n - 2.$$

That (A.5) holds follows directly from (A.1) and Lemma A.1(iii). Hence we can use the procedure in (3.4) with the polynomials in (A.3) to construct an invariant factors chain $\nu_1(\lambda)|\nu_2(\lambda)|\cdots|\nu_{n-2}(\lambda)$ of length $n - 2$. Let us define

$$S_2(\lambda) = \text{diag}\big(\nu_1(\lambda), \nu_2(\lambda), \ldots, \nu_{n-2}(\lambda)\big).$$

Then $Q(\lambda)$ and $\text{diag}(S_1(\lambda), S_2(\lambda))$ are $n \times n$ matrix polynomials with the same elementary divisors, that is, they are equivalent. By Lemma 2.3, $S_1(\lambda)$ and $S_2(\lambda)$ are respectively equivalent to real quadratic matrix polynomials $Q_1(\lambda)$ and $Q_2(\lambda)$. Hence $Q(\lambda)$ is equivalent to $\text{diag}(Q_1(\lambda), Q_2(\lambda))$ and $Q_1(\lambda)$ and $Q_2(\lambda)$ have the desired elementary divisors.

(b) $\underline{n_{1t_1} \geq 2, p = n}$. By Lemma A.1(ii) there are at least three elementary divisors equal to $\lambda - \lambda_1$. Hence, we can remove three copies of $\lambda - \lambda_1$ and $(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}}$ from the list of elementary divisors of $Q(\lambda)$ and construct

$$S_3(\lambda) = (\lambda - \lambda_1) \oplus \begin{bmatrix} (\lambda - \lambda_1)(\lambda^2 + a_1\lambda + b_1) & \lambda - \lambda_1 \\ 0 & (\lambda - \lambda_1)(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}-1} \end{bmatrix}$$

whose elementary divisors are $\lambda - \lambda_1$, $\lambda - \lambda_1$, $\lambda - \lambda_1$, and $(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}}$. The remaining elementary divisors are

(A.6)
$$\begin{aligned}
(\lambda - \lambda_1)^{m_{1j}}, & \quad j = 1: p - 3, \\
(\lambda - \lambda_i)^{m_{ij}}, & \quad j = 1: p_i, \ i = 2: r, \\
(\lambda^2 + a_1\lambda + b_1)^{n_{1j}}, & \quad j = 1: t_1 - 1 \text{ if } t_1 > 1, \\
(\lambda^2 + a_i\lambda + b_i)^{n_{ij}}, & \quad j = 1: t_i, \ i = 2: s.
\end{aligned}$$

On using (A.1) and Lemma A.1(iii) we find that

$$\max\{p - 3, p_2, \ldots, p_r, t_1 - 1, t_2, \ldots, t_s\} \leq n - 3.$$

The procedure in (3.4) allows the construction of an invariant factors chain of length $n - 3$ out of the list of polynomials in (A.6). Let $\nu_1(\lambda)|\nu_2(\lambda)|\cdots|\nu_{n-3}(\lambda)$ be such a chain and define

$$S_4(\lambda) = \mathrm{diag}(\nu_1(\lambda), \nu_2(\lambda), \ldots, \nu_{n-3}(\lambda)).$$

Hence $Q(\lambda)$ is equivalent to $\mathrm{diag}(S_3(\lambda), S_4(\lambda))$. But

$$\mathrm{diag}(S_3(\lambda), S_4(\lambda)) = \begin{bmatrix} S_1(\lambda) & X(\lambda) \\ 0 & S_5(\lambda) \end{bmatrix},$$

where $S_1(\lambda)$ is as in (A.4) and

$$X(\lambda) = \begin{bmatrix} 0 & 0 \\ (\lambda - \lambda_1) & 0 \end{bmatrix}, \quad S_5(\lambda) = \begin{bmatrix} (\lambda - \lambda_1)(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}-1} & 0 \\ 0 & S_4(\lambda) \end{bmatrix}.$$

Notice that the elementary divisors of $S_5(\lambda)$ are those in the list (A.6) together with $\lambda - \lambda_1$ and $(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}-1}$, that is, those of (A.3) when $n_{1t_1} > 1$.

Now, by Lemma 2.3 $S_1(\lambda)$ and $S_4(\lambda)$ are equivalent to 2×2 and $(n-2) \times (n-2)$ real quadratic matrix polynomials $Q_1(\lambda)$ and $Q_2(\lambda)$, respectively. Therefore $Q(\lambda)$ is equivalent to

$$\begin{bmatrix} Q_1(\lambda) & Y(\lambda) \\ 0 & Q_2(\lambda) \end{bmatrix},$$

where $Y(\lambda)$ may have degree greater than 2. If that is the case and taking into account that necessarily the leading coefficient of $Q_1(\lambda)$ is nonsingular (otherwise the degree of its determinant would be smaller than 4), we can use Lemma 2.4 to obtain a matrix equivalent to $Q(\lambda)$ with the same diagonal blocks and whose off-diagonal block is of degree smaller than 2. Such a matrix has the desired properties.

(c) $\underline{n_{1t_1} \geq 2, \, p < n}$. In this case we remove $\lambda - \lambda_1$ twice and $(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}}$ from the list of elementary divisors of $Q(\lambda)$ to obtain

(A.7)
$$\begin{aligned} (\lambda - \lambda_1)^{m_{1j}}, & \quad j = 1{:}\, p - 2, \\ (\lambda - \lambda_i)^{m_{ij}}, & \quad j = 1{:}\, p_i, \; i = 2{:}\, r, \\ (\lambda^2 + a_1\lambda + b_1)^{n_{1j}}, & \quad j = 1{:}\, t_1 - 1, \\ (\lambda^2 + a_i\lambda + b_i)^{n_{ij}}, & \quad j = 1{:}\, t_i, \; i = 2{:}\, s. \end{aligned}$$

Once more, (A.1), $p < n$, and Lemma A.1(iii) imply that $\max\{p - 2, p_2, \ldots, p_r, t_1 - 1, t_2, \ldots, t_s\} \leq n - 3$. We now use the procedure in (3.4) to construct an invariant factors chain of length $n-3$ out of the list of polynomials in (A.7). Let $\nu_1(\lambda)|\nu_2(\lambda)|\cdots|\nu_{n-3}(\lambda)$ be such a chain and define $S_7 = \mathrm{diag}(\nu_1(\lambda), \nu_2(\lambda), \ldots, \nu_{n-3}(\lambda))$. Let

$$S_6(\lambda) = (\lambda - \lambda_1) \oplus \begin{bmatrix} (\lambda - \lambda_1)(\lambda^2 + a_1\lambda + b_1) & 1 \\ 0 & (\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}-1} \end{bmatrix}$$

with elementary divisors $\lambda - \lambda_1$, $\lambda - \lambda_1$, and $(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}}$. Hence $Q(\lambda)$ is equivalent to $S_6(\lambda) \oplus S_7(\lambda)$. But

$$S_6(\lambda) \oplus S_7(\lambda) = \begin{bmatrix} S_1(\lambda) & X(\lambda) \\ 0 & S_8(\lambda) \end{bmatrix},$$

where $S_1(\lambda)$ is as in (A.4) and

$$X(\lambda) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_8(\lambda) = \begin{bmatrix} (\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}-1} & 0 \\ 0 & S_7(\lambda) \end{bmatrix}.$$

Notice that the elementary divisors of $S_8(\lambda)$ are the polynomials in the list (A.7) together with $(\lambda^2 + a_1\lambda + b_1)^{n_{1t_1}-1}$, that is, those of (A.3) when $n_{1t_1} > 1$. Now we conclude the proof as in the case $n_{1t_1} \geq 2$ and $p = n$.   □

We will need the following lemma, which is a translation into the language of matrix polynomials of a theorem by Carlson given in terms of finite generated torsion modules [5, Thm. 1].

LEMMA A.3. *Let* $T(\lambda) = \begin{bmatrix} T_1(\lambda) & X(\lambda) \\ 0 & T_2(\lambda) \end{bmatrix}$ *be a nonsingular matrix polynomial with* $T_1(\lambda) \in \mathbb{F}[\lambda]^{p \times p}$ *and* $T_2(\lambda) \in \mathbb{F}[\lambda]^{q \times q}$. *Assume that* λ_0 *is an eigenvalue of* $T(\lambda)$ *and let* $d_1 \geq d_2 \geq \cdots \geq d_r > 0$, $f_1 \geq f_2 \geq \cdots \geq f_s > 0$, *and* $g_1 \geq g_2 \geq \cdots \geq g_t > 0$ *be the exponents of the elementary divisors of* $T(\lambda)$, $T_1(\lambda)$, *and* $T_2(\lambda)$, *respectively, associated to the irreducible polynomial* $(\lambda - \lambda_0)$. *Then* $r \geq s$, $r \geq t$, $r \leq s + t$, *and*

$$d_i \geq f_i \geq d_{i+t} \quad and \quad d_i \geq g_i \geq d_{i+s}, \quad i = 1, 2, \dots.$$

In Lemma A.3 we note that if λ_0 is not an eigenvalue of $T_1(\lambda)$ ($T_2(\lambda)$), then $s = 0$ ($t = 0$, respectively). In that case $d_i = f_i$ ($d_i = g_i$) for all i, that is, $T(\lambda)$ and $T_1(\lambda)$ ($T(\lambda)$ and $T_2(\lambda)$, respectively) have the same elementary divisors at λ_0.

The procedure in the proof of the next lemma will be used to reduce the degree of the off-diagonal entries of upper triangular matrix polynomials.

LEMMA A.4. *Let* $T(\lambda) = (t_{ij}(\lambda)) \in \mathbb{F}[\lambda]^{n \times n}$ *be an upper triangular regular matrix polynomial. There are upper triangular unimodular matrices* $U(\lambda)$ *and* $V(\lambda)$ *with unit diagonal entries such that if* $\widetilde{T}(\lambda) = U(\lambda)T(\lambda)V(\lambda) = (\widetilde{t}_{ij}(\lambda))$, *then* $t_{ii}(\lambda) = \widetilde{t}_{ii}(\lambda)$ *for* $i = 1: n$ *and* $\deg(\widetilde{t}_{ij}(\lambda)) < \deg(\gcd(t_{ii}(\lambda), t_{jj}(\lambda)))$ *for* $1 \leq i < j \leq n$.

Proof. We describe a procedure to obtain $\widetilde{T}(\lambda)$. Let $j = 2$, $i = 1$.

Step 1. Compute $d_{ij}(\lambda) = \gcd(t_{ii}(\lambda), t_{jj}(\lambda))$.

Step 2. Compute the Euclidean division $t_{ij}(\lambda) = q_{ij}(\lambda)d_{ij}(\lambda) + \widetilde{t}_{ij}(\lambda)$.

Step 3. Compute polynomials $x_{ij}(\lambda)$, $y_{ij}(\lambda)$ such that $x_{ij}(\lambda)t_{ii}(\lambda) + y_{ij}(\lambda)t_{jj}(\lambda) = -q_{ij}(\lambda)d_{ij}(\lambda)$.

Step 4. Add column i multiplied by $x_{ij}(\lambda)$ to column j and row j multiplied by $y_{ij}(\lambda)$ to row i.

Step 5. If $i > 1$, then $i = i - 1$; else $j = j + 1$, $i = j - 1$. If $j \leq n$ go to Step 1; else stop.   □

A.1. Proof of Theorem 4.1. By Theorem 3.6 we already know that if $p \leq n - n_c$, then $Q(\lambda)$ is equivalent over $\mathbb{R}[\lambda]$ to a triangular quadratic matrix polynomial. In this case $T_1(\lambda)$ in (4.1) vanishes and so the number of 2×2 diagonal blocks in the "block-triangular" form of $Q(\lambda)$ is $0 = \max\{0, p + n_c - n\}$.

Let us assume from now on that $p > n - n_c$ and recall that by Lemma A.1(i) this implies that $p > 0$ and $n_c > 0$. We also assume that the leading matrix coefficient of $Q(\lambda)$ is nonsingular. We proceed by induction on n.

If $n = 1$, then either $p = 0$ or $n_c = 0$ so that $p \leq n - n_c$. Hence we start with $n = 2$. Since $p > 0$ and $n_c > 0$ the only possibility for $p > n - n_c$ is $p = 2$ and $n_c = 1$ so that the elementary divisors of $Q(\lambda)$ are $\lambda - \lambda_1$, $\lambda - \lambda_1$ and $\lambda^2 + a_1\lambda + b_1$ with $\lambda_1 \in \mathbb{R}$ and $a_1^2 - 4b_1 < 0$. The matrix $Q_1(\lambda)$ of Lemma A.2 is monic and with the same elementary divisors as $Q(\lambda)$. Thus $T_2(\lambda)$ vanishes, $T_1(\lambda) = Q_1(\lambda)$, $\rho = \max\{0, p + n_c - n\} = 1$, and the real elementary divisors of $T_1(\lambda)$ are $\lambda - \lambda_1$ twice.

Let us assume now that $n > 2$ and that the theorem holds for any real quadratic matrix polynomial of size at most $n-1$. We can assume without lost of generality that $p = p_1 \geq p_2 \geq \cdots \geq p_r$ and $t_1 \geq t_2 \geq \cdots \geq t_s$. By Lemma A.2, $Q(\lambda)$ is equivalent to a monic real quadratic block-triangular matrix polynomial

$$
(A.8) \qquad \widetilde{Q}(\lambda) = \begin{bmatrix} Q_1(\lambda) & X(\lambda) \\ 0 & Q_2(\lambda) \end{bmatrix},
$$

where $Q_1(\lambda)$ is of size 2×2 with elementary divisors $\lambda - \lambda_1, \lambda - \lambda_1$, and $\lambda^2 + a_1 \lambda + b_1$, and $Q_2(\lambda)$ is of size $(n-2) \times (n-2)$ with elementary divisors listed in (A.3). By Lemma A.2, the real elementary divisors of $\widetilde{Q}(\lambda)$ are of those of $Q_1(\lambda)$ together with those of $Q_2(\lambda)$.

Let $\widetilde{n}_c$ be the sum of the exponents of the nonreal elementary divisors of $Q_2(\lambda)$ and $\widetilde{p}$ be the maximal length of the chains of its real elementary divisors. Then by Lemma A.1(i) and (A.3),

$$
\widetilde{n}_c = n_c - 1, \quad \widetilde{p} = \widetilde{p}_1 = p - 2.
$$

Now if $\widetilde{p} = (n-2) - \widetilde{n}_c$, then, by Theorem 3.6, $Q_2(\lambda)$ is triangularizable over $\mathbb{R}[\lambda]$ and $Q(\lambda)$ is equivalent to

$$
T(\lambda) = \begin{bmatrix} T_1(\lambda) & T_3(\lambda) \\ 0 & T_2(\lambda) \end{bmatrix},
$$

where $T_1(\lambda) = Q_1(\lambda)$ and $T_2(\lambda)$ is triangular. In this case $\rho = p + n_c - n = 1$, $T_1(\lambda)$ is 2×2, and its invariant factors are $(\lambda - \lambda_1) | (\lambda - \lambda_1)(\lambda^2 + a_1 \lambda + b_1)$. The real elementary divisors of $T(\lambda)$ are the real elementary divisors of $T_1(\lambda)$ and those of $T_2(\lambda)$. Hence $T(\lambda)$ satisfies all requirements except, perhaps, that $T_3(\lambda)$ may have degree bigger than 2. In this case we can use Lemma 2.4 to reduce the degree of $T_3(\lambda)$ below 2.

If $\widetilde{p} > (n-2) - \widetilde{n}_c$, then by the induction hypothesis $Q_2(\lambda)$ in (A.8) is equivalent to

$$
\widetilde{Q}_2(\lambda) = \begin{bmatrix} T_{11}(\lambda) & Y(\lambda) \\ 0 & T_{22}(\lambda) \end{bmatrix},
$$

where $T_{22}(\lambda)$ is triangular and $T_{11}(\lambda)$ is quasi-triangular with $\widetilde{\rho} = \widetilde{p} + \widetilde{n}_c - (n-2) = p + n_c - n - 1 = \rho - 1$ diagonal blocks of size 2×2 having $(\lambda - \lambda_1) | (\lambda - \lambda_1)(\lambda^2 + a_i \lambda + b_i)$, $1 \leq i \leq \widetilde{\rho}$, as invariant factors, with polynomials $\lambda^2 + a_i \lambda + b_i$ irreducible over $\mathbb{R}[\lambda]$ and not necessarily distinct. The real elementary divisors of $\widetilde{Q}_2(\lambda)$ are those of $T_{11}(\lambda)$, i.e., $\lambda - \lambda_1$, $2\widetilde{\rho}$ times, and those of $T_{22}(\lambda)$ together with possible repetitions. Since $\widetilde{Q}_2(\lambda)$ has $p - 2$ elementary divisors that are powers of $\lambda - \lambda_1$, $T_{22}(\lambda)$ has $p - 2 - 2\widetilde{\rho}$ elementary divisors that are powers of $\lambda - \lambda_1$. Therefore, after using Lemma 2.4, if necessary, to reduce the degree of the elements in the off-diagonal blocks, we get a matrix

$$
(A.9) \qquad T(\lambda) = \begin{bmatrix} Q_1(\lambda) & X_1(\lambda) & X_2(\lambda) \\ 0 & T_{11}(\lambda) & Y(\lambda) \\ 0 & 0 & T_{22}(\lambda) \end{bmatrix} = \begin{array}{c} 2\rho \\ n-2\rho \end{array} \begin{bmatrix} \overset{2\rho}{T_1(\lambda)} & \overset{n-2\rho}{\widetilde{Y}(\lambda)} \\ 0 & T_2(\lambda) \end{bmatrix}
$$

that is quadratic and equivalent to $Q(\lambda)$. Let $(f_1, \ldots, f_u)$ be the exponents of the real elementary divisors of $T_1(\lambda)$ (which are all powers of $(\lambda - \lambda_1)$). Applying Lemma A.3

to $T_1(\lambda)$ with respect to the prime polynomial $(\lambda - \lambda_1)$ it follows that, for $(\lambda - \lambda_1)$, the number of elementary divisors of $Q_1(\lambda)$ plus the number of those of $T_{11}(\lambda)$ is not smaller that the number of elementary divisors of $T_1(\lambda)$, that is, $2 + 2\widetilde{\rho} \geq u$. Applying the same result to $T(\lambda)$ (with $T_1(\lambda)$ in the upper left position) we have that $u + p - 2 - 2\widetilde{\rho} \geq p$. Hence, $u = 2 + 2\widetilde{\rho}$. Now, by Lemma A.3, $f_i \geq 1$ for $i = 1, \ldots, 2 + 2\widetilde{\rho}$ and f_i cannot be greater than 1 because otherwise the exponent of $(\lambda - \lambda_1)$ in $\det T_1(\lambda)$ would be greater that $2 + 2\widetilde{\rho}$, contradicting that $\det T_1(\lambda) = \det Q_1(\lambda) \det T_{11}(\lambda)$. This proves (i) and (ii).

Since the real elementary divisors of $\widetilde{Q}(\lambda)$ are those of $Q_1(\lambda)$ and $\widetilde{Q}_2(\lambda)$ together with possible repetitions, the real elementary divisors of $\widetilde{Q}_2(\lambda)$ are those of $T_{11}(\lambda)$ and $T_2(\lambda)$ together with possible repetitions, and the real elementary divisors of $T_1(\lambda)$ are those of $Q_1(\lambda)$ and $T_{11}(\lambda)$ together with possible repetitions, then the real elementary divisors of $T(\lambda)$ are those of $T_1(\lambda)$ and $T_2(\lambda)$. This proves (iii). In conclusion, $T(\lambda)$ in (A.9) is the desired matrix.

Using Proposition 2.1 as in Theorems 3.3 and 3.6 we prove that the theorem also holds when the leading matrix coefficient is singular. This completes the proof of Theorem 4.1   □

A.2. Proof of Theorem 4.5. The proof of this theorem is based on the following lemma.

Lemma A.5. *Let*

$$T(\lambda) = \begin{array}{c} {\scriptstyle 2\sigma} \\ {\scriptstyle n-2\sigma} \end{array} \begin{matrix} {\scriptstyle 2\sigma} \qquad {\scriptstyle n-2\sigma} \\ \begin{bmatrix} T_1(\lambda) & T_3(\lambda) \\ 0 & T_2(\lambda) \end{bmatrix} \end{matrix}, \quad \sigma > 0,$$

be a monic quadratic matrix polynomial with $T_1(\lambda)$ quasi-triangular with σ 2×2 diagonal blocks and $T_2(\lambda)$ triangular. If $T(\lambda)$ is irreducible, then

(i) *the invariant factors of the ith 2×2 diagonal block of $T_1(\lambda)$ are $(\lambda - \lambda_1)|(\lambda - \lambda_1)(\lambda^2 + a_i\lambda + b_i)$, $i = 1 : \sigma$, where λ_1 is a real eigenvalue of $T(\lambda)$ associated to a chain of elementary divisors of maximal length and the polynomials $\lambda^2 + a_i\lambda + b_i$ are irreducible over $\mathbb{R}[\lambda]$ and not necessarily distinct;*

(ii) *$\lambda - \lambda_1, \ldots, \lambda - \lambda_1$ (2σ times) are the only real elementary divisors of $T_1(\lambda)$;*

(iii) *the real elementary divisors of $T(\lambda)$ are the real elementary divisors of $T_1(\lambda)$ and $T_2(\lambda)$ together with possible repetitions.*

Proof. We start by proving (i) and (ii). Let $B(\lambda)$ be any 2×2 diagonal block of $T_1(\lambda)$. It has been seen in the proof of Theorem 4.1 that $B(\lambda)$ is irreducible over $\mathbb{R}[\lambda]$ if and only if its elementary divisors are $\lambda - \widetilde{\lambda}$ twice and $\lambda^2 + a\lambda + b$, where $\widetilde{\lambda}$ is the real eigenvalue of $B(\lambda)$ and $a^2 - 4b < 0$. Hence, if $T(\lambda)$ is irreducible, then the elementary divisors of each 2×2 diagonal block must be of the required form.

We now show that the real eigenvalues of the 2×2 diagonal blocks are all equal to, say, λ_1. Let $B_1(\lambda)$ and $B_2(\lambda)$ be 2×2 diagonal blocks of $T(\lambda)$ with real eigenvalue λ_1 and λ_2, respectively. By Remark 4.4, $B_1(\lambda)$ and $B_2(\lambda)$ can be assumed to be consecutive blocks. If $\lambda_1 \neq \lambda_2$, then the maximal possible length of the chain of elementary divisors associated to λ_1 and λ_2 in

$$B(\lambda) = \begin{bmatrix} B_1(\lambda) & X(\lambda) \\ 0 & B_2(\lambda) \end{bmatrix}$$

is 2 for any matrix polynomial $X(\lambda)$. For $B(\lambda)$, $n = 4$, $n_c = 2$, and $n - n_c = 2 \geq p$. This means that $B(\lambda)$ is triangularizable for any $X(\lambda)$. Therefore $T(\lambda)$ is not irreducible, a contradiction.

We prove next that $(\lambda - \lambda_1), \ldots, (\lambda - \lambda_1)$ (2σ times) are the only real elementary divisors of $T_1(\lambda)$. If $(\lambda - \lambda_1)^v$ is an elementary divisor of $T_1(\lambda)$ with $v > 1$, then $T_1(\lambda)$ is equivalent to

$$
\begin{bmatrix}
(\lambda - \lambda_1)^2 & 0 & 1 & 0 \\
0 & \lambda^2 + a_1\lambda + b_1 & 0 & 0 \\
0 & 0 & (\lambda - \lambda_1)^{v-2} & 0 \\
0 & 0 & 0 & \widetilde{C}(\lambda)
\end{bmatrix},
$$

where $\widetilde{C}(\lambda)$ is any matrix polynomial whose elementary divisors are those of $T_1(\lambda)$ but $(\lambda - \lambda_1)^v$ and the irreducible $\lambda^2 + a_1\lambda + b_1$. If $\widehat{C}(\lambda) = \begin{bmatrix} (\lambda - \lambda_1)^{v-2} & 0 \\ 0 & \widetilde{C}(\lambda) \end{bmatrix}$, then $\deg \det \widehat{C}(\lambda) = \deg \det T_1(\lambda) - 4 = 4(\sigma - 1)$. So, by Lemma 2.3 and Theorem 4.1, $\widehat{C}(\lambda)$ is equivalent to a $(2\sigma - 2) \times (2\sigma - 2)$ block-triangular quadratic matrix polynomial with 2×2 and 1×1 diagonal blocks, $C(\lambda)$, such that the number of 2×2 diagonal blocks is $\sigma - 1$ at most. Hence $T_1(\lambda)$ is equivalent to

$$
\begin{bmatrix}
(\lambda - \lambda_1)^2 & 0 & x(\lambda) \\
0 & \lambda^2 + a_1\lambda + b_1 & 0 \\
0 & 0 & C(\lambda)
\end{bmatrix},
$$

and using Lemma 2.4, if necessary, we obtain a quadratic matrix polynomial equivalent to $T_1(\lambda)$ with less than σ blocks of size 2×2 in the diagonal. This is a contradiction.

It remains to show that λ_1 is the real eigenvalue of $T(\lambda)$ associated to a chain of elementary divisors of maximal length. In fact, assume that this is not the case and let p be the maximal length of the chains of elementary divisors associated to real eigenvalues. By Lemma A.3 these polynomials must be elementary divisors of $T_2(\lambda)$ because λ_1 is the only real eigenvalue of $T_1(\lambda)$. Now, $T_2(\lambda)$ is triangular of size $n - 2\sigma$ and the sum of the degrees of the quadratic elementary divisors is $n_c - \sigma$. Hence, by Theorem 3.6, $p + n_c - \sigma \le n - 2\sigma$. Thus $n > p + n_c$ and by Theorem 3.6, $T(\lambda)$ is triangularizable, again a contradiction.

(iii) As above, the elementary divisors of $T(\lambda)$ associated with real eigenvalues different from λ_1 are those of $T_2(\lambda)$. Hence it is enough to show that the elementary divisors of $T(\lambda)$ associated with λ_1 are $(\lambda - \lambda_1)$ 2σ times together with the elementary divisors of $T_2(\lambda)$ which are powers of $(\lambda - \lambda_1)$. Let $T_4(\lambda) \in \mathbb{R}[\lambda]^{q \times q}$ be the submatrix of $T_2(\lambda)$ whose diagonal elements are of the form $(\lambda - \lambda_1)(\lambda - \widetilde{\lambda}_i)$, where $\widetilde{\lambda}_i$ represents any real eigenvalue of $T(\lambda)$ including, possibly, $\widetilde{\lambda}_i = \lambda_1$. By Remark 4.4 we can assume that $T_2(\lambda) = \begin{bmatrix} T_4(\lambda) & T_7(\lambda) \\ 0 & T_8(\lambda) \end{bmatrix}$ and write

$$
T(\lambda) = \begin{bmatrix}
T_1(\lambda) & T_5(\lambda) & T_6(\lambda) \\
0 & T_4(\lambda) & T_7(\lambda) \\
0 & 0 & T_8(\lambda)
\end{bmatrix}.
$$

It remains to show that $\widetilde{T}(\lambda) = \begin{bmatrix} T_1(\lambda) & T_5(\lambda) \\ 0 & T_4(\lambda) \end{bmatrix}$ is equivalent to $\mathrm{diag}(T_1(\lambda), T_4(\lambda))$ so that the elementary divisors of $T(\lambda)$ are those of $T_1(\lambda)$ and $T_2(\lambda)$ together with possible repetitions. For this let $T_{ii}^{(1)}(\lambda)$ be the ith 2×2 diagonal block of $T_1(\lambda)$ with invariant factors $(\lambda - \lambda_1) | (\lambda - \lambda_1)(\lambda^2 + a_i\lambda + b_i)$. Let $U(\lambda) = \mathrm{diag}(U_1(\lambda), \ldots, U_\sigma(\lambda))$ and $V(\lambda) = \mathrm{diag}(V_1(\lambda), \ldots, V_\sigma(\lambda))$ be unimodular matrices with $U_i(\lambda), V_i(\lambda)$ such that

$$
D_i(\lambda) = U_i(\lambda) T_{ii}^{(1)}(\lambda) V_i(\lambda) = \mathrm{diag}\big((\lambda - \lambda_1), (\lambda - \lambda_1)(\lambda^2 + a_i\lambda + b_i)\big), \quad i = 1 : \sigma.
$$

Then

$$\widetilde{T}_1(\lambda) = U(\lambda)T_1(\lambda)V(\lambda) = \begin{bmatrix} D_1(\lambda) & X_{12}(\lambda) & \cdots & X_{1\sigma}(\lambda) \\ & D_2(\lambda) & \ddots & \vdots \\ & & \ddots & X_{\sigma-1,\sigma}(\lambda) \\ & & & D_\sigma(\lambda) \end{bmatrix}.$$

If $T_4(\lambda) = (t_{ij}^{(4)})$ with $t_{ii}^{(4)}(\lambda) = (\lambda - \lambda_1)(\lambda - \widetilde{\lambda}_i)$, then the procedure described in the proof of Lemma A.4 allows us to construct matrices $Z_1(\lambda), Z_2(\lambda) \in \mathbb{R}[\lambda]^{2\sigma \times q}$ such that

$$(A.10) \quad \widehat{T}(\lambda) = \begin{bmatrix} U(\lambda) & Z_1(\lambda) \\ 0 & I_q \end{bmatrix} \widetilde{T}(\lambda) \begin{bmatrix} V(\lambda) & V(\lambda)Z_2(\lambda) \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} \widetilde{T}_1(\lambda) & Y(\lambda) \\ 0 & T_4(\lambda) \end{bmatrix},$$

where

$$\deg(Y_{ij}(\lambda)) < \begin{cases} \deg\big(\gcd(\lambda - \lambda_1, t_{jj}^{(4)}(\lambda))\big) = \deg(\lambda - \lambda_1) \text{ if } i \text{ is odd,} \\ \deg\big(\gcd((\lambda - \lambda_1)(\lambda^2 + a_i\lambda + b_i), t_{jj}^{(4)}(\lambda))\big) = \deg(\lambda - \lambda_1) \text{ if } i \text{ is even,} \end{cases}$$

i.e., the elements of $Y(\lambda)$ are constant polynomials. We aim to show that, actually, $Y(\lambda) = 0$. In fact, consider the submatrix

$$\widehat{T}_k(\lambda) = \begin{bmatrix} \lambda - \lambda_1 & 0 & y_{2k-1,j} \\ 0 & (\lambda - \lambda_1)(\lambda^2 + a_k\lambda + b_k) & y_{2k,j} \\ 0 & 0 & (\lambda - \lambda_1)(\lambda - \widetilde{\lambda}_j) \end{bmatrix}.$$

If $y_{2k-1,j} \neq 0$ or $y_{2k,j} \neq 0$, then λ_1 has geometric multiplicity 2 and the only possible Smith form for $\widehat{T}_k(\lambda)$ is

$$(A.11) \qquad \text{diag}(1, (\lambda - \lambda_1), (\lambda - \widetilde{\lambda}_j)(\lambda - \lambda_1)^2(\lambda^2 + a_k\lambda + b_k).$$

Using the Gohberg–Lancaster–Rodman procedure in section 3, we find that (A.11) is equivalent to

$$\begin{bmatrix} (\lambda - \lambda_1)^2 & (\lambda - \lambda_1) & \\ & -(\lambda - \lambda_1)(\lambda - \widetilde{\lambda}_j) & 1 \\ & & -(\lambda^2 + a_k\lambda + b_k) \end{bmatrix}.$$

So there are unimodular matrices $\widetilde{U}_k(\lambda), \widetilde{V}_k(\lambda)$ such that $A_k(\lambda) = \widetilde{U}_k(\lambda)\widehat{T}_k(\lambda)\widetilde{V}_k(\lambda) \in \mathbb{R}[\lambda]^{3\times 3}$ is upper triangular and quadratic. By Remark 4.4 we can assume, for notational simplicity, that $k = \sigma$ and $j = 1$. Thus, if

$$\widetilde{U}(\lambda) = \text{diag}(U_1(\lambda)^{-1}, \ldots, U_{\sigma-1}^{-1}(\lambda), \widetilde{U}_\sigma(\lambda), I_{q-1}),$$
$$\widetilde{V}(\lambda) = \text{diag}(V_1(\lambda)^{-1}, \ldots, V_{\sigma-1}^{-1}(\lambda), \widetilde{V}_\sigma(\lambda), I_{q-1}),$$

then

$$\widetilde{U}(\lambda)\widehat{T}(\lambda)\widetilde{V}(\lambda) = \begin{bmatrix} T_{11}^{(1)}(\lambda) & R_{12}(\lambda) & \cdots & R_{1\,\sigma-1}(\lambda) & R_{1\,\sigma}(\lambda) & R_{1\,\sigma+1}(\lambda) \\ 0 & T_{22}^{(1)}(\lambda) & \cdots & R_{2\,\sigma-1}(\lambda) & R_{2\,\sigma}(\lambda) & R_{2\,\sigma+1}(\lambda) \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{\sigma-1\,\sigma-1}^{(1)}(\lambda) & R_{\sigma-1\,\sigma}(\lambda) & R_{\sigma-1\,\sigma+1}(\lambda) \\ 0 & 0 & \cdots & 0 & A_\sigma(\lambda) & R_{\sigma\,\sigma+1}(\lambda) \\ 0 & 0 & \cdots & 0 & 0 & T_{41}(\lambda) \end{bmatrix},$$

where $T_{41}(\lambda)$ is the upper triangular submatrix of $T_4(\lambda)$ obtained by removing its first row and column. By Lemma 2.4 we can assume that the off-diagonal blocks $R_{ij}(\lambda)$ are of degree 1 at most. It is easily seen now that $T(\lambda)$ is equivalent to a triangular quadratic matrix polynomial with less than σ diagonal blocks of size 2×2, contradicting that it is irreducible. In conclusion, $Y(\lambda) = 0$ in (A.10) and $\widetilde{T}(\lambda)$ is equivalent to $\mathrm{diag}(T_1(\lambda), T_4(\lambda))$ as desired.   □

Proof of Theorem 4.5. Let $T(\lambda)$ be an irreducible monic quasi-triangular quadratic matrix polynomial equivalent to $Q(\lambda)$ and let σ be the number of 2×2 diagonal blocks of $T(\lambda)$. First, if $T(\lambda)$ is triangular, then $\sigma = 0$ and there is nothing to prove. So, we analyze the case $\sigma > 0$ so that $Q(\lambda)$ is not triangularizable over $\mathbb{R}[\lambda]$ and $\rho = p + n_c - n > 0$. We show that $\sigma > \rho$ leads to a contradiction. In fact, if $\sigma > \rho$, then, by Theorem 4.1, $T(\lambda)$ is equivalent to a quasi-triangular matrix with ρ diagonal blocks of size 2×2, contradicting that $T(\lambda)$ is irreducible. Therefore $\sigma \le \rho$. Now, by Remark 4.4 , we can assume that

$$T(\lambda) = \left[\begin{array}{cc} T_1(\lambda) & T_3(\lambda) \\ 0 & T_2(\lambda) \end{array} \right],$$

where $T_1(\lambda)$ and $T_2(\lambda)$ have the properties of Lemma A.5. Then, $T_2(\lambda)$ is a triangular matrix of size $(n - 2\sigma) \times (n - 2\sigma)$, the number of its elementary divisors which are powers of $(\lambda - \lambda_1)$ is $p - 2\sigma$, and the sum of the degrees of its elementary divisors which are powers of irreducible quadratic polynomials over $\mathbb{R}[\lambda]$ is $n_c - \sigma$. By Theorem 3.6, $n - 2\sigma \ge (p - 2\sigma) + n_c - \sigma$, that is, $n \ge p + n_c - \sigma$, or equivalently, $\sigma \ge p + n_c - n = \rho$. Since $\sigma \le \rho$ we conclude that $\sigma = \rho$, as desired.

The converse follows now from the already proved fact that there cannot be monic quasi-triangular quadratic matrices equivalent to $Q(\lambda)$ with less than ρ diagonal blocks of size 2×2.   □

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