Vera W. de Spinadel

"Triangulature" in Andrea Palladio

At the June 1998 workshop on the architecture of Andrea Palladio, the dimensions of the rooms were much remarked. Vera Spinadel convincingly argues that Palladio used precise mathematical relationships as a basis for selecting the numerical dimensions for the rooms in this villas. The integer dimensions are demonstrated to be approximants linked to continued equations, and a particular way of deriving these integers through the use of a continued fraction expansion that approximates by excess is introduced.

In a paper published in the *Journal of the Society of Architectural Historians* [1], the Yugoslavian Branko Mitrović claims that in Palladio's plans there exist six unexplained ratios that are close to $\sqrt{3}:1$, with the closest being the four large corner rooms of the Villa Rotonda. Each corner room has dimensions 26 x 15, which differ from $\sqrt{3}:1$ by only 0.07%! This astonishingly close approximation to $\sqrt{3}$ could be a very good reason for suggesting that Palladio used, on some occasions, a system of proportions based on the height of the equilateral triangle or "triangulature".

But the numbers 26 and 15 in the drawings of the Rotonda were not chosen arbitrarily. In fact, let us look for the positive solution of the quadratic equation

(1) $$x^2 - 4x + 1 = 0$$

It can be written in the form

(2) $$x^2 = 4x - 1$$

or, dividing by x (different from zero)

$$x = 4 - \frac{1}{x}$$

Then it is possible to get the continued fraction expansion

$$x = 4 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \ddots}}}$$

where we have relaxed the condition that the terms of the continued fraction have to be positive.[2]

Let us now introduce the notation

$$x = [\,\overline{4-}\,]$$

for this purely periodic continued fraction expansion.
The positive solution of equation (1) is

$$x = 2 + \sqrt{3}$$

from which we get

$$\sqrt{3} = 2 - \cfrac{1}{4 - \cfrac{1}{4 - \ddots}} = [2-, \overline{4-}]$$

This periodic continued fraction expansion has the following rational approximants:

$$\sigma(1)=7/4=1.75$$

$$\sigma(2)=26/15=1.7333...$$

$$\sigma(3)=97/56=1.7321428...$$

$$\sigma(4)= 362/209= 1.7320574...$$

Comparing these results with the exact value $\sqrt{3}$ = 1.7320508..., we note that the second rational approximant 26/15 has two decimal figures exact. The third one has four decimal figures exact and the fourth has five decimal figures exact!

This continued fraction expansion of $\sqrt{3}$, in which the approximants are by excess, is much better than the usual “simple continued fraction” [3]:

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ddots}}}} = [1, \overline{1,2}],$$

that has the following rational approximants:

$$\sigma(1)= 2$$

$$\sigma(2) = 5/3 = 1.666...$$

$$\sigma(3) = 7/4 = 1.75$$

$$\sigma(4) = 19/11 = 1.7222...$$

It is easy to notice that in this case the fourth rational approximant has only one decimal figure exact!

This curious behavior of getting a much quicker convergence with a continued fraction expansion that approximates by excess than the simple continued fraction expansion, is extensive , by looking for the positive solutions of quadratic equations of the type

$$x^2 - nx + 1 = 0$$

to other continued fractions for $\sqrt{N}$, where $N > 0$ is an integer which is not a perfect square. This is the subject of the research I am not undertaking.

References

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Vera W. de Spinadel is a Full Consultant Professor at the Faculty of Architecture, Design and Urbanism at the University of Buenos Aires, Argentina. She is the Director of the research centre "Mathematics and Design", which comprises a team of interdisciplinary professionals working on the relations among Mathematics and Informatics with Design, where the word "design" is understood in a very broad sense (architectonic, graphic, industrial, textile, image and sound design, etc.). She organized the First and Second International Conferences on Mathematics and Design (M&D 95 and M&D 98). She is the author of several books and has published many research papers in international journals. She has received several research and development grants as well as several research and technological production prizes.