TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 2, February 2011, Pages 1101–1123
S 0002-9947(2010)05287-0
Article electronically published on September 20, 2010

TRIGONOMETRIC APPROXIMATION AND A GENERAL FORM OF THE ERDŐS TURÁN INEQUALITY

LEONARDO COLZANI, GIACOMO GIGANTE, AND GIANCARLO TRAVAGLINI

ABSTRACT. There exists a positive function $\psi(t)$ on $t \geq 0$, with fast decay at infinity, such that for every measurable set Ω in the Euclidean space and $R > 0$, there exist entire functions $A(x)$ and $B(x)$ of exponential type R, satisfying $A(x) \leq \chi_\Omega(x) \leq B(x)$ and $|B(x) - A(x)| \leqslant \psi(R \operatorname{dist}(x, \partial\Omega))$. This leads to Erdős Turán estimates for discrepancy of point set distributions in the multi-dimensional torus. Analogous results hold for approximations by eigenfunctions of differential operators and discrepancy on compact manifolds.

An infinite sequence of points $\{x_j\}_{j=1}^{+\infty}$ is uniformly distributed in the interval $0 \leq x \leq 1$ if, for every $0 \leq a < b \leq 1$,

$$\lim_{m \to +\infty} \left\{ m^{-1} \sum_{j=1}^{m} \chi_{[a,b]}(x_j) \right\} = b - a.$$

A quantitative measure of the irregularity of distribution of the points $\{x_j\}_{j=1}^{m}$ is given by the discrepancy

$$\sup_{0 \leq a < b \leq 1} \left| (b-a) - m^{-1} \sum_{j=1}^{m} \chi_{[a,b]}(x_j) \right|.$$

Upper bounds for the discrepancy are useful because they lead to upper bounds for the approximation of integrals by Riemann sums. A well-known criterion due to H. Weyl states that a sequence is uniformly distributed if and only if for every $k \neq 0$,

$$\lim_{m \to +\infty} \left\{ m^{-1} \sum_{j=1}^{m} \exp(2\pi i k x_j) \right\} = 0.$$

An upper bound for discrepancy is given by the classical inequality of P. Erdős and P. Turán:

$$\sup_{0 \leq a < b \leq 1} \left| (b-a) - m^{-1} \sum_{j=1}^{m} \chi_{[a,b]}(x_j) \right|$$
$$\leq \inf_{n=1,2,\ldots} \left\{ \frac{6}{n+1} + \frac{4}{\pi} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{n+1} \right) \left| m^{-1} \sum_{j=1}^{m} \exp(2\pi i k x_j) \right| \right\}.$$

Received by the editors February 24, 2009 and, in revised form, December 10, 2009.
2000 *Mathematics Subject Classification.* Primary 11K38, 42C15.

A proof of the above inequality relies on approximations from above and below of the characteristic functions $\chi_{[a,b]}(x)$ by trigonometric polynomials $P(x) = \sum_{k=-n}^{+n} \widehat{P}(k) \exp(2\pi i k x)$, so that

$$(b-a) - m^{-1}\sum_{j=1}^{m}\chi_{[a,b]}(x_j) \approx (b-a) - m^{-1}\sum_{j=1}^{m} P(x_j)$$
$$= (b-a) - \widehat{P}(0) - \sum_{1\leq|k|\leq n}\left(m^{-1}\sum_{j=1}^{m}\exp(2\pi i k x_j)\right)\widehat{P}(k).$$

See [2, 7, 16]. A deep study of the approximation of the characteristic function of an interval by trigonometric polynomials has been done by A. Beurling and A. Selberg, who proved that for every $0 \leq a < b \leq 1$ and $n = 0, 1, 2, \ldots$, there exist trigonometric polynomials $P_{\pm}(x)$ of degree n with

$$P_{-}(x) \leq \chi_{[a,b]}(x) \leq P_{+}(x),$$
$$\int_0^1 \left|P_{\pm}(x) - \chi_{[a,b]}(x)\right| dx = 1/(n+1).$$

See e.g. [18]. Similar extremal problems have been considered in [14], with precise estimates on the approximation, in $-\infty < x < +\infty$ with measure $|x|^{2\nu+1}\,dx$, of the function $\operatorname{sgn}(x)$ by functions of finite exponential type. A radialization of these functions then yields an analog of Selberg polynomials for approximation of characteristic functions of multi-dimensional balls, and this has been applied to Erdős Turán estimates of discrepancy. See [9, 13, 14] and also [5] for a generalization to boxes.

Here we look for a geometric analog of Erdős and Turán results in a very general framework, where the intervals in the torus are replaced by arbitrary measurable sets on a manifold, and trigonometric polynomials are replaced by finite linear combinations of eigenfunctions of the Laplace Beltrami operator. In spite of this generality, the techniques are rather simple, and they may be of some interest even in the classical Euclidean setting. Moreover, the results are optimal, up to the constants involved. In fact we shall try to pay special attention to these constants, which are quite explicit although not optimal. In this sense we acknowledge that we do not match the beauty of Beurling and Selberg results, which stems precisely from their extremal properties.

The plan of the paper is as follows. The first section is devoted to approximations from above and below of characteristic functions by entire functions of exponential type or, in the periodic setting, approximations by trigonometric polynomials. The main result of this section is the following:

Theorem 0.1. *There exists a positive function $\psi(t)$ on $t \geq 0$ with fast decay at infinity, $\psi(t) \leq c(\alpha)(1+t)^{-\alpha}$ for every α, such that for every measurable set Ω in the Euclidean space $\mathbb{R}^d$ and $R > 0$, there exist entire functions $A(x)$ and $B(x)$ of exponential type R, satisfying*

$$A(x) \leq \chi_\Omega(x) \leq B(x), \quad |B(x) - A(x)| \leqslant \psi(R \operatorname{dist}(x, \partial\Omega)).$$

Roughly speaking, the approximation $|B(x) - A(x)|$ is essentially 1 at points with distances $1/R$ from the boundary of Ω, while $|B(x) - A(x)|$ is essentially 0 at larger distances. We would like to emphasize that the function $\psi(t)$ in the theorem is independent of the set Ω and that there are no regularity assumptions on this set. If the set is regular, then there are few points at small distance from the boundary and the approximation is bad only on a small set. If the set is fractal, then there are many points at small distance from the boundary and the approximation is bad on a large set. In particular, this approximation is related to the Minkowski content of the boundary. See [8].

In the second section these approximations are applied to Erdős Turán estimates of irregularities of point distribution on a torus. In particular, following [16] and [19], we obtain explicit estimates for the discrepancy of sequences in lattices and in arithmetic progression, which improve and extend some results already in the literature. Here are some examples of the results in the second section (see, respectively, Corollaries 2.6, 2.7, and 2.11 below, where these results are proved).

Theorem 0.2. *If $0 \le \alpha \le 1$, μ is the Lebesgue measure, and Ω is a measurable set in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, let*

$$M(\alpha, \Omega) = \sup_{t>0} t^{-\alpha} \mu\left(\{\mathrm{dist}\,(x, \partial\Omega) < t\}\right)$$

(this quantity is related to the Minkowski content of $\partial\Omega$). Also, if $m^{1/d}$ is a positive integer, let $L(m) = m^{-1/d}\mathbb{Z}^d$ be the lattice points $m^{-1/d}(g_1, ..., g_d)$ with $g_j = 0, 1, ..., m^{1/d} - 1$. Then there exists a constant c such that for every such $m \ge 1$,

$$\sup_{M(\alpha,\Omega)<\gamma} \left| \mu(\Omega) - m^{-1} \sum_{x \in L(m)} \chi_\Omega(x) \right| \le c\gamma m^{-\alpha/d}.$$

Theorem 0.3. *Given $\varepsilon > 0$, for almost every x in $\mathbb{T}^d$ there exists a constant c such that for every $m > 1$,*

$$\sup_{M(\alpha,\Omega)<\gamma} \left| \mu(\Omega) - m^{-1} \sum_{j=1}^{m} \chi_\Omega(jx) \right| \le c\gamma m^{-\alpha/d} \log^{\alpha(d+1+\varepsilon)/d}(m).$$

Theorem 0.4. *Let X be a finite collection of hyperspaces of $\mathbb{R}^d$ and let $P(X)$ be the collection of all convex polyhedra in the torus $\mathbb{T}^d$ with diameter smaller than 1 and facets parallel to elements of X. Then there exists a constant c such that, given a prime number m, there exists a lattice point $g = (g_1, ..., g_d)$ in $\mathbb{Z}^d$ with $1 \le g_j \le m - 1$, such that*

$$\sup_{\Omega \in P(X)} \left| \mu(\Omega) - m^{-1} \sum_{j=1}^{m} \chi_\Omega\left(jm^{-1}g\right) \right| \le cm^{-1} \log^d(m).$$

In particular, up to a logarithmic transgression, the discrepancy of a random arithmetic progression $\{jx\}_{j=1}^{m}$ is comparable to the one of the lattice $L(m)$ with the same number of points. On the other hand, while the first two theorems applied to polyhedra with $\alpha = 1$ give the bound $m^{-1/d}$, the third theorem gives the better bound m^{-1}. This improves and extends a two dimensional result in [1, Theorem 4D], where it is proved that the discrepancy of m points with respect to polygons is dominated by $m^{-1} \log^{4+\varepsilon}(m)$.

In the third section the results obtained on the Euclidean settings are extended to compact Riemannian manifolds. We first consider approximations by linear combinations of eigenfunctions of the Laplace Beltrami operator on a compact Riemannian manifold. Then we state an analog of Erdős Turán estimates for discrepancy of point distributions on manifolds and, inspired by [17], when the manifold is a compact Lie group or homogeneous space, we consider point distributions generated by the action of a free group. In particular, using the Ramanujan bounds for eigenvalues of Hecke operators obtained in the above quoted paper, we prove the following result (see Corollary 3.4 below).

Theorem 0.5. *If $\mathcal{M} = SO(3)/SO(2)$ is the two dimensional sphere and if $\mathcal{H}$ is the free group generated by rotations of angles $\arccos(-3/5)$ around orthogonal axes, then there exists a constant c such that, if k is an integer and $\{\sigma_j\}_{j=1}^m$ is an ordering of the elements in $\mathcal{H}$ with length at most k, then for every x,*

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(\sigma_j x)\right| \leq cM(\delta,\Omega)\, m^{-\delta/(2+\delta)}\log^{2\delta/(2+\delta)}(m).$$

This result has been proved in [17] in the case of spherical caps, with a proof that relies on explicit estimates of Fourier coefficients. On the contrary, our result applies to more general domains.

The authors wish to thank Luca Brandolini for several useful discussions on the subject of the paper.

1. Approximation by entire functions

The main result in this section is the following.

Theorem 1.1. *There exists a positive function $\psi(t)$ on $t \geq 0$ with fast decay at infinity, $\psi(t) \leq c(\alpha)(1+t)^{-\alpha}$ for every α, such that for every measurable set Ω in the Euclidean space $\mathbb{R}^d$ and $R > 0$, there exist entire functions $A(x)$ and $B(x)$ of exponential type R, satisfying*

$$A(x) \leq \chi_\Omega(x) \leq B(x), \quad |B(x) - A(x)| \leqslant \psi\left(R\,\mathrm{dist}\,(x,\partial\Omega)\right).$$

Proof. By a scaling argument, the statement for the set Ω at the point x with functions of exponential type R is equivalent to the statement for the set $R\Omega$ at the point Rx with functions of exponential type 1. Hence, it suffices to prove the theorem when $R = 1$. Let $m(\xi)$ be a smooth radial function on $\mathbb{R}^d$ with $m(\xi) = 0$ if $|\xi| \geq 1/2$ and $\int_{\mathbb{R}^d} m(\xi)^2\, d\xi = 1$. Then the convolution $m * m(\xi)$ is a smooth radial function with $m * m(0) = 1$ and $m(\xi) = 0$ if $|\xi| \geq 1$. Define

$$K(x) = \int_{\mathbb{R}^d}\left(1 + |\xi|^2\right)^{-(d+1)/2} m * m(\xi) \exp(2\pi i \xi \cdot x)\, d\xi.$$

This cumbersome definition will be clarified in a series of steps.

Claim. The kernel $K(x)$ is an entire function of finite exponential type, it is positive with mean 1 on $\mathbb{R}^d$ and all its derivatives have fast decay at infinity, $\left|\partial^\beta K(x)/\partial x^\beta\right| \leq c(1+|x|)^{-\alpha}$ for every α and β.

All of this follows from the corresponding properties of the Fourier transform $\widehat{K}(\xi) = \left(1+|\xi|^2\right)^{-(d+1)/2} m * m(\xi)$. Since this Fourier transform is smooth with compact support and it is 1 at the origin, the kernel has mean 1 and all its derivatives have fast decay at infinity. Since the kernel is the convolution of the Fourier transform of $\left(1+|\xi|^2\right)^{-(d+1)/2}$, which is up to a constant $\exp(-2\pi|x|)$, and the square of the Fourier transform of $m(\xi)$, the kernel is positive. Finally, since $\widehat{K}(\xi)=0$ if $|\xi|\geq 1$, by the Paley-Wiener theorem the kernel is an entire function of finite exponential type,

$$K(x+iy)=\int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^{-(d+1)/2} m*m(\xi)\exp(2\pi i\xi\cdot(x+iy))\,d\xi,$$

$$|K(x+iy)|\leq\left(\int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^{-(d+1)/2}|m*m(\xi)|\,d\xi\right)\exp(2\pi|y|).$$

Claim. Let

$$I(t)=\int_{\{|x|\geq t\}}K(x)dx.$$

Then, for every $t\geq 0$, $I(t+1)\geq\exp(-2\pi)\,I(t)$.

Since $K(x)$ is the convolution of $c\exp(-2\pi|x|)$ and the positive function $|\widehat{m}(x)|^2$, one has

$$K(x+y)=\int_{\mathbb{R}^d}c\exp(-2\pi|x+y-z|)\,|\widehat{m}(z)|^2\,dz$$

$$\geq\exp(-2\pi|y|)\int_{\mathbb{R}^d}c\exp(-2\pi|x-z|)\,|\widehat{m}(z)|^2\,dz=\exp(-2\pi|y|)\,K(x).$$

Hence

$$I(t+1)=\int_{\{|\vartheta|=1\}}\int_{t+1}^{+\infty}K(\rho\vartheta)\rho^{d-1}d\rho d\vartheta$$

$$=\int_{\{|\vartheta|=1\}}\int_{t}^{+\infty}K((\tau+1)\vartheta)(\tau+1)^{d-1}\,d\tau d\vartheta$$

$$\geq\exp(-2\pi)\int_{\{|\vartheta|=1\}}\int_{t}^{+\infty}K(\tau\vartheta)\,\tau^{d-1}d\tau d\vartheta=\exp(-2\pi)\,I(t).$$

Claim. Define

$$G(x)=I(\operatorname{dist}(x,\partial\Omega)).$$

Then, for every x,

$$|\chi_\Omega(x)-K*\chi_\Omega(x)|\leq G(x).$$

Since $K(y)$ is positive with mean 1,

$$|\chi_\Omega(x)-K*\chi_\Omega(x)|=\left|\int_{\mathbb{R}^d}K(y)\left(\chi_\Omega(x)-\chi_\Omega(x-y)\right)dy\right|$$

$$\leq\int_{\{|y|\geq\operatorname{dist}(x,\partial\Omega)\}}K(y)dy=G(x).$$

Claim. Define

$$F(x)=I(\operatorname{dist}(x,\partial\Omega)/2).$$

Then, for every x,

$$K*G(x)\leq 2F(x).$$

Since $\operatorname{dist}(x-y,\partial\Omega) \geq \operatorname{dist}(x,\partial\Omega) - |y|$, it follows that

$$\{|z| \geq \operatorname{dist}(x-y,\partial\Omega)\} \subseteq \{|z| \geq \operatorname{dist}(x,\partial\Omega) - |y|\}.$$

Hence

$$\begin{aligned} K * G(x) &= \int_{\mathbb{R}^d} \int_{\{|z|\geq \operatorname{dist}(x-y,\partial\Omega)\}} K(y) K(z)\, dz dy \\ &\leq \int_{\{|y|\leq \operatorname{dist}(x,\partial\Omega)/2\}} \int_{\{|z|\geq \operatorname{dist}(x,\partial\Omega)/2\}} K(y) K(z)\, dz dy \\ &\quad + \int_{\{|y|\geq \operatorname{dist}(x,\partial\Omega)/2\}} \int_{\mathbb{R}^d} K(y) K(z)\, dz dy \\ &\leq 2 \int_{\{|y|\geq \operatorname{dist}(x,\partial\Omega)/2\}} K(y)\, dy. \end{aligned}$$

Claim. Let

$$\gamma^{-1} = \exp(-2\pi) \int_{\{|y|\leq 1\}} K(y)\, dy.$$

Then for every x,

$$K * G(x) \geq \gamma^{-1} G(x).$$

Since $\operatorname{dist}(x-y,\partial\Omega) \leq \operatorname{dist}(x,\partial\Omega) + |y|$, it follows that

$$\{|z| \geq \operatorname{dist}(x-y,\partial\Omega)\} \supseteq \{|z| \geq \operatorname{dist}(x,\partial\Omega) + |y|\}.$$

Hence

$$\begin{aligned} K * G(x) &= \int_{\mathbb{R}^d} \int_{\{|z|\geq \operatorname{dist}(x-y,\partial\Omega)\}} K(y) K(z)\, dz dy \\ &\geq \int_{\{|y|\leq 1\}} \int_{\{|z|\geq \operatorname{dist}(x,\partial\Omega)+1\}} K(y) K(z)\, dz dy \\ &\geq \gamma^{-1} I(\operatorname{dist}(x,\partial\Omega)). \end{aligned}$$

To conclude the proof of the theorem, define $H(x) = \gamma G(x)$ and

$$\begin{aligned} A(x) &= K * \chi_\Omega(x) - K * H(x), \\ B(x) &= K * \chi_\Omega(x) + K * H(x). \end{aligned}$$

Since the kernel $K(x)$ is an entire function of finite exponential type, then also the convolutions with this kernel are entire functions of finite exponential type. In particular, both $A(x)$ and $B(x)$ are entire functions of exponential type not larger than the one of $K(x)$. Moreover, by the above claims,

$$\begin{aligned} \chi_\Omega(x) - A(x) &= K * H(x) - (K * \chi_\Omega(x) - \chi_\Omega(x)) \geq 0, \\ B(x) - \chi_\Omega(x) &= K * H(x) - (\chi_\Omega(x) - K * \chi_\Omega(x)) \geq 0. \end{aligned}$$

Finally,

$$B(x) - A(x) = 2K * H(x) \leq 4\gamma F(x) = 4\gamma I(\operatorname{dist}(x,\partial\Omega)/2).$$

Hence, the theorem follows with $\psi(t) = 4\gamma I(t/2)$. □

It follows from the proof of the theorem that for periodic sets with respect to the integer lattice $\mathbb{Z}^d$, the above approximating entire functions are periodic too, hence they are trigonometric polynomials.

Corollary 1.2. *There exists a positive function $\psi(t)$ with fast decay at infinity, such that for every measurable set Ω in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $R = 0, 1, 2, ...$, there exist trigonometric polynomials $A(x)$ and $B(x)$ of degree R with*

$$A(x) \le \chi_\Omega(x) \le B(x), \quad |B(x) - A(x)| \leqslant \psi\left(R \operatorname{dist}(x, \partial\Omega)\right).$$

Proof. The corollary follows immediately from the theorem. However, in order to clarify what follows, we write explicitly the Fourier expansions of the trigonometric approximations. Let $\Omega = \Omega + \mathbb{Z}^d$ be a $\mathbb{Z}^d$-periodic set in $\mathbb{R}^d$. As in the proof of the theorem, for every $R > 0$, let

$$K_R(x) = \sum_{k \in \mathbb{Z}^d} R^d K(R(x+k)) = \sum_{k \in \mathbb{Z}^d} \widehat{K}(k/R) \exp(2\pi i k \cdot x),$$

$$H_R(x) = \exp(2\pi) \left(\int_{\{|y| \le 1\}} K(y)\, dy \right)^{-1} \int_{\{|y| \ge R \operatorname{dist}(x, \partial\Omega + \mathbb{Z}^d)\}} K(y)\, dy.$$

Then,

$$\begin{aligned} A(x), B(x) &= \int_{\mathbb{T}^d} K_R(y) \left(\chi_\Omega(x-y) \mp H_R(x-y)\right) dy \\ &= \sum_{k \in \mathbb{Z}^d} \widehat{K}(k/R) \left(\widehat{\chi}_\Omega(k) \mp \widehat{H}_R(k)\right) \exp(2\pi i k \cdot x). \end{aligned}$$

□

Remark 1.3. As we have said, in the above theorem the approximation $|B(x) - A(x)|$ is controlled by 1 at points with distances $1/R$ from the boundary of Ω, while $|B(x) - A(x)|$ is essentially 0 at larger distances. It follows from the inequality of S. Bernstein between the maxima of an entire function and its derivatives that this approximation is essentially optimal. Indeed, if $C(z)$ is an entire function of exponential type 1, then

$$|C(x) - C(y)| \le \sup_{z \in \mathbb{R}^d} |\nabla C(z)|\, |x-y| \le 2\pi \sup_{z \in \mathbb{R}^d} |C(z)|\, |x-y|.$$

Hence, if $A(x) \le \chi_\Omega(x) \le B(x)$ are entire functions of exponential type 1, if x is in Ω and y is outside Ω with $|x-y| \le \left(4\pi \sup_{z \in \mathbb{R}^d} \{|A(z)|, |B(z)|\}\right)^{-1}$, then $B(x) - A(y) \ge 1$ and

$$\begin{aligned} B(x) - A(x) &= (B(x) - A(y)) - (A(x) - A(y)) \ge 1 - 1/2, \\ B(y) - A(y) &= (B(x) - A(y)) - (B(x) - B(y)) \ge 1 - 1/2. \end{aligned}$$

2. Discrepancy on the torus

As stated in the Introduction, the approximation results in the previous section have simple and straightforward applications to multi-dimensional versions of the classical Erdős Turán inequality for discrepancy of point distribution. In the sequel, for simplicity, the sets Ω in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ will be periodic sets in $\mathbb{R}^d$ of the form $\Omega = \Omega^* + \mathbb{Z}^d$, with Ω^* in $\mathbb{R}^d$ with the property that $\operatorname{dist}(\Omega^*, \Omega^* + k) > \varepsilon > 0$ for every $k \in \mathbb{Z}^d - \{0\}$. With this identification, the distance of a point x from $\partial\Omega$ in $\mathbb{T}^d$ is the distance of x from $\partial\Omega^* + \mathbb{Z}^d$ in $\mathbb{R}^d$.

Theorem 2.1. *If $\{x_j\}_{j=1}^m$ is a sequence of points in the torus, if Ω is a measurable set with measure $\mu(\Omega)$, and if $H_R(x) = 4^{-1}\psi\left(2R\operatorname{dist}(x, \partial\Omega)\right)$ with $R > 0$ and $\psi(t)$ with fast decay at infinity, as in the proof of Corollary 1.2, then*

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(x_j)\right|$$
$$\leq \left|\widehat{H}_R(0)\right| + \sum_{0<|k|<R}\left(|\widehat{\chi}_\Omega(k)| + \left|\widehat{H}_R(k)\right|\right)\left|m^{-1}\sum_{j=1}^{m}\exp\left(2\pi ik\cdot x_j\right)\right|.$$

Proof. If $A(x) \leq \chi_\Omega(x) \leq B(x)$ are defined as in the proof of Corollary 1.2, then

$$A(x), B(x) = \sum_{k\in\mathbb{Z}^d}\widehat{K}\left(k/R\right)\left(\widehat{\chi}_\Omega\left(k\right) \mp \widehat{H}_R\left(k\right)\right)\exp\left(2\pi ik\cdot x\right).$$

Since $\left|\widehat{K}(\xi)\right| \leq 1$, $\widehat{K}\left(0\right) = 1$, $\widehat{K}(\xi) = 0$ if $|\xi| \geq 1$, and since $\widehat{\chi}_\Omega\left(0\right) = \mu\left(\Omega\right)$, then

$$\mu\left(\Omega\right) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(x_j) \leq \mu\left(\Omega\right) - m^{-1}\sum_{j=1}^{m}A(x_j)$$
$$= \mu\left(\Omega\right) - \sum_{k\in\mathbb{Z}^d}\widehat{K}\left(k/R\right)\left(\widehat{\chi}_\Omega\left(k\right) - \widehat{H}_R\left(k\right)\right)\left(m^{-1}\sum_{j=1}^{m}\exp\left(2\pi ik\cdot x_j\right)\right)$$
$$\leq \left|\widehat{H}_R(0)\right| + \sum_{0<|k|<R}\left(|\widehat{\chi}_\Omega(k)| + \left|\widehat{H}_R(k)\right|\right)\left|m^{-1}\sum_{j=1}^{m}\exp\left(2\pi ik\cdot x_j\right)\right|.$$

Similarly

$$-\mu\left(\Omega\right) + m^{-1}\sum_{j=1}^{m}\chi_\Omega(x_j) \leq -\mu\left(\Omega\right) + m^{-1}\sum_{j=1}^{m}B(x_j)$$
$$\leq \left|\widehat{H}_R(0)\right| + \sum_{0<|k|<R}\left(|\widehat{\chi}_\Omega(k)| + \left|\widehat{H}_R(k)\right|\right)\left|m^{-1}\sum_{j=1}^{m}\exp\left(2\pi ik\cdot x_j\right)\right|.$$

□

In order to apply the above theorem one has to estimate the exponential sums of point distributions and the Fourier transforms of domains. Motivated by the above result and by the definition of Fourier dimension of a measurable set (see e.g. Chapter 4.4 of [8]), it is possible to introduce the classes of sets whose Fourier transform have a prescribed decay at infinity. Given a measurable set Ω in the torus $\mathbb{T}^d$, assume that for some $0 \leq \alpha \leq (d+1)/2$ there exists a constant c such that for every $k \in \mathbb{Z}^d - \{0\}$,

$$\left|\int_{\mathbb{T}^d}\chi_\Omega\left(x\right)\exp\left(-2\pi ik\cdot x\right)dx\right| \leq c\left|k\right|^{-\alpha}.$$

Also, assume that for some $0 \leq \beta \leq 1$ and for every $R > 0$,

$$\left|\int_{\mathbb{T}^d}\psi\left(R\operatorname{dist}\left(x, \partial\Omega\right)\right)\exp\left(-2\pi ik\cdot x\right)dx\right| \leq \begin{cases} c\left|k\right|^{-\alpha} & \text{if } 0 < |k| < R, \\ cR^{-\beta} & \text{if } k = 0. \end{cases}$$

Denote by $F(\alpha,\beta,\Omega)$ the smallest constant c for which the above two inequalities hold. It turns out that the Fourier transform of $\psi(R\,\mathrm{dist}(x,\partial\Omega))$ is in some sense dominated by the one of $\chi_\Omega(x)$. In particular, in a number of cases the second inequality is a consequence of the first. Anyhow, in many cases it is possible to give quite precise estimates for the constants $F(\alpha,\beta,\Omega)$.

Remark 2.2. If the first of the above inequalities holds for some $\alpha>(d+1)/2$, or if the second inequality holds for some $\beta>1$, then either Ω or $\mathbb{T}^d-\Omega$ has measure zero. It suffices to show this when $(d+1)/2<\alpha<d/2+1$. For every y,

$$\int_{\mathbb{T}^d}|\chi_\Omega(x+y)-\chi_\Omega(x)|^2\,dx=\sum_{k\in\mathbb{Z}^d}|(\exp(2\pi ik\cdot y)-1)\,\widehat{\chi}_\Omega(k)|^2$$
$$\le F(\alpha,\beta,\Omega)\left(4\pi^2|y|^2\sum_{0<|k|<|y|^{-1}}|k|^{2-2\alpha}+4\sum_{|k|\ge|y|^{-1}}|k|^{-2\alpha}\right)$$
$$\le cF(\alpha,\beta,\Omega)\,|y|^{2\alpha-d}.$$

Hence for every y and every $n\ge|y|$,

$$\int_{\mathbb{T}^d}|\chi_\Omega(x+y)-\chi_\Omega(x)|\,dx$$
$$\le\sum_{j=1}^{n}\int_{\mathbb{T}^d}\left|\chi_\Omega\left(x+jn^{-1}y\right)-\chi_\Omega\left(x+(j-1)n^{-1}y\right)\right|dx$$
$$=\sum_{j=1}^{n}\int_{\mathbb{T}^d}\left|\chi_\Omega\left(x+jn^{-1}y\right)-\chi_\Omega\left(x+(j-1)n^{-1}y\right)\right|^2dx$$
$$=n\int_{\mathbb{T}^d}\left|\chi_\Omega\left(x+n^{-1}y\right)-\chi_\Omega\left(x\right)\right|^2dx\le cF(\alpha,\beta,\Omega)\,|y|^{2\alpha-d}\,n^{d+1-2\alpha}.$$

This converges to zero when n diverges. Hence, for every translation $\Omega-y=\Omega$ up to sets with measure zero. Similarly, if $\beta>1$, then either Ω or $\mathbb{T}^d-\Omega$ has measure zero. To see this, it suffices to estimate the modulus of continuity of the function

$$G(y)=\int_{\mathbb{T}^d}|\chi_\Omega(x+y)-\chi_\Omega(x)|\,dx.$$

Indeed,

$$|G(y)-G(z)|$$
$$\le\int_{\mathbb{T}^d}|\chi_\Omega(x+y)-\chi_\Omega(x+z)|\,dx\le\mu\left(\{\mathrm{dist}(x,\partial\Omega)\le|y-z|\}\right)$$
$$\le\left(\inf_{0\le t\le1}\{\psi(t)\}\right)^{-1}\int_{\mathbb{T}^d}\psi\left(|y-z|^{-1}\,\mathrm{dist}(x,\partial\Omega)\right)dx\le cF(\alpha,\beta,\Omega)\,|y-z|^{\beta}.$$

Finally, if $\beta>1$, then $G(y)$ is identically zero.

Remark 2.3. The Fourier coefficients of $\psi(R\,\mathrm{dist}(x,\partial\Omega))$ on the torus $\mathbb{T}^d$ can be evaluated by an integration over any set that tiles $\mathbb{R}^d$ via $\mathbb{Z}^d$. In particular, one can integrate on the set of points Q in $\mathbb{R}^d$ for which the distance from $\partial\Omega+\mathbb{Z}^d$ is realized precisely by the distance from $\partial\Omega$. Since $|\nabla\,\mathrm{dist}(x,\partial\Omega)|=1$ when $\mathrm{dist}(x,\partial\Omega)\neq0$, if $\partial\Omega$ has measure zero, then the coarea formula gives

$$\begin{aligned}&\left|\int_{Q\cap\{\operatorname{dist}(x,\partial\Omega)\leq\varepsilon\}}\psi\left(R\operatorname{dist}\left(x,\partial\Omega\right)\right)\exp\left(-2\pi i\xi\cdot x\right)dx\right|\\ &\leq R^{-1}\int_0^{\varepsilon R}\left|\psi\left(t\right)\right|\left|\int_{\{\operatorname{dist}(x,\partial\Omega)=R^{-1}t\}}\exp\left(-2\pi i\xi\cdot x\right)dx\right|dt.\end{aligned}$$

Moreover, since Q has measure 1,

$$\left|\int_{Q\cap\{\operatorname{dist}(x,\partial\Omega)>\varepsilon\}}\psi\left(R\operatorname{dist}\left(x,\partial\Omega\right)\right)\exp\left(-2\pi i\xi\cdot x\right)dx\right|\leq\sup_{t>\varepsilon R}\left|\psi\left(t\right)\right|.$$

In fact, one can eliminate the term $\sup_{t>\varepsilon R}\left|\psi\left(t\right)\right|$ by integrating over all Q rather than $Q\cap\{\operatorname{dist}\left(x,\partial\Omega\right)\leq\varepsilon\}$. However, in order to keep control of the level sets $\{\operatorname{dist}\left(x,\partial\Omega\right)=t\}$ it may be convenient to restrict to t small.

Remark 2.4. It follows from classical estimates on oscillatory integrals with nondegenerate critical points that if a convex body has smooth boundary with positive Gauss curvature, then

$$\begin{aligned}\left|\int_{\partial\Omega}\exp\left(-2\pi i\xi\cdot x\right)dx\right|&\leq c\left|\xi\right|^{-(d-1)/2},\\ \left|\int_{\Omega}\exp\left(-2\pi i\xi\cdot x\right)dx\right|&\leq c\left|\xi\right|^{-(d+1)/2}.\end{aligned}$$

The constant c can be bounded in terms of the smoothness and the minimum of the curvature of the boundary. See e.g. [10, 11, 12, 20]. From these estimates, with $\{\operatorname{dist}\left(x,\partial\Omega\right)=t\}$ instead of $\partial\Omega$, it easily follows that $F\left((d+1)/2,1,\Omega\right)$ is finite. If the curvature vanishes of some order at some point, then the Fourier transform in directions normal to these points has a worse decay at infinity. If some part of the boundary is completely flat, then one can guarantee only a decay of order one.

Remark 2.5. If $0\leq\alpha\leq1$, define

$$M\left(\alpha,\Omega\right)=\sup_{t>0}t^{-\alpha}\mu\left(\{\operatorname{dist}\left(x,\partial\Omega\right)<t\}\right).$$

This is related to the upper Minkowski content of $\partial\Omega$, defined by

$$\limsup_{t\to0+}t^{-\alpha}\mu\left(\{\operatorname{dist}\left(x,\partial\Omega\right)<t\}\right).$$

However, these two quantities can be quite different. In particular, if $\partial\Omega$ contains a point p, then $M\left(\alpha,\Omega\right)\geq\mu\left(\{\operatorname{dist}\left(x,p\right)<1\}\right)$. If $M\left(\alpha,\Omega\right)$ is finite, then $\partial\Omega$ has Minkowski dimension at most $d-\alpha$. The domain is regular if $\alpha=1$, and it is fractal if $0\leq\alpha<1$. The definition makes sense also when $\alpha>1$, but in this case either Ω or $\mathbb{T}^d-\Omega$ has measure zero. The proof is as in Remark 2.2. It is well known that the decay of the Fourier transform of a domain can be controlled in terms of these quantities. Indeed,

$$\begin{aligned}
&\int_{\mathbb{T}^d} \chi_\Omega(x) \exp(-2\pi i k\cdot x)dx \\
&= -\int_{\mathbb{T}^d} \chi_\Omega(x) \exp(-2\pi i k\cdot \left(x - 2^{-1}|k|^{-2} k\right))dx \\
&= 2^{-1}\int_{\mathbb{T}^d} \left(\chi_\Omega(x) - \chi_\Omega(x+2^{-1}|k|^{-2}k)\right) \exp(-2\pi i k\cdot x)dx.
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d} \chi_\Omega(x) \exp(-2\pi i k\cdot x)dx\right| \\
&\le 2^{-1}\int_{\mathbb{T}^d} \left|\chi_\Omega(x+2^{-1}|k|^{-2}k) - \chi_\Omega(x)\right| dx \\
&\le 2^{-1}\mu\left\{\operatorname{dist}(x,\partial\Omega) \le 2^{-1}|k|^{-1}\right\} \le 2^{-\alpha-1}M(\alpha,\Omega)|k|^{-\alpha}.
\end{aligned}$$

Moreover, since $\psi(t)$ is positive and has fast decay at infinity,

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d} \psi\left(R\operatorname{dist}(x,\partial\Omega)\right)\exp\left(-2\pi i k\cdot x\right)dx\right| \\
&\le \int_{\mathbb{T}^d} \psi\left(R\operatorname{dist}(x,\partial\Omega)\right)dx \le cM(\alpha,\Omega)R^{-\alpha}.
\end{aligned}$$

In particular, since $R^{-\alpha} \le |k|^{-\alpha}$ in the range $0 < |k| < R$, it follows that $F(\alpha,\alpha,\Omega) \le cM(\alpha,\Omega)$.

In the following, the above theorem and remarks will be applied to the study of discrepancies of lattices $m^{-1/d}\mathbb{Z}^d$ and arithmetic progressions $\{jx\}_{j=1}^m$ in the torus $\mathbb{R}^d/\mathbb{Z}^d$, with multiples modulo $\mathbb{Z}^d$. In particular, we will prove Theorems 0.2, 0.3 and 0.4 stated in the Introduction.

Corollary 2.6. *There exists a constant c with the following property. Let $m^{1/d}$ be a positive integer and let $L(m) = m^{-1/d}\mathbb{Z}^d$ be the lattice of points $m^{-1/d}(g_1,...,g_d)$ with $0 \le g_j \le m^{1/d}-1$. Then*

$$\left|\mu(\Omega) - m^{-1}\sum_{x\in L(m)} \chi_\Omega(x)\right| \le cF(\alpha,\beta,\Omega)m^{-\beta/(d+\beta-\alpha)}.$$

Theorem 0.2 follows from the above corollary by setting $\beta=\alpha$, replacing $F(\alpha,\alpha,\Omega)$ with $M(\alpha,\Omega)$ and observing that the constant c does not depend on the set Ω.

Proof. The sum of a geometric progression gives

$$\begin{aligned}
&m^{-1}\sum_{x\in L(m)} \exp(2\pi i k\cdot x) \\
&= \prod_{n=1}^{d}\left(m^{-1/d}\sum_{j=0}^{m^{1/d}-1}\exp\left(2\pi i m^{-1/d} j k_n\right)\right) = \begin{cases} 0 \text{ if } k\notin m^{1/d}\mathbb{Z}^d, \\ 1 \text{ if } k\in m^{1/d}\mathbb{Z}^d. \end{cases}
\end{aligned}$$

Hence, by Theorem 2.1 and the definition of $F(\alpha,\beta,\Omega)$,

$$\left|\mu(\Omega) - m^{-1}\sum_{x\in L(m)}\chi_\Omega(x)\right| \leq F(\alpha,\beta,\Omega)\left(R^{-\beta} + 2\sum_{0<\left|m^{1/d}k\right|<R}\left|m^{1/d}k\right|^{-\alpha}\right)$$
$$\leq F(\alpha,\beta,\Omega)\left(R^{-\beta} + cm^{-1}R^{d-\alpha}\right).$$

Then the choice $R = m^{1/(d+\beta-\alpha)}$ gives the desired estimate. □

Corollary 2.7. *Given $\varepsilon > 0$, for almost every x in $\mathbb{T}^d$ there exists a constant c such that for every $m > 1$ and every Ω,*

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(jx)\right| \leq cF(\alpha,\beta,\Omega)m^{-\beta/(d+\beta-\alpha)}\log^{\beta(d+1+\varepsilon)/(d+\beta-\alpha)}(m).$$

As before, Theorem 0.3 follows from the above corollary by setting $\beta = \alpha$, replacing $F(\alpha,\alpha,\Omega)$ with $M(\alpha,\Omega)$ and observing that the constant c does not depend on the set Ω.

Proof. Denoting by $\|t\|$ the distance of t to the nearest integer,

$$\left|m^{-1}\sum_{j=1}^{m}\exp(2\pi ijx\cdot k)\right| = \left|\frac{\sin(\pi mk\cdot x)}{m\sin(\pi k\cdot x)}\right| \leq \min\{1, 1/(2m\|k\cdot x\|)\}.$$

Hence, by Theorem 2.1 and the definition of $F(\alpha,\beta,\Omega)$,

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(jx)\right|$$
$$\leq F(\alpha,\beta,\Omega)\left(R^{-\beta} + 2\sum_{0<|k|<R}|k|^{-\alpha}\min\left\{1, 2^{-1}m^{-1}\|k\cdot x\|^{-1}\right\}\right)$$
$$\leq F(\alpha,\beta,\Omega)\left(R^{-\beta} + m^{-1}R^{d-\alpha}\sum_{0<|k|<R}|k|^{-d}\|k\cdot x\|^{-1}\right).$$

Finally, in [19] it is proved that for almost every x there exists a c which depends on x, such that for every $R > 0$,

$$\sum_{0<|k|<R}|k|^{-d}\|k\cdot x\|^{-1} \leq c\log^{d+1+\varepsilon}(1+R).$$

The desired result follows by choosing

$$R = m^{1/(d+\beta-\alpha)}\log^{-(d+1+\varepsilon)/(d+\beta-\alpha)}(m).$$

□

In particular, the discrepancy of a random sequence $\{jx\}_{j=1}^{m}$ with respect to any domain Ω is dominated by $cF((d+1)/2, 1, \Omega)\, m^{-2/(d+1)}\log^{2+\varepsilon}(m)$ and, by Remark 2.4, when Ω is convex with smooth boundary and positive Gauss curvature, then $F((d+1)/2, 1, \Omega)$ is finite. Similarly, by Remark 2.5, the discrepancy is dominated by $cF(1,1,\Omega)\, m^{-1/d}\log^{(d+1+\varepsilon)/d}(m)$, and when the boundary of the domain is $d-1$ dimensional, then $F(1,1,\Omega)$ is finite. These results should be compared with well-known upper and lower estimates of the discrepancy with respect to convex regions due to W. M. Schmidt and J. Beck. See e.g. [2, Theorem 15 and Corollaries 17B, 18C and 19F]. In particular, for any given compact convex set there exists an infinite sequence such that the discrepancy is bounded above by $cm^{-(d+1)/(2d)}\log^{1/2} m$. The definition of Fourier dimension does not capture polyhedra, except in the trivial case $\alpha \leq 1$. Indeed, the decay of Fourier transforms of polyhedra is not isotropic or homogeneous. Anyhow, these Fourier transforms can be computed explicitly and estimated quite precisely, and, using these estimates, one can give bounds for the discrepancy which are better than the ones obtained above.

Lemma 2.8. *If Ω is a polyhedron in $\mathbb{R}^d$ with diameter λ, then,*

$$\left|\int_\Omega \exp\left(-2\pi i\xi\cdot x\right)dx\right| \leq 2\sum_{\Omega(d)\supset\ldots\supset\Omega(1)}\prod_{j=1}^{d}\min\left\{\lambda,\left(2\pi\left|P_{\Omega(j)}\xi\right|\right)^{-1}\right\}.$$

The sum is taken over all possible decreasing chains of j dimensional faces $\Omega(j)$ of Ω, and $P_{\Omega(j)}$ is the orthogonal projection on the j dimensional subspace parallel to $\Omega(j)$.

Proof. The Fourier transform of a polyhedron can be computed explicitly, but here we are only interested in precise estimates of its size with control on all constants involved. If $\Omega(j)$ is a j dimensional face of Ω with $P_{\Omega(j)}\xi \neq 0$ and if $n(x)$ is the outgoing normal to the boundary $\partial\Omega(j)$ at the point x, the divergence theorem gives

$$\int_{\Omega(j)}\exp\left(-2\pi i\xi\cdot x\right)dx = \int_{\partial\Omega(j)}\frac{in(x)\cdot P_{\Omega(j)}\xi}{2\pi\left|P_{\Omega(j)}\xi\right|^2}\exp\left(-2\pi i\xi\cdot x\right)dx.$$

Moreover, if λ is the diameter and $\mu(\Omega(j))$ is the j dimensional measure of $\Omega(j)$, one always has the trivial estimate

$$\left|\int_{\Omega(j)}\exp\left(-2\pi i\xi\cdot x\right)dx\right| \leq \mu\left(\Omega(j)\right) \leq \lambda^j.$$

Hence, if $\Omega(j-1)$ are the $j-1$ dimensional faces of $\Omega(j)$,

$$\left|\int_{\Omega(j)}\exp\left(-2\pi i\xi\cdot x\right)dx\right|$$
$$\leq \begin{cases} \lambda^j & \text{if } 2\pi\left|P_{\Omega(j)}\xi\right| < 1/\lambda, \\ \dfrac{1}{2\pi\left|P_{\Omega(j)}\xi\right|}\displaystyle\sum_{\Omega(j-1)\subset\Omega(j)}\left|\int_{\Omega(j-1)}\exp\left(-2\pi i\xi\cdot x\right)dx\right| & \text{if } 2\pi\left|P_{\Omega(j)}\xi\right| \geq 1/\lambda. \end{cases}$$

Iterating, one can decompose the integral over $\Omega = \Omega(d)$ into a sum of integrals over chains of faces $\Omega(d) \supset \Omega(d-1) \supset \dots$, and this gives

$$\begin{aligned}
&\left|\int_\Omega \exp(-2\pi i\xi \cdot x)\,dx\right| \\
&\leq \sum_{\Omega(d)\supset\Omega(d-1)\supset\dots\supset\Omega(s)} \lambda^s \prod_{j=s+1}^{d} \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1} \\
&\quad +2 \sum_{\Omega(d)\supset\Omega(d-1)\supset\dots\supset\Omega(1)} \prod_{j=1}^{d} \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1}.
\end{aligned}$$

The first sum is over all chains of faces with $\left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1} \leq \lambda$ for $1 \leq s < j \leq d$ and $\left(2\pi \left|P_{\Omega(s)}\xi\right|\right)^{-1} > \lambda$, while the second sum is over all chains with $\left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1} \leq \lambda$ for all $1 \leq j \leq d$. Finally, in the first sum,

$$\lambda^s \prod_{j=s+1}^{d} \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1} = \prod_{j=1}^{d} \min\left\{\lambda, \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1}\right\}.$$

Indeed, since $\left|P_{\Omega(j)}\xi\right|$ is increasing in j, the terms $\min\left\{\lambda, \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1}\right\}$ are equal to λ when $1 \leq j \leq s$ and equal to $\left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1}$ when $s < j \leq d$. Similarly, in the second sum,

$$\prod_{j=1}^{d} \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1} = \prod_{j=1}^{d} \min\left\{\lambda, \left(2\pi \left|P_{\Omega(j)}\xi\right|\right)^{-1}\right\}.$$

Observe that when $\xi = 0$ the formula gives λ^d, while the exact value of the integral is the volume of Ω. □

Lemma 2.9. *Let Ω be a convex polyhedron in $\mathbb{R}^d$ with diameter λ. For any $j = 1, 2, \dots, d-1$, let $\{A(j)\}$ be the collection of all j dimensional subspaces which are intersections of a number of subspaces parallel to the faces of Ω. Finally, let $\psi(t)$ be a function on $0 \leq t < +\infty$ with fast decay at infinity. Then there exists a positive constant c, which depends on d and $\psi(t)$, but not on Ω, such that for every $R > 0$,*

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \psi\left(R \operatorname{dist}(x, \partial\Omega)\right) \exp(-2\pi i\xi \cdot x)\,dx\right| \\
&\leq c \sum_{j=0}^{d-1} \sum_{A(j)\supset\dots\supset A(1)} R^{j-d} \prod_{k=1}^{j} \min\left\{\lambda, \left(2\pi \left|P_{A(k)}\xi\right|\right)^{-1}\right\}.
\end{aligned}$$

When $j = 0$ the inner sum of products is intended to be the number of vertices of the polyhedron, and when $1 \leq j \leq d-1$ the inner sum is taken over all possible decreasing chains of j dimensional subspaces $\{A(j)\}$ and $P_{A(j)}$ is the orthogonal projection on $A(j)$.

Proof. Since $|\nabla \operatorname{dist}(x, \partial\Omega)| = 1$, the coarea formula gives

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi\left(R \operatorname{dist}(x, \partial\Omega)\right) \exp\left(-2\pi i \xi \cdot x\right) dx \\ &= \int_0^{+\infty} \left(\int_{\{\operatorname{dist}(x,\partial\Omega)=t\}} \exp\left(-2\pi i\xi \cdot x\right) dx\right) \psi\left(Rt\right) dt.\end{aligned}$$

We consider separately the level sets inside and outside Ω. The level sets $\{\operatorname{dist}(x, \partial\Omega) = t\} \cap \Omega$ are polyhedra with diameter at most λ and faces parallel to some $\{A(j)\}$. We emphasize that some of the $\{A(j)\}$ may not be parallel to any $\{\Omega(j)\}$. Anyhow, as in the previous lemma,

$$\begin{aligned}&\left|\int_0^{+\infty} \left(\int_{\{\operatorname{dist}(x,\partial\Omega)=t\}\cap\Omega} \exp\left(-2\pi i\xi \cdot x\right) dx\right) \psi\left(Rt\right) dt\right| \\ &\leq \sum_{A(d-1)\supset\ldots\supset A(1)} \left(2^d \int_0^{+\infty} |\psi(Rt)|\, dt\right) \prod_{j=1}^{d-1} \min\left\{\lambda, \left(2\pi \left|P_{A(j)}\xi\right|\right)^{-1}\right\}.\end{aligned}$$

As in the Steiner formula, the outer level sets $\{\operatorname{dist}(x, \partial\Omega) = t\} - \Omega$ are a union of sums of j dimensional faces $\{\Omega(j)\}$ and portions of $d-j-1$ dimensional spherical surfaces of radius t. Hence, if $\omega(d-j-1)$ denotes the measure of the $d-j-1$ dimensional spherical surface of unit radius,

$$\begin{aligned}&\left|\int_0^{+\infty} \left(\int_{\{\operatorname{dist}(x,\partial\Omega)=t\}-\Omega} \exp\left(-2\pi i\xi \cdot x\right) dx\right) \psi\left(Rt\right) dt\right| \\ &\quad\leq \sum_{j=0}^{d-1} \left(\omega(d-j-1) \int_0^{+\infty} |\psi(Rt)|\, t^{d-j-1} dt\right) \\ &\quad\times 2 \sum_{\Omega(j)\supset\ldots\supset\Omega(1)} \prod_{k=1}^{j} \min\left\{\lambda, \left(2\pi \left|P_{\Omega(k)}\xi\right|\right)^{-1}\right\}.\end{aligned}$$

Observe that when $\xi = 0$ the formula gives $c\left(R^{-d} + \lambda^{d-1}R^{-1}\right)$, with a constant c independent of the polyhedron. □

Theorem 2.10. *Given a finite collection of $d-1$ dimensional hyperspaces X in $\mathbb{R}^d$, let $P(X)$ be the collection of all convex polyhedra with diameter smaller than $1-\varepsilon$ and facets parallel to elements of X. If $\{A(d) \supset \ldots \supset A(1)\}$ is the collection of all possible decreasing chains of j dimensional subspaces obtained by the intersection of $\mathbb{R}^d$ and a number of hyperplanes in X, define*

$$\Phi(\xi) = \sum_{A(d)\supset\ldots\supset A(1)} \prod_{j=1}^{d} \min\left\{1, \left(2\pi \left|P_{A(j)}\xi\right|\right)^{-1}\right\}.$$

Finally, if $\{x_j\}_{j=1}^m$ is a sequence of points in the torus, define

$$\Psi(\xi) = \left|m^{-1} \sum_{j=1}^{m} \exp\left(2\pi i\xi \cdot x_j\right)\right|.$$

Then there exists a positive constant c, which depends only on the space dimension d and the upper bound $1-\varepsilon$ of diameters of polyhedra, such that for every $R>0$,

$$\sup_{\Omega\in P(X)}\left|\mu(\Omega)-m^{-1}\sum_{j=1}^{m}\chi_{\Omega}(x_j)\right|\leq c\left(R^{-1}+\sum_{0<|k|<R}\Phi\left(k\right)\Psi\left(k\right)\right).$$

Proof. This follows from Theorem 2.1 and the previous lemmas. It suffices to replace $\lambda\leq 1-\varepsilon$ with 1 and R^{j-d} with $|k|^{j-d}$ in the range $0<|k|<R$. □

For example, if X is the collection of hyperplanes $\{x_j=0\}$, then $P(X)$ is the collection of all boxes $\{a_j\leq x_j\leq b_j\}$ with diameter smaller than $1-\varepsilon$ and the above is an estimate of discrepancy with respect to boxes.

Corollary 2.11. *Given a finite collection of hyperspaces X and a prime number m, there exists a lattice point $g=(g_1,...,g_d)$ in $\mathbb{Z}^d$ with $1\leq g_j\leq m-1$, such that*

$$\sup_{\Omega\in P(X)}\left|\mu(\Omega)-m^{-1}\sum_{j=1}^{m}\chi_{\Omega}\left(jm^{-1}g\right)\right|\leq cm^{-1}\log^{d}(m).$$

The constant c depends only on the dimension and on the cardinality of the set of chains of subspaces $A\left(d\right)\supset\ldots\supset A\left(1\right)$ generated by X.

Theorem 0.4 follows from the above corollary, along with the observation that a polyhedron in the torus with facets parallel to elements of X and diameter smaller than 1 can be seen as the union of a finite number (which depends only on X) of polyhedra with facets parallel to elements of X and diameter smaller than, say, 1/2.

Proof. A preliminary result is needed.

Claim. There is a constant c, depending only on the dimension d, such that, for any decreasing chain of subspaces $A\left(d\right)\supset\ldots\supset A\left(1\right)$ and for any $R>0$,

$$\sum_{1\leq|k|\leq R}\prod_{j=1}^{d}\min\left\{1,\left(2\pi\left|P_{A(j)}k\right|\right)^{-1}\right\}\leq c\log^{d}\left(2+R\right).$$

Indeed, let $\{a_1,\ldots,a_d\}$ be an orthonormal basis of $\mathbb{R}^d$ such that $\{a_1,\ldots,a_j\}$ generates $A\left(j\right)$ for $j=1,...,d$. Since $\left|P_{A(j)}k\right|\geq|k\cdot a_j|$, it suffices to estimate

$$\sum_{1\leq|k|\leq R}\prod_{j=1}^{d}\min\left\{1,|k\cdot a_j|^{-1}\right\}.$$

Of course the idea is to replace the sum over a discrete variable with an integral over a continuous variable. Let $Q(k)$ be the cube centered at the integer point k with edges of length 1 parallel to the orthogonal axes $\{a_j\}$ and let $P(k)$ be the cube centered at k with edges of length 3. Since $|x\cdot a_j|$ is the distance of x from the hyperplane of equation $x\cdot a_j=0$, if this hyperplane crosses $P(k)$, then $|x\cdot a_j|\leq 2\sqrt{d}$ for any x in $Q(k)$, and therefore

$$\min\left\{1,|k\cdot a_j|^{-1}\right\}\leq 1=\min\left\{1,2\sqrt{d}\,|x\cdot a_j|^{-1}\right\}.$$

If this hyperplane does not cross $P(k)$, then for any x in $Q(k)$,

$$\min\left\{1, |k\cdot a_j|^{-1}\right\} = \min\left\{1, \frac{|x\cdot a_j|}{|k\cdot a_j|\,|x\cdot a_j|}\right\}$$
$$\leq \min\left\{1, \frac{|k\cdot a_j| + \sqrt{d}/2}{|k\cdot a_j|\,|x\cdot a_j|}\right\} \leq \min\left\{1, \left(1+\sqrt{d}/2\right)|x\cdot a_j|^{-1}\right\}.$$

Overall, for any k and any $x \in Q(k)$,

$$\prod_{j=1}^{d} \min\left(1, |k\cdot a_j|^{-1}\right) \leq \prod_{j=1}^{d} \min\left(1, 2\sqrt{d}\,|x\cdot a_j|^{-1}\right).$$

Hence,

$$\sum_{1\leq |k|\leq R} \prod_{j=1}^{d} \min\left\{1, |k\cdot a_j|^{-1}\right\}$$
$$\leq \sum_{1\leq |k|\leq R} \int_{Q(k)} \prod_{j=1}^{d} \min\left\{1, 2\sqrt{d}\,|x\cdot a_j|^{-1}\right\} dx$$
$$\leq \int_{\{|x\cdot a_j|\leq R+1/2\}} \prod_{j=1}^{d} \min\left\{1, 2\sqrt{d}\,|x\cdot a_j|^{-1}\right\} dx$$
$$= \left(2\int_0^{R+1/2} \min\left\{1, 2\sqrt{d}t^{-1}\right\} dt\right)^d \leq c\log^d(2+R).$$

Now comes the proof of the corollary. The sum of a geometric progression gives

$$m^{-1}\sum_{j=1}^{m} \exp\left(2\pi i j m^{-1} g\cdot k\right) = \frac{\sin\left(\pi g\cdot k\right)}{m\sin\left(\pi m^{-1} g\cdot k\right)} \exp\left(\pi i(m+1)m^{-1}g\cdot k\right).$$

This exponential sum is 0 or 1 according to $g\cdot k \neq 0 \pmod{m}$ or $g\cdot k \equiv 0 \pmod{m}$. Hence, by Theorem 2.10 with $R = m$,

$$\sup_{\Omega\in P(X)} \left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m} \chi_\Omega\left(jm^{-1}g\right)\right| \leq c\left(m^{-1} + \sum_{0<|k|<m,\ g\cdot k\equiv 0 \pmod{m}} \Phi(k)\right).$$

The heuristic behind the existence of good lattice points with the desired properties is that the ratio of k's which satisfy the congruence $g\cdot k \equiv 0 \pmod{m}$ is m^{-1} and the sum over the k's with $g\cdot k \equiv 0 \pmod{m}$ is m^{-1} times the sum over all k. This heuristic principle can be made rigorous by an averaging procedure, as in Theorem 5.7 of [16]. In order to satisfy the congruence $g_1k_1 + ... + g_dk_d \equiv 0 \pmod{m}$, if $k_i \neq 0 \pmod{m}$, then one can take g_j arbitrary for $j \neq i$, the remaining g_i being uniquely determined in the residue class. Hence, by the above claim,

$$(m-1)^{-d}\sum_{1\leq g_j\leq m-1}\left(\sum_{0<|k|<m,\ g\cdot k\equiv 0\ (\mathrm{mod}\, m)}\Phi\left(k\right)\right)$$
$$=\sum_{0<|k|<m}\Phi\left(k\right)\left((m-1)^{-d}\sum_{1\leq g_j\leq m-1,\ g\cdot k\equiv 0\ (\mathrm{mod}\, m)}1\right)$$
$$\leq (m-1)^{-1}\sum_{0<|k|<m}\Phi\left(k\right)\leq cm^{-1}\log^{d}(m).$$

Observe that the constant c is the product of the constant in the above claim and the cardinality of the set of chains of subspaces generated by X. In particular, there exists g such that

$$\sum_{0<|k|<m,\ g\cdot k\equiv 0\ (\mathrm{mod}\, m)}\Phi\left(k\right)\leq cm^{-1}\log^{d}(m).$$

□

Corollary 2.12. *Given a finite collection of hyperspaces X, then for every $\varepsilon>0$ and almost every x in $\mathbb{T}^d$ there exists a positive constant c such that for every $m>1$,*

$$\sup_{\Omega\in P(X)}\left|\mu(\Omega)-m^{-1}\sum_{j=1}^{m}\chi_{\Omega}\left(jx\right)\right|\leq cm^{-1}\log^{d+1+\varepsilon}(m).$$

Proof. As in Corollary 2.7, this follows from Theorem 2.10 and an adaptation of [19]. □

The above corollaries improve and extend a two dimensional result in [1, Theorem 4D], where it is proved that the discrepancy of m points with respect to polygons is dominated by $m^{-1}\log^{4+\varepsilon}(m)$. They should also be compared with the discrepancy of lattice points in [4].

3. Approximation and discrepancy on manifolds

Let $\mathcal{M}$ be a smooth d dimensional compact manifold without boundary, with Riemannian distance $\mathrm{dist}(x,y)$ and measure μ normalized so that $\mu\left(\mathcal{M}\right)=1$. The Laplace Beltrami operator Δ on $\mathcal{M}$ has eigenvalues $\left\{\lambda^2\right\}$, counted with appropriate multiplicity, and a complete orthonormal system of eigenfunctions $\{\varphi_\lambda(x)\}$. To every function in $L^2(\mathcal{M},d\mu)$ one can associate a Fourier transform and a Fourier series,

$$\widehat{f}(\lambda)=\int_{\mathcal{M}}f(y)\overline{\varphi_\lambda(y)}d\mu(y),\quad f(x)=\sum_{\lambda}\widehat{f}(\lambda)\varphi_\lambda(x).$$

Fourier series on compact Lie groups and symmetric spaces are examples. In particular, the eigenfunctions of the Laplace operator on the torus are trigonometric functions, and eigenfunction expansions are classical Fourier series. Similarly, eigenfunctions of the Laplace operator on the surface of a sphere are homogeneous harmonic polynomials, and eigenfunction expansions are spherical harmonic expansions. In the setting of manifolds, an analog of trigonometric polynomials is given by finite linear combinations of eigenfunctions $\sum_\lambda c_\lambda\varphi_\lambda(x)$. Indeed it can be shown that there is a close relation between approximation by functions of exponential type and by eigenfunctions. See for example [6]. The following generalizes Theorem 1.1 and Corollary 1.2.

Theorem 3.1. *Given $\alpha > 0$, there exists $\beta > 0$ such that, for every domain Ω in $\mathcal{M}$ and $R > 0$, there exist linear combinations $A(x)$ and $B(x)$ of eigenfunctions with eigenvalues at most R^2, satisfying*

$$A(x) \leq \chi_\Omega(x) \leq B(x), \quad |B(x) - A(x)| \leqslant \beta \left(1 + R \operatorname{dist}(x, \partial\Omega)\right)^{-\alpha}.$$

It is likely that a slightly more precise result holds, with a rapid decay instead of a polynomial decay $(1 + R \operatorname{dist}(x, \partial\Omega))^{-\alpha}$. However, the exponent α can be arbitrarily large, and this suffices for our applications.

Proof. The proof of this theorem is similar to the proof of Theorem 1.1, and it is based on suitable approximations of the identity adapted to the manifold, analogous to the convolution kernels in the Euclidean spaces:

$$\begin{gathered} K_R(x,y) = \sum_{\lambda < R} c_\lambda \varphi_\lambda(x) \overline{\varphi_\lambda(y)}, \\ |K_R(x,y)| \leq c(\alpha) R^d \left(1 + R \operatorname{dist}(x,y)\right)^{-\alpha}, \\ \int_{\mathcal{M}} K_R(x,y) d\mu(y) = 1. \end{gathered}$$

Moreover, these kernels are positive up to a negligible error. The construction of such kernels on Lie groups and symmetric spaces is well known, and in these cases it is possible to obtain positivity. Indeed, if a kernel has good decay and finite spectrum, then also its square has good decay and finite spectrum and a suitable normalization has mean one. We do not know whether positivity can be achieved in our general setting; however, in the sequel almost positivity will suffice. Given $m(\xi)$ as in the proof of Theorem 1.1 and

$$h(|\xi|) = \left(1 + |\xi|^2\right)^{-(d+1)/2} m * m(\xi),$$

define

$$K_R(x,y) = \sum_{\lambda} h\left(R^{-1}\lambda\right) \varphi_\lambda(x) \overline{\varphi_\lambda(y)}.$$

It is possible to prove that this kernel has an asymptotic expansion with Euclidean main term $R^d K(R \operatorname{dist}(x,y))$ and suitable control on the remainder. Although the details are not completely trivial, the techniques can be found in Chapter XII of [21], or in [3]. Finally, define

$$\begin{aligned} H_R(x) &= \beta \left(1 + R \operatorname{dist}(x, \partial\Omega)\right)^{-\alpha}, \\ A(x) &= \int_{\mathcal{M}} K_R(x,y) \left(\chi_\Omega(y) - H_R(y)\right) d\mu(y), \\ B(x) &= \int_{\mathcal{M}} K_R(x,y) \left(\chi_\Omega(y) + H_R(y)\right) d\mu(y). \end{aligned}$$

Then, as in the proof of Theorem 1.1, it is possible to show that for some $\beta > 0$ independent of Ω and R, these functions satisfy the required properties. □

Theorem 3.2. *For every sequence of points $\{x_j\}_{j=1}^m$ and domain Ω in $\mathcal{M}$ and $R>0$, if $H_R(x)$ is defined as in the proof of Theorem* 3.1*, then*

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^m \chi_\Omega(x_j)\right|$$
$$\leq \left|\widehat{H}_R(0)\right| + \sum_{0<\lambda<R}\left(|\widehat{\chi}_\Omega(\lambda)| + \left|\widehat{H}_R(\lambda)\right|\right)\left|m^{-1}\sum_{j=1}^m \varphi_\lambda(x_j)\right|.$$

Proof. This proof is completely analogous to the one of Theorem 2.1. □

Of course, the interest of the above result arises when one is able to exhibit point distributions with $m^{-1}\sum_{j=1}^m \varphi_\lambda(x_j)$ suitably small. Inspired by [17] on the problem of distributing points on a sphere, we now consider point distributions generated by the action of a free group on a homogeneous space. Let $\mathcal{G}$ be a compact Lie group, $\mathcal{K}$ a closed subgroup, and $\mathcal{M} = \mathcal{G}/\mathcal{K}$ a homogeneous space of dimension d with normalized invariant measure μ. Let $\mathcal{H}$ be a finitely generated free subgroup in $\mathcal{G}$, and assume that the action of $\mathcal{H}$ on $\mathcal{M}$ is free. Given a positive integer k, let $\{\sigma_j\}_{j=1}^m$ be an ordering of the elements in $\mathcal{H}$ with length at most k. For every function $f(x)$ on $\mathcal{M}$, define

$$Tf(x) = m^{-1}\sum_{j=1}^m f(\sigma_j x).$$

This operator is self-adjoint with norm 1, hence all its eigenvalues have modulus at most 1. Indeed, 1 is an eigenvalue, and the constants are eigenfunctions. In the following, we shall be interested in cases where all other nonconstant eigenfunctions have eigenvalues much smaller than 1. For this reason, define $\rho(m)$ as the supremum of the eigenvalues with nonconstant eigenfunctions,

$$\rho(m) = \sup_{T\varphi(x)=\nu\varphi(x),\ \varphi(x)\neq 1} |\nu|.$$

Moreover, as before, define $M(\delta,\Omega) = \sup_{t>0} t^{-\delta}\mu(\{\mathrm{dist}(x,\partial\Omega)<t\})$.

Theorem 3.3. *There exists a positive constant c such that for every point x in $\mathcal{M}$ and $R>0$,*

$$\sup_{M(\delta,\Omega)<\gamma}\left|\mu(\Omega) - m^{-1}\sum_{j=1}^m \chi_\Omega(\sigma_j x)\right| \leq c\gamma\left(R^{-\delta} + R^{(d-\delta)/2}\rho(m)\right).$$

Proof. Since the operators T and Δ commute, they have a common orthonormal system of eigenfunctions, $\Delta\varphi_\lambda(x) = \lambda^2\varphi_\lambda(x)$ and $T\varphi_\lambda(x) = T(\lambda)\varphi_\lambda(x)$. The assumption in the theorem is precisely that $|T(\lambda)| \leq \rho(m)$ if $\lambda \neq 0$. Hence, by Theorem 3.2,

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^m \chi_\Omega(\sigma_j x)\right|$$
$$\leq \left|\widehat{H}_R(0)\right| + \rho(m)\sum_{0<\lambda<R}\left(|\widehat{\chi}_\Omega(\lambda)| + \left|\widehat{H}_R(\lambda)\right|\right)|\varphi_\lambda(x)|.$$

Since $H_R(x) = \beta\left(1 + R\,\mathrm{dist}\left(x, \partial\Omega\right)\right)^{-\alpha}$, then

$$\begin{aligned}\widehat{H}_R(0) &\le \beta \int_{\mathcal{M}} \left(1 + R\,\mathrm{dist}\left(x, \partial\Omega\right)\right)^{-\alpha} d\mu(x) \\ &\le \beta\left(\mu\left(\left\{\mathrm{dist}\left(x, \partial\Omega\right) \le R^{-1}\right\}\right) + \sum_{j=0}^{+\infty} 2^{-\alpha j} \mu\left(\left\{\mathrm{dist}\left(x, \partial\Omega\right) < 2^{j+1}R^{-1}\right\}\right)\right) \\ &\le \beta\left(1 + 2^{\delta} \sum_{j=0}^{+\infty} 2^{(\delta-\alpha)j}\right) M\left(\delta, \Omega\right) R^{-\delta}.\end{aligned}$$

Similarly, by Cauchy and Bessel inequalities,

$$\begin{aligned}&\sum_{\lambda<R} \left|\widehat{H}_R(\lambda)\right| \left|\varphi_\lambda\left(x\right)\right| \le \left\{\sum_{\lambda<R} \left|\widehat{H}_R(\lambda)\right|^2\right\}^{1/2} \left\{\sum_{\lambda<R} \left|\varphi_\lambda\left(x\right)\right|^2\right\}^{1/2} \\ &\le \beta\left\{\int_{\mathcal{M}} \left(1 + R\,\mathrm{dist}\left(x, \partial\Omega\right)\right)^{-2\alpha} d\mu(x)\right\}^{1/2} \left\{\sum_{\lambda<R} \left|\varphi_\lambda\left(x\right)\right|^2\right\}^{1/2} \\ &\le \beta\left\{1 + 2^{\delta} \sum_{j=0}^{+\infty} 2^{(\delta-2\alpha)j}\right\}^{1/2} \left\{\sum_{\lambda<R} \left|\varphi_\lambda\left(x\right)\right|^2\right\}^{1/2} \sqrt{M\left(\delta, \Omega\right)} R^{-\delta/2}.\end{aligned}$$

One also gets

$$\begin{aligned}&\sum_{0<\lambda<R} \left|\widehat{\chi}_\Omega(\lambda)\right| \left|\varphi_\lambda\left(x\right)\right| \\ &\le \left\{\sum_{0<\lambda<1} \left|\widehat{\chi}_\Omega(\lambda)\right|^2\right\}^{1/2} \left\{\sum_{0<\lambda<1} \left|\varphi_\lambda\left(x\right)\right|^2\right\}^{1/2} \\ &+ \sum_{k=0}^{[\log_2(R)]} \left\{\sum_{\lambda\ge 2^k} \left|\widehat{\chi}_\Omega(\lambda)\right|^2\right\}^{1/2} \left\{\sum_{\lambda<2^{k+1}} \left|\varphi_\lambda\left(x\right)\right|^2\right\}^{1/2}.\end{aligned}$$

If $A_{2^k}(x) \le \chi_\Omega(x) \le B_{2^k}(x)$ are the approximating functions in Theorem 3.1 corresponding to $R = 2^k$, then,

$$\begin{aligned}\sum_{\lambda\ge 2^k} \left|\widehat{\chi}_\Omega(\lambda)\right|^2 &\le \int_{\mathcal{M}} \left|\chi_\Omega(x) - A_{2^k}(x)\right|^2 d\mu(x) \\ &\le \beta^2 \int_{\mathcal{M}} \left(1 + 2^k\,\mathrm{dist}\left(x, \partial\Omega\right)\right)^{-2\alpha} d\mu(x) \\ &\le \beta^2\left(1 + 2^{\delta} \sum_{j=0}^{+\infty} 2^{(\delta-2\alpha)j}\right) M\left(\delta, \Omega\right) 2^{-\delta k}.\end{aligned}$$

By classical bounds on the spectral function of an elliptic operator (see e.g. Theorem 17.5.3 in [15]),

$$\sum_{\lambda<R} \left|\varphi_\lambda\left(x\right)\right|^2 \le cR^d.$$

Hence, if α is large enough,

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(\sigma_j x)\right|$$
$$\leq cM\,(\delta,\Omega)\,R^{-\delta} + c\sqrt{M\,(\delta,\Omega)}R^{(d-\delta)/2}\rho(m).$$

Finally, observe that $\sqrt{M\,(\delta,\Omega)} \leq cM\,(\delta,\Omega)$, since $M\,(\delta,\Omega)$ is bounded below, as one sees by putting $t = 1$ in the definition of this constant. □

The following corollary is Theorem 0.5 in the Introduction, and it has been proved in [17] in the case of spherical caps.

Corollary 3.4. *If $\mathcal{M} = SO(3)/SO(2)$ is the two dimensional sphere and if $\mathcal{H}$ is the free group generated by rotations of angles $\arccos(-3/5)$ around orthogonal axes, then there exists a constant c such that, if k is an integer and $\{\sigma_j\}_{j=1}^{m}$ is an ordering of the elements in $\mathcal{H}$ with length at most k, then for every x,*

$$\left|\mu(\Omega) - m^{-1}\sum_{j=1}^{m}\chi_\Omega(\sigma_j x)\right| \leq cM\,(\delta,\Omega)\,m^{-\delta/(2+\delta)}\log^{2\delta/(2+\delta)}(m).$$

Proof. The eigenvalues of the operator T satisfy the Ramanujan bounds

$$\rho(m) = \sup_{T\varphi(x)=\nu\varphi(x),\ \varphi(x)\neq 1} |\nu| \leq cm^{-1/2}\log(m).$$

Hence, choosing $R = m^{1/(d+\delta)}\log^{-2/(d+\delta)}(m)$ in the above theorem,

$$\inf_{R>0}\left\{R^{-\delta} + R^{(d-\delta)/2}m^{-1/2}\log(m)\right\} \leq cm^{-\delta/(d+\delta)}\log^{2\delta/(d+\delta)}(m).$$

□

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Dipartimento di Matematica e Applicazioni, Edificio U5, Università di Milano - Bicocca, via R. Cozzi 53, 20125 Milano, Italia

E-mail address: `leonardo.colzani@unimib.it`

Dipartimento di Ingegneria dell'Informazione e Metodi Matematici, Università di Bergamo, viale Marconi 5, 24044 Dalmine, Italia

E-mail address: `giacomo.gigante@unibg.it`

Dipartimento di Statistica, Edificio U7, Università di Milano - Bicocca, via Bicocca degli Arcimboldi 8, 20126 Milano, Italia

E-mail address: `giancarlo.travaglini@unimib.it`