Trigonometric formulae?—No one uses them anymore

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(Received 28 November 2005; revised 8 August 2006)

Abstract

The title paraphrases a comment made during talks with local mathematics teachers in Townsville. The challenge was laid down. Where are these formulae used? Why do we think they are important? This paper covers some of the answers. In particular, these formulae were used (almost unnoticed) in a project looking at solving the Helmholtz equation on various regions in two dimensions. Some results of this project are presented.

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See http://anziamj.austms.org.au/V47EMAC2005/Sneddon for this article, Published September 20, 2006. ISSN 1446-8735

1 Introduction

The thoughts leading to this paper came from discussions that took place between members of the School of Mathematical and Physical Sciences at James Cook University and the committee of the local branch of the Queensland Association of Mathematics Teachers. As a part of those discussions we looked at the current Senior Mathematics syllabus in Queensland.

One surprise was that basic trigonometric formulae, such as that for $\sin(A+B)$ and related formulae, seem to have been relegated to just an option in the advanced mathematics subject. In the discussions that we had, a story was related about how an engineer was called in the middle of the night and asked what this formula was used for and he replied ‘nothing’. This response might appear to support the decision not to include this topic in the core syllabus. However, it came as a surprise to those present at the meeting. Unfortunately, as often happens in those situations, no one could offer a realistic application immediately. It was particularly annoying the next day when we realised that we had used the formula at an important stage in our research earlier in the year. This use of the formula had gone almost unnoticed. We shall comment on this later. In the meantime, we identified several other areas where the formulae are used.

1.1 The senior syllabus

In Queensland, the Senior Mathematics syllabus extends over the final two years of high school, years 11 and 12. There are two subjects, Mathematics B and Mathematics C. (There is a third subject, Mathematics A, which is a lower level subject, or General Mathematics.) Details of these syllabi can be found on the Queensland Studies Authority website [1].

Mathematics B covers topics such as elementary functions, elementary calculus, some integration and statistics. The Mathematics C syllabus covers group theory, real and complex numbers, matrix algebra, vectors, calculus (methods and applications) and structures and patterns (basically sequences and series). Then there is a group of optional topics, from which schools should choose just two. These optional topics are (in order) linear programming, conics, dynamics, number theory, probability models, advanced periodic and exponential functions, and other.

It is not until the second last topic that we see a reference that might cover trigonometric formulae. It is almost as if this is a 'fall back' option in case schools cannot think of any other suitable topic.

2 Why are these formulae important

Firstly, the derivation of the formula, for $\sin(A + B)$, is an interesting piece of mathematics. The proof is a neat exercise in geometry, and requires some construction and some calculation. It is typical of many types of geometric construction that are so important in physics and mechanics.

Secondly, the trigonometric formula is just as important for what it *is not*: $\sin(A+B)$ is not the same as $\sin A + \sin B$. It is of lesser importance if someone cannot quite remember the precise details of the formula. A student may not always be be sure which one is cos–cos and which is sin–cos, or remember the details of the signs. However, such a student should be aware that there is a rule for $\sin(A+B)$ and that the answer is *not* $\sin A + \sin B$. A glance at the geometry makes this clear. University mathematics staff and school teachers alike will be familiar with many other examples of this type. For example, $\sqrt{x^2+y^2}$, $1/(a+b)$, $(x+y)^2$. The important point here is that these functions are non-linear and that in reality it is a very specialised set of functions for which $f(x+y) = f(x) + f(y)$.

It is easy for *anyone* to make this mistake, but it is our perception that students are increasingly worse at these concepts. Whereas once an offending student might show some embarrassment, and the staff member might consider it such an amusing event that he or she might make a point of telling some colleagues, now students show surprise or even disbelief when told, say, that $\sqrt{x^2+y^2}$ is not equal to $x+y$. Unfortunately, staff members no longer regard it as a noteworthy event.

This is a pity. In our opinion, understanding that not all functions are linear and that some functions will have different rules for $f(x+y)$ is important for anyone learning mathematics. Even if students cannot remember the details of the rules, they should at least be aware that there are rules and know when they should check these rules.

3 But where are they *used*

The examples that we found fall into four broad areas. A qualitative discussion of some of these applications and many other applications of trigonometry can be found in Wikipedia, The Free Encyclopedia [2].

3.1 Signal processing

1. Trigonometric formulae are important in understanding how amplitude modulated signals (used in AM radio broadcasts) can be expressed in terms of the side-band frequencies. For instance

$$\cos\omega_0 t\,\cos\omega t = \frac{1}{2}\big[\cos(\omega_0+\omega)t + \cos(\omega_0-\omega)t\big].$$

   Here ω_0 is the carrier frequency and ω is the frequency of the signal. In this case a complicated function is written in terms of two much simpler functions.

2. In a similar fashion, a complicated signal can be represented in terms of its Fourier transform or the coefficients of its Fourier series. For instance, a CAT scan is one example where this happens.

   The trigonometric formulae are used in the development of the Euler formulae for the coefficients of the Fourier series. Similarly, the identity $e^{i(x+y)} = e^{ix}\times e^{iy}$, which represents the trigonometric formulae in a different form, is central in the development of the Fourier transform.

   Once the Fourier transform and the discrete Fourier transform are acknowledged as examples where the trigonometric functions and their formulae are used, then the applications are virtually limitless. It is just that these applications are often a part of software that is taken for granted.

3.2 Wave analysis

1. Standing waves can be seen as being composed of travelling waves. That is,
$$\cos x \, \cos ct = \frac{1}{2}\left[\, \cos(x+ct) + \cos(x-ct) \right].$$

2. In a similar fashion, it is sometimes helpful to express a travelling wave in terms of standing waves. For example
$$\cos(lx+my+nz-ct) = \cos(lx+my+nz)\cos ct + \sin(lx+my+nz)\sin ct\,.$$

3.3 Vibrations

The analysis of any vibrations will involve trigonometric functions. Some examples follow.

1. If different components of machinery vibrate with different frequencies, the resulting beat frequency can often be heard as a low background hum. The source of this beat frequency can be understood as a direct application of the sum–product formulae.

   A beat frequency also results when two musical notes with different frequencies are played together. These beats can be used as an aid in tuning musical instruments. (Though it is likely that the instrument tuner may be unaware of the mathematics behind these beats.)

2. It is important to be able to decompose any vibration into its normal modes. For example, the motion of a double pendulum appears reasonably complicated. However, it can readily be decomposed into its two component modes, each of which is just simple harmonic motion.

3.4 Trigonometry

The formulae are important in trigonometry. Here are two specific examples.

1. In surveying (particularly in spherical trigonometry) the formulae are used to determine if the measurements correspond to a closed figure.
2. In analysing planetary motion (or the motion of any heavenly bodies) it is necessary to go through several coordinate transformations to transform the motion in the ecliptic plane into motion expressed in terms of a reference frame fixed in the earth's surface. These transformations all rely on the trigonometric formulae.

4 The Helmholtz equation

At the time of our meeting with the mathematics teachers we had forgotten that the formulae had been used to carry out one of the steps in our research. The Helmholtz equation,

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \lambda\Psi = 0\,,$$

had arisen when considering advection–diffusion for hill slope seepage problems [3, 4, 5]. This equation typically arises when finding the motion of a vibrating membrane. In that case the eigenvalue λ is related to the frequency of the vibration. It is easy to find solutions of this equation, but not so easy to get solutions that satisfy certain boundary conditions. For example, on a square membrane whose side length is $l = 1$, if we require the solution to be zero on the boundary of the square there will be solutions in the form

$$\Psi(x, y) = \sin m\pi x\, \sin n\pi y\,,$$

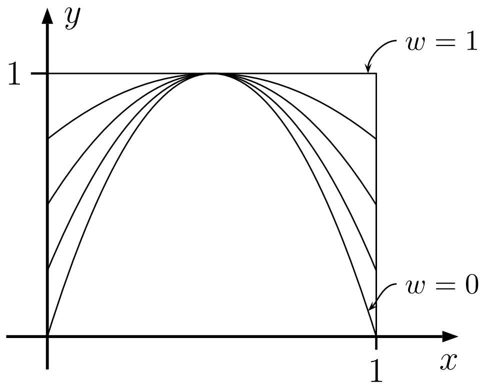


Figure 1: Graphs of the modified domain. The upper boundaries have the equations $y = w + 4(1-w)x(1-x)$ for $w = 0, 0.25, 0.5, 0.75, 1.0$.

(m and n are integers) and the value of λ will be $\pi^2(m^2+n^2)$.

We wanted solutions on a more general region. We started looking for solutions on a unit square that was modified so that the upper boundary had a parabolic shape as shown in Figure 1.

We tried to find solutions of the form

$$\Psi(x,y) = \sum_{k=1}^{N} a_k v_k(x,y)\,,$$

where the a_k are constants and the v_k are known functions that satisfy the equation, but not necessarily the boundary conditions. The aim was to find coefficients a_k so that $\Psi(x,y)$ will come close to satisfying the boundary conditions. The next step is to make an appropriate choice of the functions v_k. We initially chose $v_k = \sin m\pi x\, \sin n\pi y$, where m or n may not be integers. For a fixed λ we might choose

$$v_k = \sin k\pi x\, \sin \gamma_k \pi y\,,$$

with k an integer and $\gamma_k = \sqrt{\lambda/\pi^2 - k^2}$. This is not helpful if $k^2 > \lambda/\pi^2$. In those cases we switched to the sinh function so that

$$v_k = \sin k\pi x \, \sinh \gamma_k \pi y \, ,$$

with $\gamma_k = \sqrt{k^2 - \lambda/\pi^2}$. These functions satisfied the boundary conditions on the side and bottom boundaries. There were a sufficient number of functions to obtain a reasonable fit at the top boundary, but the nature of the sinh functions created difficulties if too many of them were required. The problem of determining the coefficients a_k became ill-conditioned.

The problem still was to find more functions that satisfied the differential equation. To overcome this we went back to the original solution and wrote it as

$$\sin m\pi x \, \sin n\pi y = \tfrac{1}{2}\big[\cos(m\pi x - n\pi y) - \cos(m\pi x + n\pi y)\big].$$

Each of these two terms satisfies the differential equation and more importantly, each one represents a standing wave at a fixed angle to the x-axis. It became clear that the functions we needed would be a set of such standing wave functions but chosen so that the directions were all equi-spaced. If there were N such functions, they could be chosen to be

$$\phi_k = \cos(m\pi x + n\pi y),$$

with

$$m = \frac{\sqrt{\lambda}}{\pi}\cos\theta_k \, , \quad n = \frac{\sqrt{\lambda}}{\pi}\sin\theta_k \, ,$$

where $\theta_k = k\pi/(2N)$, $k = 1, \ldots, N$. This choice of function guaranteed that $\pi^2(m^2 + n^2) = \lambda$ and the choice of the θ_k ensured that the directions were equi-spaced. A typical function is illustrated in Figure 2.

In practice we actually used the functions $v_k = \cos m\pi(x - 0.5)\,\sin n\pi y$ as basis functions. These functions automatically satisfied the boundary condition on the bottom boundary and were symmetric about $x = 0.5$. Thus any solutions found this way would also have this symmetry.

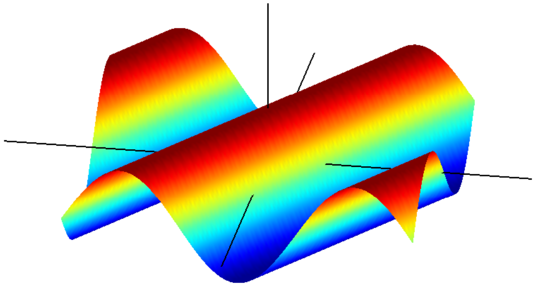
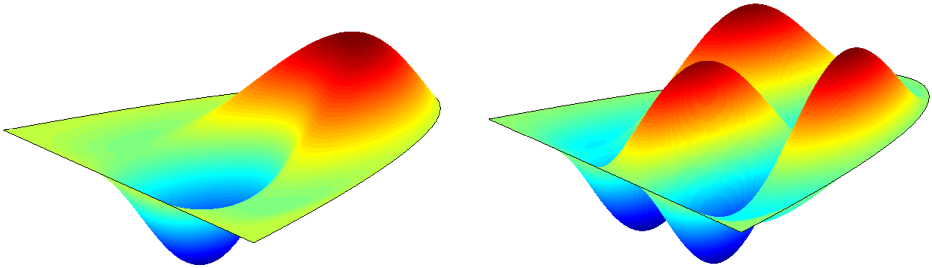


Figure 2: A typical standing wave basis function.



$\lambda = 714.93557419$ $\lambda = 1084.33325223$

Figure 3: Two of the solutions, and the corresponding values of λ, obtained for the parabolic shape $y = x(1 - x)$.

We were interested in the convergence of these different solutions on different shaped regions and how accurately the eigenvalue could be obtained. Figure 3 shows some of the solutions on the region bounded above by the parabola $y = x(1 - x)$.

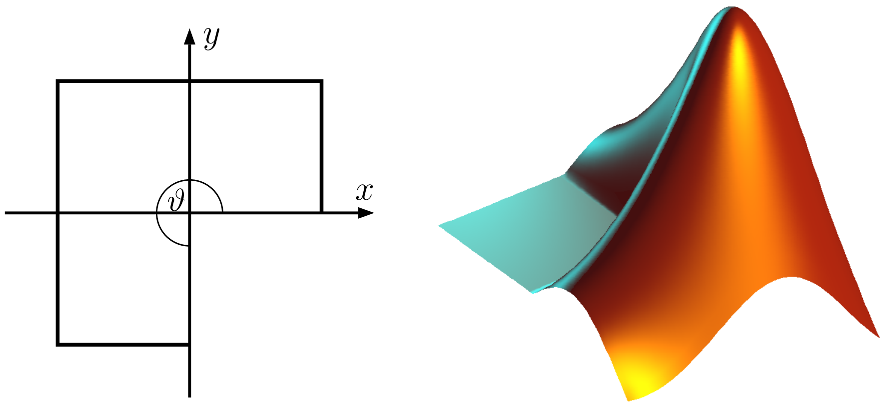


FIGURE 4: The L-shaped region and the well-known MATLAB logo.

4.1 The method of particular solutions

As an aside, Fox, Henrici and Moler [6] solved this same problem on the L-shaped region shown in Figure 4. They used the basis functions

$$v_k(x, y) = J_{\alpha k}(\sqrt{\lambda}\, r) \sin \alpha k \theta\,,$$

where $\alpha = \pi/\vartheta$ and k is a positive integer. In this case, the basis functions are expressed in terms of the polar coordinates r and θ. The function J_ν is the Bessel function of order ν. Bessel functions often appear in the solution of problems that have cylindrical symmetry. This choice of basis function guaranteed that the solution would be zero on the x-axis between $x = 0$ and $x = 1$ and on the y-axis between $y = 0$ and $y = -1$. Fox, Henrici and Moler referred to their method as the Method of Particular Solutions. The approximate solution obtained with just the first two basis functions forms the well-known MATLAB logo (Figure 4).

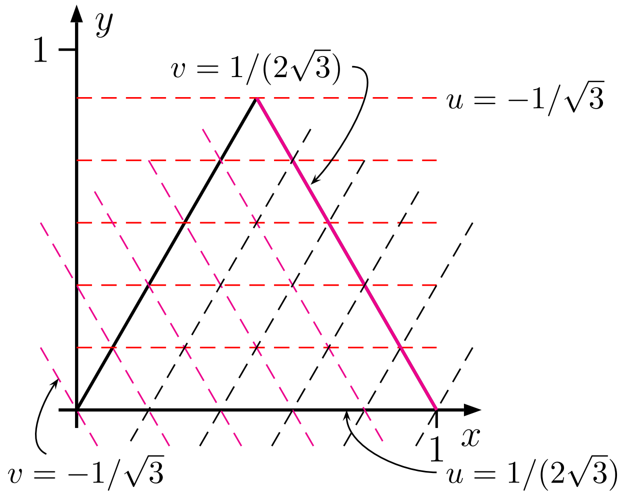


Figure 5: Triangular coordinates (u, v, w).

4.2 The equilateral triangle

Finally, this same problem can be applied to an equilateral triangle. We could use the same numerical procedure, but the problem has an exact solution in terms of trigonometric functions. The existence of an exact solution means that this example can be used to check the accuracy of the numerical methods. McCartin [7] gives a derivation of the earlier results of Lamé. In this case it is convenient to work in triangular coordinates

$$\begin{aligned}
u &= \frac{1}{2\sqrt{3}} - y\,, \\
v &= \frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) + \frac{1}{2}\left(y - \frac{1}{2\sqrt{3}}\right), \\
w &= \frac{\sqrt{3}}{2}\left(\frac{1}{2} - x\right) + \frac{1}{2}\left(y - \frac{1}{2\sqrt{3}}\right).
\end{aligned}$$

These coordinates are not independent since $u + v + w = 0$.

It can be verified that the functions

$$\phi = \sin[p\pi(u + 1/\sqrt{3})]\, \cos[q\pi(v - w)]$$

are solutions with $\lambda = \pi^2(p^2 + 3q^2)$. These solutions are symmetric about $x = 0.5$, and ϕ will be equal to zero along the x-axis provided $p = 2l/\sqrt{3}$ where l is an integer. Fortunately, we do not need an infinite sum to satisfy the boundary conditions on the other two boundaries. It turns out that three terms will be enough. Thus

$$\begin{aligned}\Psi \;=\; & \sin[p_1\pi(u + 1/\sqrt{3})]\cos[q_1\pi(v - w)] \\ & + \sin[p_2\pi(u + 1/\sqrt{3})]\cos[q_2\pi(v - w)] \\ & + \sin[p_3\pi(u + 1/\sqrt{3})])\cos[q_3\pi(v - w)].\end{aligned}$$

with $p_1 = 2l/\sqrt{3}$, $p_2 = 2m/\sqrt{3}$ and $p_3 = 2n/\sqrt{3}$. There will be additional relationships between the p's and q's needed to ensure that the remaining boundary conditions will be satisfied. The trigonometric formulae are needed to find these relationships and to put these solutions into a more appropriate form. Eventually we get the solutions

$$\begin{aligned}\Psi \;=\; & +\sin\left[\frac{4\pi}{3\sqrt{3}}(lu + mv + nw + \sqrt{3}\,l/2)\right] \\ & + \sin\left[\frac{4\pi}{3\sqrt{3}}(nu + mv + lw + \sqrt{3}\,n/2)\right] \\ & + \sin\left[\frac{4\pi}{3\sqrt{3}}(mu + nv + lw + \sqrt{3}\,m/2)\right] \\ & + \sin\left[\frac{4\pi}{3\sqrt{3}}(mu + lv + nw + \sqrt{3}\,m/2)\right] \\ & + \sin\left[\frac{4\pi}{3\sqrt{3}}(nu + lv + mw + \sqrt{3}\,n/2)\right] \\ & + \sin\left[\frac{4\pi}{3\sqrt{3}}(lu + nv + mw + \sqrt{3}\,l/2)\right].\end{aligned}$$

where l, m and n are integers such that $l + m + n = 0$. The eigenvalue is

$$\lambda = \frac{8}{9}(l^2 + m^2 + n^2).$$

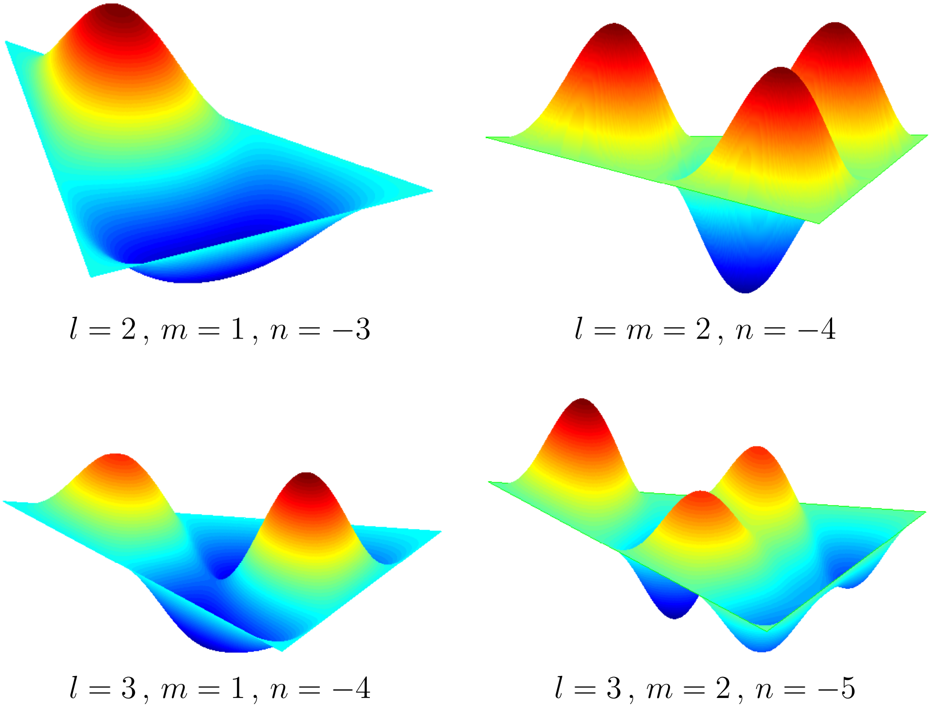


Figure 6: Various solutions of the Helmholtz equation on the triangular region.

With a simple closed form solution such as this, it is easy to plot the various solutions. Some examples are given in Figure 6.

5 Conclusion

There seems to be a difference of opinion between mathematicians and school educators about what mathematics is important. If we believe these things are important, we need to make our views known.

There are three main reasons for studying trigonometric formulae in detail at secondary school. Firstly, the formulae demonstrate fundamental properties of mathematics such as the essential non-linearity of most functions. Secondly, manipulating the formulae develops algebraic dexterity and can assist in the development of a deep understanding of these important functions. Thirdly, far from not being used anymore, these functions and the relationships between them have a wealth of applications in modern mathematics, science and finance.

Acknowledgements The authors are particularly grateful for the comments and input of the referees and the editors.

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