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TRIGONOMETRIC SUMS ASSOCIATED WITH PSEUDO-MEASURES

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The purpose of this paper is to study the structure of pseudo-measures on closed sets $E \subseteq \Gamma \equiv R/2\pi Z$ of Lebesgue measure $m(E) = 0$. In particular, we find various conditions on E so that a given pseudo-measure T supported by E is actually a measure or, at least, the first derivative of a bounded function. Such questions are obviously related to the open problem of determining if, generally, a Helson is a set of spectral synthesis.

$A(\Gamma)$ is the space of absolutely convergent Fourier series $\varphi(\gamma) = \Sigma\, a_n e^{in\gamma}$ normed by $\| \varphi \|_A = \Sigma \mid a_n \mid$; and its dual is the space of pseudo-measures $A'(\Gamma)$.

$A'(E)$ (resp., $M(E)$) is the space of pseudo-measures (resp., measures) T supported by E such that the Fourier coefficient $\widehat{T}(0) = 0$. We let $D_b(E)$ be those first order distributions T supported by E such that T is the first derivative of an L^∞ function and $\widehat{T}(0) = 0$; the bounded pseudo-measures on E are $A'_b(E) \equiv D_b(E) \cap A'(E)$.

§ 1 is devoted to notation and the statement of some formulas for pseudo measures in terms of trigonometric sums; these results are basic to what follows. Next (§ 2), we characterize a useful subspace of $M(E)$ by a Stone-Weierstrass argument. In § 3, utilizing a summability technique, we prove that certain natural subspaces of $D_b(E)$ always contain non-pseudo-measures for infinite E. Then, with a metric hypothesis on E, we derive an estimate which is useful in characterizing those E for which $A'(E) = A'_b(E)$; metric conditions are generally not sufficient to establish the boundedness of $A'(E)$-this is where we need arithmetic conditions. We give a functional analysis argument in § 4 to show the existence of a class of functions in $A(\Gamma)$ without any sort of local finite variation; and we use

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this to examine subspaces of $A(\Gamma)$ on which measures with finite support approximate pseudo-measures. In § 5, after noting that unbounded pseudo-measures exist on countable sets with a single limit point, we give a technique which shows how arithmetic conditions on E lead to $A'(E) = A'_b(E)$. The results here are preliminary. Finally (§ 6), we illustrate that properties of Helson sets lead to a large class of topologies of summability type.

I would like to thank Mr. Gordon Woodward for his helpful advice.

1. Notation and Formulas for Pseudo-Measure.

We designate the complement of E by $CE \equiv \bigcup_0 I_j$ where, with $E \subseteq [0, 2\pi)$, $I_j \equiv (\lambda_j, \gamma_j) \subseteq [0, 2\pi)$ is an open interval of length ε_j; since $m(E) = 0$, $\sum_0 \varepsilon_j = 2\pi$. For convenience, we set

$$c^{\pm}_{j,n} \equiv e^{\pm i\lambda_j n} - e^{\pm i\gamma_j n},$$

$$d^{\pm}_{j,n} \equiv c^{\pm}_{j+1,n} - c^{\pm}_{j,n};$$

generally, we drop the « + » in this notation.

Let $D_b(\Gamma)$ be the space of first order distributions T such that $T = f'$, $f \in L^\infty(\Gamma)$, and let $A'_b(\Gamma) = D_b(\Gamma) \cap A(\Gamma)$. Also define $D_1(E)$ to be those first order distributions T for which $T = f'$, $f \in L^1(\Gamma)$, $\operatorname{supp} T \subseteq E$, and $\widehat{T}(0) = 0$. It is easy to see that $f = \sum_1 k_j \chi_{I_j}$ and, as such, we generally write $T \backsim k_j$ for an element of $D_1(E)$. Besides the spaces indicated in the introdution we consider the following subspaces of $D_1(E)$:

$$D_\omega(E) \equiv \{T \backsim k_j : f \in L^p(\Gamma) \text{ for all } p < \infty\},$$

$$A'_S(E) \equiv \{T \in A'(E) : \varphi = 0 \text{ on } E,\ \varphi \in A(\Gamma), \text{ implies } \langle T, \varphi \rangle = 0\},$$

$$M_d(E) \equiv \{T \in M(E) : T \text{ is discrete}\},$$

$$G(E) \equiv \{T \backsim k_j \in D_b(E) : f(\gamma \pm) \text{ exists for all } \gamma \in \Gamma \text{ and } f \text{ has at most countably many jump discontinuities}\}.$$

We multiply [2] $S \backsim k_j$, $T \backsim h_j \in D_\omega(E)$ by

$$ST = (\Sigma_1 k_j h_j \chi_{I_j})'.$$

Now, E is *Helson* if $A'_S(E) = M(E)$, *spectral synthesis* (S) if $A'(E) = A'_S(E)$, *and strong spectral resolution* if $A'(E) = M(E)$. E is a *Dirichlet* *set* if

$$\lim_{n \to \infty} \sup_{\gamma \in E} |1 - e^{in\gamma}| = 0.$$

We set $A(E)$ to be the restrictions of $A(\Gamma)$ to E; and $A_+(E)$ to be the restrictions of absolutely convergent Taylor series to E.

Next, let $\mathcal{F}$ be the compact open sets of E so that $T \in D_1(E)$ is a finitely additive set function on $\mathcal{F}$[1 ; 2]; we norm such a T by

$$\| T \|_v \equiv \sup_{F \in \mathcal{F}} | T(F) |.$$

Also we write $I_j \leq I_k$ if, for $E \subseteq [0, 2\pi)$, $\lambda_j < \gamma_k$; $I_{j_1} \leq \ldots \leq I_{j_m}$ is a *partition P*.

For detailed proofs of the following, as well as similar results, we refer to [3, § 2].

PROPOSITION 1.1. For all $T \in A'(E)$ there is $f \in L^p(\Gamma)$, for each $p < \infty$, such that $f = \sum_1 k_j \chi_{I_j}$ a. e., $\sum e^{\delta |k_j|} \varepsilon_j < \infty$ for some $\delta > 0$, and

$$c_n \equiv \widehat{T}(n) = \frac{1}{2\pi} \sum_{j=1} k_j \, \overline{c_{j,n}} \quad . \tag{1.1}$$

PROPOSITION 1.2. For all $T \in A'(E)$, $T \backsim k_j$ and $c_n \equiv \widehat{T}(n)$,

$$k_j = \frac{1}{\varepsilon_j} \sum_n{}' \frac{c_n}{n^2} c_{j,n}.$$

PROOF. We have

$$f(\gamma) \equiv \sum_1 k_m \chi_{I_m}(\gamma) = \sum_n{}' \frac{c_n}{in} e^{in\gamma}. \qquad \text{a. e.} \tag{1.2}$$

Since $f \in L^1(\Gamma)$ and since Fourier series can be integrated term by term, we integrate both sides of (1.2) over I_j. Thus

$$k_j \varepsilon_j = \sum_n{}' \frac{c_n}{in} \int_{\lambda_j}^{\gamma_j} e^{in\gamma} \, d\gamma = \sum_n \frac{c_n}{n^2} c_{j,n}. \qquad \text{q.e.d.}$$

PROPOSITION 1.3. If $T \backsim k_j \in A'(E)$ and the partial sums $\sum_1^n k_j$ are bounded, then

$$2\pi \widehat{T}(n) = - \sum_{j=1}^{\infty} \left(\sum_{p=1}^{j} k_p \right) \overline{d_{j,n}} . \tag{1.3}$$

PROOF. By Abel's formula

$$\sum_{j=1}^{J} k_j c_{j,n}^{-} = \left(\sum_{1}^{J} k_j\right) c_{J+1,n}^{-} - \sum_{j=1}^{J}\left(\sum_{p=1}^{j} k_p\right) d_{j,n}^{-}.$$

Thus we have (1.3) by (1.1) and since $\lim_{J} c_{J+1,n}^{-} = 0$. q.e.d.

2. Structure of Measures.

Let us first recall [1,2] some characterization of $T \backsim k_j \in M(E)$ in terms of k_j. Let $T \backsim k_j \in D_1(E)$; $T \in M(E)$ if and only if any of the following equivalent conditions hold:

i. $\| T \|_v < \infty$.

ii. $\sup_{p} \sum_{1}^{m-1} | k_{n_{j+1}} - k_{n_j} | < \infty$.

iii. There is M such that if $I_{n_1} \leq \dots \leq I_{n_{2m}}$ then

$$\left| \sum_{j=1}^{m} (k_{n_{2j-1}} - k_{n_{2j}}) \right| < M.$$

PROPOSITION 2.1. Let $\{k_j\}$ be bounded by a constant C and assume

$$\sum_{1} | k_{j+1} - k_j | \equiv K < \infty.$$

Then $T \backsim k_j \in M(E)$.

PROOF. Let

$$X \equiv \{\varphi \in C^1(\Gamma) : \Sigma \, |\varphi(\lambda_j)|, \Sigma \, |\varphi(\gamma_j)| < \|\varphi\|_\infty\}.$$

X is a subalgebra of $C(\Gamma)$ which satisfies the conditions of the Stone-Weierstrass theorem.

Thus, $\overline{X} = C(\Gamma)$.

For each $\varphi \in X$, we have, for $a_j \equiv \varphi(\lambda_j) - \varphi(\gamma_j)$,

$$\left| \sum_{j=1}^{J} k_j a_j \right| = \left| \left(\sum_{1}^{J} a_j\right) k_{J+1} - \sum_{j=1}^{J} \left(\sum_{p=1}^{j} a_p\right)(k_{j+1} - k_j) \right| \leq$$

$$C \sum_{j=1}^{J} |a_j| + \sum_{j=1}^{J} \left(\sum_{p=1}^{j} |a_p|\right) |k_{j+1} - k_j| \leq 2(C + K) \|\varphi\|_\infty .$$

Since $\langle T, \varphi \rangle = \sum_1^\infty k_j a_j$, T is a continuous linear functional on a dense subspace of $C(\Gamma)$ and has support in E.

Consequently, $T \in M(E)$. q.e.d.

REMARK 1. Relative to our use of Abel's sum formula note that if $\{k_j\}$ is bounded then $\Sigma \mid k_{j+1} - k_j \mid < \infty$ if and only if $\Sigma k_j \cdot a_j$ converges for every convergent series Σa_j.

2. PROP. 2.1 generalizes the easy fact that if $\Sigma \mid k_j \mid < \infty$ then $T \backsim k_j \in M_d(E)$. A special case of PROP. 2.1 which is proved directly and simply by Schwarz's inequality, is that if $\sum_{j=1} j \mid k_{j+1} - k_j \mid < \infty$ then $T \backsim k_j \in M(E)$. It is also clear by Schwarz's inequality that if $T \backsim k_j \in M(E)$ then

$$\| T \|_1 \leq \frac{1}{J} \sum_1^J \mid k_{j+1} - k_j \mid .$$

Finally note that if $\Sigma \mid k_{j+1} - k_j \mid < \infty$ then $\lim k_j$ exists.

EXAMPLE 2.1 *a*. We first show that there are $T \backsim k_j \in M_d(E)$ such that $\Sigma \mid k_{j+1} - k_j \mid$ diverges. Take countable $E \subseteq [0, 2\pi]$ such that $\lambda_1 = \gamma_0$, $\lambda_{2j} = \gamma_{2j+2}$, $\gamma_{2j+1} = \lambda_{2j+3}$, $j = 0, 1, \ldots$, and

$$\ldots \leq I_{2j+2} \leq I_{2j} \leq \ldots I_2 \leq I_0 \leq I_1 \leq \ldots \leq I_{2j+1} \leq \ldots .$$

Setting $k_j \equiv (-1)^j \frac{1}{j}$ we have $\Sigma \mid k_{j+1} - k_j \mid = \Sigma \frac{2j+1}{j(j+1)} = \infty$ (as well as $\Sigma \mid k_j \mid = \infty$) and $f \equiv \Sigma k_j \chi_{I_j}$ a function of bounded variation. Thus $T \backsim k_j \in M_d(E)$ and

$$T = \sum_0^\infty a_{\lambda_j} \delta_{\lambda_j}, \qquad \Sigma \mid a_{\lambda_j} \mid < \infty,$$

where $a_{\lambda_0} = 2$, $a_{\lambda_{2j+1}} = 2/(2j+1)(2j+3), j \geq 0$, and $a_{\lambda_{2j}} = -2/2j(2j-2), j > 1$.

b. We must show that there are non-discrete measures $T \backsim k_j \in M(E)$ such that $\Sigma \mid k_j - k_{j+1} \mid < \infty$. Let $E \subseteq [0, 2\pi)$ be perfect and set $k_j = \frac{1}{j}$; we show supp $T = E$ so that since $\Sigma \frac{1}{j(j+1)} < \infty$ we have $T \in M(E) - M_d(E)$. Since the accessible points are dense in E it is enough to prove that each λ_m (and γ_m) is in supp T. If $\lambda_m \notin$ supp T we find $\varphi \in C(\Gamma)$ such that $\varphi = 0$ on (a neighborhood of) supp T and $\langle T, \varphi \rangle \neq 0$. To do this, first note that every subsequence of $\{k_j\}$ converges to 0 and so there is an open interval $V(\lambda_m)$ with center λ_m such that if $I_j \leq I_m$ and $I_j \cap V(\lambda_m) \neq \emptyset$

then $\frac{1}{j} < \frac{1}{m}$. Next take non-negative $\varphi \in C^1(\Gamma)$, $\operatorname{supp} \varphi \subseteq V(\lambda_m)$, $\varphi(\lambda_m) = 1$ and φ symmetric about λ_m. Thus

$$|\langle T, \varphi \rangle| = \left| \frac{1}{m} - \alpha \right|, \alpha < \frac{1}{m}.$$

c. We now observe that there are continuous measures $T \backsim k_j \in M(E)$, such that $\Sigma |k_{j+1} - k_j|$ diverges. For example the continuous Cantor function on the Cantor set is of the form $f \equiv \Sigma k_j \chi_{I_j}$ on $\bigcup_0 I_j$ and there are an infinite number of pairs k_j, k_{j+1} such that $|k_j - k_{j+1}| \geq \frac{1}{2}$. More generally, if $E \subseteq [0, 2\pi)$ has more than one limit point and $f \equiv \Sigma k_j \chi_{I_j}$ is increasing, then there is $\varepsilon > 0$ such that $|k_j - k_{j+1}| > \varepsilon$ for infinitely many j; thus there are no non trivial positive measures $T \backsim k_j$ on such sets with the property $\Sigma |k_j - k_{j+1}| < \infty$.

3. Trigonometric Sums Associated with Accessible Points.

PROPOSITION 3.1 $\sup_n \left| \sum_{j=1}^{\infty} c_{j,n} \right| \leq 2.$

PROOF. Let $\varphi \in C^1(\Gamma)$, $|\varphi| \equiv 1$, and $T \equiv -\delta_{\lambda_0} + \delta_{\gamma_0}$. Then $g' = T$ distributionally, where $g = \chi_{\Gamma - I_0}$. Now, let $f_T = \sum_1 \chi_{I_j}$ so that $f_T = g$ a. e. By definition, $\langle T, \varphi \rangle = \varphi(\gamma_0) - \varphi(\lambda_0)$; and since $f_T' = T$,

$$\langle T, \varphi \rangle = - \langle f_T, \varphi' \rangle = \sum_1^{\infty} (\varphi(\lambda_j) - \varphi(\gamma_j)),$$

where the last equality follows by the Lebesgue dominated convergence theorem.

Consequently, for $\varphi(\gamma) = e^{in\gamma}$,

$$\left| \sum_1^{\infty} (e^{in\lambda_j} - e^{in\gamma_j}) \right| = |e^{in\lambda_0} - e^{in\lambda_0}| \leq 2. \qquad \text{q.e.d.}$$

Obviously the bound of 2 in PROP. 3.1 can be refined depending on the arithmetic character of λ_0 and γ_0. Note also that for each n, $\sum_j |c_{j,n}| < \infty$.

EXAMPLE 3.1 *a.* Let E be independent. Then $\{\varepsilon_j\} \subseteq [0, 2\pi)$ is independent. Thus, by Kronecker's theorem [7, pp. 176-177], if we take any k

there is N_k so that for real α satisfying $|1 - e^{i\alpha}| > 1$ there is $n \in [0, N_k]$ for which $|e^{in\varepsilon_j} - e^{i\alpha}| < \frac{1}{2}$ if $j = 1, \dots, k$. For this n

$$|e^{in\varepsilon_j} - 1| \geq ||e^{in\varepsilon_j} - e^{i\alpha}| - |e^{i\alpha} - 1|| > \frac{1}{2}$$

if $j = 1, \dots, k$. Therefore

$$\sum_{j=1}^{\infty} |e^{in\varepsilon_j} - 1| > k/2\,; \tag{3.1}$$

and for any k we can find n such that (3.1) holds; thus

$$\sup_n \sum_j |c_{j,n}| = \infty.$$

b. Now let E be a countable Helson set, or, more generally, a set of strong spectral resolution. Obviously such a set need not be independent; for example, let $E \equiv \left\{0, \frac{2\pi}{2^j} : j = 1, \dots\right\}$. Note that if

$$\sup_n \sum_{j=1}^{\infty} |c_{j,n}| < \infty$$

then $A_b'(E) = D_b(E)$; whereas, by the hypothesis on E, $A_b'(E) = M(E)$, a contradiction (for E infinite). Consequently, once again $\sup_n \sum_j |c_{j,n}| = \infty$.

The phenomena of EXAMPLE 3.1 is general; in fact, the following lemma is straightforward.

LEMMA 3.1. Let E be infinite. Then, for any infinite sequence $\{p_j\}$ of natural numbers,

$$\sup_n \sum_{j=1}^{\infty} ||c_{p_{j+1},n}| - |c_{p_j,n}|| = \infty.$$

In the following theorem, part *a* is, of course, proved independent of LEMMA 3.1.

THEOREM 3.1 *a*.

$$\sup_n \sum_j |d_{j,n}| \tag{3.3}$$

diverges if and only it there is $T \sim k_j \in D_1(E) - A'(E)$ such that $\sum k_j$ converges.

b. There is $T \backsim k_j \in D_1(E) - A'(E)$ such that Σk_j converges for every infinite E.

PROOF. *b* is immediate from *a* and LEMMA 3.1.

a. Assume there is such a T and let (3.3) be finite. Then from PROP. 1.3 we have

$$2\pi \, | \widehat{T}(n) | \leq \sum_{j=1}^{\infty} \left(\sum_{p=1}^{j} k_p \right) | \overline{d_{j,n}} | \, ;$$

so that with our hypothesis on (3.3) we get the desired contradiction since we' ve proved $T \in A'(E)$.

For the converse assume without loss of generality that

$$\sup_{n \geq 0} \sum_j | d_{j,n} | = \infty. \tag{3.4}$$

We shall choose a sequence of j' s and n' s inductively such that for a given j_r we'll choose j_{r+1} and n_r.

Beginning with j_1 assume we have $j_1, \dots, j_r$ and $n_1, \dots, n_{r-1}$. Take $n_r > n_{r-1}$ such that

$$\sum_{j=1}^{\infty} | d_{j,n_r} | > 8rj_r + r^2 + 2r + 2,$$

by (3.4).

Note that

$$\sum_{j=1}^{j_r} | d_{j,n_r} | < 4j_r + 1.$$

Now for our n_r take $j_{r+1} > j_r$ such that

$$\sum_{j_{r+1}+1}^{\infty} | d_{j,n_r} | < 1.$$

Combining these three inequalities gives

$$\sum_{j=j_r+1}^{j_{r+1}} | d_{j,n_r} | = \sum_{j=1}^{\infty} - \sum_{j=1}^{j_r} - \left(\sum_{j=1}^{\infty} - \sum_{j=1}^{j_{r+1}} \right) > \tag{3.5}$$

$$8rj_r + r^2 + 2r + 2 - 4j_r - 1 - > 4j_r + r^2 + 2r$$

since $2r - 1 > r$.

Next we define $T \backsim k_j$.

Let $s_j \equiv \sum_1^j k_m$ and let $s_j = 0$ for $j \leq j_1$; thus define

$$k_1 = \dots = k_{j_1} = 0.$$

For $j_r < j \leq j_{r+1}$ take

$$s_j = \frac{1}{r} \frac{\overline{d_{j,n}}}{|d_{j,n_r}|},$$

noting that $s_j \to 0$.

In this manner we define all k_j. For example, let

$$k_{j_1+1} = \frac{\overline{d_{j_1+1,n_1}}}{|d_{j_1+1,n_1}|};$$

and since $s_{j_1+2n_1} = \overline{d_{j_1+2,n_1}} / | d_{j_1+2,n_1}|$ we set

$$k_{j_1+2} = sgn\ \overline{d_{j_1+2,n_1}} - \sum_{j=1}^{j_1+2} k_j .$$

Now, from PROP. 1.3,

$$2\pi \,| \widehat{T}(-n_r) | = | \sum_{j=1}^{\infty} s_j d_{j,n_r} |,$$

and so

$$2\pi \,| \widehat{T}(-n_r) | \geq || \sum_{j=j_r+1}^{j_{r+1}} | - | \sum_{j=1}^{j_r} + \sum_{j=j_{r+1}+1}^{\infty} s_j d_{j,n_r} || \geq$$

$$\frac{1}{r} \sum_{j=j_r+1}^{j_{r+1}} | d_{j,n_r} | - | \sum_{j=1}^{j_r} s_j d_{j,n_r} | - | \sum_{j=j_{r+1}+1}^{\infty} s_j d_{j,n_r} | .$$

Notice that for any domain D of summation

$$| \sum_D s_j d_{j,n_r} | \leq \sum_D | d_{j,n_r} | .$$

Consequently from (3.5)

$$2\pi \,| \widehat{T}(-n_r) | > 4j_r + r + 2 - (4j_r + 1) - 1 = r,$$

and hence $T \notin A'(E)$. q.e.d.

Independent of THEOREM 3.1, it is trivial to see that if there is $T \in D_b(E) - A'_b(E)$ then $\sup_n \sum_j | c_{j,n} | = \infty$; and a proof similar to that of the second part of THEOREM 3.1 a shows that the converse is also true.

Such T exist since every infinite E has a countably infinite Helson subset; in this regard, we further refer to [8].

PROPOSITION 3.2. If $\sum_1 \varepsilon_j e^{-\varepsilon_j} \log\left(\frac{1}{\varepsilon_j}\right) < \infty$ then

$$\sum_j \sum_n' \frac{|c_{j,n}|}{n^2} < \infty. \tag{3.6}$$

PROOF. Because $|c_{j,n}| = 2\left|\sin \frac{n\varepsilon_j}{2}\right|$ and by the Fourier series expansion of $|\sin x|$ it is sufficient to show

$$\sum_{j=1} \sum_{n=1} \frac{1}{n^2}\left(\sum_{m=1} \frac{\sin^2 \frac{mn\,\varepsilon_j}{2}}{4m^2-1}\right) < \infty.$$

Further, by an elementary calculation with residues,

$$\int_1^\infty \frac{\sin^2 \frac{xn\varepsilon_j}{2}}{4x^2-1}\,dx \leq \frac{1}{2}\int_0^\infty \frac{dx}{x^2+1} - \frac{1}{2}\int_0^\infty \frac{\cos xn\varepsilon_j}{x^2+1}\,dx = \frac{\pi}{4}\left(1 - e^{-n\varepsilon_j}\right),$$

for $n \geq 1$ and $j \geq 1$.

Thus, since we can estimate $\sin^2\left(\frac{mn\,\varepsilon_j}{2}\right)/(4m^2-1)$ in terms of

$$\int_{m-1}^{m} \frac{\sin^2 \frac{xn\,\varepsilon_j}{2}}{4x^2-1}\,dx$$

for $m \geq 2$, it is sufficient to prove

$$\sum_{j=1} \sum_{n=1} \frac{1-e^{-n\varepsilon_j}}{n^2} < \infty. \tag{3.7}$$

Noting that $\varepsilon_j \in (0, 2\pi)$ we have by the mean-value theorem that $|e^{-\varepsilon_j} - 1| \leq \varepsilon_j$ so that (3.7) reduces to showing

$$\sum_{j=1} \sum_{n=2} \frac{1-e^{-n\,\varepsilon_j}}{n^2} < \infty.$$

Letting $f(x) = (1 - e^{-x\,\varepsilon_j})/x^2$ on $[1, \infty)$ we see that $f' < 0$ so that f is decreasing, and, hence, by the integral test we need only prove

$$\sum_{j=1}^{\infty} \int_{1}^{\infty} \frac{1 - e^{-x\varepsilon_j}}{x^2}\, dx < \infty. \tag{3.8}$$

We have

$$\int_{1}^{\infty} \frac{1 - e^{-x\varepsilon}}{x^2}\, dx = \varepsilon_j \int_{\varepsilon_j}^{\infty} \frac{1 - e^{-u}}{u^2}\, du = (1 - e^{-\varepsilon_j}) -$$

$$\varepsilon_j\, e^{-\varepsilon_j} \log \varepsilon_j + \varepsilon_j \int_{\varepsilon_j}^{\infty} (\log u)\, e^{-u}\, du.$$

As is well known

$$\int_{0}^{\infty} (\log u)\, e^{-u}\, du = L,$$

Euler's constant, and so by hypothesis and the fact that

$$\Sigma\, (1 - e^{-\varepsilon_j}) < \infty,$$

we have (3.8). q.e.d.

Note that generally, by PROP. 1.1, if $T \backsim k_j \in A'(E)$ then $k_j = 0\left(\log \frac{1}{\varepsilon_j}\right)$, $j \to \infty$, whereas for E satisfying the hypothesis of PROP. 3.2, $k_j = O\left(e^{-\varepsilon_j} \log \frac{1}{\varepsilon_j}\right)$, $j \to \infty$.

In [2] it is made clear that closure of the multiplication operation of (bounded) pseudo-measures is important on Helson sets. For example, when $A'(E)$ is a Banach algebra for this multiplication not only does $A'(E) \subseteq G(E)$, as we showed in [2], but, by the open mapping theorem, $A'(E) \neq G(E)$ — for if there was equality we'd have $\overline{M(E)} = A'(E)$ since $\overline{M(E)} = G(E)$, a contradiction since $M(E) \neq G(E)$ and $M(E)$ is closed in $A'(E)$.

4. Subspaces of Bounded Variation in $A(\Gamma)$.

PROPOSITION 4.1. Given any infinite E. There is $\varphi \in A(\Gamma)$ such that

$$\sum_{j=1}^{\infty} |\, \varphi(\lambda_j) - \varphi(\gamma_j)\, | \tag{4.1}$$

diverges.

PROOF. Assume (4.1) is finite for all $\varphi \in A(\Gamma)$.

Take any $T \backsim k_j \in D_b(E)$ and define measures μ_J (on $A(\Gamma)$) by

$$\langle \mu_J, \varphi \rangle = \sum_1^J k_j(\varphi(\lambda_j) - \varphi(\gamma_j)). \tag{4.2}$$

Since (4.1) is finite we have that given $\varphi \in A(\Gamma)$ there is $K_\varphi > 0$ such that for all J, $|\langle \mu_J, \varphi \rangle| \leq K_\varphi$.

By (4.2) we consider $\varphi \in A(E)$ and so by the uniform boundedness principle $\{\mu_J\}$ is bounded in $A'_S(E)$. Hence, by Alaoglu, the fact that $\mu_J \to T$ on $C^1(\Gamma)$, and T is arbitrary in $D_b(E)$, we have $D_b(E) = A'_b(E)$.

This contradicts THEOREM 3.1. q.e.d.

REMARK *a*. PROP. 4.1 tells us something more than the well known fact that there are functions of infinite variation in $A(\Gamma)$; it tells us that locally — that is, on any given infinite set of points — there are elements of $A(\Gamma)$ with infinite variation.

b. PROP. 4.1 has some interest from the point of view of Helson sets. More precisely, if E were Helson and (4.1) were finite for all $\varphi \in A(\Gamma)$ then the argument of PROP. 4.1 is used to show $A'_b(E) = M(E)$; in fact, for $T \in A'_b(E)$ a weak * convergent subnet of $\{\mu_J\}$ converges to an element of $A'_S(E)$, and hence to a measure (for Helson sets). Thus there is some relation between the structure of $A'_b(E)$ and the variation of $A(\Gamma)$ on the accessible points of E. Of course, if an even stronger variation criterion held on $A(\Gamma)$, we could get conditions that $A'(E) = M(E)$.

Let $A_1(\Gamma)$ be the elements φ of $A(\Gamma)$ for which there is $\{\varphi_n\} \subseteq C^1(\Gamma)$ such that $\| \varphi_n - \varphi \|_A \to 0$ and

$$\sup_n \int |\varphi'_n| < \infty. \tag{4.3}$$

$A_{1+}(\Gamma)$ is the subspace of $A_1(\Gamma)$ in which the condition (4.3) is replaced by

$$\sup_n \int |\varphi'_n|^p < \infty, \text{ some } 1 < p < \infty. \tag{4.4}$$

The vector space is normed by

$$\| \varphi \| \equiv \| \varphi \|_A + K_\varphi,$$

where

$$K_\varphi \equiv \inf \left\{ \sup_n \int |\varphi'_n| : \{\varphi_n\} \subseteq C^1(\Gamma), \| \varphi_n - \varphi \|_A \to 0, \text{ and } (4.3) \right\}.$$

Because of PROP. 1.1 we define, for each $T \backsim k_j \in A'(E)$, the sequence of measures with finite support

$$\mu_J \equiv \sum_1^J k_j(\delta_{\lambda_j} - \delta_{\gamma_j}). \tag{4.5}$$

As might be expected, generally, μ_J does not converge to T in the weak $*$ topology. We do have

PROPOSITION 4.2. For all $T \backsim k_j \in A'(E)$ and for all $\varphi \in A_1(\Gamma)$,

$$\lim_J \langle \mu_J - T, \varphi \rangle = 0.$$

PROOF. Let $\{\varphi_n\} \subseteq C^1(\Gamma)$ correspond to φ, and note that

$$\langle \mu_J, \varphi_n \rangle = -\sum_1^J k_j \int_{\lambda_j}^{\gamma_j} \varphi_n'.$$

Further, $\lim_J \langle \mu_J - T, \varphi_n \rangle = 0$ since $\varphi_n \in C^1(\Gamma)$, and

$$\lim_n \langle \mu_J, \varphi - \varphi_n \rangle = 0 \text{ since } \mu_J \in A'(E).$$

Letting K be a bound for $\int |\varphi_n'|$, we have

$$|\langle T - \mu_J, \varphi_n \rangle| \leq K \sum_{J+1}^{\infty} |k_j| \varepsilon_j,$$

and so $\lim_J \langle T - \mu_J, \varphi_n \rangle = 0$ uniformly in n by PROP. 1.1. Consequently we apply the Moore-Smith theorem and have

$$\langle T, \varphi \rangle = \lim_n \langle T, \varphi_n \rangle = \lim_n \lim_J \langle \mu_J, \varphi_n \rangle = \lim_J \lim_n \langle \mu_J, \varphi_n \rangle = \lim_J \langle \mu_J, \varphi \rangle,$$

since $\| \varphi - \varphi_n \|_A \to 0$. q.e.d.

COROLLARY 4.2.1 $A_1(\Gamma) \neq A(\Gamma)$.

PROOF. If $A_1(\Gamma) = A(\Gamma)$ then every E (of measure 0) is S, a contradiction. (Note that the triadic Cantor set has non-S subsets). q.e.d.

REMARK. Note that if, in the definition of $A_1(\Gamma)$, we demanded that $\varphi_n \equiv \varphi * \varrho_n$, ϱ_n some mollifier — that is, $\varrho_n \geq 0$, $\int \varrho_n = 1$, $\varrho_n(0) \to \infty$, then it is trivial to show $A_1(\Gamma) \neq A(\Gamma)$ by the fundamental theorem of calculus.

There are several other natural subspace of $A(\Gamma)$ with bounded variation properties, with the corresponding questions of topologies, duals, category, and inter-relation, that seem interesting to investigate.

5. Bounded Pseudo-Measures.

We begin by showing that even on countable E there is no reason to expect $A'(E) = A_b'(E)$ unless E has some additional, generally arithmetic, properties.

EXAMPLE 5.1. To define E we adopt a construction of G. Salmons [8]; E will be a subset of $\left\{0, \frac{1}{n} : n = 1, \ldots\right\} \subseteq [0, 2\pi)$. We then construct an unbounded pseudo-measure on E. Let $F_n \subseteq [0, 2\pi)$ be a finite arithmetic progression with $2M_n + 1$ terms such that if $\gamma \in F_{n+1}$ then $\gamma < \lambda$ for each $\lambda \in F_n$; inductively we choose $M_n > M_{n-1}$ so that

$$\sum_{j=1}^{M_n} \frac{1}{j} \geq n^3,$$

and let $E = \overline{\bigcup F_n}$. On F_n we define a measure μ_n which has mass 0 at the « center » of F_n and mass $1/j$ $(-1/j)$ at the j — th point (of F_n) to the right (to the left) of the center. A standard calculation shows that $\| \mu_n \|_{A'} \leq 2(\pi + 1)$. Next, we calculate h_n so that $h_n' = \mu_n$ and note that $|h_n| = \sum_1^{M_n} \frac{1}{j}$ on the two intervals contiguous to the center of F_n. Hence, setting

$$\nu_k = \sum_{n=1}^{k} \frac{1}{n^2} \mu_n \quad \text{and} \quad f_k \equiv \sum_{n=1}^{k} \frac{1}{n^2} h_n,$$

we have $\| \nu_k \|_{A'} \leq 2(\pi + 1) \sum_1^k \frac{1}{n^2}$ and $|f_k| = |h_k| / k^2 \geq k$ (on the two intervals contiguous to the center of F_k).

Consequently, a subset of $\{\nu_p\}$ converges to $T \in A'(E) - M(E)$ in the weak * topology, $f_p \to f$ pointwise a. e., $f' = T$, and f is unbounded.

Proposition 5.1. $A'(E) = A'_b(E)$ if and only if

$$A'(E) \times D_1(E) \to D_1(E) \tag{5.1}$$

$$(S \backsim k_j, T \backsim h_j) \to ST \backsim k_j h_j$$

is a well-defined multiplication.

Proof. If $A'(E) = A'_b(E)$, $S \backsim k_j \in A'_b(E)$, and $T \backsim h_j \in D_1(E)$, then $\Sigma h_j k_j \chi_{I_j} \in L^1(\Gamma)$ since

$$\int |\Sigma h_j k_j \chi_{I_j}(\gamma)| \, d\gamma \leq K \int (\Sigma |h_j| \chi_{I_j}(\gamma)) \, d\gamma < \infty .$$

Conversely if $A'(E) \neq A'_b(E)$ let $T \backsim k_j \in A'(E)$ where $\lim_j |k_{n_j}| = \infty$. Without loss of generality take $|k_{n_j}| \geq j$ and define $g \equiv \Sigma h_j \chi_{I_j}$ such that $h_{n_j} = 1/(j^2 \varepsilon_{n_j})$ and $h_m = 0$ if $m \neq n_j$.

Then

$$\int |g| = \int \Sigma |h_j| \chi_{I_j}(\gamma) \, d\gamma = \sum_j \frac{1}{j^2 \varepsilon_{n_j}} \int \chi_{I_{n_j}} = \Sigma \frac{1}{j^2} = \infty.$$

On the other hand

$$\int |\Sigma k_j h_j \chi_{I_j}| \geq \int \left(\frac{1}{j \, \varepsilon_{n_j}} \chi_{I_{n_j}}(\gamma) \, d\gamma \geq \Sigma \frac{1}{j} \right),$$

a contradiction. q.e.d.

Obviously, **Prop.** 5.2 is just a usual duality property between L^∞ and L^1, and has nothing to do with $A'_b(E)$ per se.

Remark. Note that $A'(E) = A'_b(E)$ if $\sum_n' |c_{j,n}|/n^2 = 0(\varepsilon_j)$, $j \to \infty$, from **Prop.** 1.2; and that the metric condition of **Prop.** 3.2 is much weaker than this.

In [4, **Theorem** 19], Hardy and Littlewood prove that if $\varphi \in H^1$ [5, pp. 70-71] has the Fourier series $\sum_0^\infty a_n e^{in\gamma}$ then

$$\sum_{n=1}^\infty \frac{|a_n|}{n} \leq \pi \, \|\varphi\|_1 . \tag{5.2}$$

They show by counter-example that if $\varphi(\gamma) = \sum_{|n| \leq N} a_n e^{in\gamma}$ then (5.2) is not necessarily true. We shall give another type of counter example as well as showing

PROPOSITION 5.2. For all $T \in A'(\Gamma)$ there is $S \in D_b(\Gamma)$ such that $\widehat{S}(n) \equiv \widehat{T}(n)$ for $n \geq 1$.

PROOF. Given T.

A direct application of (5.2) says that if $\varphi(\gamma) = \sum_0^N a_n e^{in\gamma}$ then

$$\sum_1^N \frac{|a_n|}{n} \leq \pi \int_0^{2\pi} \left| \sum_0^N a_n e^{in\gamma} \right| d\gamma.$$

Now, if $f' = T$ we have $\widehat{f}(n) = 0\left(\frac{1}{|n|}\right)$, $|n| \to \infty$; and hence there is a constant K_T such that for all trigonometric polynomials of the form $\varphi(\gamma) \equiv \sum_1^N a_n e^{in\gamma}$

$$\left| \int f \bar{\varphi} \right| \equiv |\langle f, \varphi \rangle| \leq K_T \| \varphi \|_1 .$$

Consequently, by the Hahn-Banach theorem there is $g \in L^\infty$ such that $\langle f - g, \varphi \rangle = 0$ for all $\varphi(\gamma) = \sum_1^N a_n e^{in\gamma}$.

In particular, $\widehat{f}(n) = \widehat{g}(n)$ for all $n > 0$. q.e.d.

Because of PROP. 5.2 we say that E has *bounded halves* if for all $T \in A'(E)$ there is $S \in D_b(E)$ such that $\widehat{T}(n) = \widehat{S}(n)$ for $n \geq 1$. The question is, of course, to determine for given $E \subseteq \Gamma$ the type of subset $X \subseteq Z$ such that for all $T \in A'(E)$ there is $S \in D_b(E)$ for which $\widehat{T} = \widehat{S}$ on X. Obviously the problem is meaningful in a much more general context.

Now, assuming E has bounded halves we wish to find conditions so that $A'(E) = A'_b(E)$. Arithmetic properties definitely play a role here. In fact, using a (by now) standard approximation technique [6,10], we have

PROPOSITION 5.3. Let E be a Dirichlet set with bounded halves. Then $A'(E) = A'_b(E)$.

PROOF. Let $T \in A'(E)$ and $S \in D_b(E)$, $\widehat{S} = \widehat{T}$ for $n \geq 1$.

Observe that E Dirichlet is equivalent to

$$\lim_{n \to \infty} \sup_{\gamma \in E} |\sin n \gamma| = 0. \tag{5.3}$$

From (5.3) we know that for all $\varepsilon > 0$ there is a positive integer n_ε such that

$$\sup_{\gamma \in E} |\sin n_\varepsilon \gamma| < \frac{\varepsilon}{2} \tag{5.4}$$

and

$$\lim_{\varepsilon \to 0} n_\varepsilon = \infty .$$

Next we define the continuous ε — diminishing — M function M_ε in $[-\pi, \pi)$ to be 0 at 0 and outside $(-2\varepsilon, 2\varepsilon)$, ε at $\pm \varepsilon$, and linear otherwise.

Then from (5.4) we have for $S = g'$,

$$(\widehat{S} - \widehat{T})(2n_\varepsilon) - (\widehat{S} - \widehat{T})(0) = -\frac{i}{\pi} \langle S - T, e^{-in_\varepsilon \gamma} M_\varepsilon (\sin n_\varepsilon \gamma) \rangle,$$

since there is a neighborhood of E in which $|\sin n_\varepsilon \gamma| \leq \varepsilon$. A main feature of M_ε is that $\| M_\varepsilon \|_A \to 0$ and so, since $(\widehat{S} - \widehat{T})(2n_\varepsilon) = 0$, $(\widehat{S} - \widehat{T})(0) = 0$. A similar calculation shows $(\widehat{S} - \widehat{T})(n) = 0$ for all $n < 0$. Thus $S = T$.

q.e.d.

Note that every Kronecker set is both Helson and Dirichlet, and that there are Dirichlet sets which aren't Helson and vice-versa. Further, Dirichlet sets are not only sets of uniqueness, but Kahane [6] has shown that if E is Dirichlet then for all $T \in A'(E)$

$$\overline{\lim_{|n| \to \infty}} |\widehat{T}(n)| = \| T \|_{A'} .$$

Observe that Kronecker sets E are S [10] so that, in particular, $A'(E) = A'_b(E)$ in this case.

EXAMPLE 5.2. If the analogue of (5.2) were true for $\varphi(\gamma) = \sum_{|n| \leq N} a_n e^{in\gamma}$ then the proof of PROP. 5.2 shows that $A'(\Gamma) \subseteq D_b(\Gamma)$ which contradicts EXAMPLE 5.1.

6. Helson Sets and Summability Topologies.

Using Wik's theorem that $A(E) = A_+(E)$ characterizes Helson sets [7] we have

PROPOSITION 6.1. Let E be Helson. For all $m < 0$ there is $\sum_{n=0}^{\infty} | a_{n,m} | < \infty$ so that for each $T \in A'(E)$ we have

$$\widehat{T}(-m) = \lim_{J} \sum_{n=0}^{\infty} a_{n,m} \widehat{\mu_J}(-n), \tag{6.1}$$

where $\{\mu_J\}$ is the sequence of measure corresponding to T (as in (4.5)).

PROOF. $e^{im\gamma} = \sum_{n=0} a_{n,m} e^{in\gamma} \equiv \varphi(\gamma)$ on E, $\sum_{n=0} | a_{n,m} | < \infty$ since E is Helson. Thus, using the notation of (4.5) for $T \backsim k_j \in A'(E)$, we have

$$2\pi \widehat{\mu_J}(-m) = \langle \mu_J, \varphi \rangle ;$$

and hence $\lim_{J} \langle \mu_J, \varphi \rangle$ exists. q.e.d.

Now, if $\varphi(\gamma) \equiv \sum_{0} a_n e^{in\gamma} \in A_+(E)$ we write

$$\varphi_r(\gamma) \equiv \sum_{n=0} a_n r^n e^{in\gamma}, \; r \in (0, 1).$$

Note that $\varphi_r \in C^\infty(\Gamma)$, and hence for each $r \in (0, 1)$, $T \in A'(E)$, and $\varphi \in A_+(E)$ we have $\lim_{J} \langle \mu_J - T, \varphi_r \rangle = 0$.

PROPOSITION 6.2 Let E be Helson. Assume $T \in A'(E)$ has the property that for each $\varphi \in A_+(E)$, there exists

$$\lim_{J \to \infty} \langle \mu_J, \varphi_r \rangle, \quad \text{uniformly in } r \in \left[\frac{1}{2}, 1\right). \tag{6.2}$$

Then $T \in M(E)$.

PROOF. (6.2) allows us to use Moore-Smith so that $\langle \mu_J, \varphi \rangle$ converges for all $\varphi \in A(\Gamma)$.

Thus by the uniform boundedness principle and the fact that E is Helson we have $\{\| \mu_J \|_1\}$ bounded. Consequently by Alaoglu and PROP. 1.1, $T \in M(E)$. q.e.d.

For example, if $r = 1 - \frac{1}{n}$ then for $\varphi \backsim \sum a_n e^{in\gamma} \in A_+(E)$ and $T \in A'(E)$,

$$\langle \mu_J, \varphi_r \rangle = 2\pi \sum_{j=1}^{\infty} \left(\sum_{p=1}^{i} a_p \widehat{\mu_J}(-p) \right) \left(1 - \frac{1}{n}\right)^j \frac{1}{n},$$

noting that $1 - \frac{1}{n} = \sum_{j=1}^{\infty} \frac{1}{n} \left(1 - \frac{1}{n}\right)^j$.

Prompted by Prop. 6.2 consider diagonal sums

$$\sum_{n=0}^{\infty} a_n \widehat{\mu_J}(-n) F(J)^n$$

where $0 < F(J) < 1$ and $F(J) \to 1$.

Generally, in a dual system (X, Y) of T_2 locally convex spaces we say that a directed system $\{T_\alpha\} \subseteq X$ converges in the $\sigma\sigma(X, Y)$ topology to $T \in X$ if for all $\varphi \in Y$ there is $\{\varphi_\alpha\} \subseteq Y$ such that φ_α converges to φ and

$$\lim_\alpha \langle T_\alpha - T, \varphi_\alpha \rangle = 0.$$

Although significantly weaker than the weak * topology, it is not generally minimal [9, p. 191] and the intermediate topologies between $\sigma(X, Y)$ and $\sigma\sigma(X, Y)$ become interesting in light of Prop. 6.2, the lack of weak * convergence in § 4, and the convergence in Prop. 1.1 (in terms of (4.5)).

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