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A COMPUTER ANALYSIS OF THE TRANSIENT RESPONSE OF ELASTIC STRUCTURES

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Abstract

The computer code TRILIN employs a force method that uses prismatic beam-type elements and discrete masses for the analysis of the transient response of linearly elastic, three-dimensional, frame-type structures subjected to arbitrary loading conditions. Each beam element is capable of resisting tension, bending, and torsion. A global stiffness matrix is obtained by inverting the flexibility relationships. Modal superposition is used to solve the governing equations.

Introduction

The computer program TRILIN (Transient Response In LINear systems) has been developed to determine the transient response of linearly elastic, three-dimensional structures subjected to arbitrary loading conditions. The code has been applied to a variety of important and practical problems: among these are the Hubmobile Tower, the Packard Tower, and the Cannikin Space Frame at the nuclear test sites, as well as the Bank of Nevada and LLL's Diagnostic Chemistry Buildings.[1,2] Less obvious applications have included the determination of critical speed for a rocket-launched sampler and the analysis for the response of a large plate-like slab located in the Laser Building.

TRILIN is based on a finite-element scheme that uses uniform, beam-type elements of a linearly elastic isotropic material. The inertial properties of the elements are concentrated at the nodal points; hence, each nodal point has a diagonal inertia matrix of the same order as its degrees of freedom (up to 6). The stiffness matrix is obtained after inverting the elements' flexibility relationships by means of synthesis. We assume that the inertia and stiffness matrices are positive semidefinite. As a result, rotary inertia can be neglected, and rigid body modes can exist. Structural response is determined through solution of the eigenvalue problem and modal superposition.

The most important feature of this formulation, in addition to its simplicity, is that the displacements and stresses for the entire structure may be found simultaneously with considerable ease. This is in sharp contrast to the transfer matrix techniques discussed in Refs. 3 and 4 that give displacements and stresses at only one point and require subsequent back-substitution to obtain the entire solution. The transfer matrix technique gives an equivalent solution.

The purpose of this report is twofold: 1) to present the theory underlying TRILIN, and 2) to briefly describe the three supporting TRILIN codes (see Table 1). A User's Manual for the code is available as a separate document.[5]

Table 1. Divisions of TRILIN.

Code division	Function
SMOC	A lumped mass model with massless uniform beam-type element is used to model the structure. This code sets up the dynamic equilibrium equations and calculates the flexibility coefficients for each element. The structure is plotted in several rotated positions.
EIGEN	A disc file generated by SMOC is used to compute the stiffness matrix of the structural model. The eigenvalues and eigenvectors are then calculated for use in TRANS.
TRANS	The transient response of the structure is calculated from input ground acceleration, impulsive loading, and other time-dependent forcing functions. A mode superposition analysis is used. The stress resultants and displacements are plotted as functions of time.

Theoretical Analysis

MODELING

The first step in the analysis is to replace the actual structure by an idealized model that retains its important characteristics. The model is constructed by discretizing the structure into a finite number of linearly elastic, beam-type elements of uniform cross section that are capable of resisting tension, bending about the two principal axes in the planes of their cross sections, shearing deformation, and torsion about their centroidal axes. The elements are interconnected at locations called nodal points at which their inertial properties are concentrated. External loadings are applied and deflections are calculated at the nodal points.

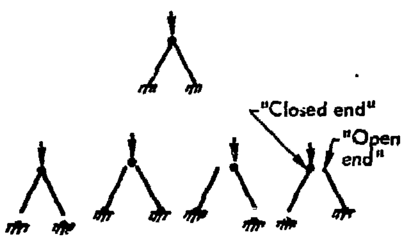

Fig. 1. (a) Structure. (b) Model configurations that are acceptable to the TRILIN code.

In this analysis the idealized model is arbitrarily broken into free bodies consisting of one node and any number of the node's intersecting elements (see Fig. 1). The governing equations based on the free bodies are then formulated. The element's end that is connected at the free body's nodal point is called the "closed end" and accordingly, the free end is called the "open end."

COORDINATE SYSTEM AND SIGN CONVENTIONS

Two sets of coordinate systems are employed. The right-handed orthogonal global system—designated by X_1, X_2, X_3—is an inertial frame of reference. A local coordinate system designated by q_{1i}, q_{2i}, q_{3i} is fixed to the closed end of the ith element, with q_{1i} pointing towards its open end. We assume that the q_{2i} and q_{3i} axes coincide with the element's principal axes of inertia. The orientation of each element is obtained by rotating it from a position parallel to the X_1 axis through an angle ψ_i in a plane parallel to the X_1X_2 plane and finally through an angle ϕ_i in a plane perpendicular to the X_1X_2 plane. The angles ψ_i and ϕ_i are right- and left-handed rotations, respectively (see Fig. 2). An X_3 directed element is obtainable by a single ϕ_i rotation of 90°. Three global coordinates a_i, b_i, and c_i locate the element's closed end.

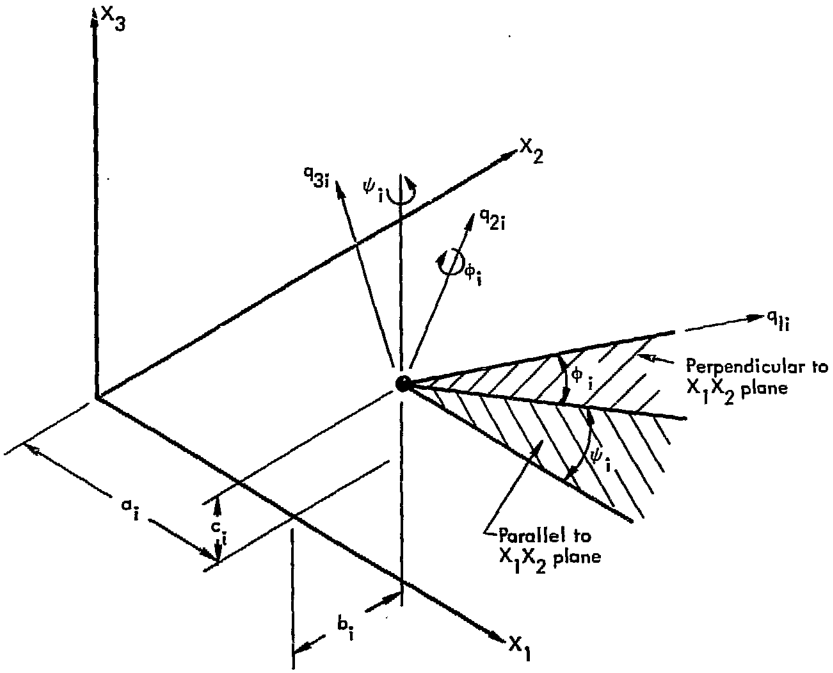

Fig. 2. Orientation of ith element in global space.

The transformation between three-dimensional vector quantities, $\bar{e}$ and $\bar{g}$ in the local and global coordinate systems, respectively, is

$$\bar{e} = T_{\phi_i} T_{\psi_i} \bar{g}. \tag{1}$$

The orthogonal matrices, T_{ψ_i} and T_{ϕ_i}, represent right- and left-handed rotations for the ith local coordinate system, respectively, and are defined by

$$T_{\psi_i} = \begin{bmatrix} \cos\psi_i & \sin\psi_i & 0 \\ -\sin\psi_i & \cos\psi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_{\phi_i} = \begin{bmatrix} \cos\phi_i & 0 & \sin\phi_i \\ 0 & 1 & 0 \\ -\sin\phi_i & 0 & \cos\phi_i \end{bmatrix} . \tag{2}$$

The global coordinates of the end points of each element are read into the SMOC code, and the angles, ψ_i and ϕ_i, are calculated internally.

Consider the ith element whose open and closed ends are located at the jth and kth nodal points, respectively. The product of the transformation matrices given in Eq. (1) may be used to form the matrix T_i such that

$$\bar{\eta}_i = T_i \bar{x}_j \tag{3}$$

$$\bar{H}_i = T_i \bar{L}_j , \tag{4}$$

where:

$$\bar{x}_j^t = [x_{1j}\, x_{2j} \ldots x_{6j}] \tag{5}$$

$$\bar{\eta}_i^t = [\eta_{1i}\, \eta_{2i} \ldots \eta_{6i}] \tag{6}$$

$$\bar{L}_i^t = [L_{1i}\, L_{2i} \ldots L_{6i}] \tag{7}$$

$$\bar{H}_i^t = [H_{1i}\, H_{2i} \ldots H_{6i}] \tag{8}$$

and the 6×6 coordinate transformation matrix T_i is

$$T_i = \begin{bmatrix} T_{\phi_i} T_{\psi_i} & 0 \\ 0 & T_{\phi_i} T_{\psi_i} \end{bmatrix} . \tag{9}$$

The first three elements in $\bar{x}_j$ represent small translational displacements of the jth nodal point in the global X_1, X_2, and X_3 directions, respectively, and the last three elements represent small right-handed rotations about the same axes. Similarly, the vector $\bar{\eta}_i$ contains the displacements and rotations in the local coordinates. The stress resultants, $\bar{L}_i$ and $\bar{H}_i$, acting on the element's open end in the global and local coordinate systems are shown in Figs. 3 and 4 oriented in their positive directions.

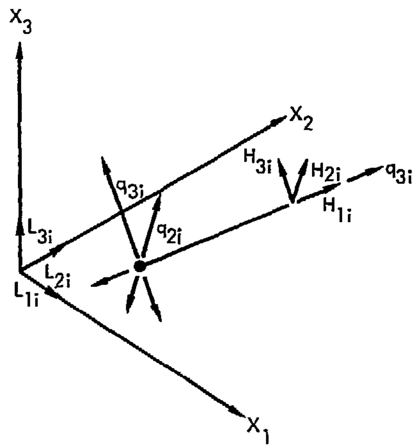

Fig. 3. Positive sign convention for shear and tension.

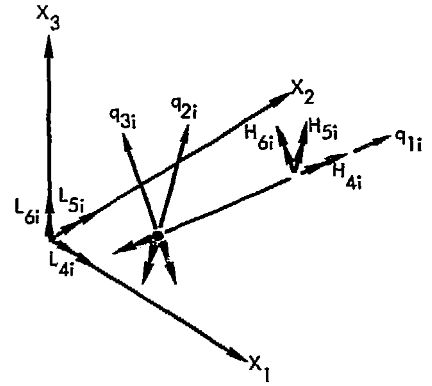

Fig. 4. Positive sign convention for moments.

DYNAMIC EQUILIBRIUM EQUATIONS

The dynamic equilibrium equations are obtained by equating the summation of forces and moments acting on a free body to the inertial forces. The summation, applied to each free body, must include 1) the forces due to the stress resultants acting on any open-ended elements terminating at the node; 2) the forces due to the resultants at the open ends of the closed-ended elements connected to the node; and 3) any external forces applied at the node.

The Euler equations[6] for the rotation of a rigid body about a point are linearized by using the principal axes and assuming that the product of the angular velocity terms are relatively small. Under these assumptions Euler's equations reduce to

$$\begin{bmatrix} L_4 \\ L_5 \\ L_6 \end{bmatrix} = \begin{bmatrix} m_4 & & \\ & m_5 & \\ & & m_6 \end{bmatrix} \begin{bmatrix} \ddot{x}_4 \\ \ddot{x}_5 \\ \ddot{x}_6 \end{bmatrix} . \tag{10}$$

This equation is rigorously true if and only if 1) the moments of inertia m_4, m_5, and m_6 are defined about the principal axes and 2) $m_4 = m_5 = m_6$.

We will now consider a general structural system discretized such that its ith nodal point has associated with it r_i closed-end elements and s_i open-end elements and is subjected to a conservative forcing function, $\overline{F}_i(t)$, given in the global coordinate system. The contribution of the jth element of the $r_i + s_i$ elements to the summation of forces in the global directions about the ith node can be expressed in the form

$$T_j^t D_j^i \overline{H}_j^i, \tag{11}$$

where the subscript i refers to the ith node and where the matrix D_j^i relates the forces acting at the jth element's open end to the forces acting on the ith nodal point. The matrix D_j^i is given in Appendix A for a general beam-type element.

Consider the simple two-dimensional example problem shown in Fig. 5. In this example, motion in the X_3 direction, rotation about the X_1 and X_2 axes, and displacement in the X_1 direction are suppressed. The problem's nonzero coordinates

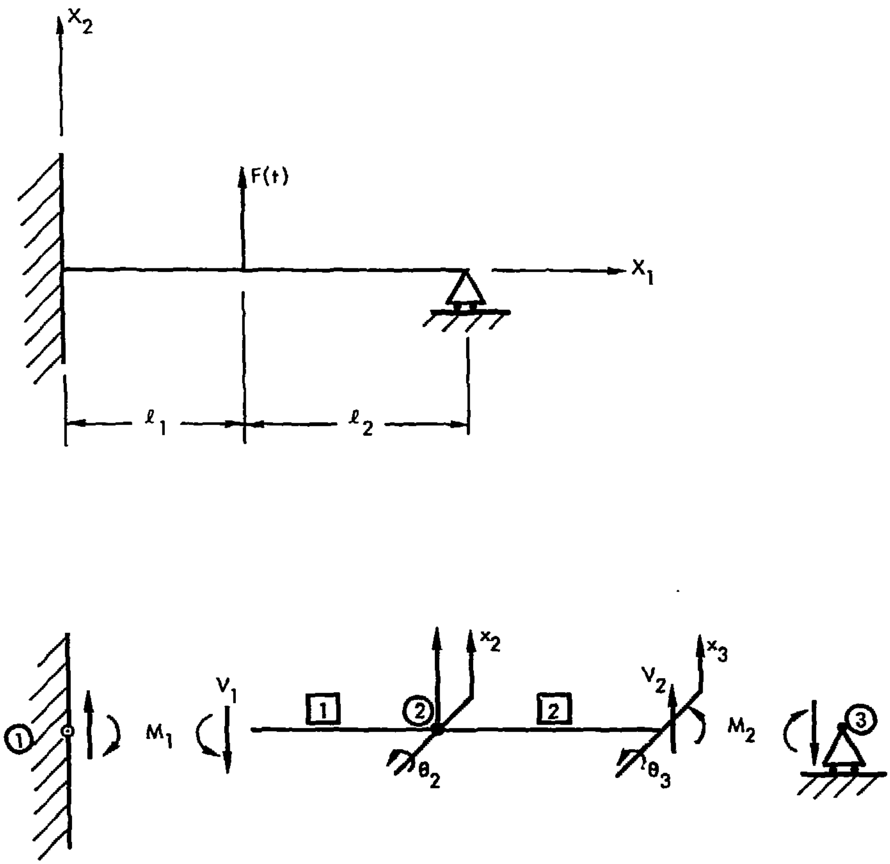

Fig. 5. Cantilever-beam model simply supported at one end.

are x_2, θ_2, θ_3. Two equations like Eq. (11) corresponding to elements 1 and 2 can be written for node 2. They are*

$$T_1^t D_1^2 \overline{H}_1^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \ell_1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ M_1 \end{bmatrix}$$

$$T_2^t D_2^2 \overline{H}_2^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \ell_2 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ M_2 \end{bmatrix} , \tag{12}$$

where ψ_1 for element 1 equals 180°. For node 3 we obtain the equation

$$T_2^t D_2^3 \overline{H}_2^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ M_2 \end{bmatrix} . \tag{13}$$

We note that the T_2 matrix is the identity matrix.

Including the contributions of the r_i closed-end plus s_i open-end elements to the force summation yields the equilibrium equations for the ith nodal point:

$$D^i H^i = m^i \ddot{x}_i - \overline{F}_i , \tag{14}$$

in which

$$D^i = \left[T_1^t D_1^i \,\vdots\, T_2^t D_2^i \,\vdots\, . \; . \; . \,\vdots\, T_{r_i+s_i}^t D_{r_i+s_i}^i \right] \tag{15}$$

$$H^{i^t} = \left[\overline{H}_1^t \overline{H}_2^t \; . \; . \; . \; \overline{H}_{r_i+s_i}^t \right] \tag{16}$$

$$m^i = \begin{bmatrix} m_{1i} & & & & & \\ & m_{2i} & & & & \\ & & m_{3i} & & & \\ & & & m_{4i} & & \\ & & & & m_{5i} & \\ & & & & & m_{6i} \end{bmatrix} . \tag{17}$$

*The 6 × 6 matrices are reduced to 2 × 2 to correspond to the two stress resultants that enter this analysis. The subscripting and notation are changed slightly to avoid unnecessary awkwardness, i.e., $V_1 = H_{21}$, $M_1 = H_{61}$, $V_2 = H_{22}$, $M_2 = H_{62}$, $x_2 = x_{22}$, $\theta_2 = x_{62}$, and $\theta_3 = x_{63}$.

The matrix m^i is the inertia matrix of the ith node. In our example problem we see that putting Eqs. (12) and (13) into the form given in Eq. (14) yields, for node 2,

$$D^2 H^2 = \left[T_1^t D_1^2 \,\vdots\, T_2^t D_2^2\right] \begin{bmatrix} \overline{H}_1^2 \\ --- \\ \overline{H}_2^2 \end{bmatrix}$$

$$= \left[\begin{array}{cc:cc} -1 & 0 & 1 & 0 \\ & & & \\ \ell_1 & 1 & \ell_2 & 1 \end{array}\right] \begin{bmatrix} V_1 \\ M_1 \\ \hdashline V_1 \\ M_2 \end{bmatrix}$$

$$= \begin{bmatrix} m_2 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_2 \\ \ddot{\theta}_2 \end{bmatrix} - \begin{bmatrix} F(t) \\ 0 \end{bmatrix} \tag{18}$$

and, for node 3,

$$D^3 H^3 = T_2 D_2^3 \overline{H}_2^3 = \begin{bmatrix} m_3 & 0 \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_3 \\ \ddot{\theta}_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{19}$$

where m_2, I_2, m_3, and I_3 correspond to m_{22}, m_{62}, m_{23}, and m_{63} in Eq. (17), respectively. Since $\ddot{x}_3 = 0$, Eq. (19) can be expressed by

$$-M_2 = I_3 \ddot{\theta}_3, \tag{20}$$

where $D^3 = -1$ and $H^3 = M_2$.

An equation similar to Eq. (14) is obtained for each of the m nodal points associated with one or more degrees of freedom. We can write these m matrix equations in the form

$$D\overline{\overline{H}} = M\ddot{\overline{x}} - \overline{F}, \tag{21}$$

where:

$$D = \begin{bmatrix} D^1 & & & & & \\ & D^2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & D^m \end{bmatrix} \tag{22}$$

$$\bar{\bar{H}}^t = \{H^{1t} H^{2t} \ldots H^{mt}\} \tag{23}$$

$$\bar{x}^t = \left[\bar{x}_1^t \bar{x}_2^t \bar{x}_3^t \ldots \bar{x}_m^t\right] \tag{24}$$

$$\bar{F}^t = \left[\bar{F}_1^t \bar{F}_2^t \ldots \bar{F}_m^t\right] \tag{25}$$

$$M = \begin{bmatrix} m^1 & & & & & \\ & m^2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & m^m \end{bmatrix} . \tag{26}$$

The n-dimensional vector $\bar{x}$ ($n \leq 6$ m) represents the n generalized unconstrained coordinates (degrees of freedom) of the structure. For the example problem, Eq. (21) becomes

$$\left[\begin{array}{cccc:c} -1 & 0 & 1 & 0 & 0 \\ \ell_1 & 1 & \ell_2 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ \hdashline M_2 \end{bmatrix} = \begin{bmatrix} m_2 & & \\ & I_2 & \\ & & I_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_2 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} - \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} . \tag{27}$$

Obviously, each stress resultant may appear twice in the vector $\bar{\bar{H}}$; hence, we introduce a rectangular connectivity matrix P_1 such that

$$\bar{\bar{H}} = P_1 \bar{H}, \tag{28}$$

where $\bar{H}$ represents the d-dimensional vector of stress resultants appearing at the open ends ordered according to the integer labels on each element. Substitution of Eq. (28) into Eq. (21) gives the dynamic equilibrium equations in the desired form, that is,

$$S\bar{H} = M\ddot{\bar{x}} - \bar{F}, \tag{29}$$

where $S = DP_1$ is an $(n \times r)$ rectangular matrix.

Equation (28) for the example problem is

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix}$$

and making the substitution into Eq. (27) yields the dynamic equilibrium equations

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ \ell_1 & 1 & \ell_2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} m_2 & & \\ & I_2 & \\ & & I_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_2 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} - \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

in the global coordinate system.

DISPLACEMENT FORCE EQUATIONS

The dynamic equilibrium equations furnish n equations, where n is the number of degrees of freedom used in the analysis of a system; however, these equations are in terms of d unknown stress resultants. Additional equations (the displacement force equations) are required to relate these stress resultants and displacements.

Consider a jth closed-end element associated with the ith node. Since the total displacement of the open end is the result of the contributions due to the rotation and translation of its closed end and the rotation and translation due to its open end loading, we write the following equation expressed in the local coordinate system:

$$T_j \overline{x}_j = \left[B_j^i \;\vdots\; C_j \right] \begin{bmatrix} T_j \overline{x}_i \\ \hline \overline{H}_j \end{bmatrix} , \tag{31}$$

which relates the open-end displacements $T_j \overline{x}_j$ to the closed-end displacements $T_j \overline{x}_i$ and open-end forces $\overline{H}_j$. The matrix C_j is the flexibility matrix of the jth element, and B_j^i relates the rigid-body translation and rotation at the jth element's open end due to the translation of the ith node [Eq. (31) is given in Appendix B for a beam element]. For the example in Fig. 6 we can write two equations like Eq. (31) for the second nodal point. For element 1 we obtain

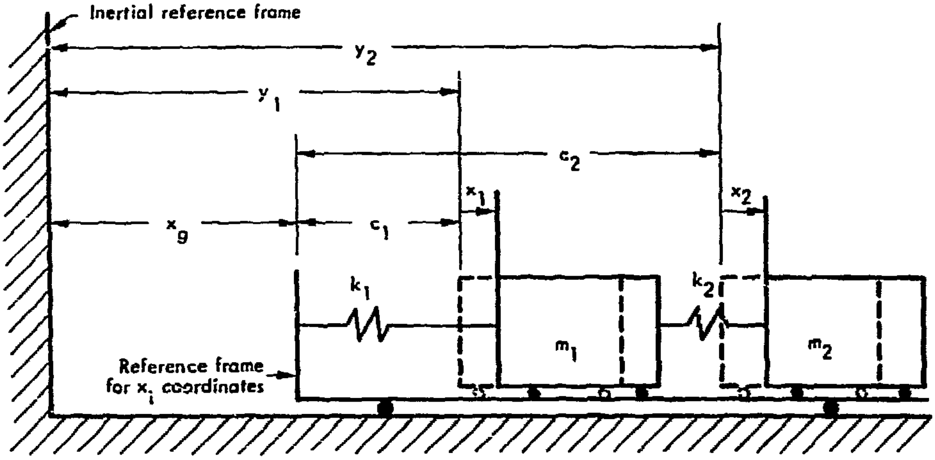

Fig. 6. Undamped spring-mass model subjected to ground excitation.

$$T_1 \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left[\begin{array}{cc:c} 1 & \ell_1 & \\ & & C_1 \\ 0 & 1 & \end{array}\right] \begin{bmatrix} T_i \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} \\ \hdashline V_1 \\ M_1 \end{bmatrix} , \tag{32}$$

and, for element 2,

$$T_2 \begin{bmatrix} x_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_3 \end{bmatrix} = \left[\begin{array}{cc:c} 1 & \ell_2 & \\ & & C_2 \\ 0 & 1 & \end{array}\right] \begin{bmatrix} T_2 \begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} \\ \hdashline V_2 \\ M_2 \end{bmatrix} . \tag{33}$$

If we denote the modulus of elasticity and second moment of area by E_i and I'_{3i}, respectively, then

$$C_i = \begin{bmatrix} \dfrac{\ell_i^3}{3E_i I'_{3i}} & \dfrac{\ell_i^2}{2E_i I'_{3i}} \\ \dfrac{\ell_i^2}{2E_i I'_{3i}} & \dfrac{\ell_i}{E_i I'_{3i}} \end{bmatrix} \qquad i = 1,2. \tag{34}$$

Equation (34) neglects shear deformation.

In general there are r_i equations like Eq. (31) at the ith node—that is, one equation for each closed-end element. Combining and rearranging these r_i matrix equations yields the equation

$$\left[B^i T^i \;\vdots\; C^i\right] \begin{bmatrix} \bar{x}_*^i \\ --- \\ H_*^i \end{bmatrix} = 0, \tag{35}$$

whose matrices are defined as follows

$$B^i = \left[\begin{array}{ccccc|cccc} B_1^i & & & & & -I & & & \\ & B_2^i & & & & & -I & & \\ & & \cdot & & & & & \cdot & \\ & & & \cdot & & & & & \cdot \\ & & & & \cdot & & & & & \cdot \\ & & & & B_{r_i}^i & & & & & -I \end{array}\right] \tag{36}$$

$$T^i = \begin{bmatrix} T_1 & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ T_{r_i} & & & & & & \\ & T_1 & & & & & \\ & & T_2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & T_{r_i} \end{bmatrix} \tag{37}$$

$$
C^i = \begin{bmatrix} C_1 & & & & \\ & C_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & C_{r_i} \end{bmatrix} \tag{38}
$$

$$
H_*^{i^t} = \left[\overline{H}_1^t \overline{H}_2^t \ldots H_{r_i}^t \right] \tag{39}
$$

$$
x_*^{i^t} = \left[\overline{x}_i^t \overline{x}_1^t \overline{x}_2^t \ldots \overline{x}_{r_i}^t \right]. \tag{40}
$$

Since it is assumed that the boundary conditions are known, only those coordinates associated with degrees of freedom are retained in Eq. (35); consequently, each element of $\overline{x}_*^i$ represents a degree of freedom.

Equation (35) for the example problem may be obtained by first stacking Eqs. (32) and (33) in the form

$$
\left[\begin{bmatrix} B_1^2 & & \vdots & -I & \\ & B_2^2 & \vdots & & -I \end{bmatrix} \begin{bmatrix} T_1 & & \\ T_2 & & \\ & T_1 & \\ & & T_2 \end{bmatrix} \;\vdots\; \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} \right] \begin{bmatrix} x_2 \\ \theta_2 \\ x_1 \\ \theta_1 \\ x_3 \\ \theta_3 \\ \hdashline V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = 0
$$

and then by eliminating the constrained coordinates x_1, θ_1, and x_3,

$$\left[\begin{array}{c:c:c:c} B_1^2 T_1 & \begin{matrix} 0 \\ 0 \end{matrix} & C_1 & 0 \\ \hdashline B_2^2 T_2 & \begin{matrix} 0 \\ -1 \end{matrix} & 0 & C_2 \end{array}\right] \left[\begin{array}{c} x_2 \\ \theta_2 \\ \theta_3 \\ \hdashline V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\right] = 0 . \tag{41}$$

If we substitute the matrices B_1^2, B_2^2, T_1 and T_2 into Eq. (41) and carry out the multiplication, we obtain

$$\left[\begin{array}{c:c} \begin{bmatrix} -1 & \ell_1 & 0 \\ 0 & 1 & 0 \\ 1 & \ell_2 & 0 \\ 0 & 1 & -1 \end{bmatrix} & \begin{bmatrix} C_1 & \\ & C_2 \end{bmatrix} \end{array}\right] \left[\begin{array}{c} x_2 \\ \theta_2 \\ \theta_3 \\ \hdashline V_1 \\ M_1 \\ V_2 \\ M_2 \end{array}\right] = 0 . \tag{42}$$

Equation (41) is expressed in the local coordinate system.

If the structural model has k nodal points, then ℓ matrix equations ($\ell \le k$) similar to Eq. (35) are obtained. Stacking them in matrix form yields the equation

$$\left[BT \;\vdots\; C\right] \left[\begin{array}{c} \overline{x}_* \\ \hdashline \overline{H} \end{array}\right] = 0, \tag{43}$$

such that B, T, and C are block diagonal matrices given by

$$B = \begin{bmatrix} B^1 & & & & \\ & B^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & B^\ell \end{bmatrix} \tag{44}$$

$$T = \begin{bmatrix} T^1 & & & & \\ & T^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & T^\ell \end{bmatrix} \tag{45}$$

$$C = \begin{bmatrix} C^1 & & & & \\ & C^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & C^\ell \end{bmatrix}, \tag{46}$$

respectively. The vector $\bar{x}_*$ is defined by

$$\bar{x}_*^t = \left[x_*^{1t} \, x_*^{2t} \, . \, . \, . \, x_*^{\ell t} \right], \tag{47}$$

and $\bar{H}$ is defined in Eq. (28). Letting $\bar{x}$ be the vector defined in Eq. (24) we form the compatibility matrix P_2 such that

$$\bar{x}_* = P_2 \bar{x}, \tag{48}$$

enabling us to write Eq. (43) in the more compact form

$$\left[R \,\vdots\, C \right] \begin{bmatrix} \bar{x} \\ --- \\ \bar{H} \end{bmatrix} = 0, \tag{49}$$

where $R = BTP_2$ is a $(d \times n)$ rectangular matrix. Equation (49) is the displacement force relationship. Since the example problem in Fig. 6 is somewhat trivial, Eq. (42) is its displacement force equation. The additional steps represented by Eqs. (43) and (49) were not required.

THE EQUATION OF MOTION

Equation (49) can be explicitly solved for $\bar{H}$ giving

$$\bar{H} = C^{-1} R \bar{x}, \tag{50}$$

and substitution of Eq. (50) into Eq. (29) yields the equations of motion for the entire structure:

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F}. \tag{51}$$

The matrix K, called the stiffness matrix, is obtained from the product

$$K = SC^{-1}R \tag{52}$$

and is symmetric, as is readily apparent, since M is a diagonal matrix (see Appendix C). Furthermore, it may be shown (see Appendix C) that

$$R = S^t, \tag{53}$$

which greatly simplifies the programming. The SMOC code, therefore, computes only the S and C matrices.

THE EIGENVALUE PROBLEM

Substitution of

$$\bar{x} = \bar{E}e^{i\omega t} \tag{54}$$

into the homogeneous portion of Eq. (51) yields the generalized eigenvalue problem

$$\left[-M\omega^2 + K\right]\bar{E} = 0, \tag{55}$$

where ω^2 and $\bar{E}$ are an eigenvalue and eigenvector, respectively.

In some practical problems it may be desirable to neglect rotational inertia, causing the inertia matrix M to be singular. When this occurs, the program rearranges the vector $\bar{E}$ and the matrices M and K, such that their symmetric properties are retained, and rewrites the eigenvalue problem in the form

$$\left[\left[\begin{array}{c|c} -M_c & 0 \\ \hline 0 & 0 \end{array}\right]\omega^2 + \left[\begin{array}{c|c} K_{pp} & K_{ps} \\ \hline K_{sp} & K_{ss} \end{array}\right]\right]\left[\begin{array}{c} \bar{E}_p \\ \hline \bar{E}_s \end{array}\right] = 0, \tag{56}$$

where the subscripts p and s refer to the coordinates associated with the positive definite portion, M_c, and the null portion of the inertia matrix, respectively. Equation (56) is easily reduced to yield the condensed eigenvalue problem given by

$$\left[-M_c\omega^2 + K_c\right]\bar{E}_p = 0, \tag{57}$$

where:

$$K_c = K_{pp} - K_{ps}K_{ss}^{-1}K_{sp}. \tag{58}$$

The vector $\bar{E}_s$ is related to $\bar{E}_p$ by the relationship

$$\overline{E}_s = -K_{ss}^{-1} K_{sp} \overline{E}_p, \tag{59}$$

and $\overline{E}$ is formed by properly combining the elements of $\overline{E}_p$ and $\overline{E}_s$.

The solution of Eq. (55) (assuming that M is nonsingular) performed in the second TRILIN code, EIGEN, yields n eigenvalues ω_i^2 and eigenvectors $\overline{E}_i$ normalized such that $\overline{E}_i^t \overline{E}_i = 1$.

Since the eignevectors are orthogonal, we obtain

$$\begin{aligned} \overline{E}_j^t M \overline{E}_i &= G_i \delta_{ij} \\ \overline{E}_j^t K \overline{E}_i &= \omega_j^2 G_i \delta_{ij}, \end{aligned} \tag{60}$$

where the subscripts i and j refer to the ith and jth modes, respectively, δ_{ij} is the Kronecker delta, G_i is a constant called the generalized mass of the ith mode, and ω_i^2 is the natural frequency of the ith mode.

The spectral matrix, Γ, is defined as

$$\Gamma = \begin{bmatrix} \omega_1^2 & & & & & \\ & \omega_2^2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \omega_n^2 \end{bmatrix}, \tag{61}$$

in which the eigenvalues are arranged in ascending order down the diagonal according to their magnitudes. The modal matrix E is given by

$$E = \left[\overline{E}_1 \overline{E}_2 \ldots \overline{E}_n\right] = \begin{bmatrix} E_{11} & \cdot & \cdot & \cdot & E_{1,n-1} & E_{1n} \\ E_{21} & E_{22} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ E_{n1} & E_{n2} & \cdot & \cdot & \cdot & E_{nn} \end{bmatrix}. \tag{62}$$

Where the column occupied by $\overline{E}_i$ corresponds to that of its respective eigenvalue, it follows from Eq. (60) that the matrices obtained from the similarity transformations

$$E^t ME = G \qquad E^t KE = G\Gamma \tag{63}$$

are diagonal.

For arbitrary excitation, the motion may be determined by superimposing the contributions of the various modes. To determine the governing equation of each mode, we introduce the coordinate transformation

$$\bar{x} = E\bar{q}, \tag{64}$$

where the vector $\bar{q}$ represents the normal coordinates of the system. By substituting Eq. (64) into Eq. (51), premultiplying by $G^{-1}E^t$, and noting the results stated in Eq. (63), we obtain the uncoupled set of equations

$$\ddot{\bar{q}} + \Gamma\bar{q} = G^{-1}E^t\bar{F}(t) = \bar{Q}(t), \tag{65}$$

in which the ith component of the vector $\bar{Q}(t)$ is called the generalized force of the ith mode.

The response of the system is found by integrating Eq. (65) (performed in the third TRILIN code, TRANS) for a given forcing function $\bar{F}(t)$ and initial conditions $\bar{q}(0)$ and $\dot{\bar{q}}(0)$ found via the transformations

$$\bar{q}(0) = E^t\bar{x}(0) \qquad \dot{\bar{q}}(0) = E^t\dot{\bar{x}}(0). \tag{66}$$

Eq. (64) is used to transform the solution $\bar{q}(t)$ back to the generalized coordinates, $\bar{x}(t)$. The stress resultants as functions of time are found from Eq. (50).

DAMPING

Up to this point in the analysis, damping has not been considered. The usual way of incorporating damping into the equations of motion in a linear analysis is to assume that it is proportional to the velocity. In matrix form the equations of motion may be written as

$$M\ddot{\bar{x}} + C^*\dot{\bar{x}} + K\bar{x} = F(t), \tag{67}$$

in which C^* is a positive semidefinite symmetric matrix of damping coefficients. Since any attempts to add damping will be somewhat arbitrary, we will require that C^* is diagonalized by the same coordinate transformation that uncouples the undamped portion of Eq. (67).

We choose C^*, therefore, such that[7]

$$E^tC^*E = 2G\gamma\Gamma^{1/2} \tag{68}$$

where γ is an n-dimensional diagonal matrix of critical damping ratios ($\gamma_i = 1$ gives critical damping in the ith mode). Transforming Eq. (67) into its normal coordinates yields the n uncoupled equations

$$\ddot{\bar{q}} + 2\gamma\Gamma^{1/2}\dot{\bar{q}} + \Gamma\bar{q} = \bar{Q}(t), \tag{69}$$

which are integrated in place of Eq. (65) when damping is included. The n γ_i's are read into the TRANS code. A typical value of γ_i for a steel structure is 0.02, that is, 2% of critical damping.

TIME-DEPENDENT NODE ACCELERATION

The type of excitation of primary concern in TRILIN is one where specified nodes are subjected to an acceleration time history. Ground motion caused by seismic input is a loading of this type and is discussed in this section.

Let $\bar{y}$ be an n-dimensional vector of generalized coordinates of the structure with respect to an inertial reference frame, and let $\bar{x}$ be the generalized coordinates with respect to a local coordinate system. It follows, therefore, that

$$\bar{y} = \bar{x}_g + \bar{x} + \bar{c}, \tag{70}$$

where $\bar{x}_g$ is an n-dimensional vector of displacements due to ground motion, and $\bar{c}$ is a vector of constants. Since the potential energy of the structure is due only to its internal energy, the ground motion contributes nothing to it and, hence, it can be expressed in terms of the $\bar{x}$ coordinates. The kinetic energy must be in terms of the $\bar{y}$ coordinates. After obtaining the kinetic and potential energy, application of Lagrange's equations yield the equations of motions in the form

$$M\ddot{\bar{y}} + K\bar{x} = 0, \tag{71}$$

and, since

$$\ddot{\bar{y}} = \ddot{\bar{x}}_g + \ddot{\bar{x}}, \tag{72}$$

it follows that

$$M\ddot{\bar{x}} + K\bar{x} = -M\ddot{\bar{x}}_g. \tag{73}$$

Calling

$$F(t) = -M\ddot{\bar{x}}_g, \tag{74}$$

we arrive at an equation similar in form to Eq. (51) that can be treated as described by Eqs. (54) to (66). It should be noted, however, that $\bar{x}$ in Eq. (73) is a vector of relative displacements.

As an example, consider the 2-degree-of-freedom undamped system shown in Fig. 6. The potential and kinetic energies are

$$V^* = \frac{1}{2}\left[K_1(y_1 - x_g - c_1)^2 + K_2(y_2 - y_1)^2\right] \tag{75}$$

$$T^* = \frac{1}{2}\left(m_1\dot{y}_1^2 + m_2\dot{y}_2^2\right). \tag{76}$$

Applying the Lagrange equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{y}_i} + \frac{\partial V^*}{\partial y_i} = 0 \tag{77}$$

yields the matrix equations

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \tag{78}$$

Since

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_g \\ x_g \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{79}$$

we obtain

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \bar{F}, \tag{80}$$

where

$$\bar{F} = -\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_g \\ \ddot{x}_g \end{bmatrix} = -\ddot{x}_g \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}. \tag{81}$$

The Three Codes of TRILIN

The computer programs SMOC, EIGEN, and TRANS are the three divisions of TRILIN.

SMOC

The first code, SMOC, accepts data defining a structural model consisting of straight beam-type elements of uniform cross section. The purpose of this code is to calculate the elements of the S and C matrices in Eqs. (29) and (49), respectively, store these matrices on a disk file for use in the EIGEN code, and draw the structure for the user. The integers associated with the elements and nodal points are used by SMOC to order the vectors $\overline{H}$ and $\widetilde{x}$.

EIGEN

The code EIGEN calculates the stiffness matrix, eigenvalues, and eigenvectors of the idealized lumped mass model, using the information stored on the disk file created by SMOC.

The inversion of C in Eq. (52) is done using the Gaussian pivot method.[8] To conserve computer storage space, the matrices S, S^t, and C are partitioned into compatible segments, which allows the stiffness matrix to be calculated in a piecewise manner. This process is facilitated by C being a block diagonal matrix. Mathematically we can write S and C in the partitioned forms

$$S = \left[S_1 \,\vdots\, S_2 \,\vdots\, \ldots \,\vdots\, S_{k'} \right] \tag{82}$$

$$C = \begin{bmatrix} C_1' & & & & \\ & C_2' & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & C_{k'}' \end{bmatrix}, \tag{83}$$

respectively, where k' is an integer and where S_i and $C_i'^{-1}$ are compatible for multiplication such that $S_i C_i'^{-1} S_i^t$ is defined. By Eqs. (52) and (53),

$$K = SC^{-1}S^t, \tag{84}$$

and therefore

$$K = \sum_{i=1}^{k'} S_i C_i'^{-1} S_i^t. \tag{85}$$

For small problems (less than 60 degrees of freedom), no partitioning is required. The partitioning is done internally in EIGEN.

The eigenvalue problem is solved in a subroutine employing Householder's triangularization, a QR algorithm to find the eigenvalues, followed by inverse iteration to obtain the eigenvectors (this technique is discussed in detail by Wilkinson and Reinsch).[9] The bandwidth of the stiffness matrix is not explicitly used; however, the calculation proceeds rapidly because of the sparseness of K.

TRANS

The code TRANS calculates the transient response of the structural model by using mode-superposition. This method is based on the fact that the modal matrix, E, in Eq. (62) may be used to reduce the coupled equations of motion to a set of uncoupled equations [that is, Eq. (69)]. Since the lower modes play a far more significant role in the response than the higher modes, the structure's response is based on these lower modes. As a result, there is a marked reduction in the number of equations to be solved. The user may select the number of modes to be used in the calculation of the response.

The transient response in each of the structural model's normal coordinates is determined in TRANS by using a simple finite-difference scheme on the integrated uncoupled modal equations [Eqs. (65) and (69)]. The elements of the vector function $\overline{F}(t)$ are replaced by straight-line segments between successive ordinates, as shown in Fig. 7, where it is assumed that these ordinates are equally spaced. We can write Eq. (69) in the following form corresponding to $t_i \leq t < t_{i+1}$:

$$\ddot{\overline{q}}(t - t_i) + 2\gamma\Gamma^{1/2}\dot{\overline{q}}(t - t_i) + \Gamma\overline{q}(t - t_i) = \overline{Q}^i + \Delta\overline{Q}^i \times (t - t_i), \tag{86}$$

where

$$\overline{Q}^i = G^{-1}E^t\overline{F}(t_i) \tag{87}$$

$$\Delta\overline{Q}^i = \frac{\overline{Q}(t_{i+1}) - \overline{Q}(t_i)}{t_{i+1} - t_i}. \tag{88}$$

Integrating Eq. (86) with the initial conditions

$$\begin{aligned} \overline{q}(0) &= \overline{q}(t_i) = \overline{q}^i \\ \dot{\overline{q}}(0) &= \dot{\overline{q}}(t_i) = \dot{\overline{q}}^i, \end{aligned} \tag{89}$$

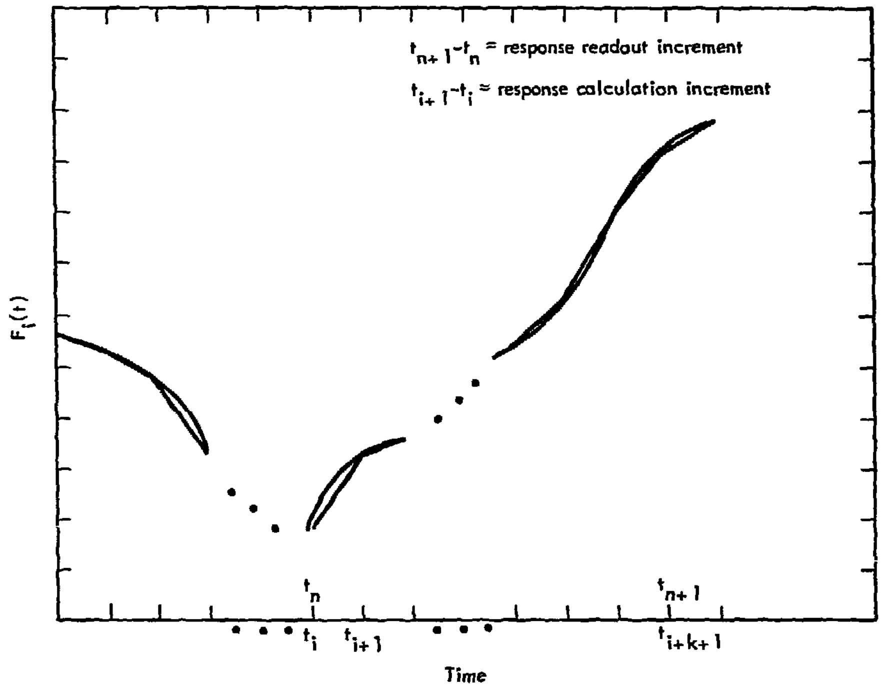

Fig. 7. The representation of the ith forcing function by a finite number of straight line segments.

we obtain, for the jth equation, the expression

$$q_j(t - t_i) = [A(\omega_j, t - t_i)] \begin{bmatrix} q^i \\ \dot{q}_j^i \\ Q_j^i \\ \Delta Q_j^i \end{bmatrix} \tag{90}$$

in which the matrix $[A(\omega_j, t - t_i)]$ is a (1×4) dimensional matrix whose elements depend on whether the modal damping in Eq. (69) is underdamped, critically damped, or overdamped. Differentiating Eq. (90) with respect to time, and setting $t = t_{i+1}$, yields

$$\bar{q}_{j*}^{i+1} = \left[A_j^{1j} \,\vdots\, A_j^{2i} \right] \begin{bmatrix} \bar{q}_{j*}^{i} \\ ----- \\ Q_j^i \\ \Delta Q_j^i \end{bmatrix}, \tag{91}$$

where:

$$\bar{q}_{j*}^{i+1} = \left[q_j^{i+1} \; \dot{q}_j^{i+1} \right]^t$$

$$\left[A_j^{1i} \,\vdots\, A_j^{2i} \right] = \left[\begin{array}{cc:cc} A_1(\omega_j, \Delta t_i) & A_2(\omega_j, \Delta t_i) & A_3(\omega_j, \Delta t_i) & A_4(\omega_j, \Delta t_i) \\ \dfrac{dA_1(\omega_j, \Delta t_i)}{dt} & \dfrac{dA_2(\omega_j, \Delta t_i)}{dt} & \dfrac{dA_3(\omega_j, \Delta t_i)}{dt} & \dfrac{dA_4(\omega_j, \Delta t_i)}{dt} \end{array} \right]$$

and $\Delta t_i = t_{i+1} - t_i$. The elements of $\left[A_j^{1i} \,\vdots\, A_j^{2i} \right]$ are given in Ref. 10. Equation (91) can be written in the equivalent form

$$q_{j*}^{i+1} = \left[Z_j^{1i} \,\vdots\, Z_j^{2i} \right] \begin{bmatrix} \bar{q}_{j*}^{i} \\ ---- \\ \bar{Q}_{j*}^{i} \end{bmatrix} \tag{92}$$

where:

$$Z_j^{1i} = A_j^{1i}$$

$$Z_j^{2i} = \begin{bmatrix} A_3(\omega_j, \Delta t_i) - A_4(\omega_j, \Delta t_i) \dfrac{1}{\Delta t_i} & A_4(\omega_j, \Delta t_i) \dfrac{1}{\Delta t_i} \\ \dfrac{dA_3(\omega_j, \Delta t_i)}{dt} - \dfrac{dA_4(\omega_j, \Delta t_i)}{dt} \dfrac{1}{\Delta t_i} & \dfrac{dA_4(\omega_j, \Delta t_i)}{dt} \dfrac{1}{\Delta t_i} \end{bmatrix}$$

$$\bar{Q}_{j*}^{i} = \left[Q_j^i \; Q_j^{i+1} \right].$$

In many problems the response readout increment, Δt_n, is much larger than the response calculation increment, Δt_i. To conserve computer time it is desirable to eliminate as many of the intermediate calculations as possible. This is done in TRANS by using a simple algorithm that permits the calculation of the response only at the response readout times, but that uses (and thus retains the accuracy of) the smaller time steps. To explain how the algorithm is derived, consider the following equations, based on Eq. (92), for the response between the time t_n and t_{n+1} (see Fig. 7):

$$\begin{aligned}
\bar{q}_{j*}^{i+2} &= \left[Z_j^{1,i+1} \,\vdots\, Z_j^{2,i+1} \right] \begin{bmatrix} \bar{q}_{j*}^{i+1} \\ ---- \\ \bar{Q}_{j*}^{i+1} \end{bmatrix} \\
&\vdots \\
\bar{q}_{j*}^{i+k+1} &= \left[Z_j^{1,i+k} \,\vdots\, Z_j^{2,i+k} \right] \begin{bmatrix} \bar{q}_{j*}^{i+k} \\ ---- \\ \bar{Q}_{j*}^{i+k} \end{bmatrix}
\end{aligned} \qquad (93)$$

Then, using forward substitution and induction, we obtain the equation

$$\bar{q}_{j*}^{n+1} = \bar{q}_{j*}^{i+k+1} = [U_{1j} \,\vdots\, U_{2j} \,\vdots\, \ldots \,\vdots\, U_{k+2,j}] \begin{bmatrix} \bar{q}_{j*}^{i} \\ ---- \\ \bar{Q}_{j*}^{i} \\ ---- \\ \vdots \\ ---- \\ \bar{Q}_{j*}^{i+k} \end{bmatrix} . \qquad (94)$$

Since the time increments are assumed to be equal,

$$\begin{aligned}
Z_j^{1,i} &= Z_j^{1,i+1} = \ldots = Z_j^{1,i+k} = Z_j^1 \\
Z_j^{2,i} &= Z_j^{2,i+1} = \ldots = Z_j^{2,i+k} = Z_j^2
\end{aligned} , \qquad (95)$$

the constants $U_{\ell j}$, $\ell = 1,2, \ldots, k+2$ are given by

$$U_{\ell j} = \begin{cases} \left(Z_j^1\right)^{k+1} & \text{if } \ell = 1 \\ \left(Z_j^1\right)^{k+2-\ell} Z_j^2 & \text{if } \ell = 2, 3, \ldots, k+1 \\ Z_j^2 & \text{if } \ell = k+2 \end{cases} .$$

The forces Q_j^{i+1}, Q_j^{i+2}, ..., Q_j^{i+k} each appear twice in the 2(k + 2) dimensional column vector in Eq. (94); hence, we introduce the condensation matrix P_3 such that

$$
\begin{bmatrix} \bar{Q}^{i}_{j*} \\ \bar{Q}^{i+1}_{j*} \\ \cdot \\ \cdot \\ \cdot \\ \bar{Q}^{i+k}_{j*} \end{bmatrix} = P_3 \begin{bmatrix} Q^{i}_{j} \\ Q^{i+1}_{j} \\ \cdot \\ \cdot \\ \cdot \\ Q^{i+k+1}_{j} \end{bmatrix} , \tag{96}
$$

and introducing Eq. (96) into Eq. (94) yields the desired equation

$$
\bar{q}^{n+1}_{j*} = U \left[\bar{q}^{n^t}_{j*} \,\vdots\, Q^{i}_{j}\, Q^{i+1}_{j}\, Q^{i+2}_{j} \ldots Q^{i+k+1}_{j} \right]^t . \tag{97}
$$

where

$$
U = [U_{1j} \,\vdots\, (U_{2j} \,\vdots\, U_3 \,\vdots \ldots \vdots\, U_{k+2,j}) P_3].
$$

The matrix P_3 merely adds columns 4 and 5, 6, and 7, ..., up to and including columns $2k + 2$ and $2k + 3$ of the $[2 \times 2(k + 2)]$ dimensional matrix in Eq. (94).

A similar technique is also used in TRANS to calculate the ground velocity and displacement for a known ground acceleration.

Concluding Remarks

A procedure has been presented to determine the transient response of linearly elastic, three-dimensional, framed-type structures subjected to arbitrary loading conditions. To keep the analysis tractable, a lumped mass formulation was used, Euler's Equations were simplified, and the flexure-torsion modes were assumed to be uncoupled. A mode superposition scheme retaining the lowest modes was used to calculate the transient response.

The computer program TRILIN based on the theory has been found to give results consistent with available closed-form "exact" solutions. The computer program contains many features unique to the Lawrence Livermore Laboratory's compiler and hardware, which makes its implementation on other systems difficult. A user's manual[5] is available.

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The contributions of the authors are as follows. A. B. Miller provided overall direction and coordination while the code was being developed. A. W. Weston[1] provided much of the original programming and was responsible with A. B. Miller for the structural analysis used within the program. D. L. Bernreuter assumed responsibility for maintenance of the code in its early stages and was responsible for many modifications. J. O. Hallquist formally developed and documented the general matrix theory of the TRILIN code and wrote the final report, using preliminary drafts by Miller, Weston, and Bernreuter. Hallquist modified the coding to simplify the code's input and the interpretation of its output.

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Appendix A

The matrix D_j^i in Eq. (11) may have one of two possible forms, depending on whether the jth element has its closed or open end at the ith node. These two possibilities are illustrated in Figs. A-1 and A-2. For the case when the jth element's closed end terminates at the ith node (Fig. A-1), D_j is

$$D_j^i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\ell_j & 0 & 1 & 0 \\ 0 & \ell_j & 0 & 0 & 0 & 1 \end{bmatrix} . \qquad \text{(A-1)}$$

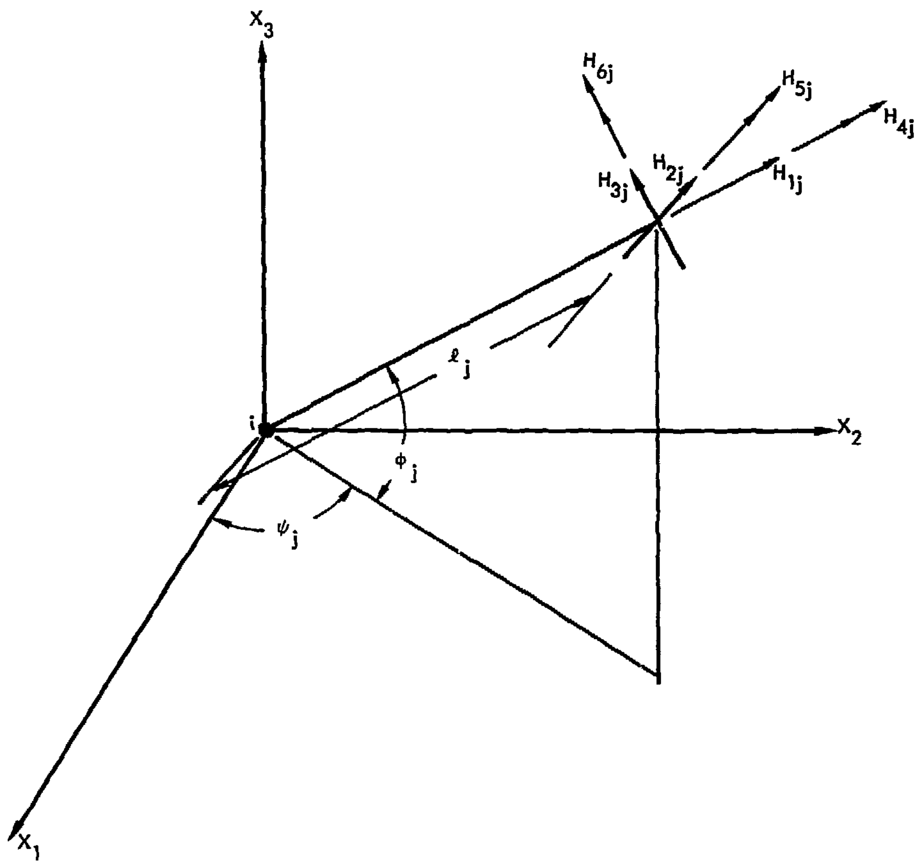

Fig. A-1. The jth element with its closed end terminated at the ith nodal point.

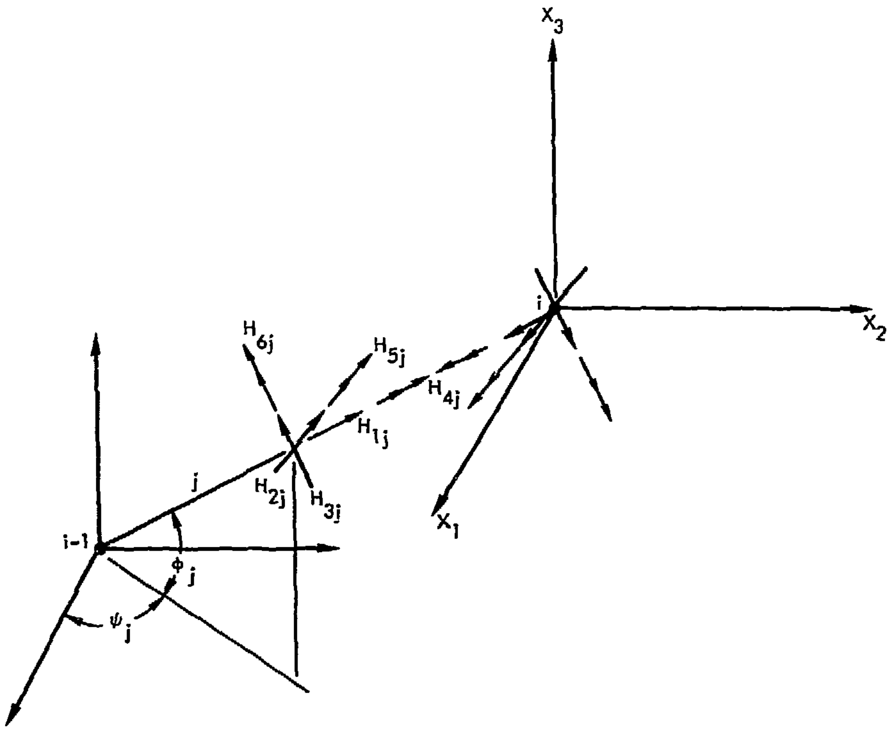

Fig. A-2. The jth element with its open end terminated at the ith nodal point.

If the jth element's open end terminates at the ith node (Fig. A-2), we see that D_j^i becomes*

$$D_j^i = -I, \tag{A-2}$$

the negative identity matrix. Obviously, the order of the D_j^i matrix will depend on the constraints placed on the system.

*In the SMOC code, this case (jth element with an open end at the ith node) is not treated in the direct manner indicated. Instead, a new transformation matrix based on an element oriented as shown in Fig. 2 but with its open and closed ends interchanged is used. This alternative approach will lead to the same final result.

Appendix B

In this Appendix we derive an explicit expression for Eq. (31) pertaining to a beam element. Since the vector on the right- and left-hand sides of Eq. (31) are well-defined, we are left with the task of giving an explicit representation of the partitioned matrix

$$\left[B_j^i \,\vdots\, C_j\right].$$

The matrix B_j^i is given by

$$B_j^i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \ell_j \\ 0 & 0 & 1 & 0 & -\ell_j & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{(B-1)}$$

and the nonzero elements of the symmetric C_j matrix are simply

$$\begin{aligned} C_{11j} &= \frac{\ell_j}{E_j A_j} & C_{22j} &= \frac{\ell_j^3}{3E_j I'_{3j}} + \frac{\ell_j \alpha_{2j}}{G'_j A_j} \\ C_{26j} = C_{62j} &= \frac{\ell_j^2}{2E_j I'_{3j}} & C_{33j} &= \frac{\ell_j^3}{3E_j I'_{2j}} + \frac{\ell_j \alpha_{3j}}{G'_j A_j} \\ C_{35j} = C_{53j} &= \frac{-\ell_j^2}{2E_j I'_{2j}} & C_{44j} &= \frac{\ell_j}{G'_j I'_{1j}} \\ C_{55j} &= \frac{\ell_j}{E_j I'_{2j}} & C_{66j} &= \frac{\ell_j}{E_j I'_{3j}} \end{aligned} \qquad \text{(B-2)}$$

for a straight isotropic beam of uniform cross section. The quantities ℓ_j, E_j, G'_j, and A_j refer to the length, modulus of elasticity, modulus of rigidity, and cross-sectional area of the jth element, respectively. The symbol α_{pj} is the ratio of maximum to average shear stress over the cross section of the jth element where p indicates the direction of shear. The second moment of area is represented by I'_{pj}, where p indicates the axis about which it is calculated and j refers to the jth element.

Appendix C

The Symmetry of the Stiffness Matrix

To show that the stiffness matrix, K, is symmetric we make the substitution,

$$\bar{x} = \bar{A}e^{\lambda t}, \tag{C-1}$$

where $\bar{A}$ is an n-dimensional vector and λ is an unknown parameter, into Eq. (50) to obtain

$$(M\lambda^2 + K)\bar{A} = \bar{F}. \tag{C-2}$$

Premultiplying Eq. (C-2) by $\bar{A}^t$ gives

$$\bar{A}^t(M\lambda^2 + K)\bar{A} = \bar{A}^t\bar{F}, \tag{C-3}$$

and taking the transpose yields

$$\bar{A}^t(M\lambda^2 + K^t)\bar{A} = \bar{F}^t\bar{A}. \tag{C-4}$$

Since $M^t = M$ and $\bar{F}^t\bar{A} = \bar{A}^t\bar{F}$, subtraction of Eqs. (C-3) and (C-4) leads to the expression

$$\bar{A}^t(K^t - K)A = 0. \tag{C-5}$$

Because the elements of $\bar{A}$ are arbitrary, we conclude that

$$K = K^t \tag{C-6}$$

and hence, that K is symmetric.

The Realtionship $R = S^t$

To demonstrate that $S^t = R$ we first note that C and C^{-1}, the flexibility matrix and its inverse, are positive definite block diagonal matrices with one block per element. By partitioning S and R we can write Eq. (52) in the form

$$[S_1\ S_2\ S_3 \ldots S_{nn}] \begin{bmatrix} C_1^{-1} & & & \\ & C_2^{-1} & & \\ & & \ddots & \\ & & & C_{nn}^{-1} \end{bmatrix} \begin{bmatrix} R_1 \\ \hdashline R_2 \\ \hdashline \vdots \\ \hdashline R_{nn} \end{bmatrix}, \tag{C-7}$$

where nn is the number of elements in the assemblage. Setting Eq. (C-7) equal to its transpose [see Eq. (C-6)] and carrying out the multiplication, we obtain the relationship

$$\sum_{i=1}^{nn} S_i C_i^{-1} R_i = \sum_{i=1}^{nn} R_i^t C_i^{-1} S_i^t. \tag{C-8}$$

But the elastic properties of each element are independent of all other elements; hence, we conclude that

$$S_j C_j^{-1} R_j = R_j^t C_j^{-1} S_j^t. \tag{C-9}$$

We will assume that the jth element has nodal points q and p at its open and closed ends, respectively, and that the displacement vector $\bar{x}$ has the form

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \hline \bar{x}_p \\ \hline \vdots \\ \hline \bar{x}_q \\ \hline \vdots \\ \bar{x}_m \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \end{matrix}\right\} u_1 \text{ elements} \\ \\ \left.\begin{matrix} \\ \end{matrix}\right\} u_2 \text{ elements} \\ \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} u_3 \text{ elements} \end{array} \tag{C-10}$$

where the summation $u_1 + u_2 + u_3$ + number of elements in the vectors $\bar{x}_q$ and $\bar{x}_p$ yield the total number of degrees of freedom, n, in the system.

It is apparent from Eqs. (14), (21), and (29) that, if C_j is a $(v \times v)$ $(v \leq 6)$ square matrix, S_j will be an $(n \times v)$ matrix of the form

$$S_j = \begin{bmatrix} 0 \\ \hdashline T_j^t D_j \\ \hdashline 0 \\ \hdashline -T_j^t I \\ \hdashline 0 \end{bmatrix}, \qquad \text{(C-11)}$$

where the sizes of the three null matrices (going down the page) are $(u_1 \times v)$, $(u_2 \times v)$, and $(u_3 \times v)$, respectively. Similarly, from Eqs. (31) and (35) we observe that

$$R_j = \left[0 \;\vdots\; B_j T_j \;\vdots\; 0 \;\vdots\; -I_j T_j \;\vdots\; 0\right], \qquad \text{(C-12)}$$

where the null matrices are $(v \times u_1)$, $(v \times u_2)$, and $(v \times u_3)$ dimensional matrices, respectively. Taking the transpose of S_j we obtain

$$S_j^t = \left[0 \;\vdots\; D_j^t T_j \;\vdots\; 0 \;\vdots\; -I_j T_j \;\vdots\; 0\right]. \qquad \text{(C-13)}$$

From Appendices A and B we note that

$$D_j^t = B_j, \qquad \text{(C-14)}$$

leading to the conclusion that

$$S_j^t = R_j, \qquad \text{(C-15)}$$

and it immediately follows from Eqs. (C-8) and (C-9) that

$$S^t = R, \qquad \text{(C-16)}$$

which is Eq. (53) in the text of this report.

Appendix D

NOMENCLATURE

Upper Case Symbols

$A(\omega_j, t - t_i)$	Coefficients for evaluating the system's displacement response from time t_i.
A_j^{1i}, A_j^{2i}	2 X 2 matrices containing the coefficients for integrating the jth modal equation over the ith time increment.
$\bar{A}$	Arbitrary vector.
B_j^i	Relates rigid body motion at the jth element's open end to the translation of the ith node.
B^i	Partitioned matrix containing a block diagonal matrix of B_j^i's and a negative identity and is associated with the ith free body.
B	A block diagonal matrix containing the B^i's.
C_i	Flexibility matrix for the ith element.
C_{jk_i}	An element of the flexibility matrix for the ith element.
C^i	Block diagonal matrix for the ith free body with the C_j matrices corresponding to the ith node's closed-end elements on the diagonal.
C	A block diagonal matrix of all C^i's.
C_i'	ith partioned segment of the C matrix.
C^*	Modal damping matrix.
D_j^i	Relates the forces acting at the jth element's open end to the forces acting on the ith nodal point.
D_i	A rectangular matrix containing the D_j^i matrices multiplied by their respective transformation matrices that transforms the forces at the ith nodal point into the global coordinate system.
D	Block diagonal matrix that contains all the D^i's on its diagonal.
$\bar{E}$	An eigenvector.
$\bar{E}_i$	Eigenvector associated with the ith normal coordinate.
$\bar{E}_p$	Portion of eigenvector corresponding to the condensed eigenvalue problem.
$\bar{E}_s$	Portion of eigenvector associated with the zero inertias.
E_i	Modulus of elasticity for the ith element.

E	Modal matrix.
$\bar{F}_i$	Vector of time-dependent forces acting on the ith node.
F	Force vector for the entire structural system.
G	Diagonal generalized mass matrix.
G_i	Generalized mass of the ith node.
G_i'	Modulus of rigidity for the ith element.
$H_{1i}, H_{2i}, \ldots H_{6}i$	The stress resultants at the open end of the ith element.
$\bar{H}_i$	Six-dimensional vector containing the six stress resultants.
$\bar{H}_j^i$	Six-dimensional vector containing the stress resultants for the jth element associated with the ith node.
H^i	Vector containing all the stress resultants acting on the ith nodal point.
$\bar{\bar{H}}$	Column vector containing the H^i's obtained for each node.
$\bar{H}$	Column vector containing each stress resultant in the system (each resultant appears only once).
H_*^i	Column vector containing all the stress resultants associated with the closed-end elements at the ith nodal point.
I	Identity matrix.
I_2, I_3	Moments of inertia used in the example problem.
I'_{p_j}	Second moment of area, where p indicates the axis about which it is calculated and j refers to the jth element.
K	Stiffness matrix.
K_c	Condensed stiffness matrix.
$K_{ss}, K_{sp}, K_{ps}, K_{pp}$	Partitioned segments of the stiffness matrix that corresponds to the null and positive definite portions of the inertia matrix.
$L_{1i}, L_{2i}, \ldots L_{6i}$	The stress resultants at the open end of the ith element in the global coordinate system.
$\bar{L}_i$	Six-dimensional containing the ith element's six stress resultants.
M	Diagonal lumped mass inertia matrix.
M_1, M_2	Internal moments used in example problem.
P_1, P_2, P_3	Connectivity and condensation matrices.
$Q_i(t)$	ith generalized force.

$\bar{Q}(t)$	Vector of generalized forces.
$\Delta\bar{Q}^i$	$\bar{Q}(t_{i+1}) - \bar{Q}(t_i)/(t_{i+1} - t_i)$.
$\bar{Q}^i_{j*}$	$(Q^i_j\ \Delta Q^i_j)$, where i and j refer to the ith time increment and the jth normal coordinate, respectively.
R	Defined in Eq. (49).
R_i	Partitioned segment of R.
S	Defined in Eq. (29) and is the transpose of R.
S_i	Partitioned segment of S
T_{ψ_i}	Coordinate transformation for the ψ_i rotation of the ith element.
T_{ϕ_i}	Coordinate transformation matrix for the ϕ_i rotation of the ith element.
T_i	6 X 6 transformation matrix for the ith element.
T^i	Matrix of transformation matrices associated with the closed end elements of the ith node.
T	Block diagonal matrix with the T^i matrices on its diagonal.
T^*	Kinetic energy.
U_{ij}	ith segment of coefficient matrix for finding the response of the j normal coordinate.
U	Defined in Eq. (97).
V^*	Potential energy.
V_1, V_2	Shear forces used in example problem.
X_1, X_2, X_3	Axes pertaining to the global coordinate system.
Z^{1i}_j, Z^{2i}_j	Defined by Eq. (92).

Lower Case Symbols

a_i, b_i, c_i	Location of the closed end of the ith element in the global coordinate system.
$\bar{c}$	Vector of constants.
d	Total number of unknown stress resultants.
$\bar{e}$	Three-dimensional vector in the local coordinate system.
$\bar{g}$	Three-dimensional vector in the global coordinate system.
m	Number of nodal points associated with one or more degrees of freedom.

m_{1i}, m_{2i}, m_{6i}	Inertias associated with the ith node.
m^i	Inertia matrix of the ith node.
m_1, m_2	Masses used in the example problem.
n	Number of degrees of freedom.
q_{1i}, q_{2i}, q_{3i}	Local coordinate system associated with the ith element.
$\bar{q}$	Vector of normal coordinates.
q_j	The ith normal coordinate.
$\bar{q}^i_{j*}$	Defined in Eq. (91).
q^i	Defined in Eq. (89).
r_i	Number of closed end elements at the ith nodal point.
s_i	Number of open end elements at the ith nodal point.
t_i	Time at start of the ith time step.
u_1, u_2, u_3	See Eq. (C-11).
v	Assumed dimension of C_i.
$\bar{x}_j$	Displacement vector of j nodal point in the global coordinate system.
x_{ij}, x_{2j}, x_{6j}	Elements of the vector $\bar{x}_j$ defined above.
$\bar{x}_g$	Vector of ground displacements.
$\bar{x}$	Vector of generalized coordinates for the entire structure.
x^i_*	Defined in Eq. (40).
$\bar{x}_*$	Vector of x^i_*'s.
$\bar{y}$	Displacement of structure in inertial reference frame.
y_i	The ith element of $\bar{y}$.

Greek Symbols

α_{pj}	Maximum-to-average shear stress over the cross section of the jth element in the p direction.
Γ	Spectral matrix.
γ	Matrix critical damping ratio.
γ_i	Damping ratio for the ith normal coordinate.
δ_{ij}	Kronecker delta.

$\bar{\eta}_i$	Deflection of the open end of the ith element in the local coordinate system.
$\eta_{1i}, \eta_{2i}, \ldots \eta_{6i}$	Elements of $\bar{\eta}_i$
$\theta_1, \theta_2, \theta_3$	Rotational coordinates for the example problem.
λ	Eigenvalue in Eq. (C-1).
ϕ_i, ψ_i	Angular coordinates that orient the ith element.
ω_i	ith natural frequency.

Superscripts

$\cdot$	Indicates time derivative.
t	Indicates transpose.
-1	Inverse.