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Tripled coincidence point results for generalized contractions in ordered generalized metric spaces

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Abstract

In this paper, we establish some tripled coincidence point results for a mixed g-monotone mapping $F : X^3 \to X$ satisfying (ψ, φ)-contractions in ordered generalized metric spaces. Also, an application and some examples are given to support our results.

Introduction and preliminaries

Banach contraction principle is one of the core subject that has been studied. It has so many different generalizations with different approaches. One of the remarkable generalizations, known as Φ-contraction, was given by Boyd and Wong [1] in 1969. In 1997, Alber and Guerre-Delabriere [2], introduced the notion of a weak φ-contraction which generalizes Boyd and Wong results, so Banach's result. Very recently, inspired from the notion of weak φ-contractions, a new concept of (ψ, φ)-contractions was introduced (see e.g. [3-7]).

Mustafa and Sims [8] introduced the notion of generalized metric spaces or simply G-metric spaces as a generalization of the concept of a metric space. Based on the concept of G-metric space, Mustafa et al. [9-11] proved several fixed point theorems for mapping satisfying different contractive condition. The study of common fixed point was initiated by Abbas and Rhoades [12]. The first result for contractive mappings in ordered G-metric spaces was obtained by Saadati et al. [13]. Bhashkar and Lakshmikantham [14] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and proved some coupled fixed point theorems in ordered metric spaces. Some authors obtained some interesting coupled fixed point theorems in G-metric spaces (see e.g. [15-18]). For more results on G-metric spaces, one can refer to the papers [9-13,15-28].

Throughout the paper, $\mathbb{N}^*$ is the set of positive integers. In 2004, Mustafa and Sims [8] introduced the concept of G-metric spaces as follows:

Definition 1. (*see* [8]). Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \text{for all} \quad x, y \in X. \tag{1}$$

Example **2**. Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 3. (*see* [8]). Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore, we say that $\{x_n\}$ is G-convergent to $x \in X$ if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 4. (see [8]). *Let (X,G) be a G-metric space. The following are equivalent:*

(1) *$\{x_n\}$ is G-convergent to x,*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$,*

(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$,*

(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 5. (*see* [8]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, i.e., $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 6. (see [8]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) *the sequence $\{x_n\}$ is G-Cauchy,*

(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.*

Definition 7. (*see* [8]). A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 8. Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if for any three G-convergent sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging to x, y and z respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Following the paper of Berinde and Borcut [29], Aydi, Karapınar and Postolache [30] introduced the following definitions:

Definition 9. (*see* [30]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$, $g : X \to X$. The mapping F is said to has the mixed g-monotone property if for any $x, y, z \in X$

$$\begin{aligned} &x_1, x_2 \in X, \quad gx_1 \leq gx_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ &y_1, y_2 \in X, \quad gy_1 \leq gy_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ &z_1, z_2 \in X, \quad gz_1 \leq gz_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \tag{2}$$

Definition 10. (*see* [30]). Let $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled coincidence point of F and g if

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz.$$

The point (gx, gy, gz) is called a point of coincidence of F and g.

Definition 11. (*see* [30]). Let $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled common fixed point of F and g if

$$F(x, y, z) = gx = x, \quad F(y, x, y) = gy = y, \quad F(z, y, x) = gz = z.$$

Definition 12. (*see* [30]). Let X be a non-empty set. Let F: $X \times X \times X \to X$ and g: $X \to X$ are such that

$$g(F(x, y, z)) = F(gx, gy, gz)$$

whenever $x, y, z \in X$, then F and g are said to be commutative.

Khan et al. [31] introduced the concept of altering distance function as follows:

Definition 13. (*altering distance function*, [31]) The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(1) ψ is continuous and non-decreasing,
(2) $\psi(t) = 0$ if and only if $t = 0$.

Let Ψ be the set of altering distances. Again, we denote by Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that

(i) φ is lower-continuous and non-decreasing,
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

The notion of a fixed point of N-order was first introduced by Samet and Vetro [32]. Later, Berinde and Borcut [29] proved some tripled fixed point results ($N = 3$) in partially ordered metric spaces (see also [33-36]). In this paper, we establish tripled coincidence point results for mappings $F : X^3 \to X$ and $g : X \to X$ involving nonlinear contractions in the setting of ordered G-metric spaces. Also, we present an application and some examples in support of our results.

Main results

Before stating our results, we give the following useful lemma.

Lemma 14. *Consider three non-negative real sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$. *Suppose there exists* $\alpha \geq 0$ *such that*

$$\lim_{n \to +\infty} \max\{a_n, b_n\} = 0 \quad \text{and} \quad \lim_{n \to +\infty} \max\{a_n, b_n, c_n\} = \alpha.$$

Then, $\limsup_{n \to +\infty} c_n = \alpha$.

Proof. First, we have $c_n \leq \max\{a_n, b_n, c_n\}$, then

$$\limsup_{n \to +\infty} c_n \leq \alpha. \tag{3}$$

For all $n \in \mathbb{N}$, we have

$$0 \leq \max\{a_n, b_n, c_n\} - c_n \leq \max\{a_n, b_n\} + c_n - c_n = \max\{a_n, b_n\},$$

which implies that $\limsup_{n\to+\infty}(\max\{a_n, b_n, c_n\} - c_n) \leq 0$. Having in mind that

$$\limsup_{n\to+\infty}(\max\{a_n, b_n, c_n\} - c_n) \geq \limsup_{n\to+\infty}\max\{a_n, b_n, c_n\} - \limsup_{n\to+\infty} c_n = \alpha - \limsup_{n\to+\infty} c_n,$$

so it follows that

$$\limsup_{n\to+\infty} c_n \geq \alpha. \tag{4}$$

By (3) and (4), we get that $\limsup_{n\to+\infty} c_n = \alpha$. □

The aim of this paper is to prove the following theorem.

Theorem 15. *Let* $(X, \leq)$ *be a partially ordered set and* (X, G) *be a G-metric space such that* (X, G) *is G-complete. Let* $F : X^3 \to X$ *and* $g : X \to X$. *Assume there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for* $x, y, z, a, b, c, u, v, w \in X$, *with* $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ *and* $gz \geq gc \geq gw$, *we have*

$$\psi(G(F(x,y,z), F(a,b,c), F(u,v,w))) \leq \psi(max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\}) - \phi(max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\}). \tag{5}$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,
(2) *F has the mixed g-monotone property,*
(3) *F is continuous,*
(4) *g is continuous and commutes with F.*

Suppose there exist $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ *and* $gz_0 \leq F(z_0, y_0, x_0)$, *then F and g have a tripled coincidence point in X, i.e., there exist x, y, z* $\in X$ *such that*

$$F(x,y,z) = gx, \quad F(y,x,y) = gy, \quad F(z,y,x) = gz.$$

Proof. Suppose $x_0, y_0, z_0 \in X$ are such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$. Since $F(X^3) \subseteq g(X)$, we can choose $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Then $gx_0 \leq gx_1$, $gy_0 \geq gy_1$ and $gz_0 \leq gz_1$. Similarly, define $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Since F has the mixed g-monotone property, we have $gx_0 \leq gx_1 \leq gx_2$, $gy_2 \leq gy_1 \leq gy_0$ and $gz_0 \leq gz_1 \leq gz_2$. Continuing this process, we can construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}) \leq gx_{n+1} = F(x_n, y_n, z_n),$$
$$gy_{n+1} = F(y_n, x_n, y_n) \leq gy_n = F(y_{n-1}, x_{n-1}, y_{n-1}),$$

and

$$gz_n = F(z_{n-1}, y_{n-1}, x_{n-1}) \leq gz_{n+1} = F(z_n, y_n, x_n).$$

If, for some integer n_0, we have $(gx_{n_0+1}, gy_{n_0+1}, gz_{n_0+1}) = (gx_{n_0}, gy_{n_0}, gz_{n_0})$, then $F(x_{n_0}, y_{n_0}, z_{n_0}) = gx_{n_0}, F(y_{n_0}, x_{n_0}, y_{n_0}) = gy_{n_0}$, and $F(z_{n_0}, y_{n_0}, x_{n_0}) = gz_{n_0}$; i.e., $(x_{n_0}, y_{n_0}, z_{n_0})$ is a tripled coincidence point of F and g. Thus we shall assume that $(gx_{n+1}, gy_{n+1}, gz_{n+1}) \neq (gx_n, gy_n, gz_n)$ for all $n \in \mathbb{N}$; i.e., we assume that either $gx_{n+1} \neq gx_n$ or $gy_{n+1} \neq gy_n$ or $gz_{n+1} \neq gz_n$. For any $n \in \mathbb{N}^*$, we have from (5)

$$\begin{aligned}\psi(G(gx_{n+1},gx_n,gx_n)) &:= \psi(G(F(x_n,y_n,z_n),F(x_{n-1},y_{n-1},z_{n-1}),F(x_{n-1},y_{n-1},z_{n-1})))\\ &\le \psi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gz_{n-1},gz_{n-1})\})\\ &\quad -\phi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\})\\ &\le \psi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1})gy_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\}),\end{aligned} \tag{6}$$

$$\begin{aligned}\psi(G(gy_n,gy_n,gy_{n+1})) &:= \psi(G(F(y_{n-1},x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1},y_{n-1}),F(y_n,x_n,y_n)))\\ &\le \psi(\max\{G(gy_{n-1},gy_{n-1},gy_n),G(gx_{n-1},gx_{n-1},gx_n)\}\\ &\quad -\phi(\max\{G(gy_{n-1},gy_{n-1},gy_n),G(gx_{n-1},gx_{n-1},gx_n)\})\\ &\le \psi(\max\{G(gy_n,gy_{n-1},gy_{n-1}),G(gx_n,gx_{n-1},gx_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\}),\end{aligned} \tag{7}$$

and

$$\begin{aligned}\psi(G(gz_{n+1},gz_n,gz_n)) &:= \psi(G(F(z_n,y_n,x_n),F(z_{n-1},y_{n-1},x_{n-1}),F(z_{n-1},y_{n-1},x_{n-1})))\\ &\le \psi(\max\{G(gz_n,gz_{n-1},gz_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gx_n,gx_{n-1},gx_{n-1})\})\\ &\quad -\phi(\max\{G(F(z_n,y_n,x_n),F(z_{n-1},y_{n-1},x_{n-1}),F(z_{n-1},y_{n-1},x_{n-1}))\})\\ &\le \psi(\max\{G(gz_n,gz_{n-1},gz_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gx_n,gx_{n-1},gx_{n-1})\}).\end{aligned} \tag{8}$$

Since $\psi:[0,+\infty) \to [0,+\infty)$ is a non-decreasing function, for $a, b, c \in [0,+\infty)$, we have $\psi(\max\{a, b, c\}) = \max\{\psi(a), \psi(b), \psi(c)\}$. Then, from (6), (7), and (8), it follows that

$$\begin{aligned}&\psi(\max\{G(gx_{n+1},gx_n,gx_n),G(gy_n,gy_n,gy_{n+1}),G(gz_{n+1},gz_n,gz_n)\})\\ &\quad = \max(\{\psi(G(gx_{n+1},gx_n,gx_n)),\psi(G(gy_n,gy_n,gy_{n+1})),\psi(G(gz_{n+1},gz_n,gz_n))\})\\ &\quad \le \psi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\}).\end{aligned}$$

The fact that ψ is non-decreasing yields that

$$\begin{aligned}&\max\{G(gx_{n+1},gx_n,gx_n),G(gy_n,gy_n,gy_{n+1}),G(gz_{n+1},gz_n,gz_n)\}\\ &\quad \le \max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\}.\end{aligned} \tag{9}$$

Thus, $\{\max\{G(gx_{n+1}, gx_n, gx_n),G(gy_n, gy_n, gy_{n+1}),G(gz_{n+1}, gz_n, gz_n)\}\}$ is a positive non-increasing sequence. Hence there exists $r \ge 0$ such that

$$\lim_{n\to+\infty} \max\{G(gx_{n+1},gx_n,gx_n),G(gy_n,gy_n,gy_{n+1}),G(gz_{n+1},gz_n,gz_n)\} = r. \tag{10}$$

Having in mind that φ is non-decreasing, then

$$\begin{aligned}&\phi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\})\\ &\qquad \ge \phi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1})\}),\end{aligned} \tag{11}$$

so from (6)-(8), we get that

$$\begin{aligned}&\psi(\max\{G(gx_{n+1},gx_n,gx_n),G(gy_n,gy_n,gy_{n+1}),G(gz_{n+1},gz_n,gz_n)\})\\ &\quad = \max(\{\psi(G(gx_{n+1},gx_n,gx_n)),\psi(G(gy_n,gy_n,gy_{n+1})),\psi(G(gz_{n+1},gz_n,gz_n))\})\\ &\quad \le \psi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1}),G(gz_n,gz_{n-1},gz_{n-1})\})\\ &\qquad -\phi(\max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_n,gy_{n-1},gy_{n-1})\}).\end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}0 &\le \max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_{n-1},gy_{n-1},gy_n)\}\\ &\le \max\{G(gx_n,gx_{n-1},gx_{n-1}),G(gy_{n-1},gy_{n-1},gy_n),G(gz_n,gz_{n-1},gz_{n-1})\},\end{aligned} \tag{13}$$

so by (10), the real sequence $\{\max\{G(gx_n, gx_{n-1}, gx_{n-1}),G(gy_{n-1}, gy_{n-1}, gy_n)\}\}$ is bounded. Thus, there exists a real number r_1 with $0 \le r_1 \le r$ and subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that

$$\lim_{k\to+\infty} \max\{G(gx_{n(k)+1}, gx_{n(k)}, gx_{n(k)}), G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1})\} = r_1. \tag{14}$$

We rewrite (12)

$$\begin{aligned}&\psi(\max\{G(gx_{n(k)+1}, gx_{n(k)}, gx_{n(k)}), G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}), G(gz_{n(k)+1}, gz_{n(k)}, gz_{n(k)})\})\\&\quad\le \psi(\max\{G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1})\})\\&\qquad - \phi(\max\{G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1})\}).\end{aligned} \tag{15}$$

Letting $k \to +\infty$ in (15), having in mind (10), (14), the continuity of ψ and the lower semi-continuity of φ, we obtain

$$\begin{aligned}\psi(r) &= \limsup_{k\to+\infty} \psi(\max\{G(gx_{n(k)+1}, gx_{n(k)}, gx_{n(k)}), G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)+1}), G(gz_{n(k)+1}, gz_{n(k)}, gz_{n(k)})\})\\&\le \limsup_{k\to+\infty} \psi(\max\{G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1})\})\\&\quad - \liminf_{k\to+\infty} \phi(\max\{G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1})\})\\&\le \psi(r) - \phi(r_1),\end{aligned}$$

which implies that $\varphi(r_1) = 0$, and using a property of φ, we find $r_1 = 0$. Thanks to Lemma 14 together with (10) and (14), it yields that

$$\begin{aligned}r &=: \lim_{k\to+\infty} \max\{G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1})\}\\&= \limsup_{k\to+\infty} G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1}).\end{aligned} \tag{16}$$

For any $k \in \mathbb{N}$, we rewrite (8) as

$$\begin{aligned}&\psi(G(gz_{n(k)+1}, gz_{n(k)}, gz_{n(k)}))\\&\quad\le \psi(\max\{G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1})\})\\&\qquad - \phi(\max\{G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1})\}).\end{aligned} \tag{17}$$

Again, letting $k \to +\infty$ in (17), having in mind (10), (16) and by the properties of ψ, φ, we obtain

$$\begin{aligned}\psi(r) &= \limsup_{k\to+\infty} \psi(G(gz_{n(k)+1}, gz_{n(k)}, gz_{n(k)}))\\&\le \limsup_{k\to+\infty} \psi(\max\{gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1})\})\\&\quad - \liminf_{k\to+\infty} \phi(\max\{G(gz_{n(k)}, gz_{n(k)-1}, gz_{n(k)-1}), G(gy_{n(k)}, gy_{n(k)-1}, gy_{n(k)-1}), G(gx_{n(k)}, gx_{n(k)-1}, gx_{n(k)-1})\})\\&\le \psi(r) - \phi(r),\end{aligned}$$

which gives that $\varphi(r) = 0$, so $r = 0$, i.e., by (10),

$$\lim_{n\to+\infty} \max\{G(gx_{n+1}, gx_n, gx_n), G(gy_n, gy_n, gy_{n+1}), G(gz_{n+1}, gz_n, gz_n)\} = 0. \tag{18}$$

Our next step is to show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are G-Cauchy sequences.

Assume the contrary, i.e., $\{gx_n\}$, $\{gy_n\}$ or $\{gz_n\}$ is not a G-Cauchy sequence, i.e.,

$$\lim_{n,m\to+\infty} G(gx_m, gx_n gx_n) \neq 0, \quad \text{or} \quad \lim_{n,m\to+\infty} G(gy_m, gy_n, gy_n) \neq 0,$$

or $\lim_{n,m\to+\infty} G(gz_m, gz_n, gz_n) \neq 0$. This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k}), G(gz_{m_k}, gz_{n_k}, gz_{n_k})\} \geq \varepsilon. \tag{19}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (19). Then

$$\max\{G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k}, gy_{n_k-1}, gy_{n_k-1}), G(gz_{m_k}, gz_{n_k-1}, gz_{n_k-1})\} < \varepsilon. \tag{20}$$

By (G5) and (20), we have

$$\begin{aligned} G(gx_{m_k}, gx_{n_k}, gx_{n_k}) &\le G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}) + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\ &< \varepsilon + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}). \end{aligned}$$

Thus, by (18) we obtain

$$\lim_{k\to+\infty} G(gx_{m_k}, gx_{n_k}, gx_{n_k}) \le \lim_{k\to+\infty} G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}) \le \varepsilon. \tag{21}$$

Similarly, we have

$$\lim_{k\to+\infty} G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \le \lim_{k\to+\infty} G(gy_{m_k}, gy_{n_k-1}, gy_{n_k-1}) \le \varepsilon. \tag{22}$$

$$\lim_{k\to+\infty} G(gz_{m_k}, gz_{n_k}, gz_{n_k}) \le \lim_{k\to+\infty} G(gz_{m_k}, gz_{n_k-1}, gz_{n_k-1}) \le \varepsilon. \tag{23}$$

Again by ($G5$) and (20), we have

$$\begin{aligned} G(gx_{m_k}, gx_{n_k}, gx_{n_k}) &\le G(gx_{m_k}, gx_{m_k-1}, gx_{m_k-1}) \\ &\quad + G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}) + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\ &\le G(gx_{m_k}, gx_{m_k-1}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{m_k}, gx_{m_k}) \\ &\quad + G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}) + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\ &< G(gx_{m_k}, gx_{m_k-1}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{m_k}, gx_{m_k}) \\ &\quad + \varepsilon + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}). \end{aligned}$$

Letting $k \to +\infty$ and using (18), we get

$$\lim_{k\to+\infty} G(gx_{m_k}, gx_{n_k}, gx_{n_k}) \le \lim_{k\to+\infty} G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}) \le \varepsilon, \tag{24}$$

$$\lim_{k\to+\infty} G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \le \lim_{k\to+\infty} G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}) \le \varepsilon, \tag{25}$$

$$\lim_{k\to+\infty} G(gz_{m_k}, gz_{n_k}, gz_{n_k}) \le \lim_{k\to+\infty} G(gz_{m_k-1}, gz_{n_k-1}, gz_{n_k-1}) \le \varepsilon. \tag{26}$$

Using (19) and (24)-(26), we have

$$\begin{aligned} &\lim_{k\to+\infty} \max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k}), G(gz_{m_k}, gz_{n_k}, gz_{n_k})\} \\ &\quad = \lim_{k\to+\infty} \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gz_{m_k-1}, gz_{n_k-1}, gz_{n_k-1}) \\ &\quad = \varepsilon. \end{aligned} \tag{27}$$

Now, using inequality (5) we obtain

$$\begin{aligned} \psi(G(gx_{m_k}, gx_{n_k}, gx_{n_k})) &= \psi(G(F(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}), F(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}))) \\ &\le \psi(\max\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(y_{m_k-1}, y_{n_k-1}, y_{n_k-1}), G(z_{m_k-1}, z_{n_k-1}, z_{n_k-1})\}) \\ &\quad - \phi(\max\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(y_{m_k-1}, y_{n_k-1}, y_{n_k-1}), G(z_{m_k-1}, z_{n_k-1}, z_{n_k-1})\}) \end{aligned} \tag{28}$$

$$\begin{aligned} \psi(G(gy_{m_k}, gy_{n_k}, gy_{n_k})) &= \psi(G(F(y_{m_k-1}, x_{m_k-1}, y_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}), F(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}))) \\ &\le \psi(\max\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(y_{m_k-1}, y_{n_k-1}, y_{n_k-1})\}) \\ &\quad - \phi(\max\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(y_{m_k-1}, y_{n_k-1}, y_{n_k-1})\}) \end{aligned} \tag{29}$$

and

$$\begin{aligned} \psi(G(gz_{m_k}, gz_{n_k}, gz_{n_k})) &= \psi(G(F(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}), F(z_{n_k-1}, y_{n_k-1}, x_{n_k-1}), F(z_{n_k-1}, y_{n_k-1}, x_{n_k-1}))) \\ &\le \psi(\max\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(y_{m_k-1}, y_{n_k-1}, y_{n_k-1}), G(z_{m_k-1}, z_{n_k-1}, z_{n_k-1})\}) \\ &\quad - \phi(\max\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), G(y_{m_k-1}, y_{n_k-1}, y_{n_k-1}), G(z_{m_k-1}, z_{n_k-1}, z_{n_k-1})\}) \end{aligned} \tag{30}$$

We deduce from (28)-(30) that

$$\begin{aligned}&\psi(\max\{G(gx_{m_k},gx_{n_k},gx_{n_k}),G(gy_{m_k},gy_{n_k},gy_{n_k}),G(gz_{m_k},gz_{n_k},gz_{n_k})\}\\&\quad=\max\{\psi(G(gx_{m_k},gx_{n_k},gx_{n_k})),\psi(G(gy_{m_k},gy_{n_k},gy_{n_k})),\psi(G(gz_{m_k},gz_{n_k},gz_{n_k}))\}\\&\quad\le\psi(\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1}),G(z_{m_k-1},z_{n_k-1},z_{n_k-1})\})\\&\qquad-\phi(\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1})\}).\end{aligned}\tag{31}$$

On the other hand, since

$$\begin{aligned}&\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1})\}\\&\quad\le\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1}),G(z_{m_k-1},z_{n_k-1},z_{n_k-1})\},\end{aligned}\tag{32}$$

then from (27),

$$\limsup_{k\to+\infty}\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1})\}\le\varepsilon.$$

Therefore, the real sequence $\{\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1})\}\}$ is bounded. Thus, up to a subsequence (still denoted the same), there exists ε_1 with $0\le\varepsilon_1\le\varepsilon$ such that

$$\lim_{k\to+\infty}\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1})\}=\varepsilon_1.\tag{33}$$

Inserting this in (31) and using (27), (33) together with the properties of ψ, φ, we get that

$$\begin{aligned}\psi(\varepsilon)&=\limsup_{k\to+\infty}\psi(\max\{G(gx_{m_k},gx_{n_k},gx_{n_k}),G(gy_{m_k},gy_{n_k},gy_{n_k}),G(gz_{m_k},gz_{n_k},gz_{n_k})\}\\&\le\limsup_{k\to+\infty}\psi(\max\{G(x_{mk-1},x_{nk-1},x_{nk-1}),G(y_{mk-1},y_{nk-1},y_{nk-1}),G(z_{mk-1},z_{nk-1},z_{nk-1})\})\\&\quad-\liminf k\to+\infty\phi(\max\{G(x_{m_k-1},x_{n_k-1},x_{n_k-1}),G(y_{m_k-1},y_{n_k-1},y_{n_k-1})\})\\&\le\psi(\varepsilon)-\phi(\varepsilon_1),\end{aligned}$$

which leads to $\varphi(\varepsilon_1)=0$, so $\varepsilon_1=0$. By this and (27), due to Lemma 14, we obtain

$$\limsup_{k\to+\infty}G(z_{m_k-1},z_{n_k-1},z_{n_k-1})=\varepsilon.$$

Combining this to (19) and (26), we find

$$\limsup_{k\to+\infty}G(z_{m_k},z_{n_k},z_{n_k})=\varepsilon.$$

Letting $k\to\infty$ in (30) and using (27), we deduce

$$\psi(\varepsilon)\le\psi(\varepsilon)-\phi(\varepsilon),$$

i.e., $\varepsilon=0$, it is a contradiction. We conclude that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are G-Cauchy sequences in the G-metric space (X, G), which is G-complete. Then, there are $x, y, z\in X$ such that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are respectively G-convergent to x, y and z, i.e., from Proposition 4, we have

$$\lim_{n\to+\infty}G(gx_n,gx_n,x)=\lim_{n\to+\infty}G(gx_n,x,x)=0,\tag{34}$$

$$\lim_{n\to+\infty}G(gy_n,gy_n,y)=\lim_{n\to+\infty}G(gy_n,y,y)=0,\tag{35}$$

$$\lim_{n\to+\infty}G(gz_n,gz_n,z)=\lim_{n\to+\infty}G(gz_n,z,z)=0.\tag{36}$$

From (34)-(36) and the continuity of g, we get thanks to Proposition 8

$$\lim_{n\to+\infty} G(g(gx_n), g(gx_n), gx) = \lim_{n\to+\infty} G(g(gx_n), gx, gx) = 0, \tag{37}$$

$$\lim_{n\to+\infty} G(g(gy_n), g(gy_n), gy) = \lim_{n\to+\infty} G(g(gy_n), gy, gy) = 0, \tag{38}$$

$$\lim_{n\to+\infty} G(g(gz_n), g(gz_n), gz) = \lim_{n\to+\infty} G(g(gz_n), gz, gz) = 0. \tag{39}$$

Since $gx_{n+1} = F(x_n, y_n, z_n)$, $gy_{n+1} = F(y_n, x_n, y_n)$ and $gz_{n+1} = F(z_n, y_n, x_n)$, so the commutativity of F and g yields that

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n), \tag{40}$$

$$g(gy_{n+1}) = g(F(y_n, x_n, y_n)) = F(gy_n, gx_n, gy_n), \tag{41}$$

$$g(gz_{n+1}) = g(F(z_n, y_n, x_n)) = F(gz_n, gy_n, gx_n). \tag{42}$$

Now we show that $F(x, y, z) = gx$, $F(y, x, y) = gy$ and $F(z, y, x) = gz$.

The mapping F is continuous, so since the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are, respectively, G-convergent to x, y and z, hence using Definition 8, the sequence $\{F(gx_n, gy_n, gz_n)\}$ is G-convergent to $F(x, y, z)$. Therefore, from (40), $\{g(gx_{n+1})\}$ is G-convergent to $F(x, y, z)$. By uniqueness of the limit, from (37) we have $F(x, y, z) = gx$.

Similarly, one finds $F(y, x, y) = gy$ and $F(z, y, x) = gz$, and this makes end to the proof. □

Corollary 16. *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space such that (X, G) is G-complete. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exists $k \in [0,1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have*

$$G(F(x,y,z), F(a,b,c), F(u,v,w)) \leq k \max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\}.$$

Assume that F and g satisfy the following conditions:

(1) *$F(X^3) \subset g(X)$,*
(2) *F has the mixed g-monotone property,*
(3) *F is continuous,*
(4) *g is continuous and commutes with F.*

Suppose there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point in X, i.e., there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy, \quad F(z,y,x) = gz.$$

Proof. If follows by taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ for all $t \geq 0$. □

Corollary 17. *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space such that (X, G) is G-complete. Let $F : X^3 \to X$ and $g : X \to X$. Assume there exists $k \in [0,1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have*

$$G(F(x,y,z),F(a,b,c),F(u,v,w)) \leq \frac{k}{3}(G(gx,ga,gu)+G(gy,gb,gv)+G(gz,gc,gw)).$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,
(2) *F has the mixed g-monotone property,*
(3) *F is continuous,*
(4) *g is continuous and commutes with F.*

Suppose there exist x_0, y_0, $z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point in X, i.e., there exist x, y, $z \in X$ such that

$$F(x,y,z)=gx, \quad F(y,x,y)=gy, \quad F(z,y,x)=gz.$$

Proof. It suffices to remark that

$$\begin{aligned} &\tfrac{k}{3}(G(gx,ga,gu)+G(gy,gb,gv)+G(gz,gc,gw)) \\ &\quad \leq k\max\{G(gx,ga,gu),G(gy,gb,gv),G(gz,gc,gw)\}. \end{aligned} \tag{43}$$

□

In the next theorem, we omit the continuity hypothesis of *F*. We need the following definition.

Definition 18. *Let $(X, \preccurlyeq)$ be a partially ordered set and G be a G-metric on X. We say that $(X, G, \leq)$ is regular if the following conditions hold:*

(i) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \leq y_n$ for all n.*

Theorem 19. *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F : X^3 \to X$ and $g: X \to X$. Assume there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for x, y, z, a, b, c, u, v, $w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have*

$$\begin{aligned} \psi(G(F(x,y,z),F(a,b,c),F(u,v,w))) &\leq \psi(\max\{G(gx,ga,gu),G(gy,gb,gv),G(gz,gc,gw)\}) \\ &\quad - \phi(\max\{G(gx,ga,gu),G(gy,gb,gv),G(gz,gc,gw)\}). \end{aligned} \tag{44}$$

Assume that $(X, G, \leq)$ is regular. Suppose that $(g(X),G)$ is G-complete, F has the mixed g-monotone property and $F(X \times X \times X) \subseteq g(X)$. Also, assume there exist x_0, y_0, $z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point in X, i.e., there exist x, y, $z \in X$ such that

$$F(x,y,z)=gx, \quad F(y,x,y)=gy, \quad F(z,y,x)=gz.$$

Proof. Proceeding exactly as in Theorem 15, we have that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are *G*-Cauchy sequences in the *G*-complete *G*-metric space $(g(X), G)$. Then, there exist *x*, *y*, $z \in X$ such that $gx_n \to gx$, $gy_n \to gy$ and $gz_n \to gz$. Since $\{gx_n\}$ and $\{gz_n\}$ are non-decreasing and $\{gy_n\}$ is non-increasing, using the regularity of $(X, G, \leq)$, we have $gx_n \leq gx$, $gz_n \leq gz$ and $gy \leq gy_n$ for all $n \geq 0$. Using (5), we get

$$\begin{aligned} \psi(G(F(x,y,z),gx_{n+1},gx_{n+1})) &= \psi(G(F(x,y,z),F(x_n,y_n,z_n),F(x_n,y_n,z_n))) \\ &\leq \psi(\max\{G(gx,gx_n,gx_n),G(gy,gy_n,gy_n),G(gz,gz_n,gz_n)\}) \\ &\quad - \phi(\max\{G(gx,gx_n,gx_n),G(gy,gy_n,gy_n),G(gz,gz_n,gz_n)\}). \end{aligned} \tag{45}$$

Letting $n \to +\infty$ in the above inequality, we obtain that

$$\psi(G(F(x,y,z),gx,gx)) \leq \psi(0) - \phi(0) = 0,$$

which implies that $G(F(x, y, z), gx, gx) = 0$, i.e., $gx = F(x, y, z)$.

Similarly, one can show that $gy = F(y, x, y)$ and $gz = F(z, y, x)$. Thus we proved that (x, y, z) is a tripled coincidence point of F and g. □

Similarly, we can state the following corollary.

Corollary 20. *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Let $F: X^3 \to X$ and $g: X \to X$. Assume there exists $k \in [0, 1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc \geq gw$, we have*

$$G(F(x,y,z),F(a,b,c),F(u,v,w)) \leq k\max\{G(gx,ga,gy),G(gy,gb,gv),G(gz,gc,gw)\}.$$

Assume that $(X, G, \leq)$ is regular. Suppose that $(g(X),G)$ is G-complete, F has the mixed g-monotone property and $F(X \times X \times X) \subseteq g(X)$. Also, assume there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point in X, i.e., there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy, \quad F(z,y,x) = gz.$$

Corollary 21. *Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space. Assume that $(X, G, \leq)$ is regular. Let $F: X^3 \to X$ and $g: X \to X$. Assume there exists $k \in [0, 1)$ such that for $x, y, z, a, b, c, u, v, w \in X$, with $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ and $gz \geq gc > gw$, we have*

$$G(F(x,y,z),F(a,b,c),F(u,v,w)) \leq \frac{k}{3}(G(gx,ga,gu) + G(gy,gb,gv) + G(gz,gc,gw)).$$

Suppose that $(g(X),G)$ is G-complete, F has the mixed g-monotone property and $F(X \times X \times X) \subseteq g(X)$. Also, assume there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 < F(z_0, y_0, x_0)$, then F and g have a tripled coincidence point in X, i.e., there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy, \quad F(z,y,x) = gz.$$

Remark **22**. Other corollaries could be derived from Theorems 15 and 19 by taking $g = Id_x$.

Now, form previous obtained results, we will deduce some tripled coincidence point results for mappings satisfying a contraction of integral type in G-metric space. Let us introduce first some notations.

We denote by Γ the set of functions $\alpha: [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) α is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$,
(ii) for all $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \alpha(s)ds > 0.$$

Let $N \in \mathbb{N}^*$ be fixed. Let $\{\alpha_i\}_{1 \leq i \leq N}$ be a family of N functions that belong to Γ. For all $t \geq 0$, we denote $(I_i)_{i=1,\ldots,N}$ as follows:

$$I_1(t) = \int_0^t \alpha_1(s)\mathrm{d}s,$$

$$I_2(t) = \int_0^{I_1(t)} \alpha_2(s)ds = \int_0^{\int_0^t \alpha_1(s)\mathrm{d}_s} \alpha_2(s)ds,$$

$$\vdots$$

$$I_N(t) = \int_0^{I_{N-1}(t)} \alpha_N(s)\mathrm{d}s.$$

We have the following result.

Theorem 23. *Let* $(X, \leq)$ *be a partially ordered set and* (X, G) *be a G-metric space such that* (X, G) *is G-complete. Let* $F : X^3 \to X$ *and* $g : X \to X$. *Assume there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for* $x, y, z, a, b, c, u, v, w \in X$, *wth* $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ *and* $gz \geq gc \geq gw$, *we have*

$$\begin{aligned} I_N(\psi(G(F(x,y,z), F(a,b,c), F(u,v,w)))) &\leq I_N(\psi(max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\})) \\ &\quad - I_N(\phi(max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\})). \end{aligned} \tag{46}$$

Assume that F and g satisfy the following conditions:

(1) $F(X^3) \subseteq g(X)$,

(2) *F has the mixed g-monotone property,*

(3) *F is continuous,*

(4) *g is continuous and commutes with F.*

Suppose there exist $x_0, y_0, z_0 \in X$ *such that* $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ *and* $gz_0 \leq F(z_0, y_0, x_0)$, *then F and g have a tripled coincidence point in X, i.e., there exist* $x, y, z \in X$ *such that*

$$F(x,y,z) = gx, \quad F(y,x,y) = gy, \quad F(z,y,x) = gz.$$

Proof. Take

$$\tilde{\phi} = I_N{}^\circ\varphi \quad \text{and} \quad \tilde{\psi} = I_N{}^\circ\psi.$$

It is easy to show that $\tilde{\psi} \in \Psi$ and $\tilde{\phi} \in \Phi$. From (46), we have

$$\begin{aligned} \tilde{\psi}(G(F(x,y,z), F(a,b,c), F(u,v,w)) &\leq \tilde{\psi}(\max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\}) \\ &\quad - \tilde{\phi}(\max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\}). \end{aligned} \tag{47}$$

Now, applying Theorem 15, we obtain the desired result. □

Similarly, we have

Theorem 24. *Let* $(X, \leq)$ *be partially ordered set and* (X, G) *be a G-metric space. Let* $F : X^3 \to X$ *and* $g: X \to X$. *Assume there exist* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for* $x, y, z, a, b, c, u, v, w \in X$, *with* $gx \geq ga \geq gu$, $gy \leq gb \leq gv$ *and* $gz \geq gc \geq gw$, *we have*

$$\begin{aligned} I_N(\psi(G(F(x,y,z), F(a,b,c), F(u,v,w)))) &\leq I_N(\psi(\max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\})) \\ &\quad - I_N(\phi(\max\{G(gx,ga,gu), G(gy,gb,gv), G(gz,gc,gw)\})). \end{aligned}$$

Assume that $(X, G, \leq)$ *is regular. Suppose that* $(g(X),G)$ *is G-complete, F has the mixed g-monotone property and* $F(X \times X \times X) \subseteq g(X)$. *Also, assume there exist* x_0, y_0, z_0

∈ X such that $gx_0 \le F(x_0, y_0, z_0)$, $gy_0 \ge F(y_0, x_0, y_0)$ *and* $gz_0 \le F(z_0, y_0, x_0)$, *then F and g have a tripled coincidence point in X, i.e., there exist x, y, z ∈ X such that*

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz.$$

Application to integral equations

In this section, we study the existence of solutions to nonlinear integral equations using the results proved in section "Main results".

Consider the integral equations in the following system

$$\begin{aligned} x(t) &= p(t) + \int_0^T S(t,s)[f(s,x(s)) + k(s,y(s)) + h(s,z(s))]\mathrm{d}s \\ y(t) &= p(t) + \int_0^T S(t,s)[f(s,y(s)) + k(s,x(s)) + h(s,y(s))]\mathrm{d}s \\ z(t) &= p(t) + \int_0^T S(t,s)[f(s,z(s)) + k(s,y(s)) + h(s,x(s))]\mathrm{d}s. \end{aligned} \tag{48}$$

We will analyze the system (48) under the following assumptions:

(i) $f, k, h: [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous,

(ii) $p : [0, T] \to \mathbb{R}$ is continuous,

(iii) $S: [0, T] \times \mathbb{R} \to [0, \infty)$ is continuous,

(iv) there exists $q > 0$ such that for all $x, y \in \mathbb{R}$, $y \ge x$,

$$\begin{aligned} &0 \le f(s,y) - f(s,x) \le q(y-x) \\ &0 \le k(s,x) - k(s,y) \le q(y-x) \\ &0 \le h(s,y) - h(s,x) \le q(y-x). \end{aligned}$$

(v) We suppose that

$$3q \sup_{t\in[0,T]} \int_0^T S(t,s)\mathrm{d}s < 1.$$

(vi) There exist continuous functions $\alpha, \beta, \gamma : [0, T] \to \mathbb{R}$ such that

$$\begin{aligned} \alpha(t) &\le p(t) + \int_0^T S(t,s)[f(s,\alpha(s)) + k(s,\beta(s)) + h(s,\gamma(s))]\mathrm{d}s \\ \beta(t) &\ge p(t) + \int_0^T S(t,s)[f(s,\beta(s)) + k(s,\alpha(s)) + h(s,\beta(s))]\mathrm{d}s \\ \gamma(t) &\le p(t) + \int_0^T S(t,s)[f(s,\gamma(s)) + k(s,\beta(s)) + h(s,\alpha(s))]\mathrm{d}s. \end{aligned}$$

We consider the space $X = C([0,T],\mathbb{R})$ of continuous functions defined on $[0,T]$ endowed with the (G-complete) G-metric given by

$$G(u,v,w)=\max_{t\in[0,T]}|u(t)-v(t)|+\max_{t\in[0,T]}|u(t)-w(t)|+\max_{t\in[0,T]}|v(t)-w(t)|\quad \text{for all } u,v,w\in X.$$

We endow X with the partial ordered $\leq$ given by: $x, y \in X$, $x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in [0, T]$.

On the other hand, we may adjust as in [37] to prove that $(X, G, \leq)$ is regular.

Our result is the following.

Theorem 25. *Under assumptions* (i)-(vi), *the system (48) has a solution in* $X^3 = (C([0, T], \mathbb{R}))^3$.

Proof. We consider the operators $F : X^3 \to X$ and $g : X \to X$ defined by

$$F(x_1,x_2,x_3)(t)=p(t)+\int_0^T S(t,s)[f(s,x_1(s))+k(s,x_2(s))+h(s,x_3(s))]ds,\quad g(x)=x\quad t\in[0,T],$$

for all x_1, x_2, x_3, $x \in X$.

First, we will prove that F has the mixed monotone property (since g is the identity on X).

In fact, for $x_1 \leq y_1$ and $t \in [0, T]$, we have

$$F(y_1,x_2,x_3)(t)-F(x_1,x_2,x_3)(t)=\int_0^T S(t,s)[f(s,y_1(s))-f(s,x_1(s))]\mathrm{d}s.$$

Taking into account that $x_1(t) \leq y_1(t)$ for all $t \in [0, T]$, so by (iv), $f(s, y_1(s)) \geq f(s, x_1(s))$. Then $F(y_1, x_2, x_3)(t) \geq F(x_1, x_2, x_3)(t)$ for all $t \in [0,T]$,i.e.,

$$F(x_1,x_2,x_3)\preceq F(y_1,x_2,x_3).$$

Similarly, for $x_2 \leq y_2$ and $t \in [0, T]$, we have

$$F(x_1,x_2,x_3)(t)-F(x_1,y_2,x_3)(t)=\int_0^T S(t,s)[k(s,x_2(s))-k(s,y_2(s))]\mathrm{d}s.$$

Having $x_2(t) \leq y_2(t)$, so by (iv), $k(s, x_2(s)) \geq k(s, y_2(s))$. Then $F(x_1, x_2, x_3)(t) \geq F(x_1, y_2, x_3)(t)$ for all $t \in [0,T]$, i.e.,

$$F(x_1,x_2,x_3)\succeq F(x_1,y_2,x_3).$$

Now, for $x_3 \leq y_3$ and $t \in [0, T]$, we have

$$F(x_1,x_2,x_3)(t)-F(x_1,y_2,x_3)(t)=\int_0^T S(t,s)[k(s,x_2(s))-k(s,y_2(s))]\mathrm{d}s.$$

Taking into account that $x_3(t) \leq y_3(t)$ for all $t \in [0, T]$, so by (iv), $h(s, x_3(s)) \geq h(s, y_3(s))$. Then $F(x_1, x_2, x_3)(t) \geq F(x_1, x_2, y_3)(t)$ for all $t \in [0, T]$, i.e.,

$$F(x_1,x_2,x_3)\succeq F(x_1,x_2,y_3).$$

Therefore, F has the mixed monotone property.

In what follows we estimate the quantity $G(F(x, y, z), F(a, b, c), F(u, v, w))$ for all $x, y, z, a, b, c, u, v, w \in X$, with $x \geq a \geq u$, $y \leq b \leq v$ *and* $z \geq c \geq w$. Since F has the mixed monotone property, we have

$$F(u,v,w) \preceq F(a,b,c) \preceq F(x,y,z).$$

We obtain

$$\begin{aligned}
&G(F(x,y,z), F(a,b,c), F(u,v,w)) \\
&\quad = \max_{t\in[0,T]} \left|F(x,y,z)(t) - F(a,b,c)(t)\right| + \max_{t\in[0,T]} \left|F(x,y,z)(t) - F(u,v,z)(t)\right| \\
&\qquad + \max_{t\in[0,T]} \left|F(u,v,w)(t) - F(a,b,c)(t)\right| \\
&\quad = \max_{t\in[0,T]} (F(x,y,z)(t) - F(a,b,c)(t)) + \max_{t\in[0,T]} (F(x,y,z)(t) - F(u,v,z)(t)) \\
&\qquad + \max_{t\in[0,T]} (F(a,b,c)(t) - F(u,v,w)(t)).
\end{aligned}$$

Note that for all $t \in [0, T]$, from (iv), we have

$$\begin{aligned}
F(x,y,z)(t) - F(a,b,c) &= \int_0^T S(t,s)[f(s,x(s)) - f(s,a(s))]\mathrm{d}s \\
&\quad + \int_0^T S(t,s)[k(s,y(s)) - k(s,b(s))]\mathrm{d}s \\
&\quad + \int_0^T S(t,s)[h(s,z(s)) - h(s,c(s))] \\
&\le q\left[\max_{s\in[0,T]} \left|x(s) - a(s)\right| + \max_{s\in[0,T]} \left|y(s) - b(s)\right|\right. \\
&\quad \left. + \max_{s\in[0,T]} \left|z(s) - c(s)\right|\right] \left(\int_0^T S(t,s)\mathrm{d}s\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\max_{t\in[0,T]} (F(x,y,z)(t) - F(a,b,c)(t)) \\
&\quad \le q\left[\max_{s\in[0,T]} \left|x(s) - a(s)\right| + \max_{s\in[0,T]} \left|y(s) - b(s)\right| + \max_{s\in[0,T]} \left|z(s) - c(s)\right|\right] \left(\sup_{t\in[0,T]} \int_0^T S(t,s)\mathrm{d}s\right).
\end{aligned} \tag{49}$$

Repeating this idea, we may get using definition of the the G-metric G

$$\begin{aligned}
&\max_{t\in[0,T]} (F(x,y,z)(t) - F(a,b,c)(t) + \max_{t\in[0,T]} (F(x,y,z)(t) - F(u,v,z)(t)) \\
&\quad + \max_{t\in[0,T]} (F(a,b,c)(t) - F(u,v,w)(t) \\
&\qquad \le q[G(x,a,u) + G(y,b,v) + G(z,c,w)] \left(\sup_{t\in[0,T]} \int_0^T S(t,s)\mathrm{d}s\right) \\
&\qquad \le 3q\left(\sup_{t\in[0,T]} \int_0^T S(t,s)\mathrm{d}s\right) \max\{G(x,a,u), G(y,b,v), G(z,c,w)\}.
\end{aligned}$$

From (v), we have $3q(\sup_{t\in[0,T]} \int_0^T S(t,s)ds) < 1$. This proves that the operator F satisfies the contractive condition appearing in Corollary 20.

Let α, β, γ be the functions appearing in assumption (vi), then by (vi), we get

$$\alpha \leq F(\alpha,\beta,\gamma), \quad \beta \geq F(\beta,\alpha,\beta), \quad \gamma \leq F(\gamma,\beta,\alpha).$$

Applying Corollary 20, we deduce the existence of x_1, x_2, $x_3 \in X$ such that

$$x_1 = F(x_1,x_2,x_3), \quad x_2 = F(x_2,x_1,x_2), \quad x_3 = F(x_3,x_2,x_1),$$

i.e., (x_1, x_2, x_3) is a solution of the system (48). □

Examples

In this section, we state two examples to support the usability of our results. Before we present our first example we worth to mention the following remark.

Remark **26**. *All our results still valid if* $(u, v, w) = (a, b, c)$.

Example **27**. *Let* $X = [0, 1]$ *with usual order. Define* $G : X \times X \times X \to X$ *by*

$$G(x,y,z) = \max\left\{|x-y|, |x-z|, |y-z|\right\}.$$

Define F: $X \times X \times X \to X$ *by*

$$F(x,y,z) = \begin{cases} 0, & y \geq \min\{x,z\}; \\ \frac{z-y}{4}, & x \geq z \geq y; \\ \frac{x-y}{4}, & z \geq x \geq y. \end{cases}$$

Also, define ψ, $\varphi : [0, +\infty) \to [0, +\infty)$ *by* $\psi(t) = t$ *and* $\phi(t) = \frac{1}{2}t$. *Then:*

a. $(X, G, \leq)$ *is a G-complete regular G-metric space.*

b. *For* $x, y, z, u, v, w \in X$ *with* $x \geq u \geq u$, $y \leq v \leq v$ *and* $z \geq w \geq w$, *we have*

$$\psi(G(F(x,y,z),F(u,v,w),F(u,v,w))) \leq \psi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}) - \phi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}).$$

c. *F has the mixed monotone property.*

Proof. To prove (b), given $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$ and $z \geq w$. Then:

Case 1: $y > \min\{x, z\}$ and $v \geq \min\{u, w\}$. Here, we have

$$\begin{aligned}\psi(G(F(x,y,z),F(u,v,w),F(u,v,w))) &= 0 \\ &\leq \psi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}) \\ &\quad - \phi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}).\end{aligned}$$

Case 2: $y \geq \min\{x, z\}$ and $u \geq w \geq v$. Here, we have $y \leq v \leq w \leq u \leq x$ and $y \leq v \leq w \leq z$. Hence $y = v = w = u = x$ *or* $y = v = w = z$. Therefore

$$\begin{aligned}\psi(G(F(x,y,z),F(u,v,w),F(u,v,w))) &= 0 \\ &\leq \psi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}) \\ &\quad - \phi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}).\end{aligned}$$

Case 3: $y \geq \min\{x, z\}$ and $w \geq u \geq v$. Here, we have $y \leq v \leq u \leq w \leq z$ and $y \leq v \leq u \leq x$. Thus $y = v = u = w = z$ or $y = v = u = x$. Therefore

$$\begin{aligned}\psi(G(F(x,y,z),F(u,v,w),F(u,v,w))) &= 0\\ &\leq \psi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\})\\ &\quad-\phi(\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}).\end{aligned}$$

Case 4: $x \geq z \geq y$ and $v \geq \min\{u, w\}$.
Suppose $w \leq v$, then $w - y \leq v - y$ and hence

$$\begin{aligned}z-y = z-w+w-y \leq z-w+v-y &= G(z,w,w)+G(y,v,y)\\ &\leq 2\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.\end{aligned}$$

Then

$$\begin{aligned}G(F(x,y,z),F(u,v,w),F(u,v,w)) &= G\left(\frac{z-y}{4},0,0\right)=\frac{z-y}{4}\\ &\leq \frac{2}{4}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\ &= \max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\ &\quad-\frac{1}{2}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.\end{aligned}$$

Suppose $v < w$, then $u \leq v < w$ and hence $u \leq v \leq w \leq z \leq x$. So

$$\begin{aligned}z-y \leq x-y &= x-u+u-y\\ &\leq (x-u)+(v-y)=G(x,u,u)+G(v,y,y)\\ &\leq 2\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}G(F(x,y,z),F(u,v,w),F(u,v,w)) &= G\left(\frac{z-y}{4},0,0\right)=\frac{z-y}{4}\\ &\leq \frac{2}{4}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\ &= \max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\ &\quad-\frac{1}{2}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.\end{aligned}$$

Case 5: $z \geq x \geq y$ and $v \geq \min\{u, w\}$.
Suppose $u \leq v$, then $u - y \leq v - y$ and hence

$$\begin{aligned}x-y = x-u+u-y \leq (x-u)+(v-y) &= G(x,u,u)+G(v,y,y)\\ &\leq 2\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}G(F(x,y,z),F(u,v,w),F(u,v,w)) &= G\left(\frac{x-y}{4},0,0\right)\\ &= \frac{1}{4}(x-y)\\ &\leq \frac{2}{4}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\ &= \max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\ &\quad-\frac{1}{2}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.\end{aligned}$$

Suppose $v < u$, then $w \le v < u$ and hence $w \le v < u \le x \le z$. So

$$\begin{aligned} x - y \le z - y = z - w + w - y &\le (z - w) + (v - y) = d(z, w) + d(y, v) \\ &\le 2 \max\{d(x, u), d(y, v), d(z, w)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} G(F(x, y, z), F(u, v, w), F(u, v, w)) &= G\left(\frac{x - y}{4}, 0, 0\right) = \frac{1}{4}(x - y) \\ &\le \frac{2}{4} \max\{G(x, u, u), G(y, v, v), G(z, w, w)\} \\ &= \max\{G(x, u, u), G(y, v, v), G(z, w, w)\} \\ &- \frac{1}{2} \max\{G(x, u, u), G(y, v, v), G(z, w, w)\}. \end{aligned}$$

Case 6: $x \ge z \ge y$ and $u \ge w \ge v$. Here, we have

$$\begin{aligned} G(F(x, y, z), F(u, v, w), F(u, v, w)) &= G\left(\frac{z - y}{4}, \frac{w - v}{4}, \frac{w - v}{4}\right) \\ &= \frac{1}{4} \left|(z - w) + (v - y)\right| \\ &\le \frac{2}{4} \max\{G(x, u, u), G(y, v, v), G(z, w, w)\} \\ &= \max\{G(x, u, u), G(y, v, v), G(z, w, w)\} \\ &- \frac{1}{2} \max\{G(x, u, u), G(y, v, v), G(z, w, w)\}. \end{aligned}$$

Case 7: $x \ge z \ge y$ and $w \ge u \ge v$. Here, we have $y \le v \le u \le w \le z \le x$. Thus,

$$\begin{aligned} G(F(x, y, z), F(u, v, w), F(u, v, w)) &= G\left(\frac{z - y}{4}, \frac{u - v}{4}, \frac{u - v}{4}\right) \\ &= \frac{1}{4} \left|(z - u) + (v - y)\right| \\ &= \frac{1}{4} \left[(z - u) + (v - y)\right] \\ &= \frac{1}{4} \left[(x - u) + (v - y)\right] \\ &= \frac{1}{4}(G(x, u, u) + G(y, v, v)) \\ &\le \frac{2}{4} \max\{G(x, u, u), G(y, v, v), G(z, w, w)\} \\ &= \max\{G(x, u, u), G(y, v, v), G(z, w, w)\} \\ &- \frac{1}{2} \max\{G(x, u, u), G(y, v, v), G(z, w, w)\}. \end{aligned}$$

Case 8: $z \ge x \ge y$ and $u \ge w \ge v$. Here, we have $y \le v \le w \le u \le x \le z$. Therefore, we have

$$\begin{aligned}
&G(F(x,y,z),F(u,v,w),F(u,v,w))\\
= G\left(\frac{x-y}{4},\frac{w-v}{4},\frac{w-v}{4}\right) &= \frac{1}{4}\left|x-y-w+v\right|\\
&\le \frac{1}{4}\left(|x-w|+\left|v-y\right|\right)\\
&\le \frac{1}{4}(z-w+\left|v-y\right|)\\
&= \frac{1}{4}(G(z,w,w)+G(y,v,v))\\
&\le \frac{2}{4}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\
&= \max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\
&- \frac{1}{2}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.
\end{aligned}$$

Case 9: $z \ge x \ge y$, $w \ge u \ge v$. Here, we have $y \le v \le u \le w \le z$. Therefore, we have

$$\begin{aligned}
G(F(x,y,z),F(u,v,w),F(u,v,w)) &= G\left(\frac{x-y}{4},\frac{u-v}{4},\frac{u-v}{4}\right)\\
&= \frac{1}{4}\left|x-y-u+v\right|\\
&\le \frac{1}{4}(|x-u|+\left|v-y\right|)\\
&= \frac{1}{4}(G(x,u,u)+G(y,v,v))\\
&\le \frac{2}{4}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\
&= \max\{G(x,u,u),G(y,v,v),G(z,w,w)\}\\
&- \frac{1}{2}\max\{G(x,u,u),G(y,v,v),G(z,w,w)\}.
\end{aligned}$$

To prove (c), let $x, y, z \in X$. To show that $F(x, y, z)$ is monotone non-decreasing in x, let $x_1, x_2 \in X$ with $x_1 \le x_2$. If $y \ge \min\{x_1, z\}$, then $F(x_1, y, z) = 0 \le F(x_2, y, z)$.

If $y < \min\{x_1, z\}$, then

$$F(x_1,y,z) = \frac{\min(x_1,z)-y}{3} \le \frac{\min\{x_2,z\}-y}{3} = F(x_2,y,z).$$

Therefore, $F(x, y, z)$ is monotone non-decreasing in x. Similarly, we may show that $F(x, y, z)$ is monotone non-decreasing in z and monotone non-increasing in y. Thus, by Theorem 19 and Remark 26, F has a tripled fixed point. Here, (0, 0,0) is the unique tripled fixed point of F.

□

Now, we state our second example to support the usability of our results for non-symmetric G-metric spaces.

Example **28**. *Let* $X = \{0, 1, 2, 3,...\}$. *Define* $G : X \times X \times X \to X$ *by*

$$G(x,y,z) = \begin{cases} x+y+z, & \text{if } x,y,z \text{ are all distinct and different from zero;}\\ x+z, & \text{if } x=y\neq z \text{ and all are different from zero;}\\ y+z+1, & \text{if } x=0, y\neq z \text{ and } y,z \text{ are different from zero;}\\ y+2, & \text{if } x=0, z=y\neq 0;\\ z+1, & \text{if } x=0, y=0, z\neq 0;\\ 0, & \text{if } x=y=x,\end{cases}$$

F: $X \times X \times X \to X$ by

$$F(x, y, z) = \begin{cases} 0, & \text{if } x \leq 3 \vee z \leq 3; \\ 1, & \text{if } x \geq 4 \wedge z \geq 4, \end{cases}$$

and g: $X \to X$ by $gx = x^2$. Also, define ψ, φ: $[0, +\infty) \to [0, +\infty)$ by $\psi(t) = t^2$ and $\varphi(t) = t$. Then

a. *$(X, G, \leq)$ is a complete nonsymmetric G-metric space.*
b. *For $x, y, z, u, v, w \in X$ with $gx \geq gu \geq gu$, $gy \leq gv \leq gv$ and $gz \geq gw \geq gw$, we have*

$$\psi(G(F(x,y,z), F(u,v,w), F(u,v,w))) \leq \psi(\max\{G(gx,gu,gu), G(gy,gv,gv), G(gz,gw,gw)\}) - \phi(\max\{G(gx,gu,gu), G(gy,gv,gv), G(gz,gw,gw)\}).$$

c. *F has the mixed g-monotone property.*
d. *$F(X \times X \times X) \subseteq gX$.*
e. *$(X, G, \leq)$ is regular.*

Proof. For (a) see Example 3.5 of Choudhury and Maity [17]. To prove (b), given x, y, z, u, v, $w \in X$ with $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$. Then: □

Case 1: $(x \leq 3 \vee z \leq 3) \wedge (u \leq 3 \vee w \leq 3)$. *Here, we have $F(x, y, z) = 0$ and $F(u, v, w) = 0$. Thus*

$$\begin{aligned}\psi(G(F(x,y,z), F(u,v,w), F(u,v,w))) &= 0 \\ &\leq \psi(\max\{G(gx,gu,gu), G(gy,gv,gv), G(gz,gw,gw)\}) \\ &\quad - \phi(\max\{G(gx,gu,gu), G(gy,gv,gv), G(gz,gw,gw)\}).\end{aligned}$$

Case 2: $(x \leq 3 \vee z \leq 3) \wedge (u \geq 4 \wedge w \geq 4)$. *Here, $x < u$ or $z < w$ which is impossible because $gx \geq gu$ and $gz \geq gw$.*

Case 3: $(x \geq 4 \wedge z \geq 4) \wedge (u \leq 3 \vee w \leq 3)$. *Here, we have $F(x, y, z) = 1$ and $F(u, v, w) = 0$. Thus*

$$\psi(G(F(x,y,z), F(u,v,w), F(u,v,w)) = \psi(G(1,0,0)) = \psi(2) = 4.$$

Also,

$$G(gx, gu, gu) = G(x^2, u^2, u^2) = \begin{cases} x^2 + 1, & \text{if } u = 0; \\ x^2 + u^2, & \text{if } u \neq 0. \end{cases}$$

In both cases, we have

$$3 \leq G(x^2, u^2, u^2) \leq \max\{G(x^2,u^2,u^2), G(y^2,v^2,v^2), G(z^2,w^2,w^2)\}.$$

Using the fact that if $a, b \in \mathbb{N}$ with $a \leq b$, then $a^2 - a \leq b^2 - b$, we deduce that

$$3^2-3 = 6 \leq (\max\{G(x^2,u^2,u^2), G(y^2,v^2,v^2), G(z^2,w^2,w^2)\})^2 - \max\{G(x^2,u^2,u^2), G(y^2,v^2,v^2), G(z^2,w^2,w^2)\}.$$

Therefore,

$$\begin{aligned}\psi(G(F(x,y,z), F(u,v,w), F(u,v,w))) &= 4 \leq 6 \\ &\leq \psi(\max\{G(x^2,u^2,u^2), G(y^2,v^2,v^2), G(z^2,w^2,w^2)\}) \\ &\quad - \phi(\max\{G(x^2,u^2,u^2), G(y^2,v^2,v^2), G(z^2,w^2,w^2)\}). \\ &= \psi(\max\{G(gx,gu,gu), G(gy,gv,gv), G(gz,gw,gw)\}) \\ &\quad - \phi(\max\{G(gx,gu,gu), G(gy,gv,gv), G(gz,gw,gw)\}).\end{aligned}$$

Case 4: $(x \geq 3 \wedge z \geq 3) \wedge (u \geq 3 \wedge w \geq 3)$. *Here, we have* $F(x, y, z) = 1$ *and* $F(u, v, w) = 1$. *Thus,*

$$\begin{aligned}\psi(G(F(x,y,z),F(u,v,w),F(u,v,w))) &= 0\\ &\leq \psi(\max\{G(gx,gu,gu),G(gy,gv,gv),G(gz,gw,gw)\})\\ &\quad - \phi(\max\{G(gx,gu,gu),G(gy,gv,gv),G(gz,gw,gw)\}).\end{aligned}$$

The proof of (c) and (d) are easy. To prove (e), let $\{x_n\}$ *be a non-decreasing sequences in X such that* $\{x_n\}$ *G-converges to x. Then* $G(x_n, x, x) \to 0$. *By definition of G, we conclude that* $x_n = x$ *for all n except finitely many. Thus* $x_n \leq x$ *for all* $n \in \mathbb{N}$. *Similarly, we show that if* $\{y_n\}$ *is a non-increasing sequence in X such that* $\{y_n\}$ *G-converges to y, then* $y_n \geq y$ *for all* $n \in \mathbb{N}$. *Thus,* $(X, G, \leq)$ *is regular.*

By Theorem 19 and Remark 26, F and g have a tripled coincidence point in X. Here, $(0, 0, 0)$ *is the tripled coincidence point of F and g.*

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Authors' contributions
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