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TRIVARIATE DENSITY OF BROWNIAN MOTION, ITS LOCAL AND OCCUPATION TIMES, WITH APPLICATION TO STOCHASTIC CONTROL

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We compute the joint density of Brownian motion, its local time at the origin, and its occupation time of $[0, \infty)$. Two derivations of the main result are offered; one is computational, whereas the other uses some of the deep properties of Brownian local time. We use the result to compute the transition probabilities of the optimal process in a stochastic control problem.

1. Introduction. We consider a Brownian motion process $\{W_t, \mathscr{F}_t; t \geq 0\}$, its local time at the origin.

$$
\begin{aligned}
L_t &:= \lim_{\varepsilon\downarrow 0}(4\varepsilon)^{-1}\mathrm{meas}\{0 \le s \le t\colon -\varepsilon < W_s < \varepsilon\} \\
&= \lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1}\mathrm{meas}\{0 \le s \le t\colon 0 \le W_s < \varepsilon\}, \quad t \ge 0,
\end{aligned}
\tag{1.1}
$$

and the occupation time $\Gamma_t = \mathrm{meas}\{0 \le s \le t\colon W_s \ge 0\}$, $t \ge 0$, of the positive half-line. For fixed $t > 0$, we shall compute the joint density of the triple (W_t, L_t, Γ_t).

Our principal result provides the desired density as

$$
P_0\{W_t \in da,\ L_t \in db,\ \Gamma_t \in d\tau\}
$$

$$
= \begin{cases}
\dfrac{b(b-a)}{\pi\tau^{3/2}(t-\tau)^{3/2}} \exp\left[-\dfrac{b^2}{2\tau} - \dfrac{(b-a)^2}{2(t-\tau)}\right] da\,db\,d\tau, \\
\qquad\qquad a < 0, \quad b > 0, \quad 0 < \tau < t, \\
\dfrac{b(b+a)}{\pi\tau^{3/2}(t-\tau)^{3/2}} \exp\left[-\dfrac{b^2}{2(t-\tau)} - \dfrac{(b+a)^2}{2\tau}\right] da\,db\,d\tau, \\
\qquad\qquad a > 0, \quad b > 0, \quad 0 < \tau < t.
\end{cases}
\tag{1.2}
$$

In Section 2 we present a simple but heavily computational derivation of this result based on the Feynman-Kac formula, and in Section 3 we sketch a probabilistic approach based on a formula of D. Williams (1969). The second approach serves as a good illustration of the philosophy "that calculations *can* be done—and very effectively at that—via the modern 'abstract' theory." (Williams, 1979).

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We note that when $a < 0$, (1.2) agrees with the joint density of $(W_t, \max_{0 \le s \le t} W_s, \arg\max_{0 \le s \le t} W_s)$ (Lévy, 1948, Shepp, 1979). This can be explained by a path decomposition argument which is the subject of a forthcoming paper.

In Section 4 we generalize (1.2) to include the case of nonzero initial condition. In Section 5 we use these results and the Girsanov transformation to compute the transition probabilities of a Brownian motion whose drift switches between two values as the process crosses a threshold. Such a diffusion arises as the optimal state trajectory in a control problem treated by Beneš, Shepp and Witsenhausen (1980).

The remainder of this section is devoted to certain remarks and corollaries regarding the marginal distributions obtainable from (1.2). First, let us note that it suffices to compute the joint density in (1.2) for $a > 0$ (or $a < 0$). This follows from the observation that, with

$$\Gamma_t^- = t - \Gamma_t = \text{meas}\{0 \le s \le t\colon W_s < 0\}, \quad t \ge 0,$$

the triples (W_t, L_t, Γ_t) and $(-W_t, L_t, \Gamma_t^-)$ are equivalent in law.

To simplify notation, we introduce

$$\ell_t = 2L_t = \lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \text{meas}\{0 \le s \le t\colon |W_s| < \varepsilon\}, \quad t \ge 0, \tag{1.3}$$

which is the local time at zero of the reflected Brownian motion $\{|W_t|; t \ge 0\}$. Integrating out τ in (1.2), we obtain the joint distribution of Brownian motion and the local time ℓ_t at zero,

$$P_0\{W_t \in da, \ell_t \in db\} = \frac{|a| + b}{\sqrt{2\pi t^3}} \exp\left[-\frac{(|a| + b)^2}{2t}\right] da\, db, \quad a \in R, \quad b > 0, \tag{1.4}$$

whence the joint distribution of reflected Brownian motion and its local time at zero (Itô and McKean, 1974, page 45)

$$P_0\{|W_t| \in da, \ell_t \in db\} = \frac{2(a + b)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(a + b)^2}{2t}\right] da\, db, \quad a > 0, \quad b > 0. \tag{1.5}$$

Integrating out a and b in (1.2), we recover P. Lévy's second arc-sine law (Lévy, 1965, Itô and McKean, 1974), which says that the density of the Brownian occupation time is

$$P_0\{\Gamma_t \in d\tau\} = \frac{d\tau}{\pi\sqrt{\tau(t - \tau)}}, \quad 0 < \tau < t. \tag{1.6}$$

The joint density of local and occupation times turns out to be

$$P_0\{\ell_t \in db,\, \Gamma_t \in d\tau\} = \frac{bt\exp(-tb^2/8\tau(t-\tau))}{4\pi\tau^{3/2}(t-\tau)^{3/2}}\, db\, d\tau, \quad b > 0, \quad 0 < \tau < t, \tag{1.7}$$

a result obtained independently by Perkins (1982, Theorem 10c) using downcrossings.

2. Analytic derivation of the trivariate density.

We begin by computing a Laplace transform related to the desired density.

LEMMA 2.1. *Let α, β, γ and a be positive. Then*

$$E_0 \int_0^\infty 1_{[a,\infty)}(W_t)\exp[-\alpha t - \beta\Gamma_t - \gamma\ell_t]\, dt = \frac{2\exp(-a\sqrt{2(\alpha+\beta)})}{\sqrt{2(\alpha+\beta)}(2\gamma + \sqrt{2\alpha} + \sqrt{2(\alpha+\beta)})}. \tag{2.1}$$

PROOF. Define, for each $x \in \mathbb{R}$,

$$u(x) = E_x \int_0^\infty 1_{[a,\infty)}(W_t)\exp[-\alpha t - \beta \operatorname{meas}\{0 \le s \le t\colon W_s \ge 0\} - \gamma\ell_t]\, dt.$$

According to the Feynman-Kac formula for elastic Brownian motion (Itô and McKean, 1974, Section 2.3 and Knight, 1981, Theorem 7.4.3), u is bounded and continuous, C^1 on $\mathbb{R}\backslash\{0\}$, C^2 on $\mathbb{R}\backslash\{0, a\}$, and satisfies

$$(\alpha + \beta 1_{[0,\infty)}(x))u(x) = \tfrac{1}{2}u''(x) + 1_{[a,\infty)}(x), \quad x \in \mathbb{R}\backslash\{0, a\},$$

$$u'(0+) - u'(0-) = 2\gamma u(0).$$

Thus, u has the form

$$u(x) = \begin{cases} A\exp(x\sqrt{2\alpha}), & x \le 0 \\ B\exp(x\sqrt{2(\alpha+\beta)}) + C\exp(-x\sqrt{2(\alpha+\beta)}), & 0 \le x \le a, \\ D\exp(-(x-a)\sqrt{2(\alpha+\beta)}) + \dfrac{1}{a+\beta}, & x \ge a, \end{cases}$$

where the constants A, B, C and D are determined by the above conditions. Solving for A, we obtain (2.1). □

The expression in (2.1) is the Laplace transform of the density (if it exists) of the pair (Γ_t, ℓ_t) on the event $\{W_t \geq a\}$. We claim this density is given by

$$P_0\{W_t \geq a;\ \ell_t \in db,\ \Gamma_t \in d\tau\}$$
$$(2.2)\qquad = \frac{(b/2)}{2\pi(t-\tau)^{3/2}\tau^{1/2}} \exp\left[-\frac{(b/2)^2}{2(t-\tau)} - \frac{(a+b/2)^2}{2\tau}\right] db\, d\tau,$$
$$a, b > 0, \quad 0 < \tau < t.$$

The verification of this claim, given in Lemma 2.2, consists of showing that this function has the Laplace transform computed in Lemma 2.1.

LEMMA 2.2. *We have*

$$\int_0^\infty \int_0^t \int_0^\infty e^{-\alpha t - \beta\tau - \gamma b}\left[\frac{(b/2)}{2\pi(t-\tau)^{3/2}\tau^{1/2}}\right]$$
$$(2.3)\qquad \cdot \exp\left[-\frac{(b/2)^2}{2(t-\tau)} - \frac{(a+b/2)^2}{2\tau}\right] db\, d\tau\, dt$$
$$= \frac{2\exp(-a\sqrt{2(\alpha+\beta)})}{\sqrt{2(\alpha+\beta)}(2\gamma + \sqrt{2\alpha} + \sqrt{2(\alpha+\beta)})}, \quad a > 0.$$

PROOF. We recall the Laplace transforms

$$(2.4)\qquad \int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{2\pi t}}\exp(-x^2/2t)\, dt = \frac{1}{\sqrt{2\lambda}}\exp(-|x|\sqrt{2\lambda}), \quad \lambda > 0,$$

$$(2.5)\qquad \int_0^\infty e^{-\lambda t}\frac{|y|}{\sqrt{2\pi t^3}}\exp(-y^2/2t)\, dt = \exp(-|y|\sqrt{2\lambda}), \quad \lambda > 0.$$

Equation (2.5) implies that for $\tau > 0$,

$$(2.6)\qquad \int_\tau^\infty e^{-\alpha t}\frac{y}{\sqrt{2\pi(t-\tau)^3}}\exp\left[-\frac{y^2}{2(t-\tau)}\right] dt$$
$$= \exp(-\alpha\tau - y\sqrt{2\alpha}); \quad \alpha, y > 0.$$

Reversing the order of integration, we can now write the left-hand side of (2.3) as

$$\int_0^\infty \int_0^\infty \int_\tau^\infty e^{-\alpha t - \beta\tau - \gamma b}\left[\frac{(b/2)}{2\pi(t-\tau)^{3/2}\tau^{1/2}}\right]\exp\left[-\frac{(b/2)^2}{2(t-\tau)} - \frac{(a+b/2)^2}{2\tau}\right] dt\, d\tau\, db$$
$$= \int_0^\infty \int_0^\infty e^{-(\alpha+\beta)\tau - \gamma b}\frac{1}{\sqrt{2\pi\tau}}\exp\left[-\frac{(a+b/2)^2}{2\tau} - \left(\frac{b}{2}\right)\sqrt{2\alpha}\right] d\tau\, db$$
$$= \int_0^\infty e^{-\gamma b}\frac{1}{\sqrt{2(\alpha+\beta)}}\exp\left[-\left(a+\frac{b}{2}\right)\sqrt{2(\alpha+\beta)} - \left(\frac{b}{2}\right)\sqrt{2\alpha}\right] db$$
$$= \frac{2\exp(-a\sqrt{2(\alpha+\beta)})}{\sqrt{2(\alpha+\beta)}(2\gamma + \sqrt{2\alpha} + \sqrt{2(\alpha+\beta)})}. \quad \square$$

PROPOSITION 2.3. *The joint density of* (W_t, L_t, Γ_t) *is given by* (1.2).

PROOF. Differentiating (2.2) with respect to a, we obtain

$$P_0\{W_t \in da, \ell_t \in db, \Gamma_t \in d\tau\}$$
$$= \frac{(b/2)(a+b/2)}{2\pi(t-\tau)^{3/2}\tau^{3/2}} \exp\left[-\frac{(b/2)^2}{2(t-\tau)} - \frac{(a+b/2)^2}{2\tau}\right] da\, db\, d\tau,$$
$$a > 0, \quad b > 0, \quad 0 < \tau < t.$$

Replacing ℓ_t by $2L_t$, $b/2$ by b, and $db/2$ by db, we obtain (1.2) for $a > 0$. For $a < 0$, we use symmetry as discussed in Section 1 to derive (1.2). □

3. Probabilistic derivation of the triviariate density. In this section we sketch a probabilistic derivation of (1.2). The details of the proof appear in Karatzas and Shreve (1983).

Let us consider the inverse occupation time

$$\Gamma^{-1}(\tau) := \inf\{t \geq 0; \Gamma_t = \tau\}, \quad t \geq 0. \tag{3.1}$$

We recall from Ikeda and Watanabe (1981), pages 122–123, that the process $W^*(\tau) := W(\Gamma^{-1}(\tau))$ is a reflected Brownian motion with local time $L^*(\tau) := L(\Gamma^{-1}(\tau))$. According to a formula of D. Williams (1969) (see also McKean, 1975, page 103),

$$E_0\{\exp(-\lambda\Gamma^{-1}(\tau)) \mid W^*(\sigma), 0 \leq \sigma \leq \tau\} = \exp(-\lambda\tau - \sqrt{2\lambda}L^*(\tau)), \quad \lambda > 0. \tag{3.2}$$

We have the joint density of $(W^*(\tau), L^*(\tau))$ from (1.5), so we are empowered to compute the density

$$P_0\{W^*(\tau) \in da, L^*(\tau) \in db, \Gamma^{-1}(\tau) \in dt\}$$
$$= \frac{b(a+b)}{\pi\tau^{3/2}(t-\tau)^{3/2}} \exp\left[-\frac{b^2}{2(t-\tau)} - \frac{(a+b)^2}{2\tau}\right] da\, db\, dt, \tag{3.3}$$
$$a > 0, \quad b > 0, \quad t > \tau.$$

To complete the derivation of (1.2) for $a > 0$, it suffices to prove the plausible equation

$$\frac{1}{dt} P_0\{W^*(\tau) \in da, L^*(\tau) \in db, \Gamma^{-1}(\tau) \in dt\}$$
$$= \frac{1}{d\tau} P_0\{W(t) \in da, L(t) \in db, \Gamma(t) \in d\tau\}, \tag{3.4}$$
$$a > 0, \quad b > 0, \quad 0 < \tau < t.$$

4. The trivariate density with nonzero initial condition. In this section we compute $P_x\{W_t \in dz, L_t \in dy, \Gamma_t \in d\tau\}$ from (1.2) for $x \neq 0$. We shall use the resulting formula to obtain the transition density in Section 5.

Define the passage time

$$T_\mu := \inf\{t \geq 0: \mu t + W_t = 0\}, \tag{4.1}$$

and set (cf. Karlin and Taylor, 1975)

$$h(s;\, x,\, \mu)\, ds := P_x\{T_\mu \in ds\}$$

$$= \frac{|x|}{\sqrt{2\pi s^3}} \exp\left[-\frac{(x+\mu s)^2}{2s}\right] ds; \quad s > 0, \quad x,\, \mu \in \mathbb{R}. \tag{4.2}$$

Because the sum of independent passage times is a passage time, we have

$$h(\cdot;\, x_1 + x_2,\, \mu) = h(\cdot;\, x_1,\, \mu) * h(\cdot;\, x_2,\, \mu); \quad x_1 x_2 > 0, \quad \mu \in \mathbb{R}. \tag{4.3}$$

In terms of h, (1.2) can be written as

$$P_0\{W_t \in da,\, L_t \in db,\, \Gamma_t \in d\tau\}$$

$$= \begin{cases} 2h(\tau;\, b,\, 0)h(t-\tau;\, b-a,\, 0), & a < 0, \quad b > 0, \quad 0 < \tau < t. \\ 2h(t-\tau;\, b,\, 0)h(\tau;\, b+a,\, 0), & a > 0, \quad b > 0, \quad 0 < \tau < t. \end{cases} \tag{4.4}$$

The strong Markov property implies that for $x \geq 0$, $a < 0$,

$$\begin{aligned} &P_x\{W_t \in da,\, L_t \in db,\, \Gamma_t \in d\tau\} \\ &\quad = P_x\{W_t \in da,\, L_t \in db,\, \Gamma_t \in d\tau,\, T_0 \leq \tau\} \\ &\quad = \int_0^\tau P_x\{W_t \in da,\, L_t \in db,\, \Gamma_t \in d\tau \mid T_0 = s\} P_x\{T_0 \in ds\} \\ &\quad = \int_0^\tau P_0\{W_{t-s} \in da,\, L_{t-s} \in db,\, \Gamma_{t-s} \in d\tau - s\} P_x\{T_0 \in ds\} \\ &\quad = 2\int_0^\tau h(\tau - s;\, b,\, 0)h(t-\tau;\, b-a,\, 0)h(s;\, x,\, 0)\, ds\, da\, db\, d\tau. \\ &\quad = 2h(\tau;\, b+x,\, 0)h(t-\tau;\, b-a,\, 0)\, da\, db\, d\tau; \\ &\qquad\qquad x \geq 0, \quad a < 0, \quad b > 0, \quad 0 < \tau < t, \end{aligned} \tag{4.5}$$

where the last equality uses (4.3). A similar computation yields

$$\begin{aligned} &P_x\{W_t \in da,\, L_t \in db,\, \Gamma_t \in d\tau\} \\ &\quad = 2h(t-\tau;\, b,\, 0)h(\tau;\, b+a+x,\, 0)\, da\, db\, d\tau; \\ &\qquad\qquad x \geq 0, \quad a > 0, \quad b > 0, \quad 0 < \tau < t. \end{aligned} \tag{4.6}$$

In the latter case, we also have

$$\begin{aligned} &P_x\{W_t \in da,\, L_t = 0,\, \Gamma_t = t\} = P_x\{W_t \in da,\, T_0 \geq t\} \\ &\quad = \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(a-x)^2}{2t}\right) - \exp\left(-\frac{(a+x)^2}{2t}\right)\right] da, \quad x \geq 0, \quad a > 0. \end{aligned} \tag{4.7}$$

Equations (4.5)–(4.7) characterize the distribution of (W_t, L_t, Γ_t) under P_x.

5. The optimal transition density in a stochastic control problem. Beneš, Shepp and Witsenhausen (1980) have treated the following stochastic control problem: to choose a nonanticipative control process u_t which minimizes the expected discounted cost

$$E_x \int_0^\infty e^{-\alpha t} \xi_t^2 \, dt,$$

subject to

$$\xi_t = x + \int_0^t u_s \, ds + W_t, \quad t \geq 0, \qquad \theta_0 \leq u_t \leq \theta_1, \quad t \geq 0,$$

where $\{W_t, \mathscr{F}_t; t \geq 0\}$ is a Brownian motion on $(\Omega, \mathscr{F}, P)$. Beneš, et al. (1980) show that the optimal control law is given by $u_t^* = \theta(\xi_t)$, $t \geq 0$, where

$$\theta(x) = \begin{cases} \theta_1, & x < \delta, \\ \theta_0, & x \geq \delta, \end{cases} \tag{5.1}$$

and

$$\delta = (\sqrt{\theta_1^2 + 2\alpha} + \theta_1)^{-1} - (\sqrt{\theta_0^2 + 2\alpha} - \theta_0)^{-1}.$$

They also compute the optimal expected cost

$$v(x) = E_x \int_0^\infty e^{-\alpha t} X_t^2 \, dt$$

corresponding to the optimally controlled diffusion X, the solution of the stochastic differential equation with two-valued drift

$$dX_t = \theta(X_t) \, dt + dW_t, \quad X_0 = x, \tag{5.2}$$

and they calculate the Laplace transform of the transition density

$$p_t(x, z) \, dz = P_x\{X_t \in dz\}$$

in the case $\delta = 0$. In the computation of this transition density, the switching point δ need not be related to θ_0 and θ_1. The transition density for other values of δ is easily obtained from the expression corresponding to $\delta = 0$ by translation.

The purpose of this section is to use the joint distribution derived in the previous section to compute the transition density $p_t(x, z)$ above as explicitly as possible. We set $\delta = 0$ throughout.

In this section we set $\Omega = C[0, \infty)$, we take $W_t \colon \Omega \to R$ to be the evaluation mapping $W_t(\omega) = \omega(t)$, we take $\mathscr{F}_t$ to be the smallest σ-algebra which makes W_s measurable for $0 \leq s \leq t$, $\mathscr{F} := \vee_{t \geq 0} \mathscr{F}_t$, and we denote by P_x the Wiener measure on $(\Omega, \mathscr{F})$ for which $P_x\{W_0 = x\} = 1$. We write E_x to denote the expectation corresponding to P_x.

For $x \in R$, define

$$\zeta(s, t) = \exp\left[\int_s^t \theta(W_u) \, dW_u - \frac{1}{2} \int_s^t \theta^2(W_u) \, du\right] \tag{5.3}$$

and with $A \in \mathscr{F}_t$, consider the probability measure:

$$\hat{P}_x(A) = E_x\{1_A \zeta(0, t)\}. \tag{5.4}$$

This definition admits a unique extension to a measure $\hat{P}_x$ on $(\Omega, \mathscr{F})$, and, according to Girsanov's Theorem (Liptser and Shiryayev, 1977, Chapter 6), under $\hat{P}_x$ the process $\{W_t, \mathscr{F}_t; t \geq 0\}$ can be regarded as a solution to (5.2). In particular,

$$p_t(x, z) = \hat{P}_x\{W_t \in dz\} = E_x\{1_{\{W_t \in dz\}} \zeta(0, t)\}. \tag{5.5}$$

In order to compute with (5.5), we must find a convenient representation for $\zeta(0, t)$. To do this, we define

$$\Theta(z) = \int_0^z \theta(y)\, dy = \begin{cases} \theta_1 z, & z \leq 0, \\ \theta_0 z, & z \geq 0. \end{cases}$$

A formal application of Itô's lemma to $\Theta(W_t)$ results in

$$\Theta(W_t) - \Theta(W_0) = \int_0^t \theta(W_u)\, dW_u + (\theta_0 - \theta_1)L_t,$$

and this step can be rigorously justified by the argument used to prove Tanaka's formula (McKean (1969), Section 3.8, or Ikeda and Watanabe (1981), page 114). On the other hand,

$$\int_0^t \theta^2(W_u)\, du = \theta_1^2 t + (\theta_0^2 - \theta_1^2)\Gamma_t.$$

In light of these facts, we can rewrite (5.5) as

$$\begin{aligned} p_t(x, z)\, dz &= \exp[\Theta(z) - \Theta(x) - \tfrac{1}{2}\theta_1^2 t] \\ &\quad \cdot \int_0^\infty \int_0^t \exp\left[(\theta_1 - \theta_0)b + \frac{1}{2}(\theta_1^2 - \theta_0^2)\tau\right] \\ &\quad \cdot P_x\{W_t \in dz, L_t \in db, \Gamma_t \in d\tau\}, \end{aligned} \tag{5.6}$$

and we can use (4.5)–(4.7) to evaluate the right-hand side of (5.6). After some manipulation, this yields

$$p_t(x, z) = \begin{cases} 2\int_0^\infty \int_0^t e^{2\theta_1 b} h(t - \tau; b - z, \theta_1) h(\tau; x + b, \theta_0)\, d\tau\, db, \\ \qquad x \geq 0, \quad z \leq 0, \\ 2\int_0^\infty \int_0^t e^{2(\theta_1 b + \theta_0 z)} h(t - \tau; b, \theta_1) h(\tau; x + b + z, \theta_0)\, d\tau\, db \\ \qquad + \dfrac{1}{\sqrt{2\pi t}}\left\{\exp\left[-\dfrac{(x - z + \theta_0 t)^2}{2t}\right]\right. \\ \qquad\qquad \left. - \exp\left[-\dfrac{(x + z - \theta_0 t)^2}{2t} - 2\theta_0 x\right]\right\}, \\ \qquad x \geq 0, \quad z > 0. \end{cases} \tag{5.7}$$

Let us now invoke explicitly the dependence on θ_0 and θ_1 by writing $p_t(x, z; \theta_0, \theta_1)$ rather than $p_t(x, z)$. The symmetry of Brownian motion results in the relation

$$p_t(x, z; \theta_0, \theta_1) = p_t(-x, -z; -\theta_1, -\theta_0), \tag{5.8}$$

and so for $x \le 0$, the transition density can be obtained from (5.7) and (5.8). We summarize this discussion with a proposition.

PROPOSITION 5.1. *Let θ be given by* (5.1), *where δ is any real number. Let X_t be the solution to the stochastic differential equation*

$$dX_t = \theta(X_t)\,dt + dW_t$$

with initial condition $X_0 = x + \delta$. If $x \ge 0$, then $P_{x+\delta}\{X_t \in dz + \delta\}$; $z \in R$, $t > 0$, is given by (5.7). *If $x \le 0$, then $P_{x+\delta}\{X_t \in dz + \delta\}$; $z \in R$, $t > 0$ is given by the right-hand side of* (5.7) *with x, z, θ_0, θ_1 replaced by $-x$, $-z$, $-\theta_1$ and $-\theta_0$, respectively.*

SPECIAL CASE. $-\theta_0 = \theta_1 = \theta$, $\delta = 0$. In this case, the integral term in the second part of (5.7) becomes

$$\begin{aligned}
2\int_0^\infty \int_0^t & e^{2\theta(b-z)} h(t-\tau; b, \theta) h(\tau; x+b+z, -\theta)\,d\tau\,db \\
&= 2\int_0^\infty \int_0^t e^{-2\theta z} h(t-\tau; b, -\theta) h(\tau; x+b+z, -\theta)\,d\tau\,db \\
&= 2\int_0^\infty e^{-2\theta z} h(t; x+2b+z, -\theta)\,db \\
&= \frac{1}{\sqrt{2\pi t^3}} e^{-2\theta z} \int_{x+z}^\infty v \exp\left[-\frac{(v-\theta t)^2}{2t}\right] dv \\
&= \frac{1}{\sqrt{2\pi t^3}} e^{-2\theta x} \int_{x+z}^\infty (v-\theta t) \exp\left[-\frac{(v-\theta t)^2}{2t}\right] dv \\
&\quad + \frac{\theta}{\sqrt{2\pi t}} e^{-2\theta z} \int_{x+z}^\infty \exp\left[-\frac{(v-\theta t)^2}{2t}\right] dv \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x+z+\theta t)^2}{2t} - 2\theta x\right] \\
&\quad + \frac{\theta}{\sqrt{2\pi t}} e^{-2\theta z} \int_{x+z}^\infty \exp\left[-\frac{(v-\theta t)^2}{2t}\right] dv.
\end{aligned}$$

It follows from (5.7) and this calculation that in this special case,

$$\begin{aligned}
p_t(x, z) = \frac{1}{\sqrt{2\pi t}} \Bigg\{ & \exp\left[-\frac{(x-z-\theta t)^2}{2t}\right] \\
& + \theta e^{-2\theta z} \int_{x+z}^\infty \exp\left[-\frac{(v-\theta t)^2}{2t}\right] dv \Bigg\},
\end{aligned} \tag{5.9}$$

$$x \ge 0, \quad z > 0.$$

A similar calculation applied to the first integral in (5.7) reveals

$$p_t(x, z) = \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left[2\theta x - \frac{(x - z + \theta t)^2}{2t} \right] + \theta e^{2\theta z} \int_{x-z}^{\infty} \exp\left[-\frac{(v - \theta t)^2}{2t} \right] dv \right\}, \quad x \geq 0, \quad z \leq 0. \tag{5.10}$$

When $\theta = 1$, we recover the expressions obtained by Shreve (1981), pages 476–477.

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REFERENCES

BENEŠ, V. E., SHEPP, L. A. and WITSENHAUSEN, H. S. (1980). Some solvable stochastic control problems. *Stochastics* **4** 134–160.

IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* North Holland, Amsterdam.

ITÔ, K. and MCKEAN, H. P. (1974). *Diffusion Processes and their Sample Paths.* Springer, New York.

KARATZAS, I. and SHREVE, S. (1983). Trivariate density of Brownian motion, its local and occupation times, with application to stochastic control. Math. Dept. Research Report 83-6, Carnegie-Mellon University, Pittsburgh, Pennsylvania.

KARLIN, S. and TAYLOR, H. (1975). *A First Course in Stochastic Processes.* Academic, New York.

KNIGHT, F. B. (1981). *Essentials of Brownian Motion and Diffusion. Math. Surveys* **18.** American Mathematical Society, Providence, Rhode Island.

LÉVY, P. (1948). *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris.

LIPTSER, R. S. and SHIRYAYEV, A. N. (1977). *Statistics of Random Processes I.* Springer-Verlag, New York.

MCKEAN, H. P. (1969). *Stochastic Integrals.* Academic, New York.

MCKEAN, H. P. (1975). Brownian local times. *Adv. Math.* **15** 91–111.

PERKINS, E. (1982). Local time is a semi-martingale. *Z. Wahrsch. verw. Gebiete* **60** 79–117.

SHEPP, L. A. (1979). The joint density of the maximum and its location for a Wiener process with drift. *J. Appl. Probab.* **16** 423–427.

SHREVE, S. E. (1981). Reflected Brownian motion in the "bang-bang" control of Brownian drift. *SIAM J. Control and Optimization* **19** 469–478.

WILLIAMS, D. (1969). Markov properties of Brownian local time. *Bull. Amer. Math. Soc.* **75** 1035–1036.

WILLIAMS, D. (1979). *Diffusions, Markov Processes, and Martingales.* Wiley, New York.

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