

TROPICAL COUNTING FROM ASYMPTOTIC ANALYSIS ON MAURER-CARTAN EQUATIONS

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ABSTRACT. Let $X = X_\Sigma$ be a toric surface and let (\check{X}, W) be its Landau-Ginzburg (LG) mirror where W is the Hori-Vafa potential as shown in their preprint. We apply asymptotic analysis to study the extended deformation theory of the LG model (\check{X}, W) , and prove that semi-classical limits of Fourier modes of a specific class of Maurer-Cartan solutions naturally give rise to tropical disks in X with Maslov index 0 or 2, the latter of which produces a universal unfolding of W . For $X = \mathbb{P}^2$, our construction reproduces Gross' perturbed potential W_n [Adv. Math. 224 (2010), pp. 169–245] which was proven to be the universal unfolding of W written in canonical coordinates. We also explain how the extended deformation theory can be used to reinterpret the jumping phenomenon of W_n across walls of the scattering diagram formed by Maslov index 0 tropical disks originally observed by Gross in the same work (in the case of $X = \mathbb{P}^2$).

1. INTRODUCTION

1.1. Background. The study of mirror symmetry for toric varieties goes back to Batyrev [5], Givental [25–27], Lian-Liu-Yau [46], Kontsevich [40], and Hori-Vafa [38]. Unlike the Calabi-Yau case, the mirror of a compact toric manifold X is given by a *Landau-Ginzburg (abbrev LG) model* (\check{X}, W) consisting of a noncompact Kähler manifold \check{X} and a holomorphic function $W : \check{X} \rightarrow \mathbb{C}$ called the *potential* [54, 55]. As a prototypical example, the LG mirror of $X = \mathbb{P}^2$ is given by $\check{X} = \{(z^0, z^1, z^2) \in \mathbb{C}^3 \mid z^0 z^1 z^2 = 1\}$ together with the restriction of $W = z^0 + z^1 + z^2$ to $\check{X} \subset \mathbb{C}^3$.

At the genus 0 level, mirror symmetry can be understood as an isomorphism between Frobenius manifolds. For a large class of examples, the construction of the *B-model Frobenius manifold* from the LG model (\check{X}, W) was carried out by Douai-Sabbah [14, 15] (see also the book [51]), generalizing the classic work of K. Saito [52]. Mirror symmetry then says that the B-model Frobenius manifold of (\check{X}, W) is isomorphic (via a possibly nontrivial mirror map) to the *A-model Frobenius manifold* constructed from the genus 0 Gromov-Witten (abbrev. GW) theory or *big* quantum cohomology of X . In the case of projective spaces this was proved by Barannikov [3].

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The geometry of this mirror symmetry can be understood using the Strominger-Yau-Zaslow (abbrev. SYZ) conjecture [53]. Namely, a natural Lagrangian torus fibration is given by the moment map $p : X \rightarrow \mathbf{P}$, and the mirror manifold \check{X} can be constructed geometrically as the moduli space of A-branes (L, ∇) consisting of a Lagrangian torus fiber L of p and a flat $U(1)$ -connection ∇ over it, or simply, as the total space of the fiberwise dual of p restricted to the interior $\text{Int}(\mathbf{P}) \subset \mathbf{P}$ [1]. The SYZ conjecture also suggests that mirror symmetry is a geometric Fourier transform; in this regard, the construction of \check{X} from X may be viewed as the “0th Fourier mode” of the mirror geometry.

The “higher Fourier modes” or “quantum corrections” come from the singular or degenerated fibers of p over the boundary $\partial\mathbf{P}$ and are captured by holomorphic disks in X with boundary on a Lagrangian torus fiber of p – this gives rise to the mirror LG potential W . Cho-Oh [11] were the first to prove, in the toric Fano case, that the so-called *Hori-Vafa potential* W [38] can be expressed in terms of counts of Maslov index 2 holomorphic disks. This was later generalized by Fukaya-Oh-Ohta-Ono [20] to all compact toric manifolds. They defined the *Lagrangian Floer potential* W^{LF} which is determined by the *obstruction cochain* \mathfrak{m}_0 in the Floer complex of a Lagrangian torus fiber of the moment map p [18, 19].

In more explicit terms, coefficients of W^{LF} are virtual counts of Maslov index 2 *stable* disks, or more precisely, *genus 0 open Gromov-Witten invariants*, and W^{LF} is a perturbation of the Hori-Vafa potential W of the form

$$W^{\text{LF}} = W + \text{correction terms}$$

because coefficients of W only encode counts of *embedded* disks (which is why $W^{\text{LF}} = W$ only when X is toric Fano). In general it is very hard to compute W^{LF} , but explicit formulas are known in a few low-dimensional examples [2, 6, 22] and when X is semi-Fano [7, 8, 28].

Using W^{LF} , one obtains an isomorphism of Frobenius algebras

$$(1.1) \quad QH^*(X) \cong \text{Jac}(W^{\text{LF}})$$

between the *small* quantum cohomology ring of X and the Jacobian ring of W , *without* going through a mirror map (or one can say that the mirror map is trivialized). To upgrade this to an isomorphism between Frobenius manifolds, Fukaya-Oh-Ohta-Ono [21] introduced the *bulk-deformed potential* W_b^{LF} as a perturbation of W^{LF} by the ambient cycles in X . In [23] they proved that (1.1) can be enhanced to an isomorphism between the A-model Frobenius manifold of X and the B-model Frobenius manifold constructed from W_b^{LF} .

Not long afterward, Gross [29, 30] constructed a very explicit perturbation of the Hori-Vafa potential $W = z^0 + z^1 + z^2$ mirror to $X = \mathbb{P}^2$ using counts of Maslov index 2 *tropical disks* in \mathbb{R}^2 (or the tropical projective plane \mathbb{TP}^2). He computed oscillatory integrals of his perturbed potential, producing beautiful tropical formulas for descendent GW invariants and proving that it is the universal unfolding of W written in canonical coordinates, thereby giving a very transparent proof of mirror symmetry for \mathbb{P}^2 via tropical geometry.

Gross’ work is closely connected with the influential Gross-Siebert program [32–35], where a key role is played by a combinatorial gadget called a *scattering diagram* which was first introduced by Kontsevich-Soibelman in [42].

On the other hand, the precise correspondence between the counting of tropical and holomorphic curves has been studied in various cases, first by Mikhalkin

[48] in dimension 2, and later by Nishinou-Siebert [50] in higher dimensions. The correspondence between tropical and holomorphic disks in toric varieties was first investigated by Nishinou [49], and more recently, clarified and refined by Hong-Lin-Zhao [37] (in the case of toric surfaces). These works indicate that tropical geometry is indeed sufficient in describing GW theory.

1.2. Asymptotic behavior of Maurer-Cartan solutions. The main goal of this paper is to explain how extended deformation theory of the LG model (\check{X}, W) can lead us naturally to Gross' perturbed potential constructed in [29]. Our main tool is asymptotic analysis on Maurer-Cartan equations and we will build on the approach developed in [10]. In a broader sense, our results are about relations between tropical disk counting on (the tropical counterpart of) a toric surface X and the extended deformation theory of its LG mirror (\check{X}, W) , where W is taken to be the Hori-Vafa potential [38].

Recall that in [10], we considered the differential-geometric deformation theory of \check{X} governed by the Kodaira-Spencer dgLa $K S_{\check{X}}^* := \Omega^{0,*}(\check{X}, T^{1,0})$ and the associated Maurer-Cartan (abbrev. MC) equation

$$(1.2) \quad \bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0.$$

An $\hbar \in \mathbb{R}_+$ parameter was introduced there to twist the complex structure of \check{X} which geometrically corresponds to the shrinking of the torus fibers in X . Following a proposal put forward by Kontsevich-Soibelman [41] and Fukaya [17], we studied Fourier expansions of a specific class of solutions of the MC equation (1.2) along fibers of $\check{p}: \check{X} \rightarrow \text{Int}(\mathbf{P})$. The main results in [10] showed that leading order terms (or semi-classical limits) of the Fourier modes of such a solution naturally give rise to a consistent scattering diagram \mathcal{D} as $\hbar \rightarrow 0$, and conversely such solutions can be constructed as sums over trees of terms with support concentrated along the walls in \mathcal{D} .

In this paper, we consider the extended Kodaira-Spencer complex given by *polyvector fields*:

$$PV^{*,*}(\check{X}) := \Omega^{0,*}(\check{X}, \wedge^* T^{1,0}).$$

This is equipped with the Dolbeault differential $\bar{\partial}$, a naturally extended Lie bracket $[\cdot, \cdot]$ and, as \check{X} is Calabi-Yau, a BV operator Δ , constituting a *differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra*. This structure is the key ingredient in the construction of the B-model Frobenius manifold in the Calabi-Yau setting [3, 4, 44].

For an LG model (\check{X}, W) , the Dolbeault differential in the dgBV algebra $PV^{*,*}$ should be replaced by the *twisted Dolbeault differential* $\bar{\partial}_W := \bar{\partial} + [W, \cdot]$, and it is natural to consider the associated *extended Maurer-Cartan equation*:

$$(1.3) \quad \bar{\partial}_W \varphi + \frac{1}{2}[\varphi, \varphi] = 0$$

for $\varphi \in PV^{*,*}(\check{X})$ (see Section 3.1). We adapt this approach with a view towards a construction of the higher genus B-model [12, 45].

Except in Sections 3.2 and 3.3, our attention will be restricted to the 2-dimensional case, so X will just be a toric surface (implicitly equipped with the toric anticanonical divisor $D = D_\infty$). To analyze the equation (1.3) associated to the mirror LG model $(\check{X} \cong (\mathbb{C}^*)^2, W)$, where W is the Hori-Vafa potential, we apply

the machinery developed in [10]. More precisely, as we will only be concerned with the leading order behavior of the MC solutions as $\hbar \rightarrow 0$, the complex $PV^{*,*}$ will be replaced by the quotient $(\mathcal{G}/\mathcal{I})^{*,*}$ of a subalgebra $\mathcal{G}^{*,*} \leq PV^{*,*}$ consisting of terms with growth control as $\hbar \rightarrow 0$ by the ideal $\mathcal{I}^{*,*}$ generated by error terms in \hbar (see Definition 3.17). We shall also employ the important notion of *asymptotic support* first introduced in [10] (see Definition 3.8).

In the 2-dimensional case, it is natural to consider deformations using n points $P_1, \dots, P_n \in \mathbb{R}^2$ in generic position.¹ In view of this, we choose an input $\Pi \in (\mathcal{G}/\mathcal{I})^{2,2}$ of the form

$$(1.4) \quad \Pi = \sum_i u_i \delta_{P_i}(\partial_1 \wedge \partial_2),$$

where u_i is a formal variable in the ring $R = R_n := \mathbb{C}[u_1, \dots, u_n] / (u_i^2 \mid 1 \leq i \leq n)$ (equipped with the maximal ideal $\mathfrak{m} = \mathfrak{m}_n := (u_1, \dots, u_n)$) which corresponds to the point P_i , $\partial_1 \wedge \partial_2$ is the canonical holomorphic bi-vector field on $(\mathbb{C}^*)^2$ and δ_{P_i} is a Dolbeault $(0, 2)$ -form with asymptotic support at P_i . The idea to work with the ring R_n is entirely motivated by the tropical geometry setup in Gross' work [29]. Also, the form δ_{P_i} should be viewed as a smoothing of a “delta-form” supported at P_i (cf. [10]).

As in [10], a solution Φ to the extended MC equation (1.3) of $(\mathcal{G}/\mathcal{I})^{*,*}$ can be constructed using Kuranishi's method [43], namely, by summing over directed ribbon weighted d -pointed k -trees (see Definition 2.6) with input Π . The MC solution Φ can then be decomposed as

$$(1.5) \quad \Phi = \Pi + \Xi^{0,0} + \Xi^{1,1},$$

where $\Xi^{i,i} \in (\mathcal{G}/\mathcal{I})^{i,i}$. A major discovery of this paper is that, as $\hbar \rightarrow 0$, the correction terms $\Xi^{1,1}$ and $\Xi^{0,0}$ give rise to tropical disks of Maslov index 0 and 2, respectively.

Theorem 1.1 (=Theorem 4.12). *For a toric surface X equipped with the Hori-Vafa potential W , there is a solution Φ to the Maurer-Cartan equation decomposed as in (1.5) such that each of the terms $\Xi^{0,0}, \Xi^{1,1}$ can be expressed as a sum over tropical disks Γ transversal to the toric divisor D_∞ whose moduli space $\overline{\mathfrak{M}}^\Gamma$ is nonempty of codimension $1 - \frac{MI(\Gamma)}{2}$ in $B_0 = \mathbb{R}^2$ (where MI denotes the Maslov index):*

$$\Xi^{0,0} = \sum_{MI(\Gamma)=2} \alpha_\Gamma \text{Mono}(\Gamma), \quad \Xi^{1,1} = \sum_{MI(\Gamma)=0} \alpha_\Gamma \text{Log}(\Theta_\Gamma);$$

here $\text{Mono}(\Gamma)$ is a holomorphic function and $\text{Log}(\Theta_\Gamma)$ is a holomorphic vector field defined explicitly for a tropical disk Γ , and α_Γ is a Dolbeault $(0, 1 - \frac{MI(\Gamma)}{2})$ -form with asymptotic support along the $(1 + \frac{MI(\Gamma)}{2})$ -dimensional tropical polyhedral subset $Q_\Gamma \subset B_0$ traced out by the stop Q of the tropical disks in the moduli space $\overline{\mathfrak{M}}^\Gamma$ (see Definition 2.9).

Furthermore, the following properties hold:

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \alpha_\Gamma|_x &= 1 && \text{for any } x \text{ in the interior } \text{Int}(Q_\Gamma) \text{ when } MI(\Gamma) = 2, \\ \lim_{\hbar \rightarrow 0} \int_\rho \alpha_\Gamma &= -1 && \text{for any } \rho \pitchfork \text{Int}_{re}(Q_\Gamma) \text{ positively when } MI(\Gamma) = 0, \end{aligned}$$

where ρ is any affine line intersecting Q_Γ positively and transversally in its relative interior $\text{Int}_{re}(Q_\Gamma)$.

¹These are the only nontrivial bulk deformations.

Following Gross [29], we define the n -pointed perturbed LG potential $W_n(Q)$ in terms of counts of Maslov index 2 tropical disks with interior marked points possibly passing through P_1, \dots, P_n and with stop at a fixed point $Q \in \mathbb{R}^2$; see Section 2 for the precise definitions. Then Theorem 1.1 gives a bijective correspondence between leading order terms of $\Xi^{0,0}$ as $\hbar \rightarrow 0$, tropical disks Γ with $MI(\Gamma) = 2$ and $\overline{\mathfrak{M}}^\Gamma \neq \emptyset$, and hence terms in the perturbed potential $W_n(Q)$:

$$\left\{ \begin{array}{c} \text{terms in the} \\ \text{expression of } \Xi^{0,0} \end{array} \right\} \xleftrightarrow{\text{Theorem 1.1}} \left\{ \begin{array}{c} \text{Tropical disks } \Gamma \\ \text{with } MI(\Gamma) = 2 \end{array} \right\} \xleftrightarrow{[29]} \left\{ \begin{array}{c} \text{terms in the} \\ \text{perturbation } W_n(Q) \end{array} \right\}.$$

In the case of $X = \mathbb{P}^2$, the tropical disks above are precisely those considered by Gross [29]. Indeed we have

$$\lim_{\hbar \rightarrow 0} \Xi^{0,0}(Q) = \sum_{MI(\Gamma)=2} \text{Mono}(\Gamma),$$

which coincides with Gross' definition of the n -pointed potential $W_n(Q)$. This explains how solutions to the extended MC equation (1.3) lead naturally to the perturbed potential W_n .

The n -pointed potential $W_n(Q)$ depends on Q , and Gross [29] proved that the dependence is dictated by wall-crossing formulas across walls of a scattering diagram \mathcal{D} constructed from the Maslov index 0 tropical disks with interior marked points possibly passing through the n marked points P_1, \dots, P_n . Theorem 1.1 gives a bijective correspondence between leading order terms of $\Xi^{1,1}$ as $\hbar \rightarrow 0$, tropical disks Γ with $MI(\Gamma) = 0$ and $\overline{\mathfrak{M}}^\Gamma \neq \emptyset$, and hence walls in the scattering diagram \mathcal{D} :

$$\left\{ \begin{array}{c} \text{terms in the} \\ \text{expression of } \Xi^{1,1} \end{array} \right\} \xleftrightarrow{\text{Theorem 1.1}} \left\{ \begin{array}{c} \text{Tropical disks } \Gamma \\ \text{with } MI(\Gamma) = 0 \end{array} \right\} \xleftrightarrow{[29]} \left\{ \begin{array}{c} \text{walls } \mathbf{w}_\Gamma = (m_\Gamma, Q_\Gamma, \Theta_\Gamma) \\ \text{in the diagram } \mathcal{D} \end{array} \right\}.$$

Remark 1.2. Underlying the above bijective correspondences is an interplay between the differential-geometric properties of the dgBV algebra $(\mathcal{G}/\mathcal{I})^{*,*}$ and the combinatorial properties of tropical disks, which has also played an important role in [10]. As will be seen in Section 3.3.1, this leads to an extended version of the tropical Lie algebra which we call the *tropical dgLa*.

1.3. Wall-crossing from Maurer-Cartan solutions. Besides giving enumerative meanings to correction terms in the MC solution Φ , Theorem 1.1, when combined with the main results in [10], has an interesting corollary which can be viewed as an alternative proof (via gauge equivalences) of the wall-crossing formulas ([29, Theorem 4.12]) for Gross' perturbed potential W_n . Let us explain the argument in this subsection.

By definition, the scattering diagram $\mathcal{D} = \mathcal{D}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ consists of walls $\mathbf{w}_\Gamma = (m_\Gamma, Q_\Gamma, \Theta_\Gamma)$, the support of each being the tropical polyhedral subset Q_Γ traced out by the stop of a tropical disk Γ (see Section 2.4.3). The intersection of these Q_Γ 's is the set $\text{Sing}(\mathcal{D}) \setminus \{P_1, \dots, P_n\}$, where $\text{Sing}(\mathcal{D})$ denotes the singular set of \mathcal{D} , and a point $j \in \text{Sing}(\mathcal{D}) \setminus \{P_1, \dots, P_n\}$ is called a *joint* in the Gross-Siebert program [35].

Restricting to a contractible open neighborhood $U = U_j \subset \mathbb{R}^2 \setminus \{P_1, \dots, P_n\}$ containing a single joint j , we have $\Phi|_U = \Xi^{0,0} + \Xi^{1,1}$ because $\delta_{P_i}|_U = 0$ and hence $\Pi|_U = 0$ in $(\mathcal{G}/\mathcal{I})^{*,*}$. By degree reasons, we see that $\Xi^{1,1}|_U$ is itself a solution to the nonextended Maurer-Cartan equation (1.2), namely, $\bar{\partial}\Xi^{1,1} + \frac{1}{2}[\Xi^{1,1}, \Xi^{1,1}] = 0$. Now [10, Theorems 1.5 and 1.6], the proofs of which were by asymptotic analysis and completely different from that of [29], imply the following statement which originally appeared in Gross [29].

Corollary 1.3 (Proposition 4.7 in [29]). *For any point $j \in \text{Sing}(\mathcal{D}) \setminus \{P_1, \dots, P_n\}$, we have $\Theta_{\gamma_j, \mathcal{D}} = \text{Id}$ for any loop γ_j around j in a sufficiently small contractible neighborhood U_j of j .*

Next we would like to work locally near a wall Q_Γ of the scattering diagram \mathcal{D} . So we consider a contractible open subset $U \subset M_{\mathbb{R}} \setminus (\{P_1, \dots, P_n\} \cup \text{Sing}(\mathcal{D}))$, which is separated into two chambers U_+ and U_- by the wall $Q_\Gamma \cap U$ as shown in Figure 1 (here U_\pm are chosen according to the orientation of the ray Q_Γ).

Results from [10, Section 4] imply that

$$(1.6) \quad \varphi := \begin{cases} \text{Log}(\Theta_\Gamma) & \text{on } U_+, \\ 0 & \text{on } U_-, \end{cases}$$

is the unique gauge solving the equation

$$(1.7) \quad e^{\text{ad}_\varphi} \bar{\partial} e^{-\text{ad}_\varphi} = \bar{\partial} - \left[\left(\frac{e^{\text{ad}_\varphi} - \text{Id}}{\text{ad}_\varphi} \right) \bar{\partial}(\varphi), \cdot \right] = \bar{\partial} + [\Xi^{1,1}, \cdot]$$

and satisfying the condition that $\varphi|_{U_-} = 0$. The Maurer-Cartan solution $\Xi^{1,1}$ behaves like a “delta-form” supported along the wall Q_Γ , while the local gauge φ in U behaves like a step-function jumping across the wall Q_Γ as shown in Figure 1.

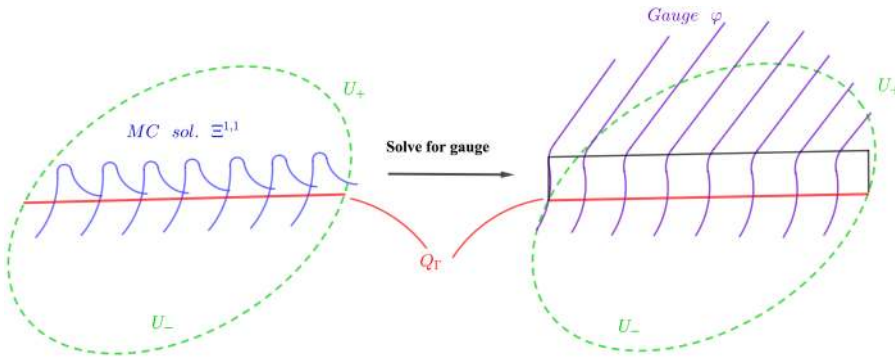


FIGURE 1. The gauge φ as step-function

When the extended MC equation (1.3) is restricted to U , we have $\Phi|_U = \Xi^{0,0} + \Xi^{1,1}$ and it decomposes into two equations:

$$\begin{aligned} \bar{\partial}\Xi^{1,1} + \frac{1}{2}[\Xi^{1,1}, \Xi^{1,1}] &= 0, \\ \bar{\partial}\Xi^{0,0} + [W, \Xi^{1,1}] + [\Xi^{0,0}, \Xi^{1,1}] &= 0. \end{aligned}$$

The second equation can be rewritten as $(\bar{\partial} + [\Xi^{1,1}, \cdot]) (W + \Xi^{0,0}) = 0$, meaning precisely that $W + \Xi^{0,0}$ is a holomorphic function with respect to the complex

structure defined by $\bar{\partial} + [\Xi^{1,1}, \cdot]$. Since $e^{\text{ad}_\varphi} \bar{\partial} e^{-\text{ad}_\varphi} = \bar{\partial} + [\Xi^{1,1}, \cdot]$, this is equivalent to saying that the function $e^{-\text{ad}_\varphi}(W + \Xi^{0,0})$, which is globally defined on U , is holomorphic with respect to the *original* Dolbeault operator $\bar{\partial}$.

Now letting $W_{n,\pm} := (W + \Xi^{0,0})|_{U_\pm}$ on U_\pm , respectively, we have

$$e^{-\text{ad}_\varphi}(W + \Xi^{0,0}) = \begin{cases} \Theta_\Gamma^{-1}(W_{n,+}) & \text{on } U_+, \\ W_{n,-} & \text{on } U_-. \end{cases}$$

So the fact that $e^{-\text{ad}_\varphi}(W + \Xi^{0,0})$ is a *globally defined* function on U implies that $\Theta_\Gamma(W_{n,-}) = W_{n,+}$. Applying this to a finite number of walls, we obtain the following wall-crossing formula which originally appeared in Gross [29].

Corollary 1.4 (Theorem 4.12 in Gross [29]). *If $Q, Q' \in \mathbb{R}^2$ are not lying on any walls in the scattering diagram \mathcal{D} , then we have*

$$(1.8) \quad W_n(Q') = \Theta_{\gamma, \mathcal{D}}(W_n(Q))$$

for any path $\gamma \subset M_\mathbb{R} \setminus \text{Sing}(\mathcal{D})$ joining Q to Q' .

1.4. Remarks. We end this introduction with a few remarks.

- (1) Just as in Gross [29], the tropical disks which appear here (see Section 2) correspond to holomorphic disks in the toric variety X which are transversal to the toric divisor D_∞ , so the tropical counts would give the correct Gromov-Witten invariants only when X is a toric Fano surface. One way to get the correct Gromov-Witten invariants in general is to apply the technique of *tropical modification* which deforms the ambient space so that the hidden curves (i.e., curves lying inside the toric divisor D_∞) can be seen; see, e.g., [39] for an introduction. We expect that our results would still hold if we replace the Hori-Vafa potential with the Lagrangian Floer potential W^{LF} , and in that case W_n would correspond to the bulk-deformed potential function defined using suitable tropical modifications.
- (2) To generalize our results to higher dimensions, one again needs a proper definition of tropical counts. In that case, we shall allow interior insertions of tropical cycles of different codimensions, instead of just points in generic position. It should be straightforward to generalize our results when only point insertions are involved. For other tropical cycles, what is missing is a description of such cycles by means of elements in the tropical dgLa as in equation (1.4).

2. TROPICAL COUNTING IN DIMENSION 2

We fix, once and for all, a rank 2 lattice M together with its dual lattice N , and write $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ for the corresponding real vector spaces. An element in M (resp., N) will be denoted by \mathfrak{m} (resp., n). Let $\Sigma \subset M_\mathbb{R}$ be a complete rational polyhedral fan and let X_Σ be the associated toric surface. We take $K := \mathcal{K}_{X_\Sigma} \cap H^2(X_\Sigma, \mathbb{Z})$ to be the monoid of integral Kähler forms, where \mathcal{K}_{X_Σ} is the Kähler cone of X_Σ . We also use $\Sigma(1)$ to denote the set of 1-dimensional cones in Σ and D_ρ to denote the toric divisor corresponding to $\rho \in \Sigma(1)$. The purpose of this section is to define the counting of tropical disks following [29, 48], with slight modifications.

2.1. The Hori-Vafa LG mirror.

Notation 2.1. If we fix a Lagrangian torus fiber L of the moment map $p : X_\Sigma \rightarrow \mathbf{P}$, then $\pi_2(X, L) \cong \sum_{\rho \in \Sigma(1)} \mathbb{Z} \cdot m_\rho$ is freely generated by the classes m_ρ 's of Maslov index 2 holomorphic disks in (X, L) , where m_ρ is the unique disk class which intersects the toric divisor D_ρ exactly once.

Let \mathcal{P} be the nonnegative cone $\pi_2(X, L)_{\geq 0} \subset \pi_2(X, L)$ generated by classes $\beta \in \pi_2(X, L)$ such that $\int_\beta \omega \geq 0$ for all $\omega \in K$, and let $\mathcal{Q} := \pi_2(X) \cap \pi_2(X, L)_{\geq 0}$ be the effective cone (or Mori cone) of X_Σ . We have $\mathcal{P}^{gp} = \pi_2(X, L)$ and $\mathcal{Q}^{gp} = \pi_2(X)$, where \mathcal{P}^{gp} and \mathcal{Q}^{gp} are the abelian groups associated to \mathcal{P} and \mathcal{Q} , respectively, and the exact sequence of monoids

$$(2.1) \quad 0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{P} \xrightarrow{\theta} M \longrightarrow 0$$

where M is identified with $\pi_1(L)$ and $\theta(\beta) := \partial\beta$ is the map taking boundary of the class $\beta \in \mathcal{P}$.

We will use m to denote an element of \mathcal{P} and \bar{m} to denote its image $\theta(m) \in M$. Note that $\theta : \mathcal{P} \rightarrow M$ maps the standard basis $\{m_\rho\} \subset \mathbb{Z}^{|\Sigma(1)|}$ to the generators of the 1-dimensional cones $\rho \in \Sigma(1)$.

Example 2.2. We consider a fan Σ in $M_\mathbb{R} \cong \mathbb{Z}^2$, with its 1-dimensional cones given by $\rho_i = \mathbb{R} \cdot m_i$ where we have $m_1 = (0, 1)$, $m_2 = (-1, 0)$, $m_3 = (0, -1)$, and $m_4 = (1, 1)$. The corresponding toric variety X_Σ is the surface F_1 . We write $\pi_2(X_\Sigma, L) = \bigoplus_{i=1}^4 \mathbb{Z} \cdot m_i$ with m_i being the unique disk class which supports a Maslov index 2 holomorphic disk intersecting exactly once with the toric boundary D_{ρ_i} . In this case \mathcal{Q} is the monoid of integral points in the cone generated by $-m_1 + m_2 + m_4$ and $m_1 + m_3$, while \mathcal{P} is the monoid of integral points in the cone generated by the m_i 's together with $-m_1 + m_2 + m_4$.

Consider the polynomial rings $\mathbb{C}[\mathcal{P}] := \mathbb{C}[z^m \mid m \in \mathcal{P}]$ and $\mathbb{C}[\mathcal{Q}] := \mathbb{C}[q^d \mid d \in \mathcal{Q}]$ and the corresponding affine toric varieties $\check{\mathcal{X}} := \text{Spec}(\mathbb{C}[\mathcal{P}])$ and $\mathcal{S} = \text{Spec}(\mathbb{C}[\mathcal{Q}])$. There is a natural morphism $\pi : \check{\mathcal{X}} \rightarrow \mathcal{S}$ induced by the map $\mathcal{Q} \rightarrow \mathcal{P}$ in (2.1). As a result we obtain a toric degeneration of the LG model over \mathcal{S} as in [29, 30].

Definition 2.3. The *Hori-Vafa potential* is the polynomial $W := \sum_{\rho \in \Sigma(1)} z^{m_\rho}$ on $\check{\mathcal{X}}$.

2.2. **Tropical disks.** Before defining tropical disks, we first introduce the underlying combinatorial structures.

Definition 2.4. A (*directed*) k -tree T consists of a finite set of vertices $\bar{T}^{[0]}$ together with a decomposition $\bar{T}^{[0]} = T_{in}^{[0]} \sqcup T^{[0]} \sqcup \{v_o\}$, where $T_{in}^{[0]}$, called the set of incoming vertices, is a set of size k and v_o is called the outgoing vertex (we also write $T_\infty^{[0]} := T_{in}^{[0]} \sqcup \{v_o\}$), a finite set of edges $\bar{T}^{[1]}$, and two boundary maps $\partial_{in}, \partial_o : \bar{T}^{[1]} \rightarrow \bar{T}^{[0]}$ (here ∂_{in} stands for incoming and ∂_o stands for outgoing), satisfying the following conditions:

- (1) Every vertex $v \in T^{[0]}$ is trivalent, and satisfies $\#\partial_o^{-1}(v) = 2$ and $\#\partial_{in}^{-1}(v) = 1$.
- (2) Every vertex $v \in T_{in}^{[0]}$ has valency one, and satisfies $\#\partial_o^{-1}(v) = 0$ and $\#\partial_{in}^{-1}(v) = 1$; we let $T^{[1]} := \bar{T}^{[1]} \setminus \partial_{in}^{-1}(T_{in}^{[0]})$.

- (3) For the outgoing vertex v_o , we have $\#\partial_o^{-1}(v_o) = 1$ and $\#\partial_{in}^{-1}(v_o) = 0$; we let $e_o := \partial_o^{-1}(v_o)$ be the outgoing edge and denote by $v_r \in T_{in}^{[0]} \sqcup T^{[0]}$ the unique vertex (which we call the root vertex) with $e_o = \partial_{in}^{-1}(v_r)$.
- (4) The *topological realization* $|\bar{T}| := (\prod_{e \in \bar{T}^{[1]}} [0, 1]) / \sim$ of the tree T is connected and simply connected; here \sim is the equivalence relation defined by identifying boundary points of edges if their images in $T^{[0]}$ are the same.

Two k -trees T_1 and T_2 are *isomorphic* if there are bijections $\bar{T}_1^{[0]} \cong \bar{T}_2^{[0]}$ and $\bar{T}_1^{[1]} \cong \bar{T}_2^{[1]}$ preserving the decomposition $\bar{T}_i^{[0]} = T_{i,in}^{[0]} \sqcup T_i^{[0]} \sqcup \{v_{i,o}\}$ and boundary maps $\partial_{i,in}$ and $\partial_{i,o}$. The set of isomorphism classes of k -trees will be denoted by \mathbb{T}_k . For a k -tree T , we abuse notation and use T (instead of $[T]$) to denote its isomorphism class.

A ribbon k -tree is a k -tree T with a cyclic ordering of $\partial_{in}^{-1}(v) \sqcup \partial_o^{-1}(v)$ for each trivalent vertex $v \in T^{[0]}$, and isomorphisms of ribbon k -trees are required to preserve this ordering.

Notation 2.5. As in [29, 30], we take n points $P_1, \dots, P_n \in M_{\mathbb{R}}$ and introduce the ring

$$R = R_n := \frac{\mathbb{C}[u_1, \dots, u_n]}{(u_i^2 \mid 1 \leq i \leq n)},$$

where we associate to each point P_i the variable u_i . This ring has the maximal ideal $\mathbf{m} = \mathbf{m}_n := (u_1, \dots, u_n)$.

Definition 2.6. A *weighted d -pointed k -tree* is a $(k + d)$ -tree Γ together with an injective map $p : \{1, \dots, d\} \hookrightarrow \partial_{in}^{-1}(\Gamma_{in}^{[0]})$ (we use p_j to denote the image $p(j) \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})$), a weight $m : \bar{\Gamma}^{[1]} \rightarrow \mathcal{P}$ (we use m_e to denote the image $m(e) \in \mathcal{P}$), and a map $u : \bar{\Gamma}^{[1]} \rightarrow R_n$ (we use u_e to denote the image $u(e) \in R_n$) satisfying the following conditions:

- (1) for every $e \in \bar{\Gamma}^{[1]}$, $u_e \in R_n$ is a monomial defined via the rule: for each $j = 1, \dots, d$, we have $u_{p_j} = u_{i_j}$ for some $i_j \in \{1, \dots, n\}$ such that $1 \leq i_1 < i_2 < \dots < i_d \leq n$, and for each $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$, we have $u_e = 1$;
- (2) for every trivalent vertex $v \in \Gamma^{[0]}$ attached with two incoming edges e_1, e_2 and an outgoing edge e_3 we require at least one of e_1, e_2 to not belong to $\partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$, and we further require $u_{e_3} = u_{e_1} \cdot u_{e_2}$ and $m_{e_3} = m_{e_1} + m_{e_2}$;
- (3) $m_e = 0$ if and only if $e \in \{p_1, \dots, p_d\}$;
- (4) for every $e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$, we have $m_e = m_\rho$ for some $\rho \in \Sigma(1)$.

Two weighted d -pointed k -trees Γ_1 and Γ_2 are said to be *isomorphic* if they are isomorphic as k -trees and the isomorphism preserves the marked points p_i 's and the weight functions m_e 's. The set of isomorphism classes of a weighted d -pointed k -trees will be denoted by $\text{WPT}_{k,d}$.

A *weighted ribbon d -pointed k -tree* is a weighted d -pointed k -tree Γ equipped with a ribbon structure on it such that if e_1 and e_2 are the incoming edges of v with outgoing edge e_3 with e_1, e_2, e_3 in cyclic ordering, then only e_1 can possibly be an edge from $\Gamma_{in}^{[0]} = \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$. Isomorphisms between these trees are defined as isomorphisms between weighted d -pointed k -trees which preserve the relevant structures. The set of isomorphism classes of a weighted ribbon d -pointed k -tree is denoted by $\text{WRT}_{k,d}$.

For a weighted d -pointed k -tree Γ (or weighted ribbon d -pointed k -tree, resp.), we abuse notation and use Γ (\mathcal{T} , resp.) instead of $[\Gamma]$ ($[\mathcal{T}]$, resp.) to stand for its isomorphism class.

Notation 2.7. Given a weighted d -pointed k -tree Γ ($d \geq 0$ and $k \geq 1$), we write $\Gamma_{in}^{[1]} := \partial_{in}^{-1}(\Gamma_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$ for the set of incoming edges excluding those which correspond to the marked points.

Given any $e \in \bar{\Gamma}^{[1]} \setminus \{p_1, \dots, p_d\}$, we let $k_e = 0$ if $\bar{m}_e = 0$, and when $\bar{m}_e \neq 0$, we let $k_e \in \mathbb{Z}_{\geq 0}$ be the unique positive integer such that $\bar{m}_e = k_e \hat{m}_e$ where $\hat{m}_e \in M$ is the primitive element. The integers $\{k_e\}_{e \in \bar{\Gamma}^{[1]} \setminus \{p_1, \dots, p_d\}}$ define the *weight function* $w_\Gamma : \bar{\Gamma}^{[1]} \setminus \{p_1, \dots, p_d\} \rightarrow \mathbb{Z}_{\geq 0}$ (hence the name “weighted k -tree”) and the formula $\bar{m}_{e_2} = \bar{m}_{e_1} + \bar{m}_{e_0}$ corresponds to the *balancing condition*, both of which appear in the original definition of tropical curves in [29, 48].

We also write m_Γ (or k_Γ) and u_Γ for the weight and monomial, respectively, associated to the unique outgoing edge e_o attached to the unique outgoing vertex v_o of Γ .

Definition 2.8. Given a d -pointed weighted k -tree Γ , we define the *multiplicity at a trivalent vertex* $v \in \Gamma_0^{[0]} := \Gamma^{[0]} \setminus \partial_o(\{p_1, \dots, p_d\})$ by

$$\text{Mult}_v(\Gamma) := |\det(\bar{m}_{e_1}, \bar{m}_{e_2})| = |\det(\bar{m}_{e_1}, \bar{m}_{e_3})| = |\det(\bar{m}_{e_2}, \bar{m}_{e_3})|,$$

where e_1, e_2 are the incoming edges and e_3 the outgoing edge attached to v . Then we define the *multiplicity* $\text{Mult}(\Gamma)$ of Γ by $\text{Mult}(\Gamma) := \prod_{v \in \Gamma_0^{[0]}} \text{Mult}_v(\Gamma)$.

Note that at a trivalent vertex $v \in \Gamma_0^{[0]}$ with incoming edges e_1, e_2 , the multiplicity $\text{Mult}_v(\Gamma) \neq 0$ if and only if $\bar{m}_{e_1}, \bar{m}_{e_2}$ are linearly independent in $M_{\mathbb{R}}$.

Given a weighted d -pointed k -tree Γ , a *realization* of Γ is defined as $|\Gamma_{\vec{s}}| := \left(\left(\bigsqcup_{e \in \partial_{in}^{-1}(\Gamma_{in}^{[0]})} (\mathbb{R}_{\leq 0})_e \right) \sqcup \left(\bigsqcup_{e \in \Gamma^{[1]}} [s_e, 0] \right) \right) / \sim$, for a set of parameters $\vec{s} := (s_e)_{e \in \Gamma^{[1]}} \in (\mathbb{R}_{< 0})^{|\Gamma^{[1]}|}$; here $(\mathbb{R}_{\leq 0})_e$ is just a copy of $\mathbb{R}_{\leq 0}$ and \sim is the equivalence relation defined by identifying boundary points of edges if their images in $\Gamma^{[0]}$ are the same. The set of realizations of Γ is parametrized by $\vec{s} \in (\mathbb{R}_{< 0})^{|\Gamma^{[1]}|}$.

Definition 2.9. A *d -pointed tropical disk* ς in $(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q)$ consists of a weighted d -pointed k -tree Γ with $u_\Gamma \neq 0$, a set of parameters $\vec{s} = (s_e)_{e \in \Gamma^{[1]}} \in (\mathbb{R}_{< 0})^{|\Gamma^{[1]}|}$, and a proper map $\varsigma : |\Gamma_{\vec{s}}| \rightarrow M_{\mathbb{R}}$ from the realization $|\Gamma_{\vec{s}}|$ of Γ to $M_{\mathbb{R}}$, satisfying the following conditions:

- (1) $\varsigma|_{(\mathbb{R}_{\leq 0})_{p_j}} \equiv P_{i_j}$ if the monomial assigned to p_j is u_{i_j} ; in particular $\varsigma|_{(\mathbb{R}_{\leq 0})_{p_j}}$ a constant map (playing the role of a marked point).
- (2) For each incoming edge $e \in \Gamma_{in}^{[1]}$, we have $\varsigma|_{(\mathbb{R}_{\leq 0})_e}(s) = \varsigma|_{(\mathbb{R}_{\leq 0})_e}(0) + s(-\bar{m}_e)$ for all $s \in \mathbb{R}_{\leq 0}$.
- (3) For each $e \in \Gamma^{[1]}$, we have $\varsigma|_{[s_e, 0]}(s) = \varsigma|_{[s_e, 0]}(0) + s(-\bar{m}_e)$ for all $s \in [s_e, 0]$ (so that the image $\text{Im}(\varsigma|_{[s_e, 0]})$ is an affine line segment with slope $-\bar{m}_e$).
- (4) The point $\varsigma(v_o) := \varsigma|_{[s_{e_o}, 0]}(0) \in M_{\mathbb{R}}$ is called the *stop* of the tropical disk ς and we require that $\varsigma(v_o) = Q$.

The *multiplicity* $\text{Mult}(\varsigma)$ of a tropical disk ς is defined as the multiplicity $\text{Mult}(\Gamma)$ of the underlying weighted d -pointed k -tree Γ . Note that $\text{Mult}(\varsigma) \neq 0$ if and only if the images of the two incoming edges at any trivalent vertex are intersecting transversally. The underlying tree Γ is said to be the *combinatorial type* of the

tropical disk ς . We use $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q)$ to denote the moduli space of tropical disks in $(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q)$ with a fixed combinatorial type Γ .

Similarly, we define a *tropical disk ς in $(\mathcal{P}, \Sigma, P_1, \dots, P_n)$* by allowing the stop Q to vary or by dropping condition (4) above, and we denote by $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ the moduli space of tropical disks in $(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ with a fixed combinatorial type Γ . In other words, $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \bigcup_Q \mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q)$. Notice that there is a natural \mathbb{R}_+ action on $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ given by translating the stop $\varsigma(v_o) = Q$ along the direction $-\bar{m}_{e_o}$, so we have a well-defined quotient $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n)/\mathbb{R}_+$, which can be regarded as the moduli space of tropical disks as the stop Q goes to infinity along the direction $-\bar{m}_{e_o}$; see [29, 30].

We further define a *tropical disk ς in (\mathcal{P}, Σ)* with a fixed combinatorial type Γ by dropping condition (4) above and replacing condition (1) by only requiring that $\varsigma|_{(\mathbb{R}_{\leq 0})_{P_j}}$ is a constant map for each $j = 1, \dots, d$.

The reader may ask why all the internal vertices $\Gamma^{[0]}$ are required to be trivalent. Indeed we have only defined *generic* tropical disks and the above moduli spaces are all noncompact. We use this approach because this suffices for the purpose of tropical counting. To compactify these moduli spaces, we need to allow the intervals $[s_e, 0]$'s corresponding to the internal edges $e \in \Gamma^{[1]}$ to shrink to zero lengths (i.e., by allowing $s_e = 0$), so that some internal vertices are allowed to be of higher valencies.

We use $\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ to denote the compactified moduli space of tropical disks Γ in (\mathcal{P}, Σ) with a fixed combinatorial type thus obtained, which gives a compactification of the union of the moduli spaces $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ as P_1, \dots, P_n vary. We use the notation $\partial\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ to stand for the set of tropical disks with at least one degenerated internal edge (i.e., $s_e = 0$ for some $e \in \Gamma^{[1]}$).

It is not hard to see that $\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \cong (\mathbb{R}_{\leq 0})^{|\Gamma^{[1]}|} \times M_{\mathbb{R}}$, where the first component $\vec{s} \in (\mathbb{R}_{\leq 0})^{|\Gamma^{[1]}|}$ parametrizes the realization $|\Gamma_{\vec{s}}|$ of Γ and the second component $M_{\mathbb{R}}$ parametrizes the stop $\varsigma(v_o)$. Its dimension is given by

$$(2.2) \quad \dim_{\mathbb{R}}(\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)) = |\Delta(\Gamma)| + d + 1,$$

where $|\Delta(\Gamma)| := k$ for a d -pointed weighted k -tree Γ . This moduli space has a natural stratification coming from the one on $(\mathbb{R}_{\leq 0})^{|\Gamma^{[1]}|}$ given naturally by the coordinate hyperplanes $s_e = 0$.

We also need to consider the partial compactification

$$\hat{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) := \left(\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \setminus \{\varsigma \mid s_{e_o} = 0\} \right) / \mathbb{R}_+,$$

and we denote by $\partial\hat{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ the set of tropical disks with $s_e = 0$ for some $e \in \Gamma^{[1]} \setminus \{e_o\}$.

Definition 2.10. We define the evaluation maps $ev_* : \overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \rightarrow M_{\mathbb{R}}$, where $*$ $\in \{1, \dots, d\} \cup \{o\}$, to be the evaluation at a marked point $ev_*(\varsigma) = \varsigma(p_*)$ when $*$ $\in \{1, \dots, d\}$, and the evaluation at the outgoing vertex $ev_*(\varsigma) = \varsigma(v_o)$ if $*$ $= o$. We put these evaluation maps together to obtain the map $\vec{ev} = (ev_1, \dots, ev_d, ev_o) : \overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \rightarrow M_{\mathbb{R}}^{d+1}$. Similarly, we have the evaluation map $\hat{ev} = (ev_1, \dots, ev_d) : \hat{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \rightarrow M_{\mathbb{R}}^d$.

Definition 2.11. We say that n distinct points P_1, \dots, P_n are in *generic position* if for any $d \leq n$, any d -tuple $(P_{i_1}, \dots, P_{i_d})$ is not lying in the image $\hat{e}v(S)$ of a stratum $S \subset \hat{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ over which the evaluation map $\hat{e}v$ is degenerated, meaning that the differential $D(\hat{e}v|_S)$ is not surjective (notice that $\hat{e}v|_S$ is an affine map and hence $D(\hat{e}v|_S)$ is a well-defined constant linear map), and this holds for any combinatorial type Γ .

We say that $n + 1$ distinct points P_1, \dots, P_n, Q are in *generic position* if the n points P_1, \dots, P_n are in generic position, and for any $d \leq n$ and any d -tuple $(P_{i_1}, \dots, P_{i_d})$, the $(d + 1)$ -tuple $(P_{i_1}, \dots, P_{i_d}, Q)$ is not lying in the image $\bar{e}v(S)$ of a stratum $S \subset \overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$ over which the evaluation map $\bar{e}v$ is degenerated and this holds for any combinatorial type Γ .

Definition 2.12. We define the *Maslov index* $MI(\Gamma)$ of a weighted d -pointed k -tree Γ by $MI(\Gamma) = 2(k - d)$, and the Maslov index $MI(\varsigma)$ of a tropical disk ς to be that of its combinatorial type Γ .

Lemma 2.13 (Lemma 2.6 in [29]). *If P_1, \dots, P_n, Q are in generic position and $MI(\Gamma) = 2r$, then $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q)$ is an $(r - 1)$ -dimensional (over \mathbb{R}) affine linear subspace of $\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \setminus \partial\overline{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$; in particular, $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q) = \emptyset$ when $r \leq 0$.*

If P_1, \dots, P_n are in generic position and

$$MI(\Gamma) = 2r,$$

then $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n)/\mathbb{R}_+$ is an r -dimensional (over \mathbb{R}) affine linear subspace of $\hat{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma) \setminus \partial\hat{\mathfrak{M}}_d^\Gamma(\mathcal{P}, \Sigma)$; in particular, we have $\mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n)/\mathbb{R}_+ = \emptyset$ when $r < 0$.

2.3. The perturbed Landau-Ginzburg potential. We now define an n -pointed LG potential as a perturbation of the Hori-Vafa mirror family (\check{X}, W) by tropical disk counts, following [29].

Definition 2.14. Given a tropical disk $\varsigma \in \mathfrak{M}_d^\Gamma(\mathcal{P}, \Sigma, P_1, \dots, P_n; Q)$ with $MI(\varsigma) = 2$, the *monomial associated to ς* is defined by

$$\text{Mono}(\varsigma) := \text{Mult}(\Gamma)z^{m_\Gamma}u_\Gamma \in \mathbb{C}[\mathcal{P}],$$

where $m_\Gamma \in \mathcal{P}$ is the weight and u_Γ is the monomial associated to the unique outgoing edge e_o of ς as in Definition 2.6.

Definition 2.15 (cf. Definition 2.7 in [29]). Fixing the points P_1, \dots, P_n, Q in generic position, we define the *n -pointed Landau-Ginzburg (LG) potential* as

$$W_n(Q) := \sum_{\varsigma} \text{Mono}(\varsigma),$$

where the sum is over all Maslov index 2 tropical disks ς in $(P_1, \dots, P_n; Q)$.

Note that the 0-pointed LG potential $W_0(Q) = W$ is precisely the Hori-Vafa potential, so the n -pointed potential $W_n(Q)$ is indeed a deformation (in the formal variables u_i 's) of W .

2.4. Scattering diagram from the Maslov index 0 disks. According to [29, 30], the dependence of the n -pointed LG potential $W_n(Q)$ on Q is governed by a scattering diagram constructed from the Maslov index 0 tropical disks. Here we recall the definition of scattering diagrams from [10, Section 3] with slight modifications; the original definition was due to Kontsevich-Soibelman [42] and can be found in [31].

2.4.1. Tropical vertex group. We consider $\mathbb{C}[\mathcal{P}] \otimes_{\mathbb{Z}} N$, whose general elements are finite linear combinations of elements of the form $z^m \otimes \check{\partial}_n$ (here $\check{\partial}_n$ is a holomorphic vector field on \check{X} associated to $n \in N$ to be defined in (3.5)). We also define the Lie bracket $[\cdot, \cdot]$ on $\mathbb{C}[\mathcal{P}] \otimes_{\mathbb{Z}} N$ by the formula:

$$(2.3) \quad [z^m \otimes \check{\partial}_n, z^{m'} \otimes \check{\partial}_{n'}] = z^{m+m'} \check{\partial}_{(\bar{m}',n)n'-(\bar{m},n')n},$$

where (\cdot, \cdot) is the natural pairing between M and N . We consider the Lie algebra $\mathfrak{g} := (\mathbb{C}[\mathcal{P}] \otimes_{\mathbb{Z}} N) \otimes_{\mathbb{C}} R_n$, where R_n is the formal power series ring R_n in Notation 2.5 equipped with its maximal ideal \mathfrak{m} .

Definition 2.16 ([31]). The *tropical Lie algebra* over $R = R_n$ is defined to be the nilpotent Lie subalgebra $\mathfrak{h} \hookrightarrow \mathfrak{g}$ given explicitly by $(\bigoplus_{m \in \mathcal{P} \setminus \{0\}} \mathbb{C} \cdot z^m \otimes (m^\perp)) \otimes_{\mathbb{C}} R \rightarrow \mathfrak{g}$. The *tropical vertex group* is defined as the exponential group of \mathfrak{h} .

Definition 2.17. Given $m \in \mathcal{P} \setminus \{0\}$ and $n \in m^\perp$, we let $\mathfrak{h}_{m,n} := (\mathbb{C}[z^m] \cdot z^m) \check{\partial}_n \otimes_{\mathbb{C}} \mathfrak{m} \hookrightarrow \mathfrak{h}$, whose general elements are of the form $\sum_{k \geq 1} \sum_I a_{k,I} z^{km} \check{\partial}_n u_I$, where $I \subset \{1, \dots, n\}$. This defines an abelian Lie subalgebra of \mathfrak{h} by (2.3).

Definition 2.18. A *wall* \mathbf{w} over R is a triple (m, Q, Θ) , where

- $m \in \mathcal{P} \setminus \{0\}$ such that \bar{m} is parallel to Q ,
- Q , called the *support* of \mathbf{w} , is a connected oriented codimension 1 convex tropical polyhedral subset of $M_{\mathbb{R}}$,²
- $\Theta \in \exp(\mathfrak{h}_{m,n_Q})$, where $n_Q \in N$ is the unique primitive element satisfying $n_Q \in (TQ)^\perp$ and $(\nu_Q, n) < 0$, and $\nu_Q \in M_{\mathbb{R}}$ here is a vector normal to Q such that the orientation of $TQ \oplus \mathbb{R} \cdot \nu_Q$ agrees with that of $M_{\mathbb{R}}$.

Definition 2.19. A *scattering diagram* \mathcal{D} over $R = R_n$ is a finite set of walls $\{(m_\alpha, Q_\alpha, \Theta_\alpha)\}_\alpha$.

Notation 2.20. For a scattering diagram \mathcal{D} , its *support* is defined as $\text{supp}(\mathcal{D}) := \bigcup_{\mathbf{w} \in \mathcal{D}} Q_{\mathbf{w}}$, and its *singular set* as $\text{Sing}(\mathcal{D}) := \bigcup_{\mathbf{w}_1 \in \mathcal{D}} \partial Q_{\mathbf{w}_1} \cup \bigcup_{\mathbf{w}_1 \pitchfork \mathbf{w}_2} (Q_{\mathbf{w}_1} \cap Q_{\mathbf{w}_2})$, where $\mathbf{w}_1 \pitchfork \mathbf{w}_2$ means transversally intersecting walls.

2.4.2. Path ordered products. An embedded path $\gamma : [0, 1] \rightarrow B_0 \setminus \text{Sing}(\mathcal{D})$ is said to be *intersecting \mathcal{D} generically* if $\gamma(0), \gamma(1) \notin \text{supp}(\mathcal{D})$, $\text{Im}(\gamma) \cap \text{Sing}(\mathcal{D}) = \emptyset$ and it intersects all the walls in \mathcal{D} transversally. Given such an embedded path γ with a sequence of real numbers $0 = t_0 < t_1 < t_2 < \dots < t_s < t_{s+1} = 1$ such that $\{\gamma(t_1), \dots, \gamma(t_s)\} = \gamma \cap \text{supp}(\mathcal{D})$, we define the *path ordered product* along γ , denoted by

$$\Theta_{\gamma, \mathcal{D}} := \Theta_{\gamma(t_s)}^{\sigma_s} \cdots \Theta_{\gamma(t_i)}^{\sigma_i} \cdots \Theta_{\gamma(t_1)}^{\sigma_1}$$

to be the product of the wall-crossing factors $\Theta_{\gamma(t_j)}$'s according to the direction of the path γ following [31], where $\sigma_j = 1$ if the orientation of $P_{i,j} \oplus \mathbb{R} \cdot \gamma'(t_i)$ agrees

²It means a connected convex subset locally defined by affine linear equations and inequalities defined over \mathbb{Q} .

with that of $M_{\mathbb{R}}$ and $\sigma_j = -1$ otherwise. For more details, we refer readers to [10, Section 3.2.1.].

Definition 2.21. A scattering diagram \mathcal{D} is said to be *consistent* if we have $\Theta_{\gamma, \mathcal{D}} = \text{Id}$, for any embedded loop γ intersecting \mathcal{D} generically. Two scattering diagrams \mathcal{D} and $\tilde{\mathcal{D}}$ are said to be *equivalent* if $\Theta_{\gamma, \mathcal{D}} = \Theta_{\gamma, \tilde{\mathcal{D}}}$ for any embedded path γ intersecting both \mathcal{D} and $\tilde{\mathcal{D}}$ generically.

2.4.3. *Maslov index 0 tropical disks.*

Definition 2.22. We define $\mathcal{D}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ to be the scattering diagram which consists of walls

$$\mathbf{w}_{\Gamma} = (m_{\Gamma}, Q_{\Gamma}, \Theta_{\Gamma})$$

for each weighted d -pointed k -tree Γ with $MI(\Gamma) = 0$ and $\mathfrak{M}_d^{\Gamma}(\mathcal{P}, \Sigma, P_1, \dots, P_n) / \mathbb{R}_+ \neq \emptyset$, where

- (1) the ray $Q_{\Gamma} \subset M_{\mathbb{R}}$ is given by the closure of the image of

$$ev_o : \mathfrak{M}_d^{\Gamma}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \rightarrow M_{\mathbb{R}}$$

- at the outgoing vertex v_o (i.e., the locus of the stop of a tropical disk ζ),
- (2) the Fourier mode $m_{\Gamma} = m_{\zeta}$ is the weight associated to the outgoing edge e_o attached to the unique outgoing vertex v_o , and
- (3) the wall-crossing automorphism Θ_{Γ} is given by the formula $\text{Log}(\Theta_{\Gamma}) = k_{\Gamma} \text{Mult}(\Gamma) z^{m_{\Gamma}} \tilde{\partial}_{n_{\Gamma}} u_{\Gamma}$, where k_{Γ} is introduced as in Notation 2.7, u_{Γ} is defined as in Definition 2.14, and $n_{\Gamma} \in N$ is the clockwise primitive normal to Q_{Γ} .

We end this section by stating two of the main results in [29] which describe how the perturbed LG potential $W_n(Q)$ jumps across the walls in the scattering diagram $\mathcal{D}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$.

Theorem 2.23 (Proposition 4.7 and Theorem 4.12 in [29]). *For any point $j \in \text{Sing}(\mathcal{D}) \setminus \{P_1, \dots, P_n\}$ and any loop γ_j around j in a sufficiently small contractible neighborhood U_j of j , we have*

$$\Theta_{\gamma_j, \mathcal{D}} = \text{Id}.$$

Furthermore, if $Q, Q' \in M_{\mathbb{R}}$ are not lying on any walls in $\mathcal{D}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$, then we have

$$W_n(Q') = \Theta_{\gamma, \mathcal{D}}(W_n(Q))$$

for any path $\gamma \subset M_{\mathbb{R}} \setminus \text{Sing}(\mathcal{D})$ joining Q to Q' .

As we have seen in the introduction, the main result of this paper (i.e., Theorem 1.1) combined with the main results of [10] can give new interpretations and alternative proofs of these two results.

3. EXTENDED DEFORMATION THEORY OF THE LG MODEL

In this section, we investigate the dgBV algebra governing the extended deformation theory of the LG model $(\check{\mathcal{X}}, W)$ and the asymptotic behavior of the Maurer-Cartan solutions when the torus fibers of the fibration $\check{p} : \check{X} \rightarrow \text{Int}(\mathbf{P})$ shrink, building on the techniques developed in [10].

3.1. The dgBV algebra coming from polyvector fields. Given an LG model (\check{X}, W) equipped with a holomorphic volume form $\check{\Omega}$, one can construct a natural differential graded Batalin-Vilkovisky (dgBV) algebra on the Dolbeault resolution of the sheaf of polyvector fields on \check{X} given by $PV^{i,j}(\check{X}) := \Omega^{0,j}(\check{X}, \wedge^i T_{\check{X}}^{1,0})$, where the degree on $PV^{i,j}(\check{X})$ is taken to be $j - i$. We briefly review this construction; see, e.g., [44].

Notation 3.1. Given local holomorphic coordinates u^1, \dots, u^n on \check{X} and an ordered subset $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, we set $d\bar{u}^I := d\bar{u}^{i_1} \wedge \dots \wedge d\bar{u}^{i_k}$, $\partial_I := \frac{\partial}{\partial u^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial u^{i_k}}$, and similarly for du^I and $\bar{\partial}_I$.

First of all, the space of smooth sections of $\wedge^* T_{\check{X}}^{1,0}$ is equipped with a natural wedge product \wedge . With a holomorphic volume form $\check{\Omega} = e^f du^1 \dots du^n$ and a polyvector field of the form ∂_I where $I = \{i_1, \dots, i_k\}$, we define $\partial_I \lrcorner \check{\Omega} := \iota_{\frac{\partial}{\partial u^{i_1}}} \dots \iota_{\frac{\partial}{\partial u^{i_k}}} \check{\Omega}$.

Definition 3.2. The BV differential $\Delta_{\check{\Omega}}$ is defined by³

$$(3.1) \quad \Delta_{\check{\Omega}} \alpha \lrcorner \check{\Omega} := \partial(\alpha \lrcorner \check{\Omega}).$$

The operation $\delta_v : \wedge^* T^{1,0} \rightarrow \wedge^{*-1} T^{1,0}$ defined by

$$(3.2) \quad \delta_v(w) := \Delta(v \wedge w) - \Delta(v) \wedge w - (-1)^k v \wedge \Delta(w)$$

is a derivation of degree $k + 1$.

Definition 3.3. We define the bracket $[\cdot, \cdot] : V \otimes V \rightarrow V$ by $[v, w] = (-1)^{|v|+1} \delta_v(w)$, where $|v|$ stands for the degree of a homogeneous element v .⁴

These structures can be extended to the Dolbeault resolution $PV^{*,*}(\check{X})$ of $\wedge^* T^{1,0}$ equipped with the twisted differential $\bar{\partial}_W = \bar{\partial} + [W, \cdot]$ and the graded commutative wedge product \wedge . In the local holomorphic coordinates u^1, \dots, u^n , writing $\alpha = \alpha_J^I d\bar{u}^J \wedge \partial_I$ (with $|I| = i$ and $|J| = j$) and $\beta = \beta_L^K d\bar{u}^L \wedge \partial_K$ (with $|K| = k$ and $|L| = l$), we have

$$\begin{aligned} \bar{\partial}(\alpha) &= \bar{\partial}(\alpha_J^I) d\bar{u}^J \wedge \partial_I; & \Delta(\alpha) &= (-1)^j d\bar{u}^J \wedge \Delta(\alpha_J^I \partial_I); \\ \alpha \wedge \beta &= (-1)^{il} \alpha_J^I \beta_L^K d\bar{u}^J \wedge d\bar{u}^L \wedge \partial_I \wedge \partial_K; & [\alpha, \beta] &= (-1)^{(i+1)l} d\bar{u}^J d\bar{u}^L [\alpha_J^I \partial_I, \beta_L^K \partial_K]. \end{aligned}$$

From these we obtain the differential graded Lie algebra (dgLa) $(PV^{*,*}[1], [\cdot, \cdot], \bar{\partial}_W)$, where $[1]$ is a degree shift. We will study the asymptotic behavior of solutions of the Maurer-Cartan equation (1.3) for degree 0 elements φ in $PV^{*,*} \otimes_{\mathbb{C}} \widehat{\mathbb{C}[Q]}_{\mathcal{S}} \otimes_{\mathbb{C}} \mathfrak{m}_n$.

Going back to our situation, by extending the exact sequence of monoids (2.1) to the associated abelian groups, we get the so-called *fan sequence* in toric geometry [13, 24]:

$$(3.3) \quad 0 \longrightarrow Q^{gp} \longrightarrow \mathcal{P}^{gp} \xrightarrow{\theta} M \longrightarrow 0.$$

³We will suppress the dependence of the BV differential on $\check{\Omega}$ whenever there is no danger of confusion.

⁴The bracket $[\cdot, \cdot]$ agrees with the well-known *Schouten-Nijenhuis Lie bracket* on smooth sections of $\wedge^* T^{1,0}$ which can be expressed as $[v_1 \wedge \dots \wedge v_k, v_1 \wedge \dots \wedge v_{k'}] = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k'}} (-1)^{i+j} [v_i, v_j] \wedge v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_k \wedge \dots \wedge \widehat{v}_j \wedge \dots \wedge v_{k'}$ and $[v_1 \wedge \dots \wedge v_k, f] = \sum_i (-1)^{k-i} v_i(f) v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge v_k$.

We have $\mathcal{P}^{gp} \cong \mathbb{Z}^{|\Sigma(1)|} \cong M \times \mathcal{Q}^{gp}$, giving a trivialization of the family over $\text{Spec}(\mathbb{C}[\mathcal{Q}^{gp}]) \subset \text{Spec}(\mathbb{C}[\mathcal{Q}])$ as

$$(3.4) \quad \check{X} \times_{\text{Spec}(\mathbb{C}[\mathcal{Q}])} \text{Spec}(\mathbb{C}[\mathcal{Q}^{gp}]) \cong T_N \times \text{Spec}(\mathbb{C}[\mathcal{Q}^{gp}]),$$

where $T_N := (N \otimes_{\mathbb{Z}} \mathbb{C})/N$ is a 2-dimensional algebraic torus.

Since \mathcal{Q} is a strictly convex polyhedral cone, there is a natural maximal ideal $\mathfrak{m} = \mathfrak{m}_{\mathcal{Q}} := \langle z^m \mid m \in \mathcal{Q} \setminus \{0\} \rangle$ in $\mathbb{C}[\mathcal{Q}]$. We consider the completion $\widehat{\mathbb{C}[\mathcal{Q}]} := \varprojlim_k \mathbb{C}[\mathcal{Q}]/\mathfrak{m}^k$ and its localization $\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}}$ at the multiplicative system $\mathcal{S} = \{z^m \mid m \in \mathcal{Q} \setminus \{0\}\}$. By taking the tensor product $\mathbb{C}[\mathcal{P}^{gp}] \otimes_{\mathbb{C}[\mathcal{Q}^{gp}]} \widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}} = \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}}$, we can treat $W \in \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}}$ as a family of LG potentials parametrized by $\widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}}$ on the (fixed) algebraic torus T_N . For the LG model $(\check{X}, W) := (T_N, W)$, we choose the local holomorphic coordinates as follows.

Notation 3.4. We fix, once and for all, a \mathbb{Z} -basis e_1, e_2 for M and identify $\mathfrak{m} = \mathfrak{m}_1 e_1 + \mathfrak{m}_2 e_2$ with $(\mathfrak{m}_1, \mathfrak{m}_2) \in \mathbb{Z}^2$. We also use $w^{\mathfrak{m}} = (w^1)^{\mathfrak{m}_1} (w^2)^{\mathfrak{m}_2}$, for $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2) \in M$, to denote a monomial on \check{X} . Notice that every $\mathfrak{m} \in M$ naturally gives a $(1, 0)$ -form $d \log(\mathfrak{m}) := d \log(w^{\mathfrak{m}})$; similarly, every $n \in N$ naturally gives a vector field $\check{\partial}_n$ satisfying $\check{\partial}_n(w^{\mathfrak{m}}) = (n, \mathfrak{m}) w^{\mathfrak{m}}$, where (\cdot, \cdot) is the natural pairing between M and N .

Equipped with the natural holomorphic volume form $\check{\Omega} := d \log w^1 \wedge d \log w^2$ on \check{X} , we obtain the triple $(\check{X}, W, \check{\Omega})$, and hence a dgBV algebra by the above discussion.

3.1.1. \hbar -family of SYZ fibrations. Following a proposal by Kontsevich-Soibelman [41] and Fukaya [17], we consider an \hbar -family of SYZ fibrations which corresponds to a large complex structure limit, so that we can apply asymptotic analysis as in the previous work [10].

We consider the log map $\text{Log} : T_N \cong (N_{\mathbb{C}}/N) \rightarrow \sqrt{-1}N_{\mathbb{R}}$ which is naturally a torus fibration. We fix a symplectic structure ω_0 on the toric surface X_{Σ} and consider the associated moment polytope $\mathbf{P} \subset \sqrt{-1}N_{\mathbb{R}}$. From the SYZ viewpoint [9, 53], the mirror manifold is obtained by dualizing the moment map on X_{Σ} , so we choose the base of the SYZ fibration to be $\check{B}_0 := \text{Int}(\mathbf{P})$ and take $\check{X} = \check{p}^{-1}(\check{B}_0)$ instead of the whole algebraic torus T_N .

Let $\{e^1, e^2\}$ be the \mathbb{Z} -basis of N dual to the chosen basis $\{e_1, e_2\}$ of M . We then let (x^1, x^2) be the oriented affine coordinates of \check{B}_0 with respect to the basis $\{e^1, e^2\}$ and let (y^1, y^2) be the affine coordinates on the torus fibers of $\check{p} : \check{X} \rightarrow \check{B}_0$.

Associated to the symplectic structure ω_0 , there is a symplectic potential $\check{\phi}$ in the action-angle coordinates written explicitly in [36]. We take $\check{\phi}$ and apply the Legendre transform $\check{L}_{\check{\phi}} : \text{Int}(\mathbf{P}) \rightarrow M_{\mathbb{R}}$ to obtain the dual integral affine manifold B_0 equipped with affine coordinates $x_1 := \frac{\partial \check{\phi}}{\partial x^1}$, $x_2 := \frac{\partial \check{\phi}}{\partial x^2}$. We prefer to work with the affine manifold B_0 because then we can deal with tropical trees instead of Morse trees, as explained in [35] (see also [10, Section 2]).

We then introduce a small $\hbar > 0$ parameter to rescale the affine coordinates on \check{B}_0 as $x^j \mapsto \hbar^{-1} x^j$, and obtain the (\hbar -dependent) holomorphic coordinates $w^j = \exp(-2\pi i(y^j + i\hbar^{-1} x^j))$ (cf. [10, Section 2]). Under these \hbar -twisted coordinates,

the holomorphic vector field $\check{\partial}_j$ is explicitly given by
 (3.5)

$$n = (n^j) \mapsto \check{\partial}_n := \sum_j n^j \check{\partial}_j = \sum_j n^j \frac{\partial}{\partial \log w^j} = \frac{i}{4\pi} \sum_j n^j \left(\frac{\partial}{\partial y^j} - i\hbar \sum_k g_{jk} \frac{\partial}{\partial x_k} \right).$$

The corresponding \hbar -dependent dgLa of polyvector fields will be denoted by $PV_{\hbar}^{*,*}$. We will consider differential forms on B_0 depending on \hbar and hence we introduce the following.

Notation 3.5. We use $\Omega_{\hbar}^*(B_0)$ (similarly for $\Omega_{\hbar}^*(U)$ for any open subset $U \subset B_0$) to denote the space of smooth sections of $\wedge^* T^*B_0$ over $B_0 \times \mathbb{R}_{>0}$, where the extra $\mathbb{R}_{>0}$ direction is parametrized by \hbar .

3.1.2. *Fourier expansions of polyvector fields.* Recall that the Fourier transform $\hat{\mathcal{F}}$

$$(3.6) \quad \hat{\mathcal{F}} : \mathbf{G}_n^{*,*} := \left(\bigoplus_{m \in \mathcal{P}} \Omega_{\hbar}^*(B_0) z^m \right) \otimes_{\mathbb{Z}} \wedge^* N \otimes_{\mathbb{C}} R_n \hookrightarrow PV_{\hbar}^{*,*} \otimes_{\mathbb{C}} \widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}} \otimes_{\mathbb{C}} R_n,$$

introduced in [10, Section 2], gives an inclusion of dg Lie subalgebras by⁵

- (1) identifying the Fourier modes $z^m \in \mathbb{C}[\mathcal{P}] \hookrightarrow \mathbb{C}[\mathcal{P}^{gp}]$ as $z^m = w^{\hat{m}} \otimes q^{\hat{m}} \in \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}}$ through the isomorphism $\mathbb{C}[\mathcal{P}^{gp}] \cong \mathbb{C}[M] \otimes_{\mathbb{C}} \widehat{\mathbb{C}[\mathcal{Q}]}_{\mathcal{S}}$,
- (2) pulling back smooth functions $f(x, \hbar)$ on B_0 to \check{X} via the equation $\hat{\mathcal{F}}(f(x, \hbar)) = \check{p}^{-1}(f(x, \hbar))$ and using the torus fibration $\check{p} : \check{X} \rightarrow B_0$,
- (3) identifying the 1-form $dx^j = \sum_k \frac{\partial^2 \phi}{\partial x_j \partial x_k} dx_k$ (where $\phi : B_0 \rightarrow \mathbb{R}$ is the Legendre dual to $\check{\phi}$) on B_0 with the $(0, 1)$ -form $\hat{\mathcal{F}}(dx^j) = \frac{\hbar}{4\pi} d \log \bar{w}^j$ on \check{X} for $j = 1, 2$,
- (4) identifying $n \in N$ with the holomorphic vector field $\check{\partial}_n$ on \check{X} by (3.5), and
- (5) extending the map skew-symmetrically.

The Dolbeaut differential $\bar{\partial}$ is identified with the deRham differential d acting on each summand $\Omega_{\hbar}^*(B_0)$ via $\hat{\mathcal{F}}$. The action of a vector field $\check{\partial}_n = (n^1, n^2)$ on $f(x, \hbar)$ by differentiation is identified as

$$(3.7) \quad \check{\partial}_n(f) = \frac{\hbar}{4\pi} \sum_{j,k} n^j \frac{\partial^2 \check{\phi}}{\partial x^j \partial x^k} \frac{\partial}{\partial x_k}(f)$$

via $\hat{\mathcal{F}}$ (recall that x^1, x^2 are affine coordinates on $\check{B}_0 \cong \text{Int}(\mathbf{P})$ while x_1, x_2 are affine coordinates on $B_0 = M_{\mathbb{R}}$).

3.2. Differential forms with asymptotic support. We will work with a dgLa constructed as a suitable quotient of a subalgebra of $\mathbf{G}_n^{*,*}$ (defined above in (3.6)), which turns out to be directly related to the tropical counting defined in Section 2. In order to do so, we need to recall the notion of *asymptotic support on a closed codimension k tropical polyhedral subset* $P \subset U$ for some convex $U \subset B_0$ which describes the behavior of differential forms $\alpha \in \Omega_{\hbar}^*(B_0)$ as $\hbar \rightarrow 0$ and also some of its basic properties from [10].

First of all, by a *tropical polyhedral subset* in U we mean a connected convex subset which is defined by *finitely many* affine linear equations or inequalities over \mathbb{Q} . For the purpose of proving the main theorem in this paper, we only need

⁵It is an inclusion since we restrict ourselves to Fourier modes m in \mathcal{P} and we take finite sums instead of infinite Fourier series.

the cases when P is either a point (whence $\dim_{\mathbb{R}}(P) = 0$) or a ray/line (whence $\dim_{\mathbb{R}}(P) = 1$) or a polyhedral domain (whence $\dim_{\mathbb{R}}(P) = 2$). However, since the new properties established in this subsection should be of independent interest and useful in a broader context, we will work with a convex open subset $U \subset B_0$ in a general (oriented) affine manifold B_0 in arbitrary dimensions.

Definition 3.6. We define $\mathcal{W}_k^{-\infty}(U) \subset \Omega_{\hbar}^k(U)$ to be the set of differential k -forms $\alpha \in \Omega_{\hbar}^k(U)$ such that for each point $q \in U$, there exists a neighborhood V of q where we have $\|\nabla^j \alpha\|_{L^\infty(V)} \leq D_{j,V} e^{-c_V/\hbar}$ for some constants c_V and $D_{j,V}$. The association $U \mapsto \mathcal{W}_k^{-\infty}(U)$ defines a sheaf over B_0 which we denote by $\mathcal{W}_k^{-\infty}$.

We also need differential forms which only blow up at polynomial orders in \hbar^{-1} .

Definition 3.7. We define $\mathcal{W}_k^\infty(U) \subset \Omega_{\hbar}^k(U)$ to be the set of differential k -forms $\alpha \in \Omega_{\hbar}^k(U)$ such that for each point $q \in U$, there exists a neighborhood V of q where we have $\|\nabla^j \alpha\|_{L^\infty(V)} \leq D_{j,V} \hbar^{-N_{j,V}}$ for some constants $D_{j,V}$ and $N_{j,V} \in \mathbb{Z}_{>0}$. The association $U \mapsto \mathcal{W}_k^\infty(U)$ defines a sheaf over B_0 which we denote by \mathcal{W}_k^∞ .

Notice that the sheaves $\mathcal{W}_k^{\pm\infty}$ in Definitions 3.6 and 3.7 are closed under the actions of $\nabla_{\frac{\partial}{\partial x}}$, the deRham differential d , and the wedge product of differential forms. We also observe the fact that $\mathcal{W}_k^{-\infty}$ is a differential graded ideal of \mathcal{W}_k^∞ . In particular, we can consider the sheaf of differential graded algebras $\mathcal{W}_*^\infty/\mathcal{W}_*^{-\infty}$, equipped with the deRham differential.

Definition 3.8. A differential k -form $\alpha \in \mathcal{W}_k^\infty(U)$ is said to have *asymptotic support on a closed codimension k tropical polyhedral subset $P \subset U$ with weight s* , denoted by $\alpha \in \mathcal{W}_P^s$ if the following conditions are satisfied:

- (1) For any $p \in U \setminus P$, there is a neighborhood $V \subset U \setminus P$ of p such that $\alpha|_V \in \mathcal{W}_k^{-\infty}(V)$ on V .
- (2) There exists a neighborhood W_P of P in U such that we can write $\alpha = h(x, \hbar)\nu_P + \eta$, where $\nu_P \in \bigwedge^k N_{\mathbb{R}}$ is the unique affine k -form which is normal to P , $h(x, \hbar) \in C^\infty(W_P \times \mathbb{R}_{>0})$ and η is an error term satisfying $\eta \in \mathcal{W}_k^{-\infty}(W_P)$ on W_P .
- (3) For any $p \in P$, there exists a sufficiently small convex neighborhood V containing p equipped with an affine coordinate system $x = (x_1, \dots, x_n)$ such that $x' := (x_1, \dots, x_k)$ parametrizes codimension k affine linear subspaces of V parallel to P , with $x' = 0$ corresponding to the subspace containing P . Within the foliation $\{(P_{V,x'})\}_{x' \in N_V}$ where $P_{V,x'} = \{(x_1, \dots, x_n) \in V \mid (x_1, \dots, x_k) = x'\}$ of V , we require that, for all $j \in \mathbb{Z}_{\geq 0}$ and multi-index $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{Z}_{\geq 0}^k$, the estimate

$$(3.8) \quad \int_{x'} (x')^\beta \left(\sup_{P_{V,x'}} |\nabla^j (\iota_{\nu_P} \alpha)| \right) \nu_P \leq D_{j,V,\beta} \hbar^{-\frac{j+s-|\beta|-k}{2}},$$

for some constant $D_{j,V,\beta}$ and some $s \in \mathbb{Z}$, where $|\beta| = \sum_l \beta_l$ is the vanishing order of the monomial $(x')^\beta = x_1^{\beta_1} \dots x_k^{\beta_k}$ along $P_{x'=0}$ and $\nu_P^\vee = \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_k}$ in this local coordinate.

Remark 3.9. Note that condition (3) in Definition 3.8 is independent of the choices of the convex neighborhood V , the transversal slice N_V , and the local affine coordinates $x = (x_1, \dots, x_n)$ (although the constant $D_{j,V,\beta}$ may depend on these choices). Therefore this condition can be checked simply by choosing a sufficiently nice neighborhood V at every point $p \in P$.

By definition, we have the nice property that

$$(3.9) \quad (x')^\beta \nabla_{\frac{\partial}{\partial x_{i_1}}} \cdots \nabla_{\frac{\partial}{\partial x_{i_j}}} \mathcal{W}_P^s(U) \subset \mathcal{W}_P^{s+j-|\beta|}(U)$$

for any affine monomial $(x')^\beta$ with vanishing order $|\beta|$ along P .

The weight s in Definition 3.8 defines the following filtration (the U dependence will be dropped whenever it is clear from the context):⁶

$$(3.10) \quad \mathcal{W}_k^{-\infty} \cdots \subset \mathcal{W}_P^{-s} \subset \cdots \subset \mathcal{W}_P^{-1} \subset \mathcal{W}_P^0 \subset \mathcal{W}_P^1 \subset \mathcal{W}_P^2 \subset \cdots \subset \mathcal{W}_P^s \subset \cdots \subset \mathcal{W}_k^\infty \subset \Omega_h^k(U).$$

This filtration keeps track of the polynomial orders of \hbar for differential k -forms with asymptotic support on P and provides a convenient tool for expressing and proving results in asymptotic analysis.

3.2.1. Behavior under d and \wedge .

Definition 3.10. A differential k -form α is said to be in $\tilde{\mathcal{W}}_k^s(U)$ if there exist finitely many polyhedral subsets P_1, \dots, P_l of codimension k such that $\alpha \in \sum_{j=1}^l \mathcal{W}_{P_j}^s(U)$; if we further have $d\alpha \in \tilde{\mathcal{W}}_{k+1}^{s+1}(U)$, then we say α is in $\mathcal{W}_k^s(U)$. We also let $\mathcal{W}_*^s(U) := \bigoplus_k \mathcal{W}_k^{s+k}(U)$ for every $s \in \mathbb{Z}$.

We have the following lemma on the compatibility between the filtration and the wedge product.

Lemma 3.11. *For two closed tropical polyhedral subsets $P_1, P_2 \subset U$ of codimension k_1, k_2 , respectively, we have $\mathcal{W}_{P_1}^s(U) \wedge \mathcal{W}_{P_2}^r(U) \subset \mathcal{W}_{P_1+P_2}^{r+s}(U)$ for any codimension $k_1 + k_2$ polyhedral subset P containing $P_1 \cap P_2$ normal to $\nu_{P_1} \wedge \nu_{P_2}$ if they intersect transversally (in particular if $\text{codim}_{\mathbb{R}}(P_1 \cap P_2) = k_1 + k_2$ we can take $P = P_1 \cap P_2$), and $\mathcal{W}_{P_1}^s(U) \wedge \mathcal{W}_{P_2}^r(U) \subset \mathcal{W}_{k_1+k_2}^{-\infty}(U)$ if their intersection is not transversal. Furthermore, we have $\mathcal{W}_{k_1}^{s_1}(U) \wedge \mathcal{W}_{k_2}^{s_2}(U) \subset \mathcal{W}_{k_1+k_2}^{s_1+s_2}(U)$. Hence $\mathcal{W}_*^0(U) \subset \mathcal{W}_*^\infty(U)$ is a dg subalgebra and $\mathcal{W}_*^{-1}(U) \subset \mathcal{W}_*^0(U)$ is a dg ideal of $\mathcal{W}_*^0(U)$, under the operations d and \wedge .*

Before giving the proof, let us clarify that when we say two closed tropical polyhedral subsets $P_1, P_2 \subset U$ of codimension k_1, k_2 are *intersecting transversally*, we mean the affine subspaces containing P_1, P_2 and of codimension k_1, k_2 , respectively, are intersecting transversally; this applies even to the case when $\partial P_i \neq \emptyset$.

Proof of Lemma 3.11. The first statement is nothing but [10, Lemma 4.22], which in turn implies the second statement as follows: Given polyhedral subsets P_1 and P_2 of codimensions k_1 and k_2 , respectively, notice that we always have some polyhedral subset P of codimension $k = k_1 + k_2$ such that $\mathcal{W}_{P_1}^{s_1}(U) \wedge \mathcal{W}_{P_2}^{s_2}(U) \subset \mathcal{W}_P^{s_1+s_2}(U)$. Therefore, we conclude that $\tilde{\mathcal{W}}_{k_1}^{s_1}(U) \wedge \tilde{\mathcal{W}}_{k_2}^{s_2}(U) \subset \tilde{\mathcal{W}}_{k_1+k_2}^{s_1+s_2}(U)$. Now, suppose $\alpha_i \in \mathcal{W}_{k_i}^{s_i}(U)$. Then we have $d\alpha_i \in \tilde{\mathcal{W}}_{k_i+1}^{s_i+1}(U)$ and therefore $(d\alpha_1) \wedge \alpha_2 \in \tilde{\mathcal{W}}_{k_1+k_2+1}^{s_1+s_2+1}(U)$;

⁶Note that the degree k of the differential forms has to be equal to the codimension of P . Also note that the sets $\mathcal{W}_k^{\pm\infty}(U)$ are independent of the choice of P .

a similar statement holds for $\alpha \wedge (d\alpha_2)$. Finally, the statements that $\mathcal{W}_*^0(U) \subset \mathcal{W}_*^\infty(U)$ is a dg subalgebra and $\mathcal{W}_*^{-1}(U) \subset \mathcal{W}_*^0(U)$ is a dg ideal follow from $\mathcal{W}_{k_1}^{s_1}(U) \wedge \mathcal{W}_{k_2}^{s_2}(U) \subset \mathcal{W}_{k_1+k_2}^{s_1+s_2}(U)$. \square

3.2.2. Behavior under integral operators. In this subsection, we study the behavior of $\mathcal{W}_P^s(U)$ under the action of an integral operator I , generalizing some of the results in [10, Section 4.2]. For a given closed tropical polyhedral subset $P \subset U$, we choose a reference tropical hyperplane $R \subset U$ which divide the domain U into $U \setminus R = U_+ \cup U_-$, together with an affine vector field v (meaning $\nabla v = 0$) not tangent to R pointing into U_+ .

By shrinking U if necessary, we assume that for any point $p \in U$, the unique flow line of v in U passing through p intersects R uniquely at a point $x \in R$. Then the time- t flow along v defines a diffeomorphism $\tau : W \rightarrow U$, $(t, x) \mapsto \tau(t, x)$, where $W \subset \mathbb{R} \times R$ is the maximal domain of definition of τ (namely, for any $x \in R$, there is a maximal time interval $I_x \subset \mathbb{R}$ so that the flow line through x has its image lying inside U). For any point $x \in R$, we denote by $\tau_x(t) := \tau(t, x)$ the flow line of v passing through x . Figure 2 illustrates the situation.

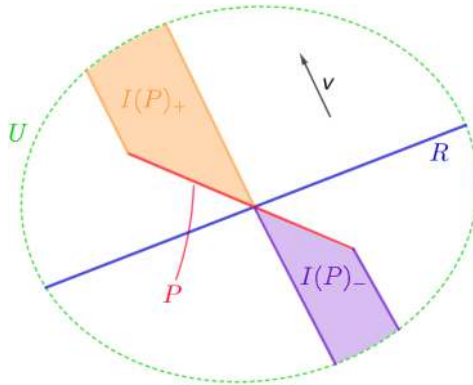


FIGURE 2. The flow along v and $I(P)$

We let $P_\pm = P \cap \overline{U}_\pm$ and define

$$(3.11) \quad I(P)_+ := (P_+ + \mathbb{R}_{\geq 0} \cdot v) \cap U; \quad I(P)_- := (P_- + \mathbb{R}_{\leq 0} \cdot v) \cap U,$$

(see again Figure 2). We also write $I(P) = I(P)_+ \cup I(P)_-$. Now we define an integral operator I by

$$(3.12) \quad I(\alpha)(t, x) := \int_0^t \iota_{\frac{\partial}{\partial s}}(\tau^*(\alpha))(s, x) ds.$$

Note that I depends on the choice of the tropical hyperplane R . We have the following lemma, which is a modification of [10, Lemma 4.23].

Lemma 3.12 (cf. Lemma 4.23 in [10]). *For $\alpha \in \mathcal{W}_P^s(U)$, we have $I(\alpha) \in \mathcal{W}_{k-1}^{-\infty}(U)$ if v is tangent to P , and $I(\alpha) \in \mathcal{W}_{I(P)_+}^{s-1}(U) + \mathcal{W}_{I(P)_-}^{s-1}(U)$ if v is not tangent to P , where $I(P)_\pm$ is defined in (3.11). Moreover, for $\alpha \in \tilde{\mathcal{W}}_k^s(U)$, we have $I(\alpha) \in \tilde{\mathcal{W}}_{k-1}^{s-1}(U)$.*

Proof. We only describe the modifications needed in order to adapt the proof of [10, Lemma 4.23]. We introduce a decomposition $\alpha = \alpha_+ + \alpha_-$ of α , where the components α_+ and α_- have asymptotic support of the same weight on P_+ and P_- respectively, using cut-offs as follows. First we consider the functions depending only on the t -coordinate given by

$$\chi_+(t) := \left(\frac{1}{\hbar\pi}\right)^{\frac{1}{2}} \int_{-\infty}^t e^{-\frac{s^2}{\hbar}} ds, \quad \chi_-(t) := 1 - \chi_+(t) = \left(\frac{1}{\hbar\pi}\right)^{\frac{1}{2}} \int_t^{\infty} e^{-\frac{s^2}{\hbar}} ds;$$

they have asymptotic support with weight 0 on $U_+ = \{t \geq 0\} \cap U$ and $U_- = \{t \leq 0\} \cap U$, respectively. Lemma 3.11 implies that the cut-offs $\alpha_{\pm} := \chi_{\pm}\alpha$ have asymptotic support with the same weight s on $U_{\pm} \cap P$, respectively. Therefore we may start by assuming $\alpha \in \mathcal{W}_P^s(U)$ with $P \subset \overline{U}_+$ and we simply write $I(P)$ to stand for $I(P)_+$. The rest of the proof is essentially the same as that of [10, Lemma 4.23]. \square

In order to understand the effect of I on $\mathcal{W}_k^s(U)$, we need the following Lemmas 3.13 and 3.14 which describe the behavior of $\mathcal{W}_P^s(U)$ under pullbacks. Following the notation in Lemma 3.12, we consider the tropical hypersurface $\mathbf{i} : R \subset U$ with an affine projection $\mathbf{p} : U \rightarrow R$ (which are explicitly given by the $\mathbf{i}(x) = (0, x)$ and $\mathbf{p}(t, x) = x$ using the affine coordinates given by τ).

Lemma 3.13. *For $\alpha \in \mathcal{W}_P^s(U)$, we have $\mathbf{i}^*(\alpha) \in \mathcal{W}_Q^s(R)$ if P intersects R transversally and Q is any polyhedral subset of R of codimension k ($= \text{codim}_{\mathbb{R}}(P \subset U)$) which contains $P \cap R$ and is normal to $\mathbf{i}^*(\nu_P)$, and $\mathbf{i}^*(\alpha) \in \mathcal{W}_k^{-\infty}(U)$ if P does not intersect R transversally. Moreover, the pull back gives a map $\mathbf{i}^* : \mathcal{W}_k^s(U) \rightarrow \mathcal{W}_k^s(R)$.*

Proof. We begin by showing the corresponding statement for $\mathcal{W}_P^s(U)$. First, we verify condition (1) of Definition 3.8. Suppose that $p \in R \setminus P$, then we can find a neighborhood V of p in $U \setminus P$ such that $\alpha|_V \in \mathcal{W}_k^{-\infty}(V)$ from the assumption that $\alpha \in \mathcal{W}_P^s(U)$. Therefore $\alpha|_{V \cap R} \in \mathcal{W}_k^{-\infty}(V \cap R)$.

For condition (2) of Definition 3.8, we first assume that P and R are not intersecting transversally. We notice that there is a neighborhood W_P of P such that α can be written as $h(x, \hbar)\nu_P + \eta$ in W_P from the assumption that $\alpha \in \mathcal{W}_P^s(U)$. Therefore we have $\mathbf{i}^*(\nu_P) = 0$ if the intersection is not transversal, and so $\mathbf{i}^*(\alpha) \in \mathcal{W}_k^{-\infty}(R)$. Suppose P and R intersect transversally, then we can take $W_{P \cap R} := \mathbf{i}^{-1}(W_P)$, and we will have $\mathbf{i}^*(\alpha)|_{W_{P \cap R}} = \mathbf{i}^*(h)\mathbf{i}^*(\nu_P) + \mathbf{i}^*(\eta)$ in $W_{P \cap R}$ with $\mathbf{i}^*(\nu_P)$ being the volume form of a normal bundle of $\mathbf{i}^{-1}(P)$ as desired for condition (2). Notice that if $Q \neq R \cap P$ and for any point $p \notin R \cap P$, there is a neighborhood V of p such that $\alpha|_{V \cap R} \in \mathcal{W}_k^{-\infty}(V \cap R)$ from our earlier discussion, and therefore condition (2) still holds for arbitrary such Q .

For condition (3), we consider a point $p \in R \cap P$ with affine coordinates

$$(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in (-\delta, \delta)^n$$

in $V \subset U$ such that $R \cap V = \{x_n = 0\}$, and $x' = (x_1, \dots, x_k)$ are parametrizing the parallel foliation $\{P_{V, x'}\}_{x' \in (-\delta, \delta)^k}$ to P in V . Then $\{P_{V, x'} \cap R\}_{x' \in (-\delta, \delta)^k}$ is the

foliation parallel to $P \cap R$ in $V \cap R$. Using the fact that $\sup_{P_{V,x'} \cap R} |\nabla^j(\iota_{\nu_P^\vee} \alpha)| \leq \sup_{P_{V,x'}} |\nabla^j(\iota_{\nu_P^\vee} \alpha)|$, we have

$$\begin{aligned} \int_{x' \in N_V} (x')^\beta \left(\sup_{P_{V,x'} \cap R} |\nabla^j(\iota_{\nu_P^\vee} \alpha)| \right) \nu_P &\leq \int_{x' \in N_V} (x')^\beta \left(\sup_{P_{V,x'}} |\nabla^j(\iota_{\nu_P^\vee} \alpha)| \right) \nu_P \\ &\leq D_{j,V,\beta} \hbar^{-\frac{j+s-|\beta|-k}{2}}, \end{aligned}$$

which is the desired estimate for condition (3) of Definition 3.8.

The statement that \mathbf{i}^* is a map from $\mathcal{W}_k^s(U)$ to $\mathcal{W}_k^s(R)$ is a direct consequence of the first statement. □

Lemma 3.14. *For $\alpha \in \mathcal{W}_P^s(R)$, we have $\mathbf{p}^*(\alpha) \in \mathcal{W}_{\mathbf{p}^{-1}(P)}^s(U)$. Moreover, the pull back gives a map $\mathbf{p}^* : \mathcal{W}_k^s(R) \rightarrow \mathcal{W}_k^s(U)$.*

Proof. For condition (1) of Definition 3.8, suppose we take $x \in U \setminus \mathbf{p}^{-1}(P)$, then we have an open subset $V \subset R \setminus P$ containing $\mathbf{p}(x)$. Therefore from the fact that $\alpha|_V \in \mathcal{W}_k^{-\infty}(V)$ (here $k = \text{codim}_{\mathbb{R}}(\mathbf{p}^{-1}(P))$) we get $\mathbf{p}^*(\alpha)|_{\mathbf{p}^{-1}(V)} \in \mathcal{W}_k^{-\infty}(\mathbf{p}^{-1}(V))$.

For condition (2) of Definition 3.8, we take a neighborhood W_P of P in R such that we can write α as $h\nu_P + \eta$ with $\eta \in \mathcal{W}_k^{-\infty}(R)$ and ν_P is the normal of P in R . We let $W_{\mathbf{p}^{-1}(P)} = \mathbf{p}^{-1}(W_P)$, and observe that $\mathbf{p}^*(\alpha) = \mathbf{p}^*(h)\mathbf{p}^*(\nu_P) + \mathbf{p}^*(\eta)$ with $\mathbf{p}^*(\nu_P)$ being normal of $\mathbf{p}^{-1}(P)$ in U which is the desired decomposition.

For condition (3), we consider a point $p \in \mathbf{p}^{-1}(P)$ with affine coordinates $(x_1, \dots, x_k, x_{k+1}, \dots, x_{n-1}) \in (-\delta, \delta)^{n-1}$ around $q := \mathbf{p}(p)$ in $V \subset R$ such that $x' = (x_1, \dots, x_k)$ are parametrizing the foliation $\{P_{V,x'}\}_{x' \in (-\delta, \delta)^k}$ parallel to P in V . Therefore, we can extend the affine coordinates as $(x_1, \dots, x_k, x_{k+1}, \dots, x_n)$ of $\mathbf{p}^{-1}(V)$ such that $\mathbf{p}(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (x_1, \dots, x_k, x_{k+1}, \dots, x_{n-1})$ in these coordinates. We notice that $\{\mathbf{p}^{-1}(P_{V,x'})\}_{x' \in (-\delta, \delta)^k}$ is the foliation parallel to $\mathbf{p}^{-1}(P)$ in $\mathbf{p}^{-1}(V)$ and we also have

$$\sup_{P_{V,x'} \cap R} |\nabla^j(\iota_{\nu_P^\vee} \alpha)| = \sup_{\mathbf{p}^{-1}(P_{V,x'})} |\nabla^j(\iota_{\mathbf{p}^*(\nu_P) \vee \mathbf{p}^*(\alpha)})|.$$

Therefore we conclude that

$$\begin{aligned} &\int_{x' \in N_{\mathbf{p}^{-1}(V)}} (x')^\beta \left(\sup_{\mathbf{p}^{-1}(P_{V,x'})} |\nabla^j(\iota_{\mathbf{p}^*(\nu_P) \vee \mathbf{p}^*(\alpha)})| \right) \mathbf{p}^*(\nu_P) \\ &= \int_{x' \in N_V} (x')^\beta \left(\sup_{P_{V,x'}} |\nabla^j(\iota_{\nu_P^\vee} \alpha)| \right) \nu_P \leq D_{j,V,\beta} \hbar^{-\frac{j+s-|\beta|-k}{2}}, \end{aligned}$$

which is the desired estimate.

The statement that \mathbf{p}^* is a map from $\mathcal{W}_k^s(R)$ to $\mathcal{W}_k^s(U)$ is a direct consequence of the first statement. □

Lemma 3.15. *For $\alpha \in \mathcal{W}_k^s(U)$, we have $I(\alpha) \in \mathcal{W}_{k-1}^{s-1}(U)$.*

Proof. Using the same notation as in Lemma 3.12, note that the integral operator I satisfies the equation $dI + Id = \text{Id} - \mathbf{p}^* \circ \mathbf{i}^*$. For a given $\alpha \in \mathcal{W}_k^s(U)$, we have $I(\alpha) \in \tilde{\mathcal{W}}_{k-1}^{s-1}(U)$ by Lemma 3.12. Making use of Lemmas 3.13 and 3.14, we have $d(I(\alpha)) = -I(d(\alpha)) + \alpha - \mathbf{p}^* \circ \mathbf{i}^*(\alpha) \in \tilde{\mathcal{W}}_k^s(U)$, which implies $I(\alpha) \in \mathcal{W}_{k-1}^{s-1}(U)$. □

Now we consider a chain of affine subspaces $\{q_0\} = U_0 \leq U_1 \cdots \leq U_n = U$ with $\dim_{\mathbb{R}}(U_j) = j$, equipped with the natural inclusions $\mathbf{i}_j : U_j \rightarrow U_{j+1}$ and affine projections $\mathbf{p}_j : U_{j+1} \rightarrow U_j$ such that the fiber of \mathbf{p}_j is tangent to a constant affine vector field v_j on U_{j+1} . Composition of the inclusion operators gives $\mathbf{i}_{i,j} : U_i \rightarrow U_j$, and similarly for the projection operator $\mathbf{p}_{i,j} : U_j \rightarrow U_i$ for $i < j$. We let $I_j : \mathcal{W}_k^s(U_{j+1}) \rightarrow \mathcal{W}_{k-1}^{s-1}(U_{j+1})$ be the integral operator defined on U_{j+1} using the vector field v_j as in the beginning of this subsection (Section 3.2.2).

We choose q_0 to be an *irrational* point in U_1 (strictly speaking it is not a tropical polyhedral subset of U_1) for later applications in Section 3.4. The definitions of $\mathbf{p}_{0,j}^*$'s are still valid if they are treated as inclusions of constant functions. Despite the fact that q_0 is irrational, the operator I_0 defines a map $\mathcal{W}_k^s(U_1) \rightarrow \mathcal{W}_{k-1}^{s-1}(U_1)$ because every $\alpha \in \mathcal{W}_1^s(U_1)$ is a finite sum of $\sum_l \alpha_l$ with $\alpha_l \in \mathcal{W}_{P_l}^s(U_1)$ for some *rational* points P_l 's on U_1 which in particular miss q_0 and therefore $I_0(P_l)$ is still a tropical subspace of U_1 .

We then define a *new* integral operator by

$$(3.13) \quad I = \mathbf{p}_{1,n}^* I_0 \mathbf{i}_{1,n}^* + \cdots + \mathbf{p}_{n-1,n}^* I_{n-2} \mathbf{i}_{n-1,n}^* + I_{n-1},$$

which is defined as $\mathcal{W}_*^s(U) \rightarrow \mathcal{W}_{*}^{s-1}(U)$, with the corresponding operator $\mathbf{i}^* := \mathbf{i}_{0,n}^*$ being the evaluation at q_0 and the operator $\mathbf{p}^* := \mathbf{p}_{0,n}^*$.

Proposition 3.16. *We have the identity $dI + \text{Id} = \text{Id} - \mathbf{p}^* \circ \mathbf{i}^*$, meaning that I is contracting the cohomology of U to that of the point q_0 .*

Proof. We first notice that $\mathbf{p}_{j+1,n}^*(dI_j + I_j d)\mathbf{i}_{j+1,n}^* = \mathbf{p}_{j+1,n}^*(\text{id}_{U_{j+1}} - \mathbf{p}_{j,j+1}^* \mathbf{i}_{j,j+1}^*)\mathbf{i}_{j+1,n}^*$, which gives $d(\mathbf{p}_{j+1,n}^* I_j \mathbf{i}_{j+1,n}^*) + (\mathbf{p}_{j+1,n}^* I_j \mathbf{i}_{j+1,n}^*)d = \mathbf{p}_{j+1,n}^* \mathbf{i}_{j+1,n}^* - \mathbf{p}_{j,n}^* \mathbf{i}_{j,n}^*$. Taking summation over $j = 0, \dots, n - 1$ gives the desired equation. \square

3.3. The tropical dgLa and its homotopy operator. From now on, we restrict ourselves to the case that $B_0 = M_{\mathbb{R}}$ with $U \subset M_{\mathbb{R}}$.

3.3.1. The tropical dgLa and the extended tropical vertex group. As an analogue of the dgLa $\mathbf{G}_n^{*,*}$ introduced in (3.6), we impose the requirement of the asymptotic behavior as $\hbar \rightarrow 0$ and replace $\Omega_{\hbar}^*(U)$ by the dg subalgebra $\mathcal{W}_*^0(U)$.

Definition 3.17. For every convex open subset $U \subset B_0$, we define a dg Lie subalgebra of $PV_{\hbar}^{*,*}|_U \otimes_{\mathbb{C}} R_n$ by

$$\mathcal{G}_n^{*,*}(U) := \hat{\mathcal{F}} \left[\left(\bigoplus_{m \in \mathcal{P}} \mathcal{W}_*^0(U) z^m \right) \otimes_{\mathbb{Z}} \wedge^* N \otimes_{\mathbb{C}} R_n \right],$$

making use of the Fourier transform (3.6). Abusing notation, we will drop the identification via Fourier transform in (3.6) and simply write

$$\mathcal{G}_n^{*,*}(U) = \left(\bigoplus_{m \in \mathcal{P}} \mathcal{W}_*^0(U) z^m \right) \otimes_{\mathbb{Z}} \wedge^* N \otimes_{\mathbb{C}} R_n.$$

Then we take the quotient by the dg Lie ideal

$$\mathcal{I}_n^{*,*}(U) := \left(\bigoplus_{m \in \mathcal{P}} \mathcal{W}_*^{-1}(U) z^m \right) \otimes_{\mathbb{Z}} \wedge^* N \otimes_{\mathbb{C}} R_n$$

to obtain

$$(\mathcal{G}/\mathcal{I})_n^{*,*}(U) := \left(\bigoplus_{m \in \mathcal{P}} (\mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U)) z^m \right) \otimes_{\mathbb{Z}} \wedge^* N \otimes_{\mathbb{C}} R_n$$

which defines a dgLa (since $\mathcal{W}_*^{-1}(U)$ is a dg ideal of $\mathcal{W}_*^0(U)$).

A general element of $\mathcal{G}_n^{i,j}$, $\mathcal{E}_n^{i,j}$, and $(\mathcal{G}/\mathcal{I})_n^{i,j}$ is a finite sum of the form

$$\sum_I \sum_{m, n_1, \dots, n_j} \alpha_{m,I}^{n_1, \dots, n_j} z^m \check{\partial}_{n_1} \wedge \dots \wedge \check{\partial}_{n_j} u_I,$$

where $I \subset \{1, \dots, n\}$ and $u_I = \prod_{i \in I} u_i$ with $\alpha_{m,I}^{n_1, \dots, n_j} \in \mathcal{W}_i^j(U)$, $\alpha_{m,I}^{n_1, \dots, n_j} \in \mathcal{W}_i^{j-1}(U)$ and $\alpha_{m,I}^{n_1, \dots, n_j} \in \mathcal{W}_i^j(U)/\mathcal{W}_i^{j-1}(U)$, respectively. We will be concerned with the Maurer-Cartan equation (1.3) of the dgLa $(\mathcal{G}/\mathcal{I})_n^{*,*}(U)$ instead of $PV_{\hbar}^{*,*}|_U \otimes_{\widehat{\mathbb{C}[Q]}_S} \otimes_{\mathbb{C}} R_n$, because we only care about the leading order behavior of the MC solutions as $\hbar \rightarrow 0$.

Making use of the holomorphic volume form $\check{\Omega}$ on \check{X} , we obtain a BV operator Δ acting on $PV_{\hbar}^{*,*}$ as in Section 3.1. The BV operator can be carried to $\mathcal{G}_n^{*,*}$ and naturally to $(\mathcal{G}/\mathcal{I})_n^{*,*}$, equipping them with dgBV structures. Explicitly, the BV operator is given by $\Delta(\alpha z^m \check{\partial}_{n_1} \wedge \dots \wedge \check{\partial}_{n_j}) = \sum_j (-1)^{|\alpha|+r-1} \check{\partial}_{n_r}(\alpha z^m) \check{\partial}_{n_1} \wedge \dots \widehat{\check{\partial}_{n_r}} \dots \wedge \check{\partial}_{n_j}$ in $\mathcal{G}_n^{*,*}$, which is further reduced to

$$\Delta(\alpha z^m \check{\partial}_{n_1} \wedge \dots \wedge \check{\partial}_{n_j}) = \sum_j (-1)^{|\alpha|+r-1} (n_r, \bar{m}) \alpha z^m \check{\partial}_{n_1} \wedge \dots \widehat{\check{\partial}_{n_r}} \dots \wedge \check{\partial}_{n_j}$$

in $(\mathcal{G}/\mathcal{I})_n^{*,*}$. This is because of the extra \hbar in the formula (3.7) giving $\check{\partial}_n(\alpha) \in \mathcal{I}_n^{*,*}$. As a consequence, the Lie bracket $[\cdot, \cdot]$ in $(\mathcal{G}/\mathcal{I})_n^{*,*}$ is given by $[\alpha z^m \check{\partial}_{n_I}, \beta z^{m'} \check{\partial}_{n_J}] = (-1)^{(|I|+1)|\beta|} \alpha \beta [z^m \check{\partial}_{n_I}, z^{m'} \check{\partial}_{n_J}]$, where $\check{\partial}_{n_I} = \check{\partial}_{n_{i_1}} \wedge \dots \wedge \check{\partial}_{n_{i_l}}$.

Definition 3.18. We call the dg Lie subalgebra $\mathcal{H}_n^{*,*} \leq (\mathcal{G}/\mathcal{I})_n^{*,*}$ defined by $\mathcal{H}_n^{*,*} := \ker(\Delta)$, which is equipped with the differential $\bar{\partial}_W$ and Lie bracket $[\cdot, \cdot]$, the *tropical dgLa*. We also call $\mathfrak{h}_n^* := \mathcal{H}_n^{*,*} \cap \ker(\bar{\partial})$ the *extended tropical Lie algebra*. The corresponding exponential group $\exp(\mathfrak{h}_n^*)$ is called the *extended tropical vertex group*.

Explicitly, we have

$$\mathfrak{h}_n^0 = \left(\bigoplus_{m \in \mathcal{P}} \mathbb{C} \cdot z^m \right) \otimes_{\mathbb{C}} R_n, \mathfrak{h}_n^1 = \left(\bigoplus_{m \in \mathcal{P}} \mathbb{C} \cdot z^m \right) \otimes_{\mathbb{Z}} m^\perp \otimes_{\mathbb{C}} R_n, \mathfrak{h}_n^2 = \mathbb{C} \cdot \check{\partial}_1 \wedge \check{\partial}_2 \otimes_{\mathbb{C}} R_n,$$

and $\mathcal{H}_n^{*,*}$ can be viewed as the Dolbeault resolution of \mathfrak{h}_n^* . We will see that solving the Maurer-Cartan equation (1.3) in $\mathcal{H}_n^{*,*}$ is intimately and directly related to tropical counting.

3.3.2. The homotopy operator. In order to solve the Maurer-Cartan equation (1.3) using Kuranishi’s method [43], we need a homotopy operator H (also called a *propagator*) to fix the gauge. Here we explain the construction of such a homotopy operator H using the operator I defined in (3.13). We will take $U = B_0 = M_{\mathbb{R}}$ and drop the dependence on U in notation in the rest of this subsection.

Notation 3.19. For each $m \in \mathcal{P}$ with the associated $\bar{m} \in M$, \bar{m} naturally gives an affine vector field on $B_0 = M_{\mathbb{R}}$ which, by abuse of notation, will also be denoted as \bar{m} . We fix an affine linear metric g_0 on $M_{\mathbb{R}}$. Then, given any real number R ,

we choose a chain of affine subspaces $\{pt\} = U_0^m \leq U_1^m \leq U_2^m = M_{\mathbb{R}}$ as follows. First, we take $v_1^m = -\bar{m}$ if $\bar{m} \neq 0$ and take v_1^m to be an arbitrary nonzero element in M if $\bar{m} = 0$. Then we set $U_1^m = \{x \mid g_0(v_1^m, x) = -R\}$ and choose U_0^m to be an irrational point on U_1^m . Such a choice defines a homotopy operator $H_m : \mathcal{W}_*^0(U) \rightarrow \mathcal{W}_{*-1}^0(U)$ using the construction in (3.13) (which was denoted by I there). We also denote the half space $\{x \mid g_0(-\bar{m}, x) \geq -R\}$ by $U_{1,+}^m$.

Definition 3.20. For each $m \in \mathcal{P}$, we define the homotopy operator $H_m : \mathcal{W}_*^0(U)z^m \rightarrow \mathcal{W}_{*-1}^0(U)z^m$ on the direct summand for each Fourier mode z^m by simply taking $H_m(\alpha z^m) := H_m(\alpha)z^m$. We also define the projection $P_m : \mathcal{W}_*^0(U)z^m \rightarrow \mathcal{W}_0^0(U_0^m)z^m$ by $P_m(\alpha z^m) := (\alpha|_{x_m^0})z^m$ at degree 0 and 0 otherwise, where $\alpha|_{x_m^0}$ is evaluation of α at the point $\{x_m^0\} = U_0^m$, and the operator $\iota_m : \mathcal{W}_0^0(U_0^m)z^m \rightarrow \mathcal{W}_*^0(U)z^m$ by $\iota_m(\alpha z^m) := \iota_m(\alpha)z^m$ at degree 0 and 0 otherwise, by setting $\iota_m : \mathcal{W}_0^0(U_0^m) \hookrightarrow \mathcal{W}_*^0(U)$ to be the embedding of constant functions over $M_{\mathbb{R}}$.

We abuse notation by treating H_m, P_m , and ι_m as acting on the spaces $\mathcal{W}_*^0(U)$ and $\mathcal{W}_0^0(U_0^m)$, respectively.

As in [10], these operators satisfy the following identity of homotopy retracting $\mathcal{W}_*^0(U)z^m$ onto its cohomology $\mathcal{W}_0^0(U_0^m)z^m = H^*(\mathcal{W}_*^0(U), d)z^m$, i.e., we have

$$(3.14) \quad \text{Id} - \iota_m P_m = dH_m + H_m d.$$

Moreover, these operators can be descended to $(\mathcal{W}_*^0(U)/\mathcal{W}_*^{-1}(U))z^m$ contracting to its cohomology $\mathbb{C} \cdot z^m \cong (\mathcal{W}_0^0(U_0^m)/\mathcal{W}_0^{-1}(U_0^m))z^m$.

Definition 3.21. We define the operators $H := \bigoplus H_m, P := \bigoplus P_m$, and $\iota := \bigoplus \iota_m$ acting on the direct sum $\bigoplus_m \mathcal{W}_*^0(U)z^m$ and its cohomology. These operators extend naturally to the tensor product $\mathcal{G}_n^{*,*}(U) = (\bigoplus_{m \in \mathcal{P}} \mathcal{W}_*^0(U)z^m) \otimes_{\mathbb{Z}} \wedge^* N \otimes_{\mathbb{C}} R_n$, and descend to the quotient $(\mathcal{G}/\mathcal{I})_n^{*,*}$. Moreover, these operators preserve $\mathcal{H}_n^{*,*}$ and hence can also be defined on $\mathcal{H}_n^{*,*}$. All of the above operators will be denoted by the same notation.

3.4. Solving the Maurer-Cartan equation. Recall that we have fixed n points P_1, \dots, P_n in generic position, with each P_i corresponding to a formal variable $u_i \in R_n$. To each P_i , we associate an input term of the form $\Pi^{(i)} = u_i \delta_{P_i}(\check{\partial}_1 \wedge \check{\partial}_2)$, where $\check{\partial}_1, \check{\partial}_2$ are the holomorphic vector fields corresponding to the basis $\{e^1, e^2\}$ of N (fixed at the beginning of Section 3.1.1), and

$$(3.15) \quad \delta_{P_i} = \frac{1}{\pi \hbar} e^{-(\eta_{i,1}^2 + \eta_{i,2}^2)/\hbar} d\eta_{i,1} \wedge d\eta_{i,2} \in \mathcal{W}_2^\infty$$

is an \hbar -dependent smoothing of the “delta-form” at P_i for some affine coordinates $(\eta_{i,1}, \eta_{i,2})$ on $M_{\mathbb{R}}$ taking the values $(0, 0)$ at P_i . We are interested in the Maurer-Cartan solutions in $\mathcal{H}_n^{*,*}$ constructed by summing over trees with input $\sum_{i=1}^n \Pi^{(i)}$.

Notice that we have $\delta_{P_i} \in \mathcal{W}_{P_i}^2$ because we can apply Lemma 3.11 to the expression

$$\delta_{P_i} = \left(\left(\frac{1}{\pi \hbar} \right)^{1/2} e^{-(\eta_{i,1}^2)/\hbar} d\eta_{i,1} \right) \wedge \left(\left(\frac{1}{\pi \hbar} \right)^{1/2} e^{-(\eta_{i,2}^2)/\hbar} d\eta_{i,2} \right),$$

and we have the following lemma from [10].

Lemma 3.22 (Lemma 4.14 in [10]). *For any affine linear function η on U , the one-form $\left(\frac{1}{\pi \hbar}\right)^{1/2} e^{-(\eta^2)/\hbar} d\eta$ has asymptotic support on the line $L := \{\eta = 0\}$ with weight 1.*

Instead of solving the Maurer-Cartan equation directly, we will solve the equation (3.16):

$$(3.16) \quad \Phi = \Pi - H([W, \Phi] + \frac{1}{2}[\Phi, \Phi]),$$

where Φ is a degree 0 element in $\mathcal{H}_n^{*,*}$, with the input

$$(3.17) \quad \Pi := \sum_{i=1}^n \Pi^{(i)}.$$

This originates from a method of Kuranishi [43] in solving the Maurer-Cartan equation of the classical Kodaira-Spencer dgLa. His method can be generalized to our current situation as follows (see, e.g., [47]).

Proposition 3.23. *Suppose that Φ satisfies the equation (3.16). Then Φ satisfies the Maurer-Cartan equation (1.3) if and only if $P([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$.*

Proof. Applying $\bar{\partial}$ to both sides of (3.16) (recall that $\bar{\partial}$ is identified with the de Rham differential d using the Fourier transform \mathcal{F} (3.6)) and using $\bar{\partial}\Pi = 0$, we obtain

$$\bar{\partial}\Phi + [W, \Phi] + \frac{1}{2}[\Phi, \Phi] = H([\bar{\partial}\Phi, W + \Phi]) + \iota \circ P([W, \Phi] + \frac{1}{2}[\Phi, \Phi]).$$

Suppose that Φ satisfies the MC equation (1.3). Then we see that $[\bar{\partial}\Phi, W + \Phi] = -[[W, \Phi] + \frac{1}{2}[\Phi, \Phi], W + \Phi] = 0$ and hence $P([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$.

For the converse, we let $\delta = \bar{\partial}\Phi + [W, \Phi] + \frac{1}{2}[\Phi, \Phi]$. It follows from the assumption $P([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$ that $\delta = H[W + \Phi, \delta] = (H \circ ad_{W+\Phi})^m(\delta)$ for any $m \in \mathbb{Z}_+$. Then by the fact that $\Phi \in \mathcal{H}_n^{*,*} \otimes \mathfrak{m}$, and the fact that ad_W is an operator of degree (1, 0), we have $\delta = 0$ by taking m large enough. \square

We notice that $P\alpha \neq 0$ only if $\alpha \in \mathcal{H}_n^{i,0}$ by its construction. When we write $\Phi = \sum_{i=0}^2 \Phi^{i,i}$ with $\Phi^{i,i} \in \mathcal{H}_n^{i,i}$, and consider the term $P([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$, we notice that $P([W, \Phi^{1,1} + \Phi^{2,2}] + \frac{1}{2}[\Phi^{1,1} + \Phi^{2,2}, \Phi^{1,1} + \Phi^{2,2}] + [\Phi^{0,0}, \Phi^{1,1} + \Phi^{2,2}]) = 0$ by degree reasons. Furthermore, we have $[W, \Phi^{0,0}] = 0 = [\Phi^{0,0}, \Phi^{0,0}]$, and therefore $P([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$. As a result, it suffices to solve the equation (3.16).

Now we look at the equation (3.16). Letting $\Xi = \Phi - \Pi$, we can solve $\Xi + H([W, \Phi] + \frac{1}{2}[\Phi, \Phi]) = 0$ iteratively in $\mathcal{H}_n^{*,*} \otimes_R (R/\mathfrak{m}^k)$ by increasing the power in \mathfrak{m}^k . We write $\Xi = \sum_{i=1} \Xi_i$ and $\Phi = \sum_i \Phi_i$ with $\Xi_i, \Phi_i \in \mathcal{H}_n^{*,*} \otimes (\mathfrak{m}^i/\mathfrak{m}^{i+1})$. We further decompose each Ξ_i and Φ_i by its degree and write $\Xi_i = \sum_{j=0}^2 \Xi_i^{j,j}$ with $\Xi_i^{j,j} \in \mathcal{H}_n^{j,j} \otimes \mathfrak{m}^i/\mathfrak{m}^{i+1}$ and similarly $\Phi_i = \sum_{j=0}^2 \Phi_i^{j,j}$.

The first order terms are simply given by $\Xi_1^{1,1} = -H[W, \Pi]$ and $\Xi_1^{0,0} = -H[W, \Xi_1^{1,1}]$. In general, the k th order equation is given by

$$(3.18) \quad \Xi_k + H[W, \Phi_k] + \sum_{j+l=k} \frac{1}{2}H[\Phi_j, \Phi_l] = 0,$$

and $\Xi_k^{j,j}$ is uniquely determined by Ξ_i with $i < k$ and $\Xi_k^{r,r}$ with $r > j$. In this way, the solution Ξ to (3.16) is uniquely determined.

There is a beautiful way to express the unique solution Ξ as a sum of terms involving the input Π over directed trees (reminiscent of a Feynman sum). To this end, let us introduce the notion of a *weighted d -pointed k -tree with ribbon structure*, whose definition originated from [16] (see also [10]).

Definition 3.24. Given a weighted ribbon d -pointed k -tree $\mathcal{T} \in \text{WRT}_{k,d}$, we align the marked points p_1, \dots, p_d (recall that marked points is itself an edge in $\partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})$) by p_{i_1}, \dots, p_{i_d} according to its cyclic ordering (or the clockwise orientation on D if we use the embedding $|\mathcal{T}| \hookrightarrow D$). We define the graded operator $\mathfrak{l}_{\mathcal{T}} : \mathcal{H}^{*,*}[2]^{\otimes d} \rightarrow \mathcal{H}^{*,*}[2]$ for input $\zeta_1, \dots, \zeta_d \in \mathcal{H}^{*,*}[2]$ by

- (1) writing $\zeta_j = \sum_{I \subset \{1, \dots, n\}} \alpha_{j,I} u_I$, and extracting the term $\alpha_{j,i_j} u_{i_j}$ in ζ_j and aligning it as the input at p_{i_j} , where $u_{i_j} \in R_n$ is the monomial associated to the marked point p_{i_j} in Definition 2.6,
- (2) aligning the term z^{m_e} at each incoming edge in $e \in \mathcal{T}_{in}^{[1]} = \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$,
- (3) applying $[\cdot, \cdot]$ at each vertex in $\mathcal{T}^{[0]}$ according to the ordering of the ribbon structure,
- (4) applying the homotopy operator $-H$ to each edge in $\mathcal{T}^{[1]}$.

We then define $\mathfrak{l}_{k,d} : \mathcal{H}^{*,*}[2]^{\otimes d} \rightarrow \mathcal{H}^{*,*}[2]$ by $\mathfrak{l}_{k,d} := \sum_{\mathcal{T} \in \text{WRT}_{k,d}} \frac{1}{2^{d-1}} \mathfrak{l}_{\mathcal{T}}$.

Setting

$$(3.19) \quad \Phi := \Pi + \Xi = \sum_{k,d \geq 1} \mathfrak{l}_{k,d}(\Pi, \dots, \Pi)$$

gives the unique solution to the equation (3.16) which is obtained by recursively solving (3.18). Note that the sum above is finite because the ideal \mathfrak{m}_n is nilpotent.

4. PROOF OF THEOREM 1.1 BY ASYMPTOTIC ANALYSIS

In this section, we prove our main result (i.e., Theorem 1.1) by using asymptotic analysis to relate the Maurer-Cartan solution $\Phi \in \mathcal{H}_n^{*,*}$, which we constructed via the sum-over-tree formula in (3.19) with the specified input Π (3.17), with the tropical disk counts defined in Section 2.

Notation 4.1. Given n points $P_1, \dots, P_n \in M_{\mathbb{R}}$ in generic position, we use $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ to denote the space $\bar{e}\bar{v}^{-1}((P_{i_1}, \dots, P_{i_d}) \times M_{\mathbb{R}})$ which gives a compactification of $\mathfrak{M}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ for any weighted ribbon tree \mathcal{T} . Here, $\bar{e}\bar{v}$ is the evaluation map defined in Definition 2.10, P_{i_j} is the point such that the monomial weight at the marked point p_j is u_{i_j} , and note that the subset $\{i_1, \dots, i_d\} \subset \{1, \dots, n\}$ is determined by the weight of \mathcal{T} .

Definition 4.2. Given a weighted ribbon d -pointed k -tree $\mathcal{T} \in \text{WRT}_{k,d}$ with $u_{\mathcal{T}} \neq 0$, we associate to each of its edges $e \in \bar{\mathcal{T}}$ a tropical polyhedral subset $Q_e \subset M_{\mathbb{R}}$ as follows. For each incoming edge $e \in \mathcal{T}_{in}^{[1]}$, we assign $Q_e = M_{\mathbb{R}}$, and for each marked point p_j we assign $Q_{p_j} = P_{i_j}$ where the monomial weight at p_j is u_{i_j} . We then inductively assign a (possibly empty) tropical polyhedral subset Q_e to each edge $e \in \mathcal{T}^{[1]}$ by the following rule:

If e_1 and e_2 are two incoming edges meeting at a vertex v with an outgoing edge e_3 for which Q_{e_1} and Q_{e_2} are defined beforehand, we set $Q_{e_3} := (Q - \mathbb{R}_{\geq 0} \bar{m}_{e_3})$ if both Q_{e_1} and Q_{e_2} are nonempty and they intersect transversally at $Q := Q_{e_1} \cap Q_{e_2}$, and $Q_{e_3} := \emptyset$ otherwise.

We denote the tropical polyhedral subset associated to the unique outgoing edge e_o by $Q_{\mathcal{T}}$.

We start with a combinatorial lemma concerning the tropical polyhedral subset $Q_{\mathcal{T}}$.

Lemma 4.3. *If $MI(\mathcal{T}) < 0$, then both $Q_{\mathcal{T}}$ and $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ are empty. For $MI(\mathcal{T}) = 0$ or 2 and $\text{Mult}(\mathcal{T}) \neq 0$, ev_o is a diffeomorphism onto its image and we have $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$, which is of dimension $\frac{MI(\mathcal{T})}{2} + 1$ if $Q_{\mathcal{T}} \neq \emptyset$.*

Proof. We prove by induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$, i.e., when there are no trivalent vertices. Then the only possible trees are the ones with a unique edge e . In this case we have $MI(\mathcal{T}) = 2$ and $Q_{\mathcal{T}} = M_{\mathbb{R}}$, and the lemma holds automatically.

For the induction step, suppose we have a tree \mathcal{T} with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2 . We split \mathcal{T} at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points, respectively. Then we have the decomposition

$$(4.1) \quad \left(\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n)_{ev_o} \times_{ev_o} \overline{\mathfrak{M}}_{d_2}^{\mathcal{T}_2}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \right) \times \mathbb{R}_{\geq 0} \cdot (-\bar{m}_{\mathcal{T}}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n),$$

and there are two cases to consider.

The first case is when one of the incoming edges, say e_2 , is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. In this case \mathcal{T}_2 is not a weighted tree in the sense of Definition 2.6, but we can still take $Q_{\mathcal{T}_2}$ to be the point P_{e_2} associated to e_2 .

If $MI(\mathcal{T}_1) \leq 0$, then by the induction hypothesis and the generic assumption (Definition 2.11), $Q_{\mathcal{T}_1}$ cannot intersect $Q_{\mathcal{T}_2}$ transversally and hence $Q_{\mathcal{T}} = \emptyset$. On the other hand we have $MI(\mathcal{T}) < 0$, so $\mathfrak{M}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$.

If $MI(\mathcal{T}_1) = 2$, then $Q_{\mathcal{T}_1}$ intersect $Q_{\mathcal{T}_2}$ transversally at $Q_{\mathcal{T}_2}$ automatically if $Q_{\mathcal{T}_2}$ lies on $Q_{\mathcal{T}_1}$, and otherwise both $Q_{\mathcal{T}} = \mathfrak{M}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$. In this case $MI(\mathcal{T}) = 0$, $\text{Mult}(\mathcal{T}) = \text{Mult}(\mathcal{T}_1)$ and $m_{\mathcal{T}} = m_{\mathcal{T}_1}$. Assuming $\text{Mult}(\mathcal{T}) = \text{Mult}(\mathcal{T}_1) \neq 0$, we have $Q_{\mathcal{T}_1} = (ev_o)_*(\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ by the induction hypothesis and the above decomposition becomes

$$\left(\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \cap (ev_{\mathcal{T}_1, o})^{-1}(Q_{\mathcal{T}_2}) \right) \times \mathbb{R}_{\geq 0} \cdot (-\bar{m}_{\mathcal{T}_1}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n),$$

implying that $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$, and hence the dimension of $Q_{\mathcal{T}}$ is exactly given by $\frac{MI(\mathcal{T})}{2} + 1$ ⁷.

The second case is when both \mathcal{T}_1 and \mathcal{T}_2 have $k_1, k_2 \geq 1$. In this case we have $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$ and the two moduli spaces $\overline{\mathfrak{M}}_{d_i}^{\mathcal{T}_i}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ have dimensions $MI(\mathcal{T}_i)/2 + 1$, respectively, if they are nonempty. Using the decomposition in equation (4.1), we notice that if $\overline{\mathfrak{M}}_{d_i}^{\mathcal{T}_i}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$ for $i = 1$ or 2 , then $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n)) = \emptyset$. So $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n) = \emptyset$ if $MI(\mathcal{T}) < 0$. Therefore it remains to consider the cases when $MI(\mathcal{T}_i) = 0, 2$ and $MI(\mathcal{T}) = 0, 2$.

Assuming $\text{Mult}(\mathcal{T}) = \text{Mult}(\mathcal{T}_1)\text{Mult}(\mathcal{T}_2)\text{Mult}_{v_r}(\mathcal{T}) \neq 0$, from the induction hypothesis we have $Q_{\mathcal{T}_i} = ev_o(\overline{\mathfrak{M}}_{d_i}^{\mathcal{T}_i}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ for $i = 1, 2$. Since we have $\text{Mult}_{v_r}(\mathcal{T}) \neq 0$, $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ can only intersect transversally. Therefore,

⁷We indeed have $(ev_{\mathcal{T}_1, o})^{-1}(Q_{\mathcal{T}_2}) \in \mathfrak{M}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ due to the generic assumption on P_1, \dots, P_n 's.

if $\text{Mult}(\mathcal{T}) \neq 0$, then we have $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ intersecting transversally and $Q_{\mathcal{T}} = Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2} - \mathbb{R}_{\geq 0} \bar{m}_{\mathcal{T}} = (ev_o)_*(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ from the decomposition (4.1) and the Definition 4.2 for $Q_{\mathcal{T}}$. Finally, by the generic assumption on P_1, \dots, P_n , $Q_{\mathcal{T}}$ has dimension $\frac{MI(\mathcal{T})}{2} + 1$ whenever it is nonempty. \square

Lemma 4.4. *There exists a large enough $R > 0$ such that the half space $U_{1,+}^m$ in Notation 3.19 contains $\text{Sing}(\mathcal{D})$ and also the tropical polyhedral subset $Q_{\mathcal{T}}$ for any \mathcal{T} with $m_{\mathcal{T}} = m$, $MI(\mathcal{T}) = 0$ or 2 , $\text{Mult}(\mathcal{T}) \neq 0$, $u_{\mathcal{T}} \neq 0$ and with at least one marked point.*

Proof. The existence of a fixed R depends on the finiteness of the total number of weighted ribbon trees \mathcal{T} (for arbitrary number of marked points, and $k = |\mathcal{T}_{in}^{[0]}|$) with $MI(\mathcal{T}) = 0, 2$, $\text{Mult}(\mathcal{T}) \neq 0$, and $u_{\mathcal{T}} \neq 0$. We prove by induction on the number N of vertices in $\mathcal{T}^{[0]}$ the existence of $R_N > 0$ satisfying the lemma for all \mathcal{T} with $|\mathcal{T}^{[0]}| \leq N$.

The initial case concerns the tree \mathcal{T} with a unique internal vertex v_r , with two incoming edges e_1 and e_2 , and one outgoing edge e_o in clockwise orientation. Furthermore, we have $e_1 \in \mathcal{T}_{in}^{[1]}$ and e_2 is an edge corresponding to a marked point with monomial weight u_{e_2} . In this case we have $MI(\mathcal{T}) = 0$ and $Q_{\mathcal{T}} = P_{e_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}}$ which is lying in $U_{1,+}^m$ as we required $\text{Sing}(\mathcal{D}) \subset U_{1,+}^m$ when we chose $U_{1,+}^m$ in Notation 3.19.

For the induction step, suppose we have a tree \mathcal{T} with $|\mathcal{T}^{[0]}| = N + 1$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2 . We split \mathcal{T} at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points, respectively. There are two cases to consider (as in the proof of Lemma 4.3).

This first case is when one of the incoming edges, say e_2 , is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. We let $Q_{\mathcal{T}_2} = P_{e_2}$ be the corresponding marked point. From the proof of Lemma 4.3, we know that we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$ for $Q_{\mathcal{T}} \neq \emptyset$. In this case $Q_{\mathcal{T}} = P_{e_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}}$ and we have $Q_{\mathcal{T}} \subset U_{1,+}^m$ by the same reason as in the initial step.

In the second case we have both \mathcal{T}_1 and \mathcal{T}_2 having $k_1, k_2 \geq 1$, and we have $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Assuming $Q_{\mathcal{T}} \neq \emptyset$, then one of the $Q_{\mathcal{T}_1}, Q_{\mathcal{T}_2}$ is a ray or a line, and we assume that it is $Q_{\mathcal{T}_1}$, with $MI(\mathcal{T}_2) = 0, 2$. Therefore for any point $x \in Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2}$ we have the relations $g_0(-\bar{m}_{\mathcal{T}_1}, x) \geq -R_N$ and $g_0(-\bar{m}_{\mathcal{T}_2}, x) \geq -R_N$, and hence $g_0(-\bar{m}_{\mathcal{T}}, x) \geq -2R_N$.⁸ Therefore by taking $R_{N+1} = 2R_N$, we have $g_0(-\bar{m}_{\mathcal{T}}, x) \geq -R_{N+1}$ and hence $Q_{\mathcal{T}} = Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}} \subset U_{1,+}^m$ for $\bar{m} \neq 0$ as desired. \square

We are now ready to prove the key lemma which relates our Maurer-Cartan solution with the locus $Q_{\mathcal{T}}$ traced out by the stops of the tropical disks introduced in Definition 4.2.

Notation 4.5. Given a weighted ribbon d -pointed k -tree \mathcal{T} , we define a differential form $\alpha_{\mathcal{T}} \in \mathcal{W}_*^0$ as follows. First we align the marked points p_1, \dots, p_d (recall that a marked point is itself an edge in $\partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})$) by p_{i_1}, \dots, p_{i_d} according to its cyclic ordering. Then $\alpha_{\mathcal{T}}$ is the output of the following procedure:

⁸Here g_0 is the linear metric introduced in Notation 3.19.

- (1) aligning $\delta_{P_{i_j}}$ as the input at the edge corresponding to the marked point p_{i_j} , if the monomial weight associated to p_{i_j} is u_{i_j} ,
- (2) aligning the constant 1 at each incoming edge in $e \in \mathcal{T}_{in}^{[1]} = \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]}) \setminus \{p_1, \dots, p_d\}$,
- (3) applying the wedge product \wedge at each vertex in $\mathcal{T}^{[0]}$ according to the ordering of the ribbon structure,
- (4) applying the homotopy operator $-H$ to each edge in $\mathcal{T}^{[1]}$.

Definition 4.6. Given a weighted ribbon d -pointed k -tree $\mathcal{T} \in \text{WRT}_{k,d}$ with $\text{Mult}(\mathcal{T}) \neq 0$, we set $(-1)^{\chi(\mathcal{T})} := \prod_{v \in \mathcal{T}^{[0]}} (-1)^{\chi(\mathcal{T},v)}$, where $(-1)^{\chi(\mathcal{T},v)}$ is defined by the rules (with the convention that $(-1)^{\chi(\mathcal{T})} = 1$ if $\mathcal{T}^{[0]} = \emptyset$): if v is connected to a marked point we set $\chi(\mathcal{T},v) = 0$, and $(-1)^{\chi(\mathcal{T},v)}$ is defined inductively along the tree \mathcal{T} for each trivalent vertex v not connecting to any marked point p_i 's (attached to two incoming edges e_1, e_2 and one outgoing edge e_3 so that e_1, e_2, e_3 are arranged in the clockwise orientation) by comparing the orientation of the ordered basis $\{-\bar{m}_{e_1}, -\bar{m}_{e_2}\}$ with that of B_0 .

Lemma 4.7. Let $\mathcal{T} \in \text{WRT}_{k,d}$ be a weighted ribbon d -pointed k -tree. Then we have $\iota_{\mathcal{T}}(\Pi, \dots, \Pi)$

$$= \begin{cases} 0 & \text{if } MI(\mathcal{T}) \neq 0, 2 \text{ or } Q_{\mathcal{T}} = \emptyset \text{ or } \text{Mult}(\mathcal{T}) = 0, \\ (-1)^{\chi(\mathcal{T})} \alpha_{\mathcal{T}} \text{Mult}(\mathcal{T}) z^{m_{\mathcal{T}}} u_{\mathcal{T}} & \text{if } MI(\mathcal{T}) = 2 \text{ and } Q_{\mathcal{T}} \neq \text{frm}[o]--\emptyset \text{ and } \text{Mult}(\mathcal{T}) \neq 0, \\ (-1)^{\chi(\mathcal{T})} \alpha_{\mathcal{T}k_{\mathcal{T}}} \text{Mult}(\mathcal{T}) z^{m_{\mathcal{T}}} \check{\delta}_{n_{Q_{\mathcal{T}}}} u_{\mathcal{T}} & \text{if } MI(\mathcal{T}) = 0 \text{ and } Q_{\mathcal{T}} \text{frm}[o]--\neq \emptyset \text{ and } \text{Mult}(\mathcal{T}) \neq 0 \end{cases}$$

in $\mathcal{H}_n^{*,*}$, where $\alpha_{\mathcal{T}} \in \mathcal{W}_{Q_{\mathcal{T}}}^{s_{\mathcal{T}}}$ in which $s_{\mathcal{T}} := 1 - \frac{MI(\mathcal{T})}{2}$, and $n_{Q_{\mathcal{T}}}$ is the clockwise oriented normal to the ray or line $Q_{\mathcal{T}}$ when $Q_{\mathcal{T}} \neq \emptyset$ in the case $MI(\mathcal{T}) = 0$.

Proof. First of all, from Notation 4.5 we can see that the degree of the form $\alpha_{\mathcal{T}}$ is exactly given by $s_{\mathcal{T}}$ which can only be 0 or 1 since the operator associated to the outgoing edge is a homotopy operator and it decreases the degree by 1. Therefore we notice that $\alpha_{\mathcal{T}} \neq 0$ except when $MI(\mathcal{T}) = 0$ or 2.

Once again, we prove by induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$ and the only possible trees are the ones with a unique edge e . In this case, we have $MI(\mathcal{T}) = 2$, $Q_{\mathcal{T}} = M_{\mathbb{R}}$, and $\iota_{\mathcal{T}}(\Pi, \dots, \Pi) = z^{m_{\mathcal{T}}}$, so the lemma holds.

For the induction step, suppose we have a tree \mathcal{T} with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2 . We split \mathcal{T} at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points, respectively. As before, there are two possible scenarios.

This first case is when one of the incoming edges, say e_2 , is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. In this case we let $P_{e_2} = Q_{\mathcal{T}_2}$ to be the marked point associated to e_2 . The proof of Lemma 4.3 shows that we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$ in order to have $Q_{\mathcal{T}} \neq \emptyset$, and $Q_{\mathcal{T}} \neq \emptyset$ if and only if $P_{e_2} \in Q_{\mathcal{T}_1}$. By the induction hypothesis we have $\iota_{\mathcal{T}_1}(\Pi, \dots, \Pi) = (-1)^{\chi(\mathcal{T}_1)} \alpha_{\mathcal{T}_1} \text{Mult}(\mathcal{T}_1) z^{m_{\mathcal{T}_1}} u_{\mathcal{T}_1}$ with $\alpha_1 \in \mathcal{W}_{Q_{\mathcal{T}_1}}^0$. Therefore we have

$$\iota_{\mathcal{T}}(\Pi, \dots, \Pi) = -(-1)^{\chi(\mathcal{T}_1)} \text{Mult}(\mathcal{T}_1) H(\alpha_{\mathcal{T}_1} \wedge \delta_{P_{e_2}}) [z^{m_{\mathcal{T}_1}}, \check{\delta}_1 \wedge \check{\delta}_2] u_{\mathcal{T}_1} u_{e_2},$$

where $u_{\mathcal{T}_1} u_{e_2} = u_{\mathcal{T}}$.

Now Lemma 3.11 implies that $\alpha_1 \wedge \delta_{P_{e_2}} \in \mathcal{W}_{P_{e_2}}^2$. By our choice we have $P_{e_2} \in U_{1,+}^{m_{\mathcal{T}}}$ and hence applying Lemma 3.15, we get $\alpha_{\mathcal{T}} = -H(\alpha_1 \wedge \delta_{P_{e_2}}) \in \mathcal{W}_{Q_{\mathcal{T}}}^1$, where

$Q_{\mathcal{T}} = P_{e_2} - \mathbb{R}_{\geq 0} \cdot \bar{m}_{\mathcal{T}}$ as in Definition 4.2. Furthermore, we have $[z^{m_{\mathcal{T}_1}}, \check{\partial}_1 \wedge \check{\partial}_2] = (e_2^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_2 - (e_1^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_1$, where e_1^*, e_2^* is the dual basis to e_1, e_2 introduced in Notation 3.4. As in Notation 2.7, we can write $k_{\mathcal{T}}\hat{m}_{\mathcal{T}} = \bar{m}_{\mathcal{T}}$ for some primitive $\hat{m}_{\mathcal{T}} \in M$. Since we have $m_{\mathcal{T}_1} = m_{\mathcal{T}}$, we find that $(e_2^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_2 - (e_1^*, \bar{m}_{\mathcal{T}_1})\check{\partial}_1 = k_{\mathcal{T}}n_{Q_{\mathcal{T}}}$. Together with the fact that $\chi(\mathcal{T}) = \chi(\mathcal{T}_1)$ and $\text{Mult}(\mathcal{T}) = \text{Mult}(\mathcal{T}_1)$, we obtain the desired identity in this case.

In the second case we have both \mathcal{T}_1 and \mathcal{T}_2 having $k_1, k_2 \geq 1$, and $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Assuming $Q_{\mathcal{T}} \neq \emptyset$, then one of $Q_{\mathcal{T}_1}, Q_{\mathcal{T}_2}$, say $Q_{\mathcal{T}_1}$, is a ray or a line, and they intersect transversally. There are two subcases depending on whether $MI(\mathcal{T}_2) = 0$ or 2.

We first assume that $MI(\mathcal{T}_2) = 0$. Then we can write

$$\iota_{\mathcal{T}_i}(\Pi, \dots, \Pi) = (-1)^{\chi(\mathcal{T}_i)} \text{Mult}(\mathcal{T}_i) \alpha_{\mathcal{T}_i} z^{m_{\mathcal{T}_i}} \check{\partial}_{n_i} u_{\mathcal{T}_i},$$

where we abbreviate $n_i = k_{\mathcal{T}_i} n_{Q_{\mathcal{T}_i}}$ and $n_{Q_{\mathcal{T}_i}}$ is the primitive clockwise oriented normal to $Q_{\mathcal{T}_i}$. Therefore we have

$$\begin{aligned} \iota_{\mathcal{T}}(\Pi, \dots, \Pi) &= -(-1)^{\chi(\mathcal{T}_1) + \chi(\mathcal{T}_2)} \text{Mult}(\mathcal{T}_1) \text{Mult}(\mathcal{T}_2) H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) [z^{m_{\mathcal{T}_1}} \check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}} \check{\partial}_{n_2}] u_{\mathcal{T}_1} u_{\mathcal{T}_2}. \end{aligned}$$

Using Lemma 3.11 we see that $\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2} \in \mathcal{W}_{Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2}}^2$ in the case that $Q_{\mathcal{T}_1}$ and $Q_{\mathcal{T}_2}$ are intersecting transversally (otherwise the product is $0 \in \mathcal{H}_n^{0,*}$). Applying Lemma 4.4 together with Lemma 3.15, we get $\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \in \mathcal{W}_{Q_{\mathcal{T}}}^1$. Furthermore, we have $[z^{m_{\mathcal{T}_1}} \check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}} \check{\partial}_{n_2}] = z^{m_{\mathcal{T}}} ((\bar{m}_{\mathcal{T}_2}, n_{\mathcal{T}_1})\check{\partial}_{n_2} - (\bar{m}_{\mathcal{T}_1}, n_{\mathcal{T}_2})\check{\partial}_{n_1})$, and $(\bar{m}_{\mathcal{T}_2}, n_{\mathcal{T}_1})\check{\partial}_{n_2} - (\bar{m}_{\mathcal{T}_1}, n_{\mathcal{T}_2})\check{\partial}_{n_1} = \det(\bar{m}_{e_1}, \bar{m}_{e_2})(n_{\mathcal{T}_1} + n_{\mathcal{T}_2})$. If $\{-\bar{m}_{e_1}, -\bar{m}_{e_2}\}$ is positively oriented, then $\det(\bar{m}_{e_1}, \bar{m}_{e_2}) > 0$ and $n_{\mathcal{T}_1} + n_{\mathcal{T}_2} = k_{\mathcal{T}} n_{Q_{\mathcal{T}}}$, where $k_{\mathcal{T}}$ is introduced in Notation 2.7, and $\det(\bar{m}_{e_1}, \bar{m}_{e_2}) = \text{Mult}_{v_r}(\mathcal{T})$. Notice that switching to the assumption that $\{-\bar{m}_{e_1}, -\bar{m}_{e_2}\}$ is negatively oriented would result in a minus sign in $\det(\bar{m}_{e_1}, \bar{m}_{e_2})$ and hence contribute an extra $(-1)^{\chi(\mathcal{T}, v_r)}$ in the formula (i.e., in this case $\chi(\mathcal{T}, v_r) = 1$). Combining with the fact that $\text{Mult}(\mathcal{T}) = \text{Mult}(\mathcal{T}_1) \text{Mult}_{v_r}(\mathcal{T})$, $(-1)^{\chi(\mathcal{T})} = (-1)^{\chi(\mathcal{T}_1)} (-1)^{\chi(\mathcal{T}, v_r)}$, we obtain the desired formula.

In the second subcase we assume that $MI(\mathcal{T}_2) = 2$, so by the induction hypothesis we have $\iota_{\mathcal{T}_2}(\Pi, \dots, \Pi) = (-1)^{\chi(\mathcal{T}_2)} \alpha_{\mathcal{T}_2} \text{Mult}(\mathcal{T}_2) z^{m_{\mathcal{T}_2}} u_{\mathcal{T}_2}$. Therefore we have

$$\begin{aligned} \iota_{\mathcal{T}}(\Pi, \dots, \Pi) &= -(-1)^{\chi(\mathcal{T}_1) + \chi(\mathcal{T}_2)} \text{Mult}(\mathcal{T}_1) \text{Mult}(\mathcal{T}_2) H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) [z^{m_{\mathcal{T}_1}} \check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}} \check{\partial}_{n_2}] u_{\mathcal{T}_1} u_{\mathcal{T}_2}, \end{aligned}$$

where we absorb the $k_{\mathcal{T}_1}$ into $n_1 = k_{\mathcal{T}_1} n_{Q_{\mathcal{T}_1}}$ again. Applying Lemmas 4.4, 3.11, and 3.15 as in the previous subcase, we obtain that $\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \in \mathcal{W}_{Q_{\mathcal{T}}}^0$. Furthermore, we have $[z^{m_{\mathcal{T}_1}} \check{\partial}_{n_1}, z^{m_{\mathcal{T}_2}} \check{\partial}_{n_2}] = \det(\bar{m}_{\mathcal{T}_1}, \bar{m}_{\mathcal{T}_2}) z^{m_{\mathcal{T}}} = (-1)^{\chi(\mathcal{T}, v_r)} \text{Mult}_{v_r}(\mathcal{T})$ which gives us the desired identity. \square

Next we would like to take a closer look at the differential form $\alpha_{\mathcal{T}}$ defined in Notation 4.5.

Definition 4.8 (cf. Definition 5.29 in [10]). We attach a differential form ν_e on $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$ to each $e \in \bar{\mathcal{T}}^{[1]}$ recursively by the rules: $\nu_e := 1$ for each incoming edge $e \in \partial_{in}^{-1}(\mathcal{T}_{in}^{[0]})$; $\nu_{e_3} = (-1)^{|\nu_{e_1}| |\nu_{e_2}|} \nu_{e_1} \wedge \nu_{e_2} \wedge ds_{e_3}$ (here $|\nu_{e_2}|$ is the cohomological degree of ν_{e_2}) if v is an internal vertex with incoming edges $e_1, e_2 \in \mathcal{T}_0$ and outgoing edge e_3 such that e_1, e_2, e_3 is clockwise oriented.

We let $\nu_{\mathcal{T}}$ be the differential form attached to the unique outgoing edge $e_o \in \mathcal{T}^{[1]}$, which defines a volume form or orientation on $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$.

Given a weighted ribbon d -pointed k -tree \mathcal{T} with $MI(\mathcal{T}) = 0$ with $Q_{\mathcal{T}} \neq \emptyset$, which is either a ray or a line, we let $\eta_{\mathcal{T}}$ be the unique affine function on $M_{\mathbb{R}}$ such that $\eta_{\mathcal{T}} = 0$ on $Q_{\mathcal{T}}$ and $\eta_{\mathcal{T}}$ takes positive values on the anticlockwise oriented normal to $Q_{\mathcal{T}}$.

Lemma 4.9. *For a weighted ribbon d -pointed k -tree \mathcal{T} with $MI(\mathcal{T}) = 0, 2$ and $Q_{\mathcal{T}} \neq \emptyset$ and $\text{Mult}(\mathcal{T}) \neq 0$, there exists some $c > 0$ such that*

$$(ev_{\mathcal{T}, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}, i_d})^*(d\eta_1 d\eta_2) = \begin{cases} (-1)^{\chi(\mathcal{T})} c \nu_{\mathcal{T}} + \varepsilon & \text{if } MI(\mathcal{T}) = 2, \\ (-1)^{\chi(\mathcal{T})} c \nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}} + \varepsilon & \text{if } MI(\mathcal{T}) = 0, \end{cases}$$

where ε satisfies $\iota_{\nu_{\mathcal{T}}^{\vee}} \varepsilon = 0$ (here $\nu_{\mathcal{T}}^{\vee}$ is a top polyvector field dual to $\nu_{\mathcal{T}}$ over the component $\mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$) and η_1, η_2 are the affine coordinates on $M_{\mathbb{R}}$ with respect to the oriented basis e_1, e_2 introduced in Notation 3.4.

Proof. First of all, notice that both $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$ and $M_{\mathbb{R}}^d$ are affine manifolds and $\bar{e}\bar{v}$ is affine linear. So all the differential forms appearing in this lemma are affine differential forms. Therefore it suffices to check the equality at a point in $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$. Also since $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma) \cong \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|} \times M_{\mathbb{R}}$, we can always write

$$(ev_{\mathcal{T}, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}, i_d})^*(d\eta_1 d\eta_2) = \begin{cases} c' \nu_{\mathcal{T}} + \varepsilon & \text{if } MI(\mathcal{T}) = 2, \\ c' \nu_{\mathcal{T}} \wedge \alpha + \varepsilon & \text{if } MI(\mathcal{T}) = 0 \end{cases}$$

for some $c' \in \mathbb{R}$, and some 1-form $\alpha \in \Omega^1(M_{\mathbb{R}})$ with $\iota_{\nu_{\mathcal{T}}^{\vee}} \varepsilon = 0$. We need to show that α is a constant multiple of $d\eta_{\mathcal{T}}$ and the constant $c' = (-1)^{\chi(\mathcal{T})} c$ for some $c > 0$.

In the case $MI(\mathcal{T}) = 0$ with $Q_{\mathcal{T}} \neq \emptyset$, the moduli space $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n)$ is a 1-dimensional affine subspace of $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$. We take any path ς lying inside $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \subset \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$. Since $ev_{\mathcal{T}, i_j} \circ \varsigma$ is a constant map for any $j = 1, \dots, d$, we have $\iota_{\varsigma'}((ev_{\mathcal{T}, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}, i_d})^*(d\eta_1 d\eta_2)) = 0$, where ς' is the affine vector field on $\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$ induced by ς . On the other hand, $(ev_o)_*(\varsigma')$ is tangent to $Q_{\mathcal{T}} = ev_o(\overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$. So α must be a constant multiple of $d\eta_{\mathcal{T}}$ and we can write $(ev_{\mathcal{T}, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}, i_d})^*(d\eta_1 d\eta_2) = c' \nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}} + \varepsilon$ for some constant c' .

We now prove that c' is of the form $(-1)^{\chi(\mathcal{T})} c$ for some $c > 0$ by induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$ and the only possible trees are those with a unique edge e . Since there are no evaluation maps, we adopt the convention that the left hand side of the equality in the lemma is equal to 1 to make the statement true in this case.

For the induction step, suppose we have a tree \mathcal{T} with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2 . We split \mathcal{T} at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points, respectively. There are two possible cases.

This first case is when one of the incoming edges, say e_2 , is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. As in the proof of Lemma 4.3, we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$. We use the identification $\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma) \xrightarrow{ev_{\mathcal{T}_1, o}} \times_{\tau_{e_o}}$

$(\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$, under which the evaluation map $ev_{\mathcal{T},o} : \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma) \rightarrow M_{\mathbb{R}}$ is identified as the projection to the last coordinate of the product on the left hand side, and the evaluation at the marked point e_2 is identified as the projection τ_{e_o} to the second factor of $\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}$. We have

$$(ev_{\mathcal{T}_1, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}_1, i_{d_1}})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_1)} c\nu_{\mathcal{T}_1} + \varepsilon_{\mathcal{T}_1}$$

for some $c > 0$ by the induction hypothesis. Since $MI(\mathcal{T}) = 0$, $Q_{\mathcal{T}}$ is a ray or a line. We take an affine path ϱ in $M_{\mathbb{R}}$ transversal to $Q_{\mathcal{T}}$ parametrized by the affine coordinate $\eta_{\mathcal{T}}$. Then restricting to $\mathbb{R}_{\leq 0} \times \varrho$, we have $ev_{\mathcal{T}, i_d}^*(d\eta_1 d\eta_2) = \tau_{e_o}^*(d\eta_1 d\eta_2) = ds_{e_o} \wedge d\eta_{\mathcal{T}}$, where s_{e_o} is the coordinate on $\mathbb{R}_{\leq 0}$ associated to the outgoing edge e_o . Putting these together we have $(-1)^{\chi(\mathcal{T}_1)} c\nu_{\mathcal{T}_1} \wedge ds_{e_o} \wedge d\eta_{\mathcal{T}} = (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}} \wedge d\eta_{\mathcal{T}}$.

In the second case we have both \mathcal{T}_1 and \mathcal{T}_2 having $k_1, k_2 \geq 1$, and we have $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Assuming $Q_{\mathcal{T}} \neq \emptyset$, then one of $Q_{\mathcal{T}_1}, Q_{\mathcal{T}_2}$, say $Q_{\mathcal{T}_1}$, must be a ray or a line. There are two subcases depending on whether $MI(\mathcal{T}_2) = 0$ or $MI(\mathcal{T}_2) = 2$.

We first assume that $MI(\mathcal{T}_2) = 0$. In this case we have $MI(\mathcal{T}) = 0$, and both $|\mathcal{T}_1^{[1]}|, |\mathcal{T}_2^{[1]}|$ are odd, and hence so is $|\mathcal{T}_1^{[1]}| + |\mathcal{T}_2^{[1]}|$. Similar to the previous case, we use the identification $\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma) \times_{ev_{\mathcal{T}_1, o}} \times_{ev_{\mathcal{T}_2, o}} \overline{\mathfrak{M}}_{d_2}^{\mathcal{T}_2}(\mathcal{P}, \Sigma) \times_{ev_{\mathcal{T}_2, o}} \times_{\tau_{e_o}} (\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$. From the induction hypothesis we have the relation $(ev_{\mathcal{T}_a, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}_a, i_{d_1}})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_a)} c_a \nu_{\mathcal{T}_a} \tau_{e_o}^*(d\eta_{\mathcal{T}_a}) + \varepsilon_{\mathcal{T}_a}$ with $c_a > 0$ and $\iota_{\nu_{\mathcal{T}_a}} \varepsilon_{\mathcal{T}_a} = 0$ for $a = 1, 2$. Taking their product we get

$$\begin{aligned} & (ev_{\mathcal{T}, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}, i_d})^*(d\eta_1 d\eta_2) \\ &= (-1)^{\chi(\mathcal{T}_1) + \chi(\mathcal{T}_2) + |\mathcal{T}_1^{[1]}| + |\mathcal{T}_2^{[1]}|} \nu_{\mathcal{T}_1} \wedge \nu_{\mathcal{T}_2} \tau_{e_o}^*(d\eta_{\mathcal{T}_1} \wedge d\eta_{\mathcal{T}_2}) + \varepsilon, \end{aligned}$$

where $c := c_1 c_2 > 0$ and $\iota_{\nu_{\mathcal{T}}} \varepsilon = 0$. Furthermore, we have $\tau_{e_o}^*(d\eta_{\mathcal{T}_1} \wedge d\eta_{\mathcal{T}_2}) = (-1)^{\chi(\mathcal{T}, v)} ds_{e_o} \wedge d\eta_{\mathcal{T}}$, where s_{e_o} is the coordinate on $\mathbb{R}_{\leq 0}$ associated to the outgoing edge e_o . Putting these together we obtain the desired identity.

Now assuming $MI(\mathcal{T}_2) = 2$, we have $(ev_{\mathcal{T}_2, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}_2, i_{d_1}})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_2)} c_2 \nu_{\mathcal{T}_2} + \varepsilon_{\mathcal{T}_2}$ instead. In this case, $|\mathcal{T}_1^{[1]}| + |\mathcal{T}_2^{[1]}|$ is even and $\nu_{\mathcal{T}_2}$ is an even degree differential form. Therefore we obtain

$$\begin{aligned} & (ev_{\mathcal{T}, i_1})^*(d\eta_1 d\eta_2) \cdots (ev_{\mathcal{T}, i_d})^*(d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T}_1) + \chi(\mathcal{T}_2)} c\nu_{\mathcal{T}_1} \wedge \nu_{\mathcal{T}_2} \tau_{e_o}^*(d\eta_{\mathcal{T}_1}) + \varepsilon \\ &= (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}_1} \wedge \nu_{\mathcal{T}_2} \wedge ds_{e_o} + \varepsilon' = (-1)^{\chi(\mathcal{T})} c\nu_{\mathcal{T}} + \varepsilon' \end{aligned}$$

using the fact that $\tau_{e_o}^*(d\eta_{\mathcal{T}_1}) = (-1)^{\chi(\mathcal{T}, v)} ds_{e_o} + \beta$ for some 1-form β on $M_{\mathbb{R}}$. Notice that switching the roles of \mathcal{T}_1 and \mathcal{T}_2 would yield the same result. This completes the proof of the lemma. \square

Lemma 4.10. *For $MI(\mathcal{T}) = 0, 2$ with $Q_{\mathcal{T}} \neq \emptyset$ and $\text{Mult}(\mathcal{T}) \neq 0$, we have the identity*

$$\alpha_{\mathcal{T}} = (-1)^{k+d-1} (ev_o)_* \left((ev_{i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{i_d})^*(\delta_{P_{k_d}}) \right),$$

where $ev_* : \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma) \rightarrow M_{\mathbb{R}}$'s are the evaluation maps introduced in Definition 2.10, and the orientation on fibers of ev_o is defined similarly as in Definition 4.8 (notice that $(-1)^{k+d-1} = (-1)^{MI(\mathcal{T})/2-1}$).

Proof. We prove by using induction on the number of vertices in $\mathcal{T}^{[0]}$. The initial case is when $\mathcal{T}^{[0]} = \emptyset$ and the only possible trees are the ones with a unique edge e , for which the statement is trivial.

For the induction step, suppose we have a tree \mathcal{T} with $\mathcal{T}^{[0]} \neq \emptyset$ with the unique root vertex $v_r \in \mathcal{T}^{[0]}$ connecting to the outgoing edge e_o with two incoming edges e_1 and e_2 . We split \mathcal{T} at v_r to obtain two trees $\mathcal{T}_1, \mathcal{T}_2$ with outgoing edges e_1, e_2 and k_1, k_2 incoming edges, d_1 and d_2 marked points, respectively. There are two possible cases.

This first case is when one of the incoming edges, say e_2 , is an edge corresponding to a marked point so that $k_2 = 0$ and $d_2 = 1$. As in the proof of Lemma 4.3, we must have $MI(\mathcal{T}_1) = 2$ and $MI(\mathcal{T}) = 0$. In this case we let $P_{e_2} = Q_{\mathcal{T}_2}$ be the marked point associated to e_2 . The induction hypothesis says that $\alpha_{\mathcal{T}_1} = (-1)^{k+d-2}(ev_{\mathcal{T}_1,o})_* \left((ev_{\mathcal{T}_1,i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_1,i_{d_1}})^*(\delta_{P_{k_{d_1}}}) \right)$, which is a function with asymptotic support on $Q_{\mathcal{T}_1}$. Then we have

$$\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \delta_{P_{e_2}}) = - \int_{-\infty}^0 \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \delta_{P_{e_2}})$$

in $\mathcal{W}_*^0/\mathcal{W}_*^{-1}$, where $\tau_{e_o} : \mathbb{R} \times M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}$ is the flow associated to $-\bar{m}_{\mathcal{T}}$. This equality holds because we have $P_{e_2} \in U_{1,+}^{m_{\mathcal{T}}}$ and hence any integral over a domain not intersecting P_{e_2} gives 0 in $\mathcal{W}_*^0/\mathcal{W}_*^{-1}$.

Writing $\bar{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma)_{ev_{\mathcal{T}_1,o} \times \tau_{e_o}}(\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \bar{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$, where the evaluation map $ev_{\mathcal{T},o} : \bar{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma) \rightarrow M_{\mathbb{R}}$ is identified as the projection to the last factor in the product on the left hand side, and the evaluation at the marked point e_2 is identified as τ_{e_o} on $\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}$. Then we have

$$\begin{aligned} & - \int_{-\infty}^0 \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \delta_{P_{e_2}}) \\ &= (-1)^{k+d-1} \int_{-\infty}^0 \tau_{e_o}^* \left((ev_{\mathcal{T}_1,o})_* \left((ev_{\mathcal{T}_1,i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_1,i_{d_1}})^*(\delta_{P_{k_{d_1}}}) \right) \wedge \delta_{P_{e_2}} \right) \\ &= (-1)^{k+d-1} \int_{-\infty}^0 \left(\int_{\mathbb{R}_{\leq 0}^{|\mathcal{T}_1|}} (ev_{\mathcal{T},i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T},i_{d-1}})^*(\delta_{P_{k_{d-1}}}) \right) \wedge ev_{i_d}^*(\delta_{P_{e_2}}) \\ &= (-1)^{k+d-1} (ev_{\mathcal{T},o})_* \left((ev_{\mathcal{T},i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T},i_d})^*(\delta_{P_{e_2}}) \right). \end{aligned}$$

In the second case we have both \mathcal{T}_1 and \mathcal{T}_2 having $k_1, k_2 \geq 1$ and $MI(\mathcal{T}) = MI(\mathcal{T}_1) + MI(\mathcal{T}_2)$. Making use of Lemma 4.4 again, we notice that by comparing the domain of integration intersecting $Q_{\mathcal{T}_1} \cap Q_{\mathcal{T}_2}$ we have $\alpha_{\mathcal{T}} = -H(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) = - \int_{-\infty}^0 \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2})$, where τ_{e_o} is the flow of $-\bar{m}_{\mathcal{T}}$.

Notice that we have $\overline{\mathfrak{M}}_{d_1}^{\mathcal{T}_1}(\mathcal{P}, \Sigma) \times_{ev_{\mathcal{T}_1, o}} \times_{ev_{\mathcal{T}_2, o}} \overline{\mathfrak{M}}_{d_2}^{\mathcal{T}_2}(\mathcal{P}, \Sigma) \times_{ev_{\mathcal{T}_2, o}} \times_{\tau_{e_o}} (\mathbb{R}_{\leq 0} \times M_{\mathbb{R}}) = \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma)$, and therefore we obtain

$$\begin{aligned} & - \int_{-\infty}^0 \tau_{e_o}^*(\alpha_{\mathcal{T}_1} \wedge \alpha_{\mathcal{T}_2}) \\ &= (-1)^{k+d-1} \int_{-\infty}^0 \tau_{e_o}^* \left((ev_{\mathcal{T}_1, o})_* \left((ev_{\mathcal{T}_1, i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_1, i_{d_1}})^*(\delta_{P_{k_{d_1}}}) \right) \right. \\ & \quad \left. \wedge (ev_{\mathcal{T}_2, o})_* \left((ev_{\mathcal{T}_2, i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}_2, i_{d_2}})^*(\delta_{P_{k_{d_2}}}) \right) \right) \\ &= (-1)^{k+d-1} \int_{-\infty}^0 (-1)^{|\mathcal{T}_1^{[1]}| + |\mathcal{T}_2^{[1]}|} \int_{\mathbb{R}_{\leq 0}^{|\mathcal{T}_1| + |\mathcal{T}_2|}} \left((ev_{\mathcal{T}, i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}, i_d})^*(\delta_{P_{k_d}}) \right) \\ &= (-1)^{k+d-1} (ev_{\mathcal{T}, o})_* \left((ev_{\mathcal{T}, i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{\mathcal{T}, i_d})^*(\delta_{P_{k_d}}) \right). \end{aligned}$$

□

Lemma 4.10 allows us to compute the contribution of $\alpha_{\mathcal{T}}$ explicitly as follows.

Lemma 4.11. *For $MI(\mathcal{T}) = 2$ with $Q_{\mathcal{T}} \neq \emptyset$ and $\text{Mult}(\mathcal{T}) \neq 0$, and for any point x in the interior $\text{Int}(Q_{\mathcal{T}})$, we have $\lim_{\hbar \rightarrow 0} \alpha_{\mathcal{T}}|_x = (-1)^{\chi(\mathcal{T})}$. For $MI(\mathcal{T}) = 0$ with $Q_{\mathcal{T}} \neq \emptyset$ and $\text{Mult}(\mathcal{T}) \neq 0$, and for an arbitrary embedded path $\varrho : (a, b) \rightarrow M_{\mathbb{R}}$ intersecting the relative interior $\text{Int}_{re}(Q_{\mathcal{T}})$ transversally and positively (here positive means the orientation of $\{-\bar{m}_{\mathcal{T}}, \varrho'\}$ agrees with that of $M_{\mathbb{R}}$), we have $\lim_{\hbar \rightarrow 0} \int_{\varrho} \alpha_{\mathcal{T}} = (-1)^{\chi(\mathcal{T})+1}$.*

Proof. We begin with $MI(\mathcal{T}) = 2$. In this case, $k + d - 1$ is even so we have the identity $\alpha_{\mathcal{T}} = (ev_o)_* \left((ev_{i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{i_d})^*(\delta_{P_{k_d}}) \right)$. Fixing a point $x \in \text{Int}(Q_{\mathcal{T}})$, we consider the evaluation map $\hat{e}v_x : \overline{\mathfrak{M}}_d^{\mathcal{T}}(\mathcal{P}, \Sigma) \cap ev_o^{-1}(x) \rightarrow M_{\mathbb{R}}^d$ which pulls back the volume form $\prod^d d\eta_1 \wedge d\eta_2$ to $(-1)^{\chi(\mathcal{T})} cv_{\mathcal{T}}$, and in particular $\hat{e}v_x$ is a diffeomorphism onto its image (notice that $\hat{e}v_x$ is affine linear). We let $C_x := \text{Im}(\hat{e}v_x) \subset M_{\mathbb{R}}^d$. Then we have

$$(ev_o)_* \left((ev_{i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{i_d})^*(\delta_{P_{k_d}}) \right)|_x = (-1)^{\chi(\mathcal{T})} \int_{C_x} \delta_{P_{k_1}} \wedge \cdots \wedge \delta_{P_{k_d}}.$$

Using the fact that $x \in \text{Int}(Q_{\mathcal{T}})$ and the assumption that P_1, \dots, P_n are in generic position (Definition 2.11), we see that $(P_{k_1}, \dots, P_{k_d}) \in \text{Int}(C_x)$. Together with the explicit form of δ_{P_i} 's in (3.15), we have $\lim_{\hbar \rightarrow 0} \int_{C_x} \delta_{P_{k_1}} \wedge \cdots \wedge \delta_{P_{k_d}} = 1$.

For $MI(\mathcal{T}) = 0$, $k + d - 1$ is odd. We consider $\mathcal{I}_{\varrho} := \bigcup_{t \in (a, b)} \mathcal{I}_{\varrho}(t)$, where we write $\mathcal{I}_x := \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|} \times \{x\} \cong \mathbb{R}_{\leq 0}^{|\mathcal{T}^{[1]}|}$ and treat $\nu_{\mathcal{T}}$ as a volume element on each \mathcal{I}_x . Similar to the previous case we consider $\hat{e}v_{\varrho} : \mathcal{I}_{\varrho} \rightarrow M_{\mathbb{R}}^d$ which gives $\hat{e}v_{\varrho}^*(\prod^d d\eta_1 d\eta_2) = (-1)^{\chi(\mathcal{T})} cv_{\mathcal{T}} \wedge d\eta_{\mathcal{T}}$. Therefore we have

$$\int_{\varrho} \alpha_{\mathcal{T}} = (-1) \int_{\mathcal{I}_{\varrho}} (ev_{i_1})^*(\delta_{P_{k_1}}) \cdots (ev_{i_d})^*(\delta_{P_{k_d}}) = (-1)^{\chi(\mathcal{T})+1} \int_{C_{\varrho}} \delta_{P_{k_1}} \wedge \cdots \wedge \delta_{P_{k_d}}.$$

Again using the generic assumption on the points P_1, \dots, P_n , we get $(P_{k_1}, \dots, P_{k_d}) \in \text{Int}(C_{\varrho})$ and therefore $\lim_{\hbar \rightarrow 0} \int_{C_{\varrho}} \delta_{P_{k_1}} \wedge \cdots \wedge \delta_{P_{k_d}} = 1$. □

For a weighted d -pointed k -tree Γ with $MI(\Gamma) = 0, 2$ and $Q_{\Gamma} \neq \emptyset$ (notice that the definition of the polyhedral subset Q_{Γ} does not depend on the ribbon structure),

since the monomial weights u_{k_j} 's at the marked points p_{i_j} 's are all distinct, there are exactly 2^{d-1} ribbon structures (up to isomorphisms) on Γ . Notice that $\iota_{\mathcal{T}}(\Pi, \dots, \Pi)$ does not depend on the ribbon structure as well because $\Pi \in \mathcal{H}_n^{2,2}$ and Π commute with even elements in $\mathcal{H}_n^{*,*}$ (one can also see from Lemmas 4.7 and 4.11 that the terms $(-1)^{\chi(\mathcal{T})}$, which depend on the ribbon structure, indeed cancel with each other).

Therefore for each weighted d -pointed k -tree Γ , we can fix an arbitrary ribbon tree \mathcal{T} whose underlying tree $\underline{\mathcal{T}}$ is Γ , and write

$$\iota_{k,d}(\Pi, \dots, \Pi) := \sum_{\Gamma \in \text{WPT}_{k,d}} \iota_{\mathcal{T}}(\Pi, \dots, \Pi).$$

By setting $\alpha_{\Gamma} := (-1)^{\chi(\mathcal{T})} \alpha_{\mathcal{T}}$ and combining Lemmas 4.7 and 4.11, we obtain our main theorem.

Theorem 4.12. *The Maurer-Cartan solution $\Phi \in \mathcal{H}_n^{*,*}$ constructed in (3.19) is of the form*

$$\Phi = \Pi + \Xi^{0,0} + \Xi^{1,1},$$

with $\Xi^{i,i} \in \mathcal{H}_n^{i,i}$ for $i = 0, 1, 2$, and both correction terms $\Xi^{0,0}$ and $\Xi^{1,1}$ can be expressed as a sum over tropical disks:

$$\begin{aligned} \Xi^{0,0} &= \sum_{k,d} \sum_{\substack{\Gamma \in \text{WPT}_{k,d}, MI(\Gamma)=2 \\ \overline{\mathfrak{M}}_d^{\Gamma}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \neq \emptyset}} \alpha_{\Gamma} \text{Mult}(\Gamma) z^{m_{\Gamma}} u_{\Gamma}, \\ \Xi^{1,1} &= \sum_{k,d} \sum_{\substack{\Gamma \in \text{WPT}_{k,d}, MI(\Gamma)=0 \\ \overline{\mathfrak{M}}_d^{\Gamma}(\mathcal{P}, \Sigma, P_1, \dots, P_n) \neq \emptyset}} \alpha_{\Gamma} k_{\Gamma} \text{Mult}(\Gamma) z^{m_{\Gamma}} \check{\partial}_{n_{Q_{\Gamma}}} u_{\Gamma}, \end{aligned}$$

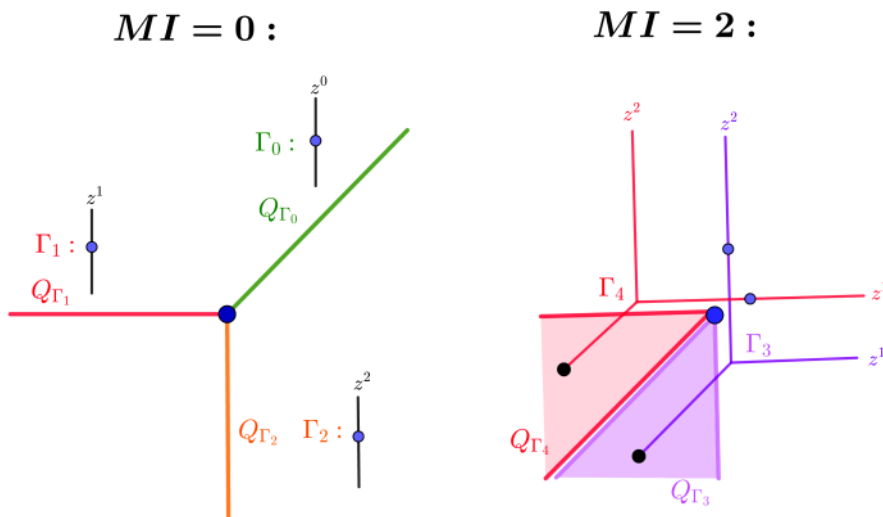
where $\text{WPT}_{k,d}$ is the set of isomorphism classes of weighted d -pointed k -trees introduced in Definition 2.6. Furthermore, in the above expressions we have $\alpha_{\Gamma} \in \mathcal{W}_{Q_{\Gamma}}^{s_{\Gamma}}$, where $Q_{\Gamma} = \text{ev}_o(\overline{\mathfrak{M}}_d^{\Gamma}(\mathcal{P}, \Sigma, P_1, \dots, P_n))$ is of codimension $s_{\Gamma} := 1 - \frac{MI(\Gamma)}{2}$ in $M_{\mathbb{R}}$, and

$$\begin{aligned} \lim_{\hbar \rightarrow 0} \alpha_{\Gamma}|_x &= 1 \quad \text{for any } x \in \text{Int}(Q_{\Gamma}) \text{ when } MI(\Gamma) = 2, \\ \lim_{\hbar \rightarrow 0} \int_{\varrho} \alpha_{\Gamma} &= -1 \quad \text{for any } \varrho \pitchfork \text{Int}_{re}(Q_{\Gamma}) \text{ positively when } MI(\Gamma) = 0. \end{aligned}$$

Example 4.13. We give an example of the locus Q_{Γ} traced out by weighted 1-pointed k -trees Γ in the case $n = 1$, i.e., when there is only 1 marked point. For a tree Γ with $MI(\Gamma) = 0$, the only possibility is that $k = 1$ and there are precisely 3 such trees $\Gamma_0, \Gamma_1, \Gamma_2$ as shown in Figure 3 together with the corresponding 1-dimensional loci $Q_{\Gamma_0}, Q_{\Gamma_1}, Q_{\Gamma_2}$. For the case $MI(\Gamma) = 2$, we have $k = 2$, and there are 6 such trees. Two of them, which we call Γ_3 and Γ_4 , with the same attached monomial $\text{Mono}(\Gamma) = z^1 z^2$, are shown in Figure 3. Note that the boundary between Q_{Γ_3} and Q_{Γ_4} is not a wall in \mathcal{D} although the moduli space jumps across it. This is because the attached monomial $\text{Mono}(\Gamma)$ does not jump across the boundary, and this agrees with the fact that Φ is simply a holomorphic function outside the support $\text{Supp}(\mathcal{D})$.

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FIGURE 3. Tropical disks and their moduli spaces for $n = 1$

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