Tropical Effective Primary and Dual Nullstellensätze*

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Abstract

Tropical algebra is an emerging field with a number of applications in various areas of mathematics. In many of these applications appeal to tropical polynomials allows to study properties of mathematical objects such as algebraic varieties and algebraic curves from the computational point of view. This makes it important to study both mathematical and computational aspects of tropical polynomials.

In this paper we prove tropical Nullstellensatz and moreover we show effective formulation of this theorem. Nullstellensatz is a next natural step in building algebraic theory of tropical polynomials and effective version is relevant for computational aspects of this field.

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1 Introduction

A *min-plus* or *tropical semiring* is defined by the set $\mathbb{K}$, which can be $\mathbb{R}$, $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$, $\mathbb{Q}$ or $\mathbb{Q}_\infty = \mathbb{Q} \cup \{+\infty\}$ endowed with two operations *tropical addition* $\oplus$ and *tropical multiplication* $\odot$ defined in the following way:

$$x \oplus y = \min\{x, y\}, \qquad x \odot y = x + y.$$

Tropical polynomials are a natural analog of classical polynomials. In classical terms it can be expressed in the form $f(\vec{x}) = \min_i M_i(\vec{x})$, where each $M_i(\vec{x})$ is a linear polynomial (a tropical monomial) in variables $\vec{x} = (x_1, \ldots, x_n)$, and all coefficients of all M_i are nonnegative integers except a free coefficient which can be any element of $\mathbb{K}$.

The degree of a tropical monomial M is the sum of its coefficients (except the free coefficient) and the degree of a tropical polynomial f denoted by $\deg(f)$ is the maximal

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degree of its monomials. A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial f if the minimum $\min_i\{M_i(\vec{a})\}$ is either attained on at least two different monomials M_i or is infinite. We defer a more detailed definitions on the basics of min-plus algebra to Preliminaries.

Tropical polynomials have appeared in various areas of mathematics and found many applications (see, for example, [13, 20, 24, 21, 22, 12]). One of the most important advantage of tropical algebra is that it makes some properties of classical mathematical objects computationally accessible [26, 13, 20, 24]. One of the main goals of min-plus mathematics is to build a theory of tropical polynomials which would help to work with them and would possibly lead to new results in the related areas. Computational reasons, on the other hand, make it important to keep the theory maximally computationally efficient.

The best studied so far is the case of linear tropical polynomials and systems of linear tropical polynomials. For them the analog of the large part of the theory of classical linear polynomials was established. This includes studies of tropical analogs of the rank of a matrix and the independence of vectors [4, 15, 1], the analog of the determinant of a matrix and its properties [22], the analog of Gauss triangular form [8]. Also the solvability problem for tropical linear systems was studied from the complexity point of view. Interestingly, it turned out to be polynomially equivalent to a well known mean payoff games problem [10]. Thus, this problem lies in $\mathsf{NP} \cap \mathsf{coNP}$, but is not known to be in P.

For tropical polynomials of arbitrary degree less is known. In [23] the radical of a tropical ideal was explicitly described. In [26] it was shown that solvability problem for tropical polynomial systems is NP-complete.

Along with tropical polynomials there were also studied min-plus polynomials. Min-plus polynomial is an expression of the form $\min_i M_i(\vec{x}) = \min_j L_j(\vec{x})$, where M_i and L_j are tropical monomials. A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial if $\min_i M_i(\vec{a}) = \min_j L_j(\vec{a})$.

Min-plus polynomials were studied mainly for its connections to dynamic programming (see [3, 16]). As in the case of tropical polynomials here the best studied case is the case of linear min-plus polynomials [3]. Also in [10] the connection of min-plus and tropical linear polynomials was established.

As for the min-plus polynomials of arbitrary degree much less is known. We are only aware of the result on the computational complexity of the system of min-plus polynomials: paper [11] shows that this problem is NP-complete.

Our results

The next natural step in developing of the theory of tropical polynomials would be an analog of classical Nullstellensatz, the theorem which for the classical polynomials constitutes one of the cornerstones of algebraic geometry. Concerning the tropical Nullstellensatz, the problem was already addressed in the paper [7]. In this paper there was established a general idea to approach this theorem in the tropical case through the dual formulation. Moreover, in [7] there was formulated a conjecture (which we restate below as Conjecture 3) capturing the formulation of the tropical dual Nullstellensatz and this conjecture was proven for the case of polynomials of 1 variable. Previously in [25] tropical dual Nullstellensatz was established for a pair of polynomials ($k = 2$) in 1 variable relying on the classical resultant and on the Kapranov's theorem [5, 25].

More specifically, in [7] there was considered a Macaulay matrix of the system of tropical polynomials $F = \{f_1, \dots, f_k\}$. This matrix can be easily constructed from F: we just consider all polynomials $f_i + M_j$ (in classical notation) of degree at most N, where N is a parameter and M_j is a tropical monomial. We put the coefficients of these polynomials in the rows of the matrix, where columns of the matrix correspond to monomials. Empty entries of the matrix we fill with ∞. The resulting matrix we denote by C_N. In [7] it was conjectured that the system of polynomials F has a solution iff the tropical linear system with the matrix C_N has a solution, and moreover N can be bounded by some function on n, k and the degree of polynomials in F (this refers to effectiveness).

In this paper we prove this conjecture. Moreover, we show an effective version of the theorem. That is, we pose bounds on N and provide examples showing that they are close to tight. These bounds are relevant for computational aspects of tropical polynomial systems. Surprisingly, it turns out that the cases of tropical semiring with and without ∞ differ dramatically. More specifically, in the case of tropical semirings $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{Q}$ we show that F has a solution iff the tropical linear system with the matrix C_N has a solution, where $N = (n+2) \cdot k \cdot d$, d is the maximal degree of polynomials in F, k is the number of polynomials in F and n is the number of variables. For the case of tropical semirings $\mathbb{K} = \mathbb{R}_\infty$ and $\mathbb{K} = \mathbb{Q}_\infty$ we show a similar result, but with $N = (Cd)^{\min(n,k)}$ for some constant C. Thus for the case without ∞ the bound on N is polynomial in n, k, d and for the case with ∞ the bound on N is still polynomial in d, but is exponential in n and k. We give examples showing that our bounds on N are qualitatively optimal, that is the difference of the values of N in these cases is not an artifact of the proof, but is unavoidable. However, quantitatively there is a gap between upper and lower bounds, see Section 3 for details.

Regarding the substantial gap between the required degree in the finite and infinite cases we observe there is a similar situation for classical Nullstellensatz. Indeed, we show that in case of semiring $\mathbb{R}$ the bound in a tropical effective Nullstellensatz depends on the sum of the degrees of the polynomials, while in case of larger semiring $\mathbb{R}_\infty$ the bound depends on the product of the degrees (Theorems 4 and 9). We recall that for systems of classical polynomials over an algebraically closed field the bound on the effective Nullstellensatz depends on the sum of the degrees of polynomials in homogeneous (projective) case [18, 19] while the bound depends on the product of the degrees for arbitrary polynomials (affine case) [6, 17].

Next we show the primary version of tropical Nullstellensatz. We view Nullstellensatz as a duality[1] result for systems of polynomials: if there is *no* solution to the system of polynomials then some positive property holds (something *does* exist). In the classical case this positive property is the containment of 1 in the ideal generated by polynomials (over algebraically closed field). The naive analog does not hold for the tropical case. Indeed, for example, in the tropical ideal generated by the system of tropical polynomials $\{\min(x, 0), \min(x, 1)\}$ there are no polynomials with only one monomial and thus there is no polynomial 0. Basically, the point is that in the tropical semiring there is no subtraction, so in any algebraic combination of polynomials no monomials cancel out. To overcome this difficulty we introduce the notion of *nonsingular* tropical algebraic combination of tropical polynomials (see the definition in Preliminaries; here we only note that the property is simple and straightforward to check). For the primary tropical Nullstellensatz we show that there is no solution to tropical linear system F iff there is a nonsingular tropical algebraic combination of polynomials in F of degree at most N. We show this result for both cases of tropical semiring with and without ∞ and the value N in both cases corresponds to the size of Macaulay matrix in the tropical dual Nullstellensatz.

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[1] To avoid a confusion we note that the word 'dual' is used in two different meanings. First, we use it in the term "dual Nullstellensatz" as opposed to standard version of Nullstellensatz. This means that dual Nullstellensatz is obtained from standard Nullstellensatz by (linear) duality. Second, we use the word 'dual' in term "duality result" to denote the general type of results. Since standard Nullstellensatz is a duality result itself, applying linear duality to it results in a non-duality result. Thus, dual Nullstellensatz is not a duality result.

To establish primary Nullstellensatz we need a duality for tropical linear systems. We show this duality result as a sidestep. However we note that this results is heavily based on already known results [2] and is a simple corollary of them.

We also prove similar results for the case of min-plus polynomials. As a sidestep of our analysis we show the close connection between tropical and min-plus systems of polynomials. We argue that these two models are very closely connected and that this connection can be used to establish new results in tropical algebra. The observation is that some results (like linear duality) are easier to obtain for min-plus polynomials and then translate to tropical polynomials, and some other results (like Nullstellensatz) on the other hand are easier to obtain for tropical polynomials and then translate to min-plus polynomials. In our opinion it is fruitful for further development of the theory to consider both models simultaneously.

Our techniques

We use the general approach of the paper [7] to Nullstellensatz through dual formulation.

To establish the dual Nullstellensatz we use methods of discrete geometry dealing with integer polyhedra. First we obtain dual Nullstellensatz for the case without ∞. The case with ∞ requires much more additional work.

To obtain primary Nullstellensatz we apply the duality results for linear tropical polynomials. We note that these results rely on the completely different combinatorial techniques, namely on the connection to mean payoff games [2].

Other works on tropical Nullstellensatz

In paper [14] there was established Nullstellensatz for tropical semiring augmented with additional elements (called ghosts). This result is in the line with other results [24] trying to capture tropical mathematics by the means of the classical ones. However, tropical semiring augmented with ghosts constitutes (logically) a completely different model compared to usual tropical semiring. Thus our results are incomparable with the one of the paper [14].

We also note that the paper [23] (which has Nullstellensatz in the title) takes completely different view on Nullstellensatz. We consider Nullstellensatz as a result on the solvability of system of polynomials, and paper [23] views Nullstellensatz as a result on the structure of the radical of a tropical ideal. As it can be easily seen, for example, from our results during the translation from classical world to the tropical one, the connection between these two objects changes drastically (cf. with example $F = \{\min(x, 0), \min(x, 1)\}$ above). Thus our results are incomparable with the results of [23] as well.

The rest of the paper is organized as follows. In Section 2 we introduce main definitions. In Section 3 we present tropical and min-plus dual Nullstellensätze. In Section 4 we present tropical and min-plus primary Nullstellensätze. In Section 5 we present our results on tropical and min-plus linear duality. In Section 6 we present results on connection between tropical and min-plus polynomial systems.

Due to the space constraint in this extended abstract many proofs are omitted. They can be found in the full version of the paper [9].

2 Preliminaries

2.1 Min-plus algebra

Tropical and min-plus polynomials

A *min-plus* or *tropical semiring* is defined by the set $\mathbb{K}$, which can be $\mathbb{R}$, $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$, $\mathbb{Q}$ or $\mathbb{Q}_\infty = \mathbb{Q} \cup \{+\infty\}$ endowed with two operations, *tropical addition* $\oplus$ and *tropical multiplication* $\odot$ defined in the following way:

$$x \oplus y = \min\{x, y\}, \qquad x \odot y = x + y.$$

Below we mainly consider $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{R}_\infty$. The proofs however literally translate to the cases of $\mathbb{Q}$ and $\mathbb{Q}_\infty$.

The tropical (or min-plus) monomial in variables $x_1, \ldots, x_n$ is defined as

$$M = c \odot x_1^{\odot i_1} \odot \ldots \odot x_n^{\odot i_n}, \tag{1}$$

where c is an element of the semiring $\mathbb{K}$ and $i_1, \ldots, i_n$ are nonnegative integers. In usual notation the monomial is

$$M = c + i_1 x_1 + \ldots + i_n x_n.$$

The degree of the monomial is defined as the sum $i_1 + \ldots + i_n$. We denote $\vec{x} = (x_1, \ldots, x_n)$ and for $I = (i_1, \ldots, i_n)$ we introduce the notation

$$\vec{x}^I = x_1^{\odot i_1} \odot \ldots \odot x_n^{\odot i_n}.$$

A tropical polynomial is the tropical sum of tropical monomials

$$f = \bigoplus_i M_i,$$

or in usual notation $f = \min_i M_i$. The degree of the tropical polynomial f denoted by $\deg(f)$ is the maximal degree of its monomials. A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial f if the minimum $\min_i\{M_i(\vec{a})\}$ is either attained on at least two different monomials M_i or is infinite.

A min-plus polynomial is an expression of the form

$$\bigoplus_i M_i(\vec{x}) = \bigoplus_j L_j(\vec{x}),$$

where M_i, L_j are min-plus monomials. The degree of min-plus polynomial is the maximal degree among monomials M_i and L_j over all i, j. A point $\vec{a} \in \mathbb{K}^n$ is a root of this polynomial if the equality holds for $\vec{x} = \vec{a}$.

Linear polynomials

An important special case of tropical and min-plus polynomials are linear polynomials. They can be defined as general tropical polynomials of degree 1. However, it is convenient to denote by a linear polynomial an expression of the form

$$\min_{1 \le j \le n} \{a_j + x_j\}.$$

That is we assume that all variables are presented exactly once. The tropical linear system

$$\min_{1 \le j \le n} \{a_{ij} + x_j\}, \; 1 \le i \le m, \tag{2}$$

can be naturally associated with its matrix $A \in \mathbb{K}^{m \times n}$. We will also use a matrix notation $A \odot \vec{x}$ for such system.

Analogously min-plus linear systems

$$\min_{1 \le j \le n} \{a_{ij} + x_j\} = \min_{1 \le j \le n} \{b_{ij} + x_j\},\ 1 \le i \le m,$$

can be associated with a pair of matrices A and B corresponding to the left-hand side and the right-hand side of an equation. We will also write min-plus linear system in a matrix form as $A \odot \vec{x} = B \odot \vec{x}$. It will be also convenient to consider min-plus linear systems of (componentwise) inequalities $A \odot \vec{x} \le B \odot \vec{x}$. It is not hard to see that their expressive power is the same as of equations.

▶ **Lemma 1.** *Given a min-plus system of linear equations it is easy to construct an equivalent system of min-plus linear inequalities and visa versa.*

Proof. Indeed, each min-plus linear equation $L_1(\vec{x}) = L_2(\vec{x})$ is equivalent to the pair of min-plus inequalities $L_1(\vec{x}) \ge L_2(\vec{x})$ and $L_1(\vec{x}) \le L_2(\vec{x})$. On the other hand min-plus linear inequality $L_1(\vec{x}) \le L_2(\vec{x})$ is equivalent to the min-plus equation $L_1(\vec{x}) = \min(L_1(\vec{x}), L_2(\vec{x}))$. It is not hard to see that the last equation can be transformed to the form of min-plus linear equation. ◀

There is one more important convention we make concerning the case of tropical semiring with infinity. For two matrices $A, B \in \mathbb{R}_\infty^{n \times m}$ we say that the system $A \odot \vec{x} < B \odot \vec{x}$ has a solution if there is $\vec{x} \in \mathbb{R}_\infty^m$ such that for each row of the system if one of sides is finite, then strict inequality holds, but also the case where both sides are equal to ∞ is allowed (informally, we can say that $\infty < \infty$).

We also consider non-homogeneous tropical linear systems

$$\min_{1 \le j \le n} \{a_{ij} + x_j, a_i\},\ 1 \le i \le m. \tag{3}$$

This system can be naturally associated to the matrix $A \in \mathbb{K}^{m \times (n+1)}$ and written in the matrix form as $A \odot (\vec{x}, 0)$. Analogously, we can consider non-homogeneous min-plus linear systems $A \odot (\vec{x}, 0) \le B \odot (\vec{x}, 0)$. We note that over $\vec{x} \in \mathbb{R}^n$ the tropical system $A \odot (\vec{x}, 0)$ is solvable iff homogeneous system $A \odot \vec{x}'$ is solvable, where $\vec{x}' = (\vec{x}, x_{n+1})$. Indeed, we can add the same number to all coordinates of the solution of the latter system to make $x_{n+1} = 0$. The same is true for min-plus case. But the same is not true over $\mathbb{R}_\infty$: homogeneous system always has a solution (just let $\vec{x} = (\infty, \dots, \infty)$), but non-homogeneous system does not always have a solution.

3 Tropical and Min-plus Dual Nullstellensatz

▶ **Definition 2.** For a given system of tropical polynomials $F = \{f_1, \dots, f_k\}$ in n variables we introduce its infinite Macaulay matrix C. The columns of C correspond to nonnegative integer vectors $I \in \mathbb{Z}_+^n$ and the rows of C correspond to the pairs (j, J), where $1 \le j \le k$ and $J \in \mathbb{Z}_+^n$. For given I and (j, J) we let the entry $c_{(j,J),I}$ be equal to the coefficient of the monomial $\vec{x}^I$ in the polynomial $\vec{x}^J \odot f_j$ (if there is no such monomial in the polynomial we assume that the entry is equal to $+\infty$). By C_N we denote the finite submatrix of the matrix C consisting of the columns I such that $i_1 + \ldots + i_n \le N$ and the rows which have all their finite entries in these columns. The tropical linear system associated with C_N will be of interest to us. Over $\mathbb{R}_\infty$ we consider non-homogeneous system with the matrix C_N. The column corresponding to constant monomial is a non-homogeneous column.

For the system of min-plus polynomials $F = \{f_1 = g_1, \dots, f_k = g_k\}$ we analogously introduce the pair of matrices C and D corresponding to the left-hand sides and the right-hand sides of polynomials respectively. In the same way we introduce matrices C_N, D_N and the corresponding linear systems $C_N \odot \vec{y} = D_N \odot \vec{y}$. Analogously, for the case of $\mathbb{R}_\infty$ we consider non-homogeneous systems.

In the paper [7] there were conjectured three forms of the tropical dual Nullstellensatz theorem. We state the most strong of them, effective Nullstellensatz theorem.

▶ **Conjecture 3** ([7]). *There is a function N of n and of $\deg(f_i)$ for $1 \le i \le k$ such that the system of polynomials F has a common tropical root iff the tropical linear system corresponding to the matrix C_N has a solution.*

Note that the classical analog of this statement is precisely the effective Nullstellensatz theorem in the dual form (see [7] for the detailed discussion).

In [7] the conjecture was proven for the case of $n = 1$. In this paper we prove the general case of the conjecture.

▶ **Theorem 4** (Tropical Dual Nulstellensatz). *Consider the system of tropical polynomials $F = \{f_1, \dots, f_k\}$ of n variables. Denote by d_i the degree of the polynomial f_i and let $d = \max_i d_i$.*

(i) *Over semiring $\mathbb{R}$ the system F has a solution iff the Macaulay tropical linear system $C_N \odot \vec{y}$ for*

$$N = (n+2)\,(d_1 + \ldots + d_k)$$

has a solution.

(ii) *Over semiring $\mathbb{R}_\infty$ the system F has a solution iff the non-homogeneous Macaulay tropical linear system $C_N \odot \vec{y}$ for*

$$N = poly(n,k)\,(2d)^{\min(n,k)}$$

has a solution.

Proof sketch. We describe the proof idea here. We concentrate on the case $\mathbb{R}$. For the case of $\mathbb{R}_\infty$ much more additional work is required.

Throughout the proof we consider rows of the matrix C_N, solutions to $C_N \odot \vec{y}$, coefficients of tropical polynomials f_i. All of them can be viewed as vectors $\vec{a} = \{a_I\}_I$ which coordinates are labeled by $I \in D$ for some $D \subseteq \mathbb{Z}_+^n$. We further consider these vectors as the set of points $\{(I, a_I)\}_I$ in $(n+1)$-dimensional space. It is convenient to consider the first n dimension as horizontal and the last one as vertical.

We next consider, what does it mean for the vector $\vec{a}$ to be a solution to one of the tropical linear equations $\vec{c} \odot \vec{y}$ of the system $C_N \odot \vec{y}$. By the definition this means that the value $c_I + a_I$ is minimized for at least two different I. It is not hard to see that equivalently this means that there is such $t \in \mathbb{R}$ that $\{-a_I + t\}_I$ lies below $\{c_I\}_I$ and has at least two common points with it. That is, we can adjust the set of points corresponding to $\{-a_I\}_I$ moving it along the vertical line in such a way that it lies below the set of points corresponding to the equation and has at least two common points with it. Thus we obtain geometrical interpretation of tropical solutions.

Next, we note that if we talk about solution to polynomial f, we can still consider the polynomial as a set of points $\{(I, f_I)\}_I$, where f_I is a coefficient of the monomial $\vec{x}^I$ in f, but now the solution corresponds not to an arbitrary set of points $\{a_I\}_I$ but to a hyperplane.

Now it is not hard to capture the goal in geometric terms. We know that there is a solution to $C_N \odot \vec{y}$. We need to show that then there is a hyperplane solution to this system (or to system F, which is the same in the case of hyperplanes).

An interesting feature of our proof is how we obtain a hyperplane solution. The natural way would be to start with a non-hyperplane solution $\vec{a}$ and somehow modify it to make it a hyperplane. This was an approach of paper [7] for the case $n = 1$. However, it is not clear how to do it for $n > 1$. Instead we actually find the hyperplane solution inside the system F.

For this for each polynomial f_j in F we consider the corresponding set of points $\{(I, f_{j,I})\}_I$, add to them all points (I, t) for all I and $t \geq f_{j,I}$ and consider the convex hull of this set of points. As a result we obtain a polytope P_j which is called extended Newton polytope. Note that P_j is infinite in the vertical direction.

Then we consider a new polytope P_0. It can be expressed by the following formula:

$$P_0 = (n + 2) \cdot (P_1 + \ldots + P_k),$$

where all operations are in the sense of the Minkowski sum. It turns out that the solution to the system F can be found in P_0. Namely, one of the facets of P_0 is a solution.

To see the idea behind this let the bottom of P_0 be the set of lowest points of P_0, first n coordinates of which are integer. Informally, the bottom of P_0 is a discrete version of the set of its non-vertical facets. Note that the bottom of P_0 can be considered as a vector $\{b_I\}_I$. First, it turns out (and it is not hard to show) that $\{b_I\}_I$ is a tropical linear combination of the rows of C_N. This means that a solution $\vec{a}$ to $C_N \odot \vec{y}$ is also a solution to a tropical linear equation given by $\vec{b}$. Second observation is that it can be shown that for any i we can translate P_i to any place inside P_0. Now we can consider the set of points $\{(I, b_I)\}_I$ and adjust the set $\{(I, a_I)\}_I$ in such a way that it is below $\{(I, b_I)\}_I$ and has at least two points in common with it. We consider one of these points and move P_i for arbitrary i to this point and inside of P_0. Then P_i will lie above $\{(I, b_I)\}_I$. On the other hand it will correspond to one of the rows of C_N and thus $\vec{a}$ will be a solution to it. Thus P_i and $\vec{a}$ will have two points in common and they will also be common points of $\vec{b}$ which lie between $\vec{a}$ and P_i. Thus P_i will have two common points with $\vec{b}$, that is the bottom of P_0 is a solution to P_i. A more careful analysis along these lines shows that actually, one of the facets of P_0 is a solution to all polynomials in F. ◀

We show dual Nullstellensatz for min-plus case.

▶ **Theorem 5** (Min-Plus Dual Nullstellensatz). *Consider the system of min-plus polynomials $F = \{f_1 = g_1, \ldots, f_k = g_k\}$ of n variables. Denote by d_i the degree of the polynomial $f_i = g_i$ and let $d = \max_i d_i$.*

(i) *Over semiring $\mathbb{R}$ the system F has a solution iff the Macaulay min-plus linear system $C_N \odot \vec{y} = D_N \odot \vec{y}$ for*

$$N = (n + 2)\,(d_1 + \ldots + d_k)$$

has a solution.

(ii) *Over semiring $\mathbb{R}_\infty$ the system F has a solution iff the non-homogeneous Macaulay min-plus linear system $C_N \odot \vec{y} = D_N \odot \vec{y}$ for*

$$N = poly(n, k)\,(2d)^{\min(n,k)}$$

has a solution.

The proof of this theorem is based on the application of the connection between tropical and min-plus polynomial systems, which we describe below, to tropical Dual Nullstellensatz.

We provide examples showing that our bounds on N are qualitatively tight. Namely for the semiring $\mathbb{R}$ we consider the following family F of $(n+1)$ tropical polynomials of degree d:

$$\begin{aligned} f_1 &= 0 \oplus 0 \odot x_1, \\ f_{i+1} &= 0 \odot x_i^{\odot d} \oplus 0 \odot x_{i+1}, \ i \in [n-1] \\ f_{n+1} &= 0 \oplus 1 \odot x_n. \end{aligned}$$

It is not hard to see that this system has no solutions. Indeed, if there is a solution, then from f_1 we can see that $x_1 = 0$, then from f_2 we can see that $x_2 = 0$ etc., from f_n we can see that $x_n = 0$. However from f_{n+1} we have that $x_n = -1$ which is a contradiction.

On the other hand, we show that the Macaulay tropical system with the matrix $C_{(d-1)(n-1)}$ corresponding to the system F has a solution.

For the semiring $\mathbb{R}_\infty$ for any $d > 1$ we consider the following system F of tropical polynomials of variables $x_1, \ldots, x_n, y$.

$$\begin{aligned} f_1 &= 0 \odot x_1 \odot y \oplus 0, \\ f_{i+1} &= 0 \odot x_i^{\odot d} \oplus 0 \odot x_{i+1}, \text{for } i = 1, \ldots, n-1, \\ f_{n+1} &= 0 \odot x_{n-1}^{\odot d} \oplus 1 \odot x_n. \end{aligned}$$

This system clearly has no solutions. Indeed, we can consecutively show that all coordinates of a solution should be finite and then the polynomials f_n and f_{n+1} give a contradiction.

On the other hand, we show that non-homogeneous Macaulay system $C_{d^{n-1}-1} \odot \vec{y}$ has a solution.

Both of these examples translate to min-plus setting straightforwardly. The details of the proofs are omitted

We note that quantitatively there is a room for improvement between our lower and upper bounds on N. The gap is more substantial in the case of semiring $\mathbb{R}$. Assuming for the sake of simplicity that $n = k$ our upper bound gives approximately $N \leq dn^2$ and our lower bound gives $N \geq dn$. Thus we can formulate an open problem.

▶ **Open Problem 6.** *Close the gap between the upper and the lower bound on N in the tropical Nullstellensatz.*

4 Primary Tropical and Min-Plus Nullstellensatz

Next we establish Nullstellensatz in a more standard primary form.

We start with a more intuitive min-plus Nullstellensatz.

▶ **Theorem 7** (Min-Plus Primary Nullstellensatz)**.** *Consider the system of min-plus polynomials $F = \{f_1 = g_1, \ldots, f_k = g_k\}$ of n variables. Denote by d_i the degree of the polynomial $f_i = g_i$ and let $d = \max_i d_i$.*

Over semiring $\mathbb{R}$ the system F has no solution iff we can construct an algebraic min-plus combination $f = g$ of degree at most

$$N = (n+2)(d_1 + \ldots + d_k)$$

of them such that for each monomial $M = x_1^{\odot j_1} \odot \ldots \odot x_n^{\odot j_n}$ its coefficient in f is greater than its coefficient in g. In algebraic combination $f = g$ we allow to use not only polynomials $f_i = g_i$, but also $g_i = f_i$.

Over semiring $\mathbb{R}_\infty$ the system F has no solution iff we can construct an algebraic combination $f = g$ of degree at most

$$N = \mathit{poly}(n,k)\,(2d)^{\min(n,k)}$$

of them such that for each monomial $M = x_1^{\odot j_1} \odot \ldots \odot x_n^{\odot j_n}$ its coefficient in f is greater than its coefficient in g and with additional property that the constant term in g is finite.

Proof. We present a proof for the case $\mathbb{R}$.

We will use the min-plus linear duality (Lemma 10 below) for the proof of this theorem.

By Theorem 1 the system of polynomials F has no solution over $\mathbb{R}$ iff the corresponding Macaulay linear system

$$C_N \odot \vec{y} = D_N \odot \vec{y}$$

has no finite solution. By Lemma 1 this system is equivalent to the system of min-plus inequalities

$$\begin{pmatrix} C_N \\ D_N \end{pmatrix} \odot \vec{x} \leq \begin{pmatrix} D_N \\ C_N \end{pmatrix} \odot \vec{x}.$$

By Lemma 10 the fact that this system has no finite solution is equivalent to the fact that the dual system

$$\begin{pmatrix} D_N^T & C_N^T \end{pmatrix} \odot \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix} < \begin{pmatrix} C_N^T & D_N^T \end{pmatrix} \odot \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix}$$

has a solution in $\mathbb{R}_\infty^n$ (here we allow for both sides to be infinite in some rows; note that we have to use linear duality over $\mathbb{R}_\infty$ since C_N and D_N might have infinite entries).

This system can be interpreted back in terms of polynomials. Indeed, note that now the columns of the matrices correspond to the equations of F multiplied by some $\vec{x}^J$ and rows correspond to some monomials $\vec{x}^I$. Thus the solution to the system corresponds to the sum of equations of F multiplied by some monomials, such that each coefficient of the sum on the left side is smaller than the coefficient of the sum on the right side. The fact that we allow both sides to be infinite in some row corresponds to the fact that some monomials might be not presented in the sum. The fact that we allow infinite coordinates in the solution correspond to the fact that we do not have to use all polynomials of $\vec{x}^I f_j = \vec{x}^I g_j$ in algebraic combination. ◀

For the tropical case we will need the following definition.

▶ **Definition 8.** For the system of tropical polynomials $f_1, \ldots, f_k$ and tropical monomials $M_1, \ldots, M_m$ the algebraic combination

$$g = \bigoplus_{j=1}^{m} g_j,$$

where

$$g_j = M_j \odot f_{i_j},$$

is called nonsingular if the following two properties hold:

- for each monomial M of g there is a (unique) $1 \leq l(M) \leq m$ such that the coefficient of M at polynomial $g_{l(M)}$ is less than the coefficients of M at all other polynomials g_j for $j \neq l(M)$;
- for different M and M' we have $l(M) \neq l(M')$.

Now we can formulate tropical Nullstellenstz in a primary form.

▶ **Theorem 9** (Tropical Primary Nullstellensatz). *Consider the system of tropical polynomials $F = \{f_1 = g_1, \ldots, f_k = g_k\}$ of n variables. Denote by d_i the degree of the polynomial f_i and let $d = \max_i d_i$.*

The system F has no solution over $\mathbb{R}$ iff there is a nonsingular algebraic combination g for it of degree at most

$$N = (n + 2)\,(d_1 + \ldots + d_k)$$

The system F has no solution over $\mathbb{R}_\infty$ iff there is a nonsingular algebraic combination g for it of degree at most

$$N = poly(n, k)\,(2d)^{\min(n,k)}$$

and with finite constant monomial.

For the proofs of the last two theorems we apply min-plus and tropical linear duality (which we describe below) to min-plus and tropical dual Nullstellensätze respectively. The idea is simple. By dual Nullstellensatz we have that there is solution for system F iff there is a solution to Macaulay linear system. Applying linear duality to this system we get that there is a solution for F iff there is no solution to some new (tropical or min-plus) linear system. Finally, we interpret this solution back in terms of polynomials and obtain primary Nullstellensätze.

5 Linear Duality

We show the following simple formulation of min-plus duality.

▶ **Lemma 10.** *For two matrices $A, B \in \mathbb{R}^{n\times m}$ exactly one of the following is true.*
1. *There is a solution to $A \odot \vec{x} \le B \odot \vec{x}$.*
2. *There is a solution to $B^T \odot \vec{y} < A^T \odot \vec{y}$.*

For two matrices $A, B \in \mathbb{R}_\infty^{n\times m}$ exactly one of the following is true.
1. *There is a solution $\vec{x} \ne (\infty, \ldots, \infty)$ to $A \odot \vec{x} \le B \odot \vec{x}$.*
2. *There is a finite solution to $B^T \odot \vec{y} < A^T \odot \vec{y}$.*

For two matrices $A, B \in \mathbb{R}_\infty^{n\times m}$ exactly one of the following is true.
1. *There is a finite solution to $A \odot \vec{x} \le B \odot \vec{x}$.*
2. *There is a solution $\vec{y} \ne (\infty, \ldots, \infty)$ to $B^T \odot \vec{y} < A^T \odot \vec{y}$.*

We show similar result for tropical duality.

▶ **Lemma 11.** *For a matrix $A \in \mathbb{R}^{n\times m}$ exactly one of the following is true.*
1. *There is a solution to $A \odot \vec{x}$.*
2. *There is $\vec{z}$ such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once and for each two rows the minimums are in different columns.*

For a matrix $A \in \mathbb{R}_\infty^{n\times m}$ exactly one of the following is true.
1. *There is a finite solution to $A \odot \vec{x}$.*
2. *There is $\vec{z}$ such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once or is equal to ∞ and for each two rows the (unique) minimums are in different columns.*

For a matrix $A \in \mathbb{R}_\infty^{n\times m}$ exactly one of the following is true.
1. *There is a solution to $A \odot \vec{x}$.*
2. *There is a finite $\vec{z}$ such that in each row of $A^T \odot \vec{z}$ the minimum is attained only once and for each two rows the minimums are in different columns.*

The idea for the proof of min-plus linear duality is a connection to mean payoff games established in [2]. Once this connection is known the proof of min-plus linear duality is rather simple. However, we are not aware of a presentation of this (or similar) result in the literature, so due to the simplicity of the formulation and the fact that we use this result to obtain primary Nullstellensatz, we decide to include it in the paper.

For the proof of tropical linear duality we use connection between tropical and min-plus polynomials and deduce the tropical duality from the min-plus duality. However, we note that it also can be shown directly using the analysis of the paper [8].

6 Tropical vs. Min-plus

We also establish the connection between tropical and min-plus polynomial systems.

▶ **Lemma 12.** *For both $\mathbb{R}$ and $\mathbb{R}_\infty$ given a system of tropical polynomials we can construct a system of min-plus polynomials over the same set of variables and with the same set of solutions.*

In the other direction we do not have such a simple connection, but we can still prove the following lemma.

▶ **Lemma 13.** *For any system of min-plus polynomials F over n variables there is a system of tropical polynomials T over $2n$ variables and an injective linear transformation $H\colon \mathbb{R}^n_\infty \to \mathbb{R}^{2n}_\infty$ such that the image of the solutions of F coincides with the solution set of T. The same is true over semiring $\mathbb{R}$.*

The proofs of these lemmas follow the lines of the proof of the analogous statement for the case of linear polynomials in the paper [10].

References

1 Marianne Akian, Stephane Gaubert, and Alexander Guterman. Linear independence over tropical semirings and beyond. *Contemporary Mathematics*, 495:1–33, 2009.

2 Marianne Akian, Stephane Gaubert, and Alexander Guterman. Tropical polyhedra are equivalent to mean payoff games. *International Journal of Algebra and Computation*, 22(1), 2012.

3 Peter Butkovič. *Max-linear Systems: Theory and Algorithms*. Springer, 2010.

4 M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. *Combinatorial and computational geometry*, 52:213–242, 2005.

5 Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. Non-archimedean amoebas and tropical varieties. *Journal fur die reine und angewandte Mathematik (Crelles Journal)*, 2006.601:139–157, 2007.

6 Marc Giusti, Joos Heintz, and Juan Sabia. On the efficiency of effective Nullstellensätze. *Computational complexity*, 3(1):56–95, 1993.

7 Dima Grigoriev. On a tropical dual Nullstellensatz. *Advances in Applied Mathematics*, 48(2):457 – 464, 2012.

8 Dima Grigoriev. Complexity of solving tropical linear systems. *Computational Complexity*, 22(1):71–88, 2013.

9 Dima Grigoriev and Vladimir V. Podolskii. Tropical effective primary and dual Nullstellensätze. *CoRR*, abs/1409.6215, 2014.

10 Dima Grigoriev and Vladimir V. Podolskii. Complexity of tropical and min-plus linear prevarieties. *Computational complexity, to appear*, pages 1–34, 2015.

11 Dima Grigoriev and Vladimir Shpilrain. Tropical cryptography. *Communications in Algebra*, 42(6):2624–2632, 2014.

12 Birkett Huber and Bernd Sturmfels. A polyhedral method for solving sparse polynomial systems. *Mathematics of Computation*, 64:1541–1555, 1995.

13 I. Itenberg, G. Mikhalkin, and E.I. Shustin. *Tropical Algebraic Geometry*. Oberwolfach Seminars. Birkhäuser, 2009.

14 Zur Izhakian. Tropical algebraic sets, ideals and an algebraic Nullstellensatz. *International Journal of Algebra and Computation*, 18(06):1067–1098, 2008.

15 Zur Izhakian and Louis Rowen. The tropical rank of a tropical matrix. *Communications in Algebra*, 37(11):3912–3927, 2009.

16 Stasys Jukna. Lower bounds for tropical circuits and dynamic programs. *Electronic Colloquium on Computational Complexity (ECCC)*, 21:80, 2014.

17 János Kollár. Sharp effective Nullstellensatz. *J. Amer. Math. Soc.*, 1:963–975, 1988.

18 D. Lazard. Algèbre linéaire sur $K[X_1, \ldots, X_n]$ et élimination. *Bull. Soc. Math. France*, 105(2):165–190, 1977.

19 D. Lazard. Resolution des systemes d'equations algebriques. *Theoret. Comput. Sci.*, 15(1):77–110, 1981.

20 Diane Maclagan and Bernd Sturmfels. *Introduction to Tropical Geometry*, volume 161 of *AMS Graduate Studies in Mathematics*. AMS, to appear, 2015.

21 Grigory Mikhalkin. Amoebas of algebraic varieties and tropical geometry. In Simon Donaldson, Yakov Eliashberg, and Mikhael Gromov, editors, *Different Faces of Geometry*, volume 3 of *International Mathematical Series*, pages 257–300. Springer US, 2004.

22 Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. First steps in tropical geometry. *Idempotent Mathematics and Mathematical Physics, Contemporary Mathematics*, 377:289–317, 2003.

23 Eugenii Shustin and Zur Izhakian. A tropical Nullstellensatz. *Proceedings of the American Mathematical Society*, 135(12):3815–3821, 2007.

24 Bernd Sturmfels. *Solving Systems of Polynomial Equations*, volume 97 of *CBMS Regional Conference in Math.* American Mathematical Society, 2002.

25 Luis Felipe Tabera. Tropical resultants for curves and stable intersection. *Revista Matemática Iberoamericana*, 24(3):941–961, 04 2008.

26 Thorsten Theobald. On the frontiers of polynomial computations in tropical geometry. *J. Symb. Comput.*, 41(12):1360–1375, 2006.