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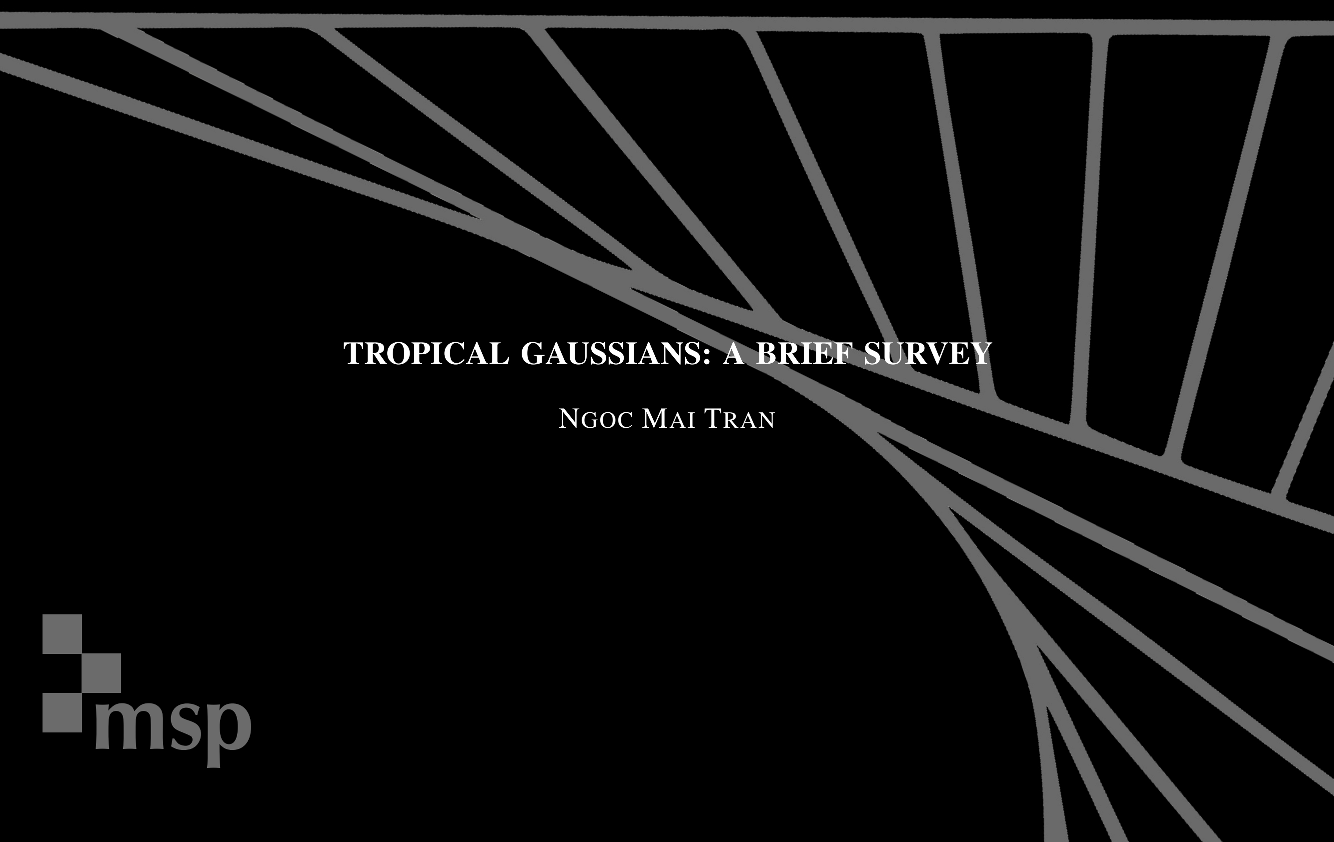

TROPICAL GAUSSIANS: A BRIEF SURVEY

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TROPICAL GAUSSIANS: A BRIEF SURVEY

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We review the existing analogues of the Gaussian measure in the tropical semiring and outline various research directions.

1. Introduction

Tropical mathematics has found many applications in both pure and applied areas, as documented by a growing number of monographs on its interactions with various other areas of mathematics: algebraic geometry [Baker and Payne 2016; Gross 2011; Huh 2018; Maclagan and Sturmfels 2015], discrete event systems [Baccelli et al. 1992; Butkovič 2010], large deviations and calculus of variations [Kolokoltsov and Maslov 1997; Puhalskii 2001], and combinatorial optimization [Joswig $\geq$ 2020]. At the same time, new applications are emerging in phylogenetics [Monod et al. 2018; Yoshida et al. 2019; Page et al. 2020], statistics [Hook 2017], economics [Baldwin and Klemperer 2019; Crowell and Tran 2016; Elsner and van den Driessche 2004; Gursoy et al. 2013; Joswig 2017; Shiozawa 2015; Tran 2013; Tran and Yu 2019], game theory, and complexity theory [Allamigeon et al. 2018; Akian et al. 2012]. There is a growing need for a systematic study of probability distributions in tropical settings. Over the classical algebra, the Gaussian measure is arguably the most important distribution to both theoretical probability and applied statistics. In this work, we review the existing analogues of the Gaussian measure in the tropical semiring. We focus on the three main characterizations of the classical Gaussians central to statistics: invariance under orthonormal transformations, independence and orthogonality, and stability. We show that some notions do not yield satisfactory generalizations, others yield the classical geometric or exponential distributions, while yet others yield completely different distributions. There is no single notion of a 'tropical Gaussian measure' that would satisfy multiple tropical analogues of the different characterizations of the classical Gaussians. This is somewhat expected, for the interaction between geometry and algebra over the tropical semiring is rather different from that over $\mathbb{R}$. Different branches of tropical mathematics lead to different notions of a tropical Gaussian, and it is a worthy goal to fully explore all the options. We conclude with various research directions.

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Keywords: Gaussian, normal distribution, tropical semiring, p-adic, idempotent probability.

2. Three characterizations of the classical Gaussian

The Gaussian measure $\mathcal{N}(\mu, \Sigma)$, also called the normal distribution with mean $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n\times n}$ is the probability distribution with density

$$f_{\Sigma,\mu}(x) \propto \exp(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)), \quad x \in \mathbb{R}^n.$$

Let $\mathbf{I}$ denote the identity matrix, and $\mathbf{0} \in \mathbb{R}^n$ the zero vector. Measures $\mathcal{N}(\mathbf{0}, \Sigma)$ are called centered Gaussians, while $\mathcal{N}(\mathbf{0}, \mathbf{I})$ is the standard Gaussian. Any Gaussian can be standardized by an affine linear transformation.

Lemma 2.1. *Let $\Sigma = U\Lambda U^\top$ be the eigendecomposition of Σ. Then $X \sim \mathcal{N}(\mu, \Sigma)$ if and only if $(U\Lambda^{1/2})^{-1}(X-\mu) \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.*

The standard Gaussian has two important properties. First, if X is a standard Gaussian in $\mathbb{R}^n$, then its coordinates $X_1, \dots, X_n$ are n independent and identically distributed (i.i.d) random variables. Second, for any orthonormal matrix A, $AX \overset{d}{=} X$. These two properties completely characterize the standard Gaussian [Kallenberg 2002, Proposition 11.2]. This result was first formalized in dimension three by Maxwell [1860] when he studied the distribution of gas particles, though the essence of his argument was made by Herschel [1850] ten years earlier, as pointed out in [Bryc 1995, p10].

Theorem 2.2 (Maxwell). *Let $X_1, \dots, X_n$ be i.i.d univariate random variables, where $n \geq 2$. Then the distribution of $X = (X_1, \dots, X_n)$ is spherically symmetric iff the X_i's are centered Gaussians on $\mathbb{R}$.*

From a statistical perspective, Lemma 2.1 and Theorem 2.2 reduces working with data from the Gaussian measure to doing linear algebra. In particular, if data points come from a Gaussian measure, then they are the affine linear transformation of data points from a standard Gaussian, whose coordinates are always independent regardless of the orthonormal basis that it is represented in. These properties are fundamental to principal component analysis, an important statistical technique whose tropical analogue is actively being studied [Yoshida et al. 2019].

There are numerous other characterizations of the Gaussian measure whose ingredients are only orthogonality and independence, see [Bogachev 1998, §1.9] and references therein. One famous example is Kac's theorem [1939]. It is a special case of the Darmois–Skitovich theorem [Darmois 1953; Skitovich 1953], which characterizes Gaussians (not necessarily centered) in terms of independence of linear combinations. A multivariate version of this theorem is also known; see [Kagan et al. 1972].

Theorem 2.3 (Darmois–Skitovich). *Let $X_1, \dots, X_n$ be independent univariate random variables. Then the X_i's are Gaussians if and only if there exist $\alpha, \beta \in \mathbb{R}^n$, $\alpha_i, \beta_i \neq 0$ for all $i = 1, \dots, n$, such that $\sum_i \alpha_i X_i$ and $\sum_i \beta_i X_i$ are independent.*

Another reason for the wide applicability of Gaussians in statistics is the Central Limit Theorem. An interesting historical account of its development can be found in [Kallenberg 2002, §4]. From the Central Limit Theorem, one can derive yet other characterizations of the Gaussian, such as the distribution which maximizes entropy subject to a fixed variance [Barron 1986]. The appearance of the Gaussian in the Central Limit Theorem is fundamentally linked to its characterization as the unique 2-stable distribution.

This is expressed in the following theorem by Pólya [1923]. There are a number of variants of this theorem; see [Bogachev 1998; Bryc 1995] and discussions therein.

Theorem 2.4 (Pólya). *Suppose $X, Y \in \mathbb{R}^n$ are independent random variables. Then X, Y and $(X+Y)/\sqrt{2}$ have the same distribution iff this distribution is the centered Gaussian.*

3. Tropical analogues of Gaussians

3.1. *Tropicalizations of p-adic Gaussians.* Evans [2001] used Kac's Theorem as the definition of Gaussians to extend them to local fields. Local fields are finite algebraic extensions of either the field of p-adic numbers or the field of formal Laurent series with coefficients drawn from the finite field with p elements [Evans 2001]. In particular, local fields come with a tropical valuation val, and thus one can define a tropical Gaussian to be the tropicalization of the Gaussian measure on a local field. A direct translation of [Evans 2001, Theorem 4.2] shows that the tropicalization of the one-dimensional p-adic Gaussian is the classical geometric distribution.

Proposition 3.1 (tropicalization of the p-adic Gaussian). *For a prime $p \in \mathbb{N}$, let X be a $\mathbb{Q}_p$-valued Gaussian with index $k \in \mathbb{Z}$. Then* $\mathrm{val}(X)$ *is a random variable supported on* $\{k, k+1, k+2, \dots\}$, *and it is distributed as* $k + \mathrm{geometric}(1-p^{-1})$. *That is,*

$$\mathbb{P}(\mathrm{val}(X) = k+s) = p^{-s}(1-p^{-1}) \textit{ for } s = 0, 1, 2, \dots.$$

Proof. Recall that a nonzero rational number $r \in \mathbb{Q}\backslash\{0\}$ can be uniquely written as $r = p^s(a/b)$ where a and b are not divisible by p, in which case the valuation of r is $|r| := p^{-s}$. The completion of $\mathbb{Q}$ under the metric $(x, y) \mapsto |x-y|$ is the field of p-adic numbers, denoted $\mathbb{Q}_p$. The tropical valuation of r is $\mathrm{val}(r) := s$. By [Evans 2001, Theorem 4.2], the family of $\mathbb{Q}_p$-valued Gaussians is indexed by $\mathbb{Z}$. For each $k \in \mathbb{Z}$, there is a unique $\mathbb{Q}_p$-valued Gaussian supported on the ball $p^k\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \le p^{-k}\}$. Furthermore, the Gaussian is the normalized Haar measure on this support. As $p^k\mathbb{Z}_p$ is made up of p translated copies of $p^{k+1}\mathbb{Z}_p$, which in turn is made up of p translated copies of $p^{k+2}\mathbb{Z}_p$, a direct computation yields the density of $\mathrm{val}(X)$. □

There is a large and growing literature surrounding probability on local fields, or more generally, analysis on ultrametric spaces. They have found diverse applications, from spin glasses, protein dynamics, and genetics, to cryptography and geology; see the recent comprehensive review [Dragovich et al. 2017] and references therein. The p-adic Gaussian was originally defined as a step towards building Brownian motions on $\mathbb{Q}_p$ [Evans 2001]. It would be interesting to use tools from tropical algebraic geometry to revisit and expand results involving random p-adic polynomials, such as the expected number of zeroes in a random p-adic polynomial system [Evans 2006], or properties of determinants of matrices with i.i.d p-adic Gaussians [Evans 2002]. Previous work on random p-adic polynomials from a tropical perspective tends to consider systems with uniform valuations [Avendaño and Ibrahim 2011]. Proposition 3.1 hints that to connect the two literatures, the geometric distribution may be more suitable.

3.2. ***Gaussians via tropical linear algebra.*** Consider arithmetic done in the tropical algebra $(\overline{\mathbb{R}}, \oplus, \odot)$, where $\overline{\mathbb{R}}$ is $\mathbb{R}$ together with the additive identity. In the max-plus algebra $(\overline{\mathbb{R}}, \overline{\oplus}, \odot)$ where $a \overline{\oplus} b = \max(a, b)$, for instance, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. In the min-plus algebra $(\overline{\mathbb{R}}, \underline{\oplus}, \odot)$ where $a \underline{\oplus} b = \min(a, b)$, we have $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. To avoid unnecessary technical details, in this section we focus on vectors taking values in $\mathbb{R}$ instead of $\overline{\mathbb{R}}$.

Tropical linear algebra was developed by several communities with different motivations. It evolved as a linearization tool for certain problems in discrete event systems, queueing theory and combinatorial optimization; see the monographs [Baccelli et al. 1992; Butkovič 2010], as well as the recent survey [Komenda et al. 2018] and references therein. A large body of work focuses on using the tropical setting to find analogous versions of classical results in linear algebra and convex geometry. Many fundamental concepts have rich tropical analogues, including the spectral theory of matrices [Akian et al. 2006; Baccelli et al. 1992; Butkovič 2010], linear independence and projectors [Allamigeon et al. 2011; Akian et al. 2011; Butkovič et al. 2007; Sergeev 2009], separation and duality theorems in convex analysis [Briec and Horvath 2008; Cohen et al. 2004; Gaubert and Katz 2011; Nitica and Singer 2007], matrix identities [Gaubert 1996; Hollings and Kambites 2012; Morrison and Tran 2016; Simon 1994], matrix rank [Chan et al. 2011; Develin et al. 2005; Izhakian and Rowen 2009; Shitov 2011], and tensors [Butkovic and Fiedler 2018; Tsukerman 2015]. Another research direction focuses on the combinatorics of objects arising in tropical convex geometry, such as polyhedra and hyperplane arrangements [Akian et al. 2012; Develin and Sturmfels 2004; Joswig and Loho 2016; Joswig et al. 2007; Sturmfels and Tran 2013; Tran 2017]. These works have close connections to matroid theory and are at the interface of tropical linear algebra and tropical algebraic geometry [Ardila and Develin 2009; Fink and Rincón 2015; Giansiracusa and Giansiracusa 2018; Hampe 2015; Loho and Smith 2020].

Despite the rich theory of tropical linear algebra, in this section we shall show that there is currently no satisfactory way to define the tropical Gaussian as a classical probability measure based on the characterizations of Gaussians via orthogonality and independence as in Section 2. This is somewhat surprising, for there are good analogues of norms and orthogonal decomposition in the tropical algebra. In hindsight, the main difficulty stems from the fact that such tropical analogues are compatible with tropical arithmetic, while classical measure theory was developed with the usual algebra. In Section 3.3 we consider the idempotent probability measure theory, where there is a well-defined Gaussian measure complete with a quadratic density function analogous to the classical case.

The natural definition for tropical linear combinations of $v_1, \ldots, v_m \in \mathbb{R}^n$ is the set of vectors of the form

$$[v_1, \ldots, v_m] := \{a_1 \odot v_1 \oplus \cdots \oplus a_m \odot v_m \text{ for } a_1, \ldots, a_m \in \mathbb{R}\}, \tag{1}$$

where scalar-vector multiplication is defined pointwise. That is, for $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$, $a \odot v \in \mathbb{R}^n$ is the vector with entries

$$(a \odot v)_i = a + v_i \text{ for } i = 1, \ldots, n.$$

We shall also write $a + v$ for $a \odot v$, with the convention scalar-vector addition is defined pointwise.

For finite m, $V := [v_1, \dots, v_m]$ is always a compact set in $\mathbb{TP}^{n-1} := \mathbb{R}^n/\mathbb{R}\mathbf{1}$ [Develin and Sturmfels 2004]. Unfortunately, this means one cannot hope to have finitely many vectors to 'tropically span' $\mathbb{R}^m$. Nonetheless, there is a well-defined analogue orthogonal projection in the tropical algebra. Associated to a tropical polytope $V := [v_1, \dots, v_m]$ defined by (1) is the canonical projector $P_V : \mathbb{R}^n \to V$ that plays the role of the orthogonal projection onto V [Cohen et al. 2004]. This projection is compatible with the projective Hilbert metric d_H [Cohen et al. 2001; 2004], in the sense that $P_V(x)$ is a best-approximation under the projective Hilbert metric of x by points in V [Cohen et al. 2004; Akian et al. 2011]. When V is a polytrope, that is, a tropical polytope that is also classically convex, then P_V can be written as a tropical matrix-vector multiplication. This is analogous to classical linear algebra, where best-approximations in the Euclidean distance can be written as a matrix-vector multiplication.

In the max-plus algebra, the projective Hilbert metric is defined by

$$d_H(x, y) = \max_{i,j\in[n]} (x_i - y_i + y_j - x_j).$$

It induces the Hilbert projective norm $\|\cdot\|_H : \mathbb{R}^m \to \mathbb{R}$ via $\|x\|_H = d_H(x, 0)$. Since $d_H(x, y) = \max_i(x_i - y_i) - \min_j(x_j - y_j)$, one finds that

$$\|x\|_H = \|x - \min_i x_i\|_\infty.$$

This formulation shows that the projective Hilbert norm plays the role of the ℓ_∞-norm on $\mathbb{TP}^{n-1}$. The appearance of ℓ_∞, instead of ℓ_2, agrees with the conventional 'wisdom' that generally in the tropical algebra, ℓ_2 is replaced by ℓ_∞ [Evans 2001].

To generalize Maxwell's characterization of the classical Gaussians, we need a concept of orthogonality. One could attempt to mimic orthogonality via the orthogonal decomposition theorem, as done in [Evans 2001] for the case of local fields discussed in Section 3.1. Namely, over a normed space $(\mathcal{Y}, \|\cdot\|)$ over some field K, say that $y_1, \dots, y_m \in \mathcal{Y}$ are orthogonal if and only if for all $\alpha_i \in K$, the norm of the vector $\sum_i \alpha_i y_i$ equals the norm of the vector $(|\alpha_1|\|y_1\|, \dots, |\alpha_m|\|y_m\|)$, that is,

$$\Big\|\sum_i \alpha_i y_i\Big\| = \big\|(|\alpha_1|\|y_1\|, \dots, |\alpha_m|\|y_m\|)\big\|. \tag{2}$$

In the Euclidean case, this is the Pythagorean identity

$$\Big\|\sum_i \alpha_i y_i\Big\|_2 = \Big(\sum_i |\alpha_i|^2\|y_i\|^2\Big)^{1/2},$$

for example. The ℓ_∞-norm, unfortunately, does not work well with the usual notion of independence in probability. In the Hilbert projective norm, (2) can be interpreted either as

$$\|\max_i(\alpha_i + y_i)\|_H = \max_i \|\alpha_i + y_i\|_H - \min_i \|\alpha_i + y_i\|_H = \max_i \|y_i\|_H - \min_i \|y_i\|_H \tag{3}$$

or

$$\|\max_i(\alpha_i + y_i)\|_H = \max_i(\alpha_i + \|y_i\|_H) - \min_i(\alpha_i + \|y_i\|_H). \tag{4}$$

Unfortunately, neither formulation give a satisfactory notion of orthogonality. In (3), as the norm is projective, the coefficients α_i have disappeared from the RHS. This does not support the notion that over an orthogonal set of vectors in the classical sense, computing the norm of linear combinations is the same as computing norm of the vector of coefficients. In (4), for sufficiently large α_1, the RHS increases without bound whereas the LHS is bounded, and thus equality cannot hold for all $\alpha_i \in \mathbb{R}$ over *any* generating set of y_i's.

The Darmois–Skitovich characterization for Gaussians also does not generalize well. Note that the additive identity in $(\overline{\mathbb{R}}, \oplus, \odot)$ is either $-\infty$ or $+\infty$, so the condition that $\alpha_i, \beta_i \neq 0$ becomes redundant. The following lemma states that the any compact distribution will satisfy the Darmois–Skitovich condition.

Lemma 3.2. *Let $X_1, \dots, X_n$ be independent random variables on $\mathbb{R}^n$. Then there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\bigoplus_{i=1}^n \alpha_i \odot X_i$ and $\bigoplus_{i=1}^n \beta_i \odot X_i$ are independent if and only if $X_1, \dots, X_n$ have compact support.*

Proof. Let us sketch the proof for $n = 2$ under the min-plus algebra. Let $X = (X_1, X_2) \in \mathbb{R}^2$ and $Y = (Y_1, Y_2) \in \mathbb{R}^2$ be two independent variables. Define $\bar{F}_X, \bar{F}_Y : \mathbb{R}^2 \to [0, 1]$ via $\bar{F}_X(t) = \mathbb{P}(X \geq t)$ and $\bar{F}_Y(t) = \mathbb{P}(Y \geq t)$. Fix $\alpha, \beta \in \mathbb{R}^2$. For $t \in \mathbb{R}^2$,

$$\begin{aligned}
\mathbb{P}(\alpha_1 \odot X \oplus \alpha_2 \odot Y \geq t) &= \mathbb{P}(\min(\alpha_1 + X, \alpha_2 + Y) \geq t) && \text{by definition} \\
&= \mathbb{P}(X \geq t - \alpha_1)\mathbb{P}(Y \geq t - \alpha_2) && \text{by independence} \\
&= \bar{F}_X(t - \alpha_1)\bar{F}_Y(t - \alpha_2).
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
&\mathbb{P}(\alpha_1 \odot X \oplus \alpha_2 \odot Y \geq t, \beta_1 \odot X \oplus \beta_2 \odot Y \geq t) \\
&\qquad = \mathbb{P}(\min(\alpha_1 + X, \alpha_2 + Y) \geq t, \min(\beta_1 + X, \beta_2 + Y) \geq t) && \text{by definition} \\
&\qquad = \mathbb{P}(X \geq t - \alpha_1, X \geq t - \beta_1)\mathbb{P}(Y \geq t - \alpha_2, Y \geq t - \beta_2) && \text{by independence} \\
&\qquad = \min(\bar{F}_X(t - \alpha_1), \bar{F}_X(t - \beta_1)) \min(\bar{F}_Y(t - \alpha_2), \bar{F}_Y(t - \beta_2)).
\end{aligned}$$

Therefore, for $\alpha_1 \odot X \oplus \alpha_2 \odot Y$ and $\beta_1 \odot X \oplus \beta_2 \odot Y$ to be independent, for all $t \in \mathbb{R}^2$, we need

$$\begin{aligned}
\bar{F}_X(t-\alpha_1)\bar{F}_X(t-\beta_1)\bar{F}_Y(t-\alpha_2)\bar{F}_Y(t-\beta_2) &= \min(\bar{F}_X(t-\alpha_1), \bar{F}_X(t-\beta_1)) \min(\bar{F}_Y(t-\alpha_2), \bar{F}_Y(t-\beta_2)) \cdot \\
&\qquad \max(\bar{F}_X(t-\alpha_1), \bar{F}_X(t-\beta_1)) \max(\bar{F}_Y(t-\alpha_2), \bar{F}_Y(t-\beta_2)) \\
&= \min(\bar{F}_X(t-\alpha_1), \bar{F}_X(t-\beta_1)) \min(\bar{F}_Y(t-\alpha_2), \bar{F}_Y(t-\beta_2)).
\end{aligned}$$

But $\bar{F}_X$ and $\bar{F}_Y$ are nonincreasing functions taking values between 0 and 1. So

$$\bar{F}_X(t-\alpha_1)\bar{F}_X(t-\beta_1)\bar{F}_Y(t-\alpha_2)\bar{F}_Y(t-\beta_2) \leq \min(\bar{F}_X(t-\alpha_1), \bar{F}_X(t-\beta_1)) \min(\bar{F}_Y(t-\alpha_2), \bar{F}_Y(t-\beta_2)),$$

and equality holds if and only if

$$\min(\bar{F}_X(t - \alpha_1), \bar{F}_X(t - \beta_1)) = 0, \text{ or } \min(\bar{F}_Y(t - \alpha_2), \bar{F}_Y(t - \beta_2)) = 0.$$

As either of these scenarios must hold for each $t \in \mathbb{R}^2$, we conclude that X and Y must have compact supports. Conversely, suppose that X and Y have compact supports. Then one can choose $\alpha_1 = \beta_2 = 0$

and $\alpha_2 = \beta_1$ be a sufficiently large number, so that

$$\alpha_1 \odot X \oplus \alpha_2 \odot Y = X, \text{ and } \beta_1 \odot X \oplus \beta_2 \odot Y = Y.$$

In this case, the Darmois–Skitovich condition holds trivially, as desired. □

Now consider Pólya's condition. Here the Gaussian is characterized via stability under addition. When addition is replaced by minimum, it is well-known that this leads to the classical exponential distribution. One such characterization, which generalizes to distributions on arbitrary lattices, is the following [Bryc 1995, Theorem 3.4.1].

Theorem 3.3. *Suppose X, Y are independent and identically distributed nonnegative random variables. Then this distribution is the exponential if and only if for all $a, b > 0$ such that $a + b = 1$, $\min(X/a, Y/b)$ has the same distribution as X.*

By considering $\log(X)$ and $\log(Y)$, one could restate this theorem in terms of the min-plus algebra, though the condition $a + b = 1$ does not have an obvious tropical interpretation. This shows that the tropical analogue of Gaussian is the classical exponential distribution.

3.3. *Gaussians in idempotent probability.* Idempotent probability is a branch of idempotent analysis, which is functional analysis over idempotent semirings [Kolokoltsov and Maslov 1997]. Idempotent semirings are characterized by the additive operation being idempotent, that is, $a \oplus a = a$. The tropical semirings used in the previous sections are idempotent, but there are others, such as the Boolean semiring in semigroup theory. Idempotent analysis was developed by Litvinov, Maslov and Shipz [Litvinov et al. 1998] in relation to problems of calculus of variations. Closely related are the work on large deviations [Puhalskii 2001], which has found applications in queueing theory, as well as fuzzy measure theory and logic [Dubois and Prade 2000; Wang and Klir 1992]. The work we discussed in this section is based on that of Akian, Quadrat and Viot and coauthors [Akian et al. 2011; 1994], whose goal was to develop idempotent probability as a theory completely in parallel to classical probability. Following their convention, we work over the min-plus algebra.

All fundamental concepts of probability have an idempotent analogue, see [Akian et al. 1994] and references therein. For a flavor of this theory, we compare the concept of a measure. In classical settings, a probability measure μ is a map from the σ-algebra on a space Ω to $\mathbb{R}_{\geq 0}$ that satisfies three properties: (i) $\mu(\varnothing) = 0$, (ii) $\mu(\Omega) = 1$, and (iii) for a countable sequence (E_i) of pairwise disjoint sets,

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i).$$

The analogous object in the min-plus probability is the cost measure $\mathbb{K}$ defined by three axioms: (i) $\mathbb{K}(\varnothing) = +\infty$, (ii) $\mathbb{K}(\Omega) = 0$, and

$$\mathbb{K}\Big(\bigcup_i E_i\Big) = \bigoplus_i \mathbb{K}(E_i) = \inf_i \mathbb{K}(E_i).$$

Idempotent probability is rich and has interesting connections with dynamic programming and optimization. For instance, tropical matrix-vector multiplication can be interpreted as an update step in a Markov chain, so the Bellman equation plays the analogue of the Kolmogorov–Chapman equation. Most notably, the classical quadratic form $(x-y)^2/2\sigma^2$ defines a stable distribution [Akian et al. 1994]. Furthermore, it is the unique density that is invariant under the Legendre–Fenchel transform [Akian et al. 1994], which is the tropical analogue of the Fourier transform [Kolokoltsov and Maslov 1997]. This is in parallel to the characterization of the scaled version of the Gaussian density $x \mapsto \exp(-\pi x^2)$ being invariant under the Fourier transform. While $x \mapsto \exp(-\pi x^2)$ is not the unique function to possess this property [Duffin 1948], the fact that the Fourier transform of a $\mathcal{N}(\mu, \sigma^2)$ univariate Gaussian has the form $x \mapsto \exp(i\mu x - \frac{x^2}{2})$ is frequently employed to prove independence of linear combinations of Gaussians. Under this light, one can regard the idempotent measure correspond to the density $(x-y)^2/2\sigma^2$ to be the idempotent analogue of the classical Gaussian.

4. Open directions

4.1. ***Tropical curves, metric graphs and Gaussians via the Laplacian operator.*** From the perspective of stochastic analysis, the Gaussian measure can be characterized as the unique invariant measure for the Ornstein–Uhlenbeck semigroup [Bogachev 1998, §1]. This semigroup is a powerful tool in proving hypercontractivity and log-Sobolev inequalities. In particular, the Gaussian density can be characterized as the function that satisfies such inequalities with the best constants [Bogachev 1998]. One useful characterization of the Ornstein–Uhlenbeck semigroup is by its generator, whose definition has two ingredients: a Laplacian operator and a gradient operator ∇. Let us elaborate. Let γ be a centered Gaussian measure on $\mathbb{R}^n$. The Ornstein–Uhlenbeck semigroup $(T_t, t \geq 0)$ is defined on $L^2(\gamma)$ by the Mehler formula

$$T_t h(x) = \int_{\mathbb{R}^n} h(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\,\gamma(dy), \quad t > 0$$

and T_0 is the identity operator. It characterizes the Gaussian measure in the following sense [Bogachev 1998, §1].

Lemma 4.1. *γ is the unique invariant probability measure for $(T_t, t \geq 0)$.*

One can arrive at this semigroup without the Mehler's formula as follows. Let $\mathcal{D} = \{h \in L^2(\gamma) : \lim_{t\to 0} \frac{T_t h - h}{t}$ exists in the norm of $L^2(\gamma)\}$. (Recall that $L^2(\gamma)$ is the space of square integrable functions with respect to the measure γ). The linear operator L defined on $\mathcal{D}$ by

$$Lh = \lim_{t\to 0} \frac{T_t h - h}{t}$$

is called the generator of the semigroup $(T_t, t \geq 0)$. The generator of the Ornstein–Uhlenbeck semigroup is given by

$$Lh(x) = \Delta h(x) - \langle x, \nabla h\rangle = \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(x) - \sum_{i=1}^n x_i \frac{\partial h}{\partial x_i}(x).$$

This generator uniquely specifies the semigroup. Importantly, the two ingredients needed to define L are the Laplacian operator Δ, and the gradient operator ∇. Thus the semigroup can be defined on Riemannian manifolds, for instance. This opens up ways to define Gaussians on tropical curves.

In tropical algebraic geometry, an abstract tropical curve is a metric graph [Mikhalkin and Zharkov 2008]. There are some minor variants: with vertex weights [Brannetti et al. 2011; Chan 2012], or just the compact part [Baker and Faber 2006]. An embedded tropical curve is a balanced weighted one-dimensional complex in $\mathbb{R}^n$. There are several constructions of tropical curves. In particular, they arise as limits of amoebas through a process called Maslov dequantization in idempotent analysis [Litvinov et al. 1998]. Tropical algebraic geometry took off with the landmark paper of Mikhalkin [2005], who used tropical curves to compute Gromov–Witten invariants of the plane $\mathbb{P}^2$ [Maclagan and Sturmfels 2015]. Since then, tropical curves, and more generally, tropical varieties, have been studied in connection to mirror and symplectic geometry [Gross 2011]. Another heavily explored aspect of tropical curves is their divisors and Riemann–Roch theory [Baker and Norine 2007; Baker and Payne 2016; Gathmann and Kerber 2008; Mikhalkin and Zharkov 2008]. This theory is connected to chip-firing and sandpiles, which were initially conceived as deterministic models of random walks on graphs [Cooper and Spencer 2006].

Metric graphs are Riemannian manifolds with singularities [Baker and Faber 2006]. Brownian motions defined on metric graphs, heat semigroups on graphs, and graph Laplacians are an active research area [Kostrykin et al. 2012; Post 2009]. As of now, however, the author is unaware of an analogue of the Ornstein–Uhlenbeck semigroup and its invariant measure on graphs. It would also be interesting to study what Brownian motion on graphs reveals about tropical curves and their Jacobians.

4.2. ***Further open directions.*** The natural ambient space for doing tropical convex geometry is not $\mathbb{R}^m$, but $\mathbb{TP}^{n-1}$, where a vector $x \in \mathbb{R}^m$ is identified with all of its scalar multiples $a \odot x$. Probability theory on classical projective spaces relies on group representation [Benoist and Quint 2014]. Unfortunately, there is no satisfactory tropical analogue of the general linear group. Every invertible $n \times n$ matrix with entries in $\overline{\mathbb{R}}$ is the composition of a diagonal matrix and a permutation of the standard basis of $\overline{\mathbb{R}}^n$ [Kolokoltsov and Maslov 1997]. We note that several authors have studied tropicalization of special linear group over a field with valuation [Joswig et al. 2007; Werner 2011]. It would be interesting to see whether this can be utilized to define probability measures on $\mathbb{TP}^{n-1}$.

Another approach is to 'fix' the main difficulty with the idempotent algebra, namely, the lack of the additive inverse. Some authors have put back the additive inverse and developed a theory of linear algebra in this new algebra, called the supertropical algebra [Izhakian and Rowen 2010]. It would be interesting to study matrix groups and their actions under this algebra, and in particular, pursue the definition of Gaussians as invariant measures under actions of the orthogonal group.

4.3. ***Beyond Gaussians.*** In a more applied direction, $\mathbb{TP}^{n-1}$ is a natural ambient space to study problems in economics, network flow and phylogenetics. Thus one may want an axiomatic approach to finding distributions on $\mathbb{TP}^{n-1}$ tailored for specific applications. For instance, in shape-constrained density estimation, log-concave multivariate totally positive of ordered two (MTP2) distributions are those whose density $f : \mathbb{R}^d \to \mathbb{R}$ is log-concave and satisfies the inequality

$$f(x)f(y) \leq f(x \vee y) f(x \wedge y) \quad \text{for all } x, y \in \mathbb{R}^d.$$

A variety of distributions belong to this family. Requiring that such inequalities hold for all $x, y \in \mathbb{TP}^{n-1}$ leads to the stronger condition of $L^\natural$-concavity

$$f(x)f(y) \leq f((x+\alpha\mathbf{1}) \vee y) f(x \wedge (y-\alpha\mathbf{1})) \qquad \text{for all } x, y \in \mathbb{R}^d, \alpha \geq 0.$$

A Gaussian distribution is log-concave MTP2 if and only if the inverse of its covariance matrix is an M-matrix [Lauritzen et al. 2019]. Only diagonally dominant Gaussians are $L^\natural$-concave [Murota 2003, §2]. This subclass of densities has nice properties that make them algorithmically tractable in Gaussian graphical models [Malioutov et al. 2006; Weiss and Freeman 2001]. In particular, density estimation for $L^\natural$-concave distributions is significantly easier than for log-concave MTP2 [Robeva et al. 2018]. It would be interesting to pursue this direction to define distributions on the space of phylogenetic trees.

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