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Tropical refined curve counting from higher genera and lambda classes

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Abstract Block and Göttsche have defined a q-number refinement of counts of tropical curves in $\mathbb{R}^2$. Under the change of variables $q = e^{iu}$, we show that the result is a generating series of higher genus log Gromov–Witten invariants with insertion of a lambda class. This gives a geometric interpretation of the Block-Göttsche invariants and makes their deformation invariance manifest.

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Contents

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List of symbols

i	The standard square root of -1 in $\mathbb{C}$
q	The formal refined variable in Block-Göttsche invariants
u	A formal variable keeping track of the genus in generating series of Gromov–Witten invariants, related to q by $q = e^{iu}$
A_*	A Chow group
A^*	An operatorial cohomology Chow group, see [16]
Δ	A balanced collection of vectors in $\mathbb{Z}^2$, of cardinality $\|\Delta\|$
X_Δ	The toric surface defined by Δ
β_Δ	The curve class defined by Δ
n	A number of points in $(\mathbb{C}^*)^2$ or $\mathbb{R}^2$
$g_{\Delta,n}$	The integer $n+1-\|\Delta\|$
P	A set of n points P_j in $(\mathbb{C}^*)^2$
p	A set of n points p_j in $\mathbb{R}^2$
Γ	A graph, often source of a parametrized tropical curve
$V(\Gamma)$	The set of vertices of Γ, of cardinality $\|V(\Gamma)\|$
$E(\Gamma)$	The set of edges of Γ, of cardinality $\|E(\Gamma)\|$
$E_f(\Gamma)$	The set of bounded edges of Γ, of cardinality $\|E_f(\Gamma)\|$
$E_\infty(\Gamma)$	The set of bounded edges of Γ, of cardinality $\|E_\infty(\Gamma)\|$
h	A parametrized tropical curve
$m(V)$	The multiplicity of a vertex V
$w(E)$	The weight of an edge E
$\overline{M}_{g,n,\Delta}$	A moduli space of genus g stable log maps
$N_g^{\Delta,n}$	A genus g log Gromov–Witten invariant
$N_{\mathrm{trop}}^{\Delta,n}(q)$	A refined tropical curve count
$T_{\Delta,p}$	A finite set of genus $g_{\Delta,n}$ parametrized tropical curves
$T_{\Delta,p}^g$	A finite set of genus g parametrized tropical curves
$\overline{\mathcal{M}}$	A monoid
$\mathrm{pt}_{\overline{\mathcal{M}}}$	The log point of ghost monoid $\overline{\mathcal{M}}$
Σ	The tropicalization functor
X_0	The central fiber of a toric degeneration of X_Δ
P^0	A set of n points P_j^0 in X_0, degeneration of P
X_{Δ_V}	An irreducible component of X_0
$N_{g,h}^{\Delta,n}$	A genus g log Gromov–Witten invariant marked by h
$N_{g,V}^{1,2}$	A genus g log Gromov–Witten invariant attached to a vertex V with a preferred choice of edges
$N_{g,V}$	A genus g log Gromov–Witten invariant attached to a vertex V
$F_V(u)$	A generating series of log Gromov–Witten invariants attached to a vertex V

$F_m(u)$	A generating series of log Gromov–Witten invariants attached to a vertex of multiplicity m

1 Introduction

Tropical geometry gives a combinatorial way to approach problems in complex and real algebraic geometry. An early success of this approach is Mikhalkin's correspondence theorem [34], proved differently and generalized by Nishinou and Siebert [38], between counts of complex algebraic curves in complex toric surfaces and counts with multiplicity of tropical curves in $\mathbb{R}^2$. Our main result, Theorem 1, is an extension to a correspondence between some generating series of higher genus log Gromov–Witten invariants of toric surfaces and counts with q-multiplicity of tropical curves in $\mathbb{R}^2$.

Counts of tropical curves in $\mathbb{R}^2$ with q-multiplicity were introduced by Block and Göttsche [8]. The usual multiplicity of a tropical curve is defined as a product of integer multiplicities attached to the vertices. Block and Göttsche remarked that one can obtain a refinement by replacing the multiplicity m of a vertex by its q-analogue

$$[m]_q := \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{-\frac{m-1}{2}} \left(1 + q + \cdots + q^{m-1}\right).$$

The q-multiplicity of a tropical curve is then the product of the q-multiplicities of the vertices. The count with q-multiplicity of tropical curves specializes for $q = 1$ to the ordinary count with multiplicity. This definition is done at the tropical level so is combinatorial in nature and its geometric meaning is *a priori* unclear.

Let Δ be a balanced collection of vectors in $\mathbb{Z}^2$ and let n be a non-negative integer.[1] This determines a complex toric surface X_Δ and a counting problem of virtual dimension zero for complex algebraic curves in X_Δ of some genus $g_{\Delta,n}$, of some class β_Δ, satisfying some tangency conditions with respect to the toric boundary divisor, and passing through n points of X_Δ in general position. Let $N^{\Delta,n} \in \mathbb{N}$ be the solution to this counting problem. According to Mikhalkin's correspondence theorem, $N^{\Delta,n}$ is a count with multiplicity of tropical curves in $\mathbb{R}^2$, and so it has a Block-Göttsche refinement $N^{\Delta,n}(q) \in \mathbb{N}[q^{\pm\frac{1}{2}}]$.

For every $g \geqslant g_{\Delta,n}$, we consider the same counting problem as before—same curve class, same tangency conditions—but for curves of genus g. The virtual dimension is now $g - g_{\Delta,n}$. To obtain a number, we integrate a class of degree $g - g_{\Delta,n}$, the lambda class $\lambda_{g-g_{\Delta,n}}$, over the virtual fundamental class

[1] Precise definitions are given in Sect. 2.

of a corresponding moduli space of stable log maps. For every $g \geqslant g_{\Delta,n}$, we get a log Gromov–Witten invariant $N_g^{\Delta,n} \in \mathbb{Q}$.

Theorem 1 *For every Δ balanced collection of vectors in $\mathbb{Z}^2$, and for every non-negative integer n such that $g_{\Delta,n} \geqslant 0$, we have the equality*

$$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = N^{\Delta,n}(q) \left((-i)\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\right)^{2g_{\Delta,n}-2+|\Delta|}$$

of power series in u with rational coefficients, where

$$q = e^{iu} = \sum_{n \geqslant 0} \frac{(iu)^n}{n!},$$

and $|\Delta|$ is the cardinality of Δ.

Remarks

- According to Theorem 1, the knowledge of the Block-Göttsche invariant $N^{\Delta,n}(q)$ is equivalent to the knowledge of the log Gromov–Witten invariants $N_g^{\Delta,n}$ for all $g \geqslant g_{\Delta,n}$. This provides a geometric meaning to Block-Göttsche invariants, independent of any choice of tropical limit, making their deformation invariance manifest.
- Given a family $\pi : \mathcal{C} \to B$ of nodal curves, the Hodge bundle $\mathbb{E}$ is the rank g vector bundle over B whose fiber over $b \in B$ is the space $H^0(C_b, \omega_{C_b})$ of sections of the dualizing sheaf ω_{C_b} of the curve $C_b = \pi^{-1}(b)$. The lambda classes are classically [36] the Chern classes of the Hodge bundle:

  $$\lambda_j := c_j(\mathbb{E}).$$

  The log Gromov–Witten invariants $N_g^{\Delta,n}$ are defined by an insertion of $(-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}}$ to cut down the virtual dimension from $g - g_{\Delta,n}$ to zero.
- One can interpret Theorem 1 as establishing integrality and positivity properties for higher genus log Gromov–Witten invariants of X_Δ with one lambda class inserted.
- The change of variables $q = e^{iu}$ makes the correspondence of Theorem 1 quite non-trivial. In particular, it cannot be reduced to an easy enumerative correspondence. It is essential to have a virtual/non-enumerative count on the Gromov–Witten side: for g large enough, most of the contributions to $N_g^{\Delta,n}$ come from maps with contracted components.
- In Theorem 6, we present a generalization of Theorem 1 where some intersection points with the toric boundary divisor can be fixed.

– One could ask for a generalization of Theorem 1 including descendant log Gromov–Witten invariants, i.e. with insertion of psi classes. In the simplest case of a trivalent vertex with insertion of one psi class, it is possible to reproduce the numerator $q^{\frac{m}{2}} + q^{-\frac{m}{2}}$ of the multiplicity introduced by Göttsche and Schroeter [18] in the context of refined broccoli invariants, in a way similar to how we reproduce the numerator $q^{\frac{m}{2}} - q^{-\frac{m}{2}}$ of the Block-Göttsche multiplicity in Theorem 1. This will be described in some further work.

1.1 Relation with previous and future works

1.1.1 q-Analogues

It is a general principle in mathematics, going back at least to Heine's introduction of q-hypergeometric series in 1846, that many "classical" notions have a q-analogue, recovering the classical one in the limit $q \to 1$. The Block-Göttsche refinement of the tropical curve counts in $\mathbb{R}^2$ is clearly an example of this principle. In many other examples, it is well known that it is a good idea to write $q = e^{\hbar}$, the limit $q \to 1$ becoming the limit $\hbar \to 0$. From this point of view, the change of variable $q = e^{iu}$ in Theorem 1 is maybe not so surprising.

1.1.2 Göttsche–Shende refinement by Hirzebruch genus

Whereas the specialization of Block-Göttsche invariants at $q = 1$ recovers a count of complex algebraic curves, the specialization $q = -1$ recovers in some cases a count of real algebraic curves in the sense of Welschinger [43]. This strongly suggests a motivic interpretation of the Block-Göttsche invariants and indeed one of the original motivations of Block and Göttsche was the fact that, under some ampleness assumptions, the refined tropical curve counts should coincide with the refined curve counts on toric surfaces defined by Göttsche and Shende [19] in terms of Hirzebruch genera of Hilbert schemes. Using motivic integration, Nicaise, Payne and Schroeter [37] have reduced this conjecture to a conjecture about the motivic measure of a semialgebraic piece of the Hilbert scheme attached to a given tropical curve.

Our approach to the Block-Göttsche refined tropical curve counting is clearly different from the Göttsche–Shende approach: we interpret the refined variable q as coming from the resummation of a genus expansion whereas they interpret it as a formal parameter keeping track of the refinement from some Euler characteristic to some Hirzebruch genus.

The Göttsche–Shende refinement makes sense for surfaces more general than toric ones, as do the higher genus log Gromov–Witten invariants with one lambda class inserted. So one might ask if Theorem 1 can be extended to more general surfaces, as a relation between Göttsche–Shende refined invariants and generating series of higher genus log Gromov–Witten invariants. Combining known results about Göttsche–Shende refined invariants [19] and higher genus Gromov–Witten invariants, [11,33], one can show that it is indeed the case for K3 and abelian surfaces. In particular, Theorem 1 is not an isolated fact but part of a family of similar results. The case of a log Calabi-Yau surface obtained as complement of a smooth anticanonical divisor in a del Pezzo surface, and its relation with, in physics terminology, a worldsheet definition of the refined topological string of local del Pezzo threefolds, will be discussed in a future work.

1.1.3 Tropical vertex

Filippini and Stoppa [15] have related refined tropical curve counting to the q-version of the tropical vertex of [23], i.e. of the 2-dimensional Kontsevich-Soibelman scattering diagram. Combined with the main result of the present paper, we get an enumerative interpretation of the q-version of the tropical vertex. Details will be given in a separate publication [9]. With this enumerative interpretation, it is possible to give an higher genus generalization of the Gross-Hacking-Keel [22] mirror symmetry construction for log Calabi-Yau surfaces [10].

Using the connection with the q-version of the tropical vertex, Filippini and Stoppa [15] have related refined tropical curve counting to refined Donaldson–Thomas theory of quivers. This story was the initial motivation for the work eventually leading to the present paper. Applications of the present paper in this context will be discussed elsewhere.

1.1.4 MNOP

The change of variables $q = e^{iu}$ is reminiscent of the MNOP Gromov–Witten/Donaldson–Thomas (DT) correspondence on threefold [31,32]. The log Gromov–Witten invariants $N_g^{\Delta,n}$ can be rewritten as $\mathbb{C}^*$-equivariant log Gromov–Witten invariants of the threefold $X_\Delta \times \mathbb{C}$, where $\mathbb{C}^*$ acts by scaling on $\mathbb{C}$, see Lemma 7 of [33]. If a log DT theory and a log MNOP correspondence were developed, this would predict that the generating series of $N_g^{\Delta,n}$ is a rational function in $q = e^{iu}$, which is indeed true by Theorem 1. But it would not be enough to imply Theorem 1 because the relation between log DT invariants and Block-Göttsche invariants is *a priori* unclear. In fact, the Göttsche–Shende conjecture and the result of Filippini and Stoppa suggest that Block-Göttsche invariants are refined DT invariants whereas the MNOP correspondence involves unrefined DT invariants. This topic will be discussed in more details elsewhere.

1.1.5 BPS integrality

When the log Gromov–Witten invariants of $X_\Delta \times \mathbb{C}$ coincide with ordinary Gromov–Witten invariants of $X_\Delta \times \mathbb{C}$, which is probably the case if $|v| = 1$ for every $v \in \Delta$ and if the toric boundary divisor of X_Δ is positive enough, then the integrality implied by Theorem 1 coincides with the BPS integrality predicted by Pandharipande in [41], and proved via symplectic methods by Zinger in [44], for generating series of Gromov–Witten invariants of a threefold and of curve class intersecting positively the anticanonical divisor.

1.1.6 Mikhalkin refined real count

Mikhalkin has given in [35] an interpretation of some particular Block-Göttsche invariants in terms of counts of real curves. We do not understand the relation with our approach in terms of higher genus log Gromov–Witten invariants. We merely remark that both for us and for Mikhalkin, it is the numerator of the Block-Göttsche multiplicities which appears naturally.

1.1.7 Parker theory of exploded manifolds

This paper owes a great intellectual debt towards the paper [42] of Brett Parker, where a correspondence theorem between tropical curves in $\mathbb{R}^3$ and some generating series of curve counts in exploded versions of toric threefold is proved. Indeed, a conjectural version of Theorem 1 was known to the author around April 2016[2] but it was only after the appearance of [42] in August 2016 that it became clear that this result should be provable with existing technology. In particular, the idea to reduce the amount of explicit computations by exploiting the consistency of some gluing formula (see Sect. 8) follows [42]. In the present paper, we use the theory of log Gromov–Witten invariants because of the algebraic bias of the author, but it should be possible to write a version in the language of exploded manifolds.

1.2 Plan of the paper

In Sect. 2, we fix our notations and we describe precisely the objects involved in the formulation of Theorem 1. In Sect. 3, we review some gluing and vanishing properties of the lambda classes.

The next five Sections form the proof of Theorem 1.

[2] And was for example presented at the Workshop: Curves on surfaces and threefold, EPFL, Lausanne, 21 June 2016.

The first step of the proof, described in Sect. 4, is an application of the decomposition formula of Abramovich, Chen, Gross, Siebert [3] to the toric degeneration of Nishinou, Siebert [38]. This gives a way to write our log Gromov–Witten invariants as a sum of contributions indexed by tropical curves.

In the second step of the proof, described in Sects. 6 and 7, we prove a gluing formula which gives a way to write the contribution of a tropical curve as a product of contributions of its vertices. Here, gluing and vanishing properties of the lambda classes reviewed in Sect. 3, combined with a structure result for non-torically transverse stable log maps proved in Sect. 5, play an essential role. In particular, we only have to glue torically transverse stable log maps and we don't need to worry about the technical issues making the general gluing formula in log Gromov–Witten theory difficult (see Abramovich, Chen, Gross, Siebert [4]).

After the decomposition and gluing steps, what remains to do is to compute the contribution to the log Gromov–Witten invariants of a tropical curve with a single trivalent vertex. The third and final step of the proof of Theorem 1, carried out in Sect. 8, is the explicit evaluation of this vertex contribution. Consistency of the gluing formula leads to non-trivial relations between these vertex contributions, which enable us to reduce the problem to particularly simple vertices. The contribution of these simple vertices is computed explicitly by reduction to Hodge integrals previously computed by Bryan and Pandharipande [12] and this ends the proof of Theorem 1.

In Appendix A, we present for the sake of concreteness an explicit example.

2 Precise statement of the main result

2.1 Toric geometry

Let Δ be a balanced collection of vectors in $\mathbb{Z}^2$, i.e. a finite collection of vectors in $\mathbb{Z}^2 - \{0\}$ summing to zero.[3] Let $|\Delta|$ be the cardinality of Δ. For $v \in \mathbb{Z}^2 - \{0\}$, let $|v|$ the divisibility of v in $\mathbb{Z}^2$, i.e. the largest positive integer k such that we can write $v = kv'$ with $v' \in \mathbb{Z}^2$. Then the balanced collection Δ defines the following data by standard toric geometry.

- A projective[4] toric surface X_Δ over $\mathbb{C}$, whose fan has rays $\mathbb{R}_{\geqslant 0} v$ generated by the vectors $v \in \mathbb{Z}^2 - \{0\}$ contained in Δ. We denote ∂X_Δ the toric boundary divisor of X_Δ.

[3] A given element of $\mathbb{Z}^2 - \{0\}$ can appear several times in Δ. Here we follow the notation used by Itenberg and Mikhalkin in [25].

[4] This is true only if the elements in Δ are not all collinear. If they are, we replace X_Δ by a toric compactification whose choice will be irrelevant for our purposes.

– A curve class β_Δ on X_Δ, whose polytope is dual to Δ. If ρ is a ray in the fan of X_Δ, we write D_ρ for the prime toric divsisor of X_Δ dual to ρ and Δ_ρ the set of elements $v \in \Delta$ such that $\mathbb{R}_{\geqslant 0} v = \rho$. Then we have

$$\beta_\Delta . D_\rho = \sum_{v \in \Delta_\rho} |v|,$$

and these intersection numbers uniquely determine β_Δ. The total intersection number of β_Δ with the toric boundary divisor ∂X_Δ is given by

$$\beta_\Delta . (-K_{X_\Delta}) = \sum_{v \in \Delta} |v|.$$

– Tangency conditions for curves of class β_Δ with respect to the toric boundary divisor of X_Δ. We say that a curve C is of type Δ if it is of class β_Δ and if for every ray ρ in the fan of X_Δ, the curve C intersects D_ρ in $|\Delta_\rho|$ points with multiplicities $|v|$, $v \in \Delta_\rho$. Similarly, we have a notion of stable log map of type Δ.
– An asymptotic form for a parametrized tropical curve $h \colon \Gamma \to \mathbb{R}^2$ in $\mathbb{R}^2$. We say that a parametrized tropical curve in $\mathbb{R}^2$ is of type Δ if it has $|\Delta|$ unbounded edges, with directions v and with weights $|v|$, $v \in \Delta$.

2.2 Log Gromov–Witten invariants

The moduli space of n-pointed genus g stable maps to X_Δ of class β_Δ intersecting properly the toric boundary divisor ∂X_Δ with tangency conditions prescribed by Δ is not proper: a limit of curves intersecting ∂X_Δ properly does not necessarily intersect ∂X_Δ properly. A nice compactification of this space is obtained by considering stable log maps. The idea is to allow maps intersecting ∂X_Δ non-properly, but to remember some additional information under the form of log structures, which give a way to make sense of tangency conditions even for non-proper intersections. The theory of stable log maps has been developed by Gross and Siebert [24], and Abramovich and Chen [2,14]. By stable log maps, we always mean basic stable log maps in the sense of [24]. We refer to Kato [26] for elementary notions of log geometry.

We consider the toric divisorial log structure on X_Δ and use it to view X_Δ as a log scheme. Let $\overline{M}_{g,n,\Delta}$ be the moduli space of n-pointed genus g stable log maps to X_Δ of type Δ. By n-pointed, we mean that the source curves are equipped with n marked points *in addition* to the marked points keeping track of the tangency conditions with respect to the toric boundary divisor. We consider that the latter are notationally already included in Δ.

By the work of Gross, Siebert [24] and Abramovich, Chen [2,14], $\overline{M}_{g,n,\Delta}$ is a proper Deligne-Mumford stack[5] of virtual dimension

$$\mathrm{vdim}\ \overline{M}_{g,n,\Delta}=g-1+n+\beta_\Delta.(-K_{X_\Delta})-\sum_{v\in\Delta}(|v|-1)=g-1+n+|\Delta|,$$

and it admits a virtual fundamental class

$$[\overline{M}_{g,n,\Delta}]^{\mathrm{virt}}\in A_{\mathrm{vdim}\ \overline{M}_{g,n,\Delta}}(\overline{M}_{g,n,\Delta},\mathbb{Q}).$$

The problem of counting n-pointed genus g curves passing though n fixed points has virtual dimension zero if

$$\mathrm{vdim}\ \overline{M}_{g,n,\Delta}=2n,$$

i.e. if the genus g is equal to

$$g_{\Delta,n}:=n+1-|\Delta|.$$

In this case, the corresponding count of curves is given by

$$N^{\Delta,n}:=\langle\tau_0(\mathrm{pt})^n\rangle_{g_{\Delta,n},n,\Delta}:=\int_{[\overline{M}_{g_{\Delta,n},n,\Delta}]^{\mathrm{virt}}}\prod_{j=1}^n \mathrm{ev}_j^*(\mathrm{pt}),$$

where $\mathrm{pt}\in A^2(X_\Delta)$ is the class of a point and ev_j is the evaluation map at the j-th marked point.

According to Mandel and Ruddat [30], Mikhalkin's correspondence theorem can be reformulated in terms of these log Gromov–Witten invariants. Our refinement of the correspondence theorem will involve curves of genus $g\geqslant g_{\Delta,n}$.

For $g>g_{\Delta,n}$, inserting n points is no longer enough to cut down the virtual dimension to zero. The idea is to consider the Hodge bundle $\mathbb{E}$ over $\overline{M}_{g,n,\Delta}$. If

[5] Moduli spaces of stable log maps have a natural structure of log stack. The structure of log stack is particularly important to treat correctly evaluation morphisms in log Gromov–Witten theory in general, see [1]. In this paper, we always consider these moduli spaces as stacks over the category of schemes, not as log stacks, and we will always work with naive evaluation morphisms between stacks, not log stacks. This will be enough for us. See the remark at the end of Sect. 4.2 for some justification.

$\pi : \mathcal{C} \to \overline{M}_{g,n,\Delta}$ is the universal curve, of relative dualizing[6] sheaf ω_π, then

$$\mathbb{E} := \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{M}_{g,n,\Delta}$. The Chern classes of the Hodge bundle are classically [36] called the lambda classes and denoted as

$$\lambda_j := c_j(\mathbb{E}),$$

for $j = 0, \ldots, g$. Because the virtual dimension of $\overline{M}_{g,n,\Delta}$ is given by

$$\operatorname{vdim} \overline{M}_{g,n,\Delta} = g - g_{\Delta,n} + 2n,$$

inserting the lambda class $\lambda_{g-g_{\Delta,n}}$ and n points will cut down the virtual dimension to zero, so it is natural to consider the log Gromov–Witten invariants with one lambda class inserted

$$\begin{aligned} N_g^{\Delta,n} &:= \langle (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \tau_0(\mathrm{pt})^n \rangle_{g,n,\Delta} \\ &:= \int_{[\overline{M}_{g,n,\Delta}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \prod_{j=1}^{n} \mathrm{ev}_j^*(\mathrm{pt}). \end{aligned}$$

Our refined correspondence result, Theorem 5, gives an interpretation of the generating series of these invariants in terms of refined tropical curve counting.

2.3 Tropical curves

We refer to Mikhalkin [34], Nishinou, Siebert [38], Mandel, Ruddat [30], and Abramovich, Chen, Gross, Siebert [3] for basics on tropical curves. Each of these references uses a slightly different notion of parametrized tropical curve. We will use a variant of [3], Definition 2.5.3, because it is the one which is the most directly related to log geometry. It is easy to go from one to the other.

For us, a graph Γ has a finite set $V(\Gamma)$ of vertices, a finite set $E_f(\Gamma)$ of bounded edges connecting pairs of vertices and a finite set $E_\infty(\Gamma)$ of legs attached to vertices that we view as unbounded edges. By edge, we refer to a bounded or unbounded edge. We will always consider connected graphs.

A parametrized tropical curve $h : \Gamma \to \mathbb{R}^2$ is the following data:

- A non-negative integer $g(V)$ for each vertex V, called the genus of V.

[6] The dualizing line bundle of a nodal curve coincides with the log cotangent bundle up to some twist by marked points and so is a completely natural object from the point of view of log geometry.

- A bijection of the set $E_\infty(\Gamma)$ of unbounded edges with

$$\{1, \dots, |E_\infty(\Gamma)|\},$$

where $|E_\infty(\Gamma)|$ is the cardinality of $E_\infty(\Gamma)$.
- A vector $v_{V,E} \in \mathbb{Z}^2$ for every vertex V and E an edge adjacent to V. If $v_{V,E}$ is not zero, the divisibility $|v_{V,E}|$ of $v_{V,E}$ in $\mathbb{Z}^2$ is called the weight of E and is denoted $w(E)$. We require that $v_{V,E} \neq 0$ if E is unbounded and that for every vertex V, the following balancing condition is satisfied:

$$\sum_E v_{V,E} = 0,$$

where the sum is over the edges E adjacent to V. In particular, the collection Δ_V of non-zero vectors $v_{\Delta,E}$ for E adjacent to V is a balanced collection as in Sect. 2.1.
- A non-negative real number $\ell(E)$ for every bounded edge of E, called the length of E.
- A proper map $h\colon \Gamma \to \mathbb{R}^2$ such that
  - If E is a bounded edge connecting the vertices V_1 and V_2, then h maps E affine linearly on the line segment connecting $h(V_1)$ and $h(V_2)$, and $h(V_2) - h(V_1) = \ell(E) v_{V_1,E}$.
  - If E is an unbounded edge of vertex V, then h maps E affine linearly to the ray $h(V) + \mathbb{R}_{\geqslant 0} v_{V,E}$.

The genus g_h of a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is defined by

$$g_h := g_\Gamma + \sum_{V \in V(\Gamma)} g(V),$$

where g_Γ is the genus of the graph Γ.

We fix Δ a balanced collection of vectors in $\mathbb{Z}^2$, as in Sect. 2.1, and we fix a bijection of Δ with $\{1, \dots, |\Delta|\}$. We say that a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is of type Δ if there exists a bijection between Δ and $\{v_{V,E}\}_{E \in E_\infty(\Gamma)}$ compatible with the fixed bijections to

$$\{1, \dots, |\Delta|\} = \{1, \dots, |E_\infty(\Gamma)|\}.$$

Remark that

$$\sum_{E \in E_\infty(\Gamma)} v_{V,E} = 0$$

by the balancing condition.

We say that a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is n-pointed if we have chosen a distribution of the labels $1, \dots, n$ over the vertices of Γ, a vertex having the possibility to have several labels. Vertices without any label are said to be unpointed whereas those with labels are said to be pointed. For $j = 1, \dots, n$, let V_j be the pointed vertex having the label j. Let $p = (p_1, \dots, p_n)$ be a configuration of n points in $\mathbb{R}^2$. We say that a n-pointed parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ passes through p if $h(V_j) = p_j$ for every $j = 1, \dots, n$. We say that a n-pointed parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ passing through p is rigid if it is not contained in a non-trivial family of n-pointed parametrized tropical curves passing through p of the same combinatorial type.

Proposition 2 *For every balanced collection Δ of vectors in $\mathbb{Z}^2$, and n a non-negative integer such that $g_{\Delta,n} \geqslant 0$, there exists an open dense subset $U_{\Delta,n}$ of $(\mathbb{R}^2)^n$ such that if $p = (p_1, \dots, p_n) \in U_{\Delta,n}$ then $p_j \neq p_k$ for $j \neq k$ and if $h\colon \Gamma \to \mathbb{R}^2$ is a rigid[7] n-pointed parametrized tropical curve of genus $g \leqslant g_{\Delta,n}$ and of type Δ passing through p, then*

- *$g = g_{\Delta,n}$.*
- *We have $g(V) = 0$ for every vertex V of Γ. In particular, the graph Γ has genus $g_{\Delta,n}$.*
- *Images by h of distinct vertices are distinct.*
- *No edge is contracted to a point.*
- *Images by h of two distinct edges intersect in at most one point.*
- *Unpointed vertices are trivalent.*
- *Pointed vertices are bivalent.*

Proof This is essentially Proposition 4.11 of Mikhalkin [34], which itself is essentially some counting of dimensions. In [34], there is no genus attached to the vertices but if we have a parametrized tropical curve of genus $g \leqslant g_{\Delta,n}$ with some vertices of non-zero genus, the underlying graph has genus strictly less than g and so strictly less than $g_{\Delta,n}$, which is impossible by Proposition 4.11 of [34] for p general enough. □

Proposition 3 *If $p \in U_{\Delta,n}$, then the set $T_{\Delta,p}$ of rigid n-pointed genus $g_{\Delta,n}$ parametrized tropical curves $h\colon \Gamma \to \mathbb{R}^2$ of type Δ passing through p is finite.*

Proof This is Proposition 4.13 if Mikhalkin [34]: there are finitely many possible combinatorial types for a parametrized tropical curve as in Proposition 2, and for a fixed combinatorial type, the set of such tropical curves passing

[7] Here, the rigidity assumption is only necessary to forbid contracted edges. It happens to be the natural assumption in the general form of the decomposition formula of [3], as explained and used in Sect. 4.3.

through p is a zero dimensional intersection of a linear subspace with an open convex polyhedron, so is a point. □

Lemma 4 *Let $h\colon \Gamma \to \mathbb{R}^2$ be a parametrized tropical curve in $T_{\Delta,p}$. Then Γ has $2g_{\Delta,n} - 2 + |\Delta|$ trivalent vertices.*

Proof By definition of $T_{\Delta,p}$, the graph Γ is of genus $g_{\Delta,n}$ and its vertices are either trivalent or bivalent. Replacing the two edges adjacent to each bivalent vertex by a unique edge, we obtain a trivalent graph $\hat{\Gamma}$ with the same genus and the same number of unbounded edges as Γ. Let $|V(\hat{\Gamma})|$ be the number of vertices of $\hat{\Gamma}$ and let $|E_f(\hat{\Gamma})|$ be the number of bounded edges of $\hat{\Gamma}$. A count of half-edges using that $\hat{\Gamma}$ is trivalent gives

$$3|V(\hat{\Gamma})| = 2|E_f(\hat{\Gamma})| + |\Delta|.$$

By definition of the genus, we have

$$1 - g_{\Delta,n} = |V(\hat{\Gamma})| - |E_f(\hat{\Gamma})|.$$

Eliminating $|E_f(\hat{\Gamma})|$ from the two previous equalities gives the desired formula and so finishes the proof of Lemma 4. □

For $h\colon \Gamma \to \mathbb{R}^2$ a parametrized tropical curve in $\mathbb{R}^2$ and V a trivalent vertex of adjacent edges E_1, E_2 and E_3, the multiplicity of V is the integer defined by

$$m(V) := |\det(v_{V,E_1}, v_{V,E_2})|.$$

Thanks to the balancing condition

$$v_{V,E_1} + v_{V,E_2} + v_{V,E_3} = 0,$$

we also have

$$m(V) = |\det(v_{V,E_2}, v_{V,E_3})| = |\det(v_{V,E_3}, v_{V,E_1})|.$$

For $(h\colon \Gamma \to \mathbb{R}^2) \in T_{\Delta,p}$, the multiplicity of h is defined by

$$m_h := \prod_{V \in V^{(3)}(\Gamma)} m(V),$$

where the product is over the trivalent, i.e. unpointed, vertices of Γ.

We denote $N_{\mathrm{trop}}^{\Delta,p}$ the count with multiplicity of n-pointed genus $g_{\Delta,n}$ parametrized tropical curves of type Δ passing through p, i.e.

$$N_{\mathrm{trop}}^{\Delta,p} := \sum_{h \in T_{\Delta,p}} m_h .$$

This tropical count with multiplicity has a natural refinement, first suggested by Block and Göttsche [8]. We can replace the integer valued multiplicity m_h of a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ by the $\mathbb{N}[q^{\pm\frac{1}{2}}]$-valued multiplicity

$$m_h(q) := \prod_{V \in V^{(3)}(\Gamma)} \frac{q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \prod_{V \in V^{(3)}(\Gamma)} \left(\sum_{j=0}^{m(V)-1} q^{-\frac{m(V)-1}{2}+j} \right),$$

where the product is taken over the trivalent vertices of Γ. The specialization $q = 1$ recovers the usual multiplicity:

$$m_h(1) = m_h .$$

Counting the parametrized tropical curves in $T_{\Delta,p}$ as above but with q-multiplicities, we obtain a refined tropical count

$$N_{\mathrm{trop}}^{\Delta,p}(q) := \sum_{h \in T_{\Delta,p}} m_h(q) \in \mathbb{N}[q^{\pm\frac{1}{2}}],$$

which specializes to the tropical count $N_{\mathrm{trop}}^{\Delta,p}$ at $q = 1$:

$$N_{\mathrm{trop}}^{\Delta,p}(1) = N_{\mathrm{trop}}^{\Delta,p} .$$

2.4 Unrefined correspondence theorem

Let Δ be a balanced collection of vectors in $\mathbb{Z}^2$, as in Sect. 2.1, and let n be a non-negative integer and $p \in U_{\Delta,n}$. Then we have some log Gromov–Witten count $N^{\Delta,n}$ of n-pointed genus $g_{\Delta,n}$ curves of type Δ passing through n points in the toric surface X_Δ (see Sect. 2.2), and we have some count with multiplicity $N_{\mathrm{trop}}^{\Delta,n}$ of n-pointed genus $g_{\Delta,n}$ tropical curves of type Δ passing through n points $p = (p_1, \ldots, p_n)$ in $\mathbb{R}^2$ (see Sect. 2.3). The (unrefined) correspondence theorem then takes the simple form

$$N^{\Delta,n} = N_{\mathrm{trop}}^{\Delta,p} .$$

The result proved by Mikhalkin [34] and generalized by Nishinou, Siebert [38] is an equality between the tropical count $N_{\text{trop}}^{\Delta,n}$ and an enumerative count of algebraic curves. The fact that this enumerative count coincides with the log Gromov–Witten count $N^{\Delta,n}$ is proved by Mandel and Ruddat in [30].

2.5 Refined correspondence theorem

The Block-Göttsche refinement from $N^{\Delta,p}$ to $N^{\Delta,p}(q)$, reviewed in Sect. 2.3, is done at the tropical level so is combinatorial in nature and its geometric meaning is a priori unclear.

The main result of the present paper is a new non-tropical interpretation of Block-Göttsche invariants in terms of the higher genus log Gromov–Witten invariants with one lambda class inserted $N_{\Delta,n}^{g}$ that we introduced in Sect. 2.2. In particular, this geometric interpretation is independent of any tropical limit and makes the tropical deformation invariance of Block-Göttsche invariants manifest.

More precisely, we prove a refined correspondence theorem, already stated as Theorem 1 in the Introduction.

Theorem 5 *For every Δ balanced collection of vectors in $\mathbb{Z}^2$, for every non-negative integer n such that $g_{\Delta,n} \geqslant 0$, and for every $p \in U_{\Delta,n}$, we have the equality*

$$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = N_{\text{trop}}^{\Delta,p}(q) \left((-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\right)^{2g_{\Delta,n}-2+|\Delta|}$$

of power series in u with rational coefficients, where

$$q = e^{iu} = \sum_{n \geqslant 0} \frac{(iu)^n}{n!}.$$

Remarks

- The change of variables $q = e^{iu}$ makes the above correspondence quite non-trivial. In particular, in contrast to its unrefined version, it cannot be reduced to a finite to one enumerative correspondence. It is essential to have a virtual/non-enumerative count on the Gromov–Witten side: for g large enough, most of the contributions to $N_g^{\Delta,n}$ come from maps with contracted components.
- The refined tropical count has the symmetry $N_{\text{trop}}^{\Delta,n}(q) = N_{\text{trop}}^{\Delta,n}(q^{-1})$ and so, after the change of variables $q = e^{iu}$, is a even power series in u. In

particular, as

$$(-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \in u\mathbb{Q}[\![u^2]\!],$$

the tropical side of Theorem 5 lies in

$$u^{2g_{\Delta,n}-2+|\Delta|}\mathbb{Q}[\![u^2]\!],$$

as does the Gromov–Witten side. Taking the leading order terms on both sides in the limit $u \to 0$, $q \to 1$, we recover the unrefined correspondence theorem $N^{\Delta,n} = N^{\Delta,p}_{\mathrm{trop}}$.

– By Lemma 4, we know that $2g_{\Delta,n} - 2 + |\Delta|$ is the number of trivalent vertices of a parametrized tropical curve in $T_{\Delta,p}$. In particular, the tropical side of Theorem 5 can be obtained directly by considering only the numerators of the Block-Göttsche multiplicities, i.e. Theorem 5 can be rewritten

$$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = \sum_{h \in T_{\Delta,p}} \prod_V (-i) \left(q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}} \right),$$

where $q = e^{iu}$.

2.6 Fixing points on the toric boundary

It is possible to generalize Theorem 5 by fixing the position of some of the intersection points with the toric boundary divisor. Let Δ^F be a subset of Δ and let

$$\mathrm{ev}_{\Delta^F} : \overline{M}_{g,n,\Delta} \to (\partial X_\Delta)^{|\Delta^F|}$$

be the evaluation map at the intersection points with the toric boundary divisor ∂X_Δ indexed by the elements of Δ^F.

The problem of counting n-pointed genus g curves of type Δ passing through n given points of X_Δ and with fixed position of the intersection points with ∂X_Δ indexed by Δ^F, has virtual dimension zero if the genus is equal to

$$g_{\Delta,n}^{\Delta^F} := n + 1 - |\Delta| + |\Delta^F|.$$

For every $g \geqslant g_{\Delta,n}^{\Delta^F}$, we define the invariants

$$N_{g,\Delta^F}^{\Delta,n} := \int_{[\overline{M}_{g,n,\Delta}]^{virt}} (-1)^{g-g_{\Delta,n}^{\Delta^F}} \lambda_{g-g_{\Delta,n}^{\Delta^F}} \mathrm{ev}_{\Delta^F}^*(r^{|\Delta^F|}) \prod_{j=1}^n \mathrm{ev}_j^*(\mathrm{pt}),$$

where $r \in A^1(\partial X_\Delta)$ is the class of a point on ∂X_Δ.

We can consider the corresponding tropical problem. Fix a generic configuration $x = (x_v)_{v \in \Delta^F}$ of points in $\mathbb{R}^2$ and say that a tropical curve of type Δ is of type (Δ, Δ^F) if the unbounded edges in correspondence with Δ^F asymptotically coincide with the half-lines $x_v + \mathbb{R}_{\geqslant 0} v$, $v \in \Delta^F$.

We define a refined tropical count

$$N_{\mathrm{trop}, \Delta^F}^{\Delta, p, x}(q) \in \mathbb{N}[q^{\pm \frac{1}{2}}],$$

by counting with q-multiplicity the tropical curves of genus $g_{\Delta,n}^{\Delta^F}$ and of type (Δ, Δ^F) passing through a generic configuration $p = (p_1, \dots, p_n)$ of n points in $\mathbb{R}^2$.

The following result is the generalization of Theorem 5 to the case of non-empty Δ^F.

Theorem 6 *For every Δ balanced collection of vectors in $\mathbb{Z}^2$, for every Δ^F subset of Δ and for every n non-negative integer such that $g_{\Delta,n}^{\Delta^F} \geqslant 0$, we have the equality*

$$\sum_{g \geqslant g_{\Delta,n}^{\Delta^F}} N_{g,\Delta^F}^{\Delta,n} u^{2g-2+|\Delta|} = \left(\prod_{v \in \Delta^F} \frac{1}{|v|} \right) N_{\mathrm{trop}}^{\Delta,p,x}(q) \left((-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \right)^{2g_{\Delta,n}^{\Delta^F} - 2 + |\Delta|}$$

of power series in u with rational coefficients, where $q = e^{iu}$.

The proof of Theorem 6 is entirely parallel to the proof of Theorem 5 (Theorem 1 of the Introduction). The required modifications are discussed at the end of Sect. 8.4.

3 Gluing and vanishing properties of lambda classes

In this Section, we review some well-known facts: a gluing result for lambda classes, Lemma 7, and then a vanishing result, Lemma 8.

Lemma 7 *Let B be a scheme over $\mathbb{C}$. Let Γ be a graph, of genus g_Γ, and let $\pi_V : \mathcal{C}_V \rightarrow B$ be prestable curves over B indexed by the vertices V of Γ. For every edge E of Γ, connecting vertices V_1 and V_2, let $s_{E,1}$ and $s_{E,2}$ be smooth sections of π_{V_1} and π_{V_2} respectively. Let $\pi : \mathcal{C} \rightarrow B$ be the prestable curve* *over B obtained by gluing together the sections $s_{V_1,E}$ and $s_{V_2,E}$ corresponding to a same edge E of Γ. Then, we have an exact sequence*

$$0 \to \bigoplus_{V \in V(\Gamma)} (\pi_V)_* \omega_{\pi_V} \to \pi_* \omega_\pi \to \mathcal{O}^{\oplus g_\Gamma} \to 0,$$

where ω_{π_V} and ω_π are the relative line bundles.

Proof Let $s_E \colon B \to \mathcal{C}$ be the gluing sections. Then we have an exact sequence

$$0 \to \mathcal{O}_\mathcal{C} \to \bigoplus_{V \in V(\Gamma)} \mathcal{O}_{\mathcal{C}_V} \to \bigoplus_{E \in E(\Gamma)} \mathcal{O}_{s_E(B)} \to 0.$$

Applying $R\pi_*$, we obtain an exact sequence

$$\begin{aligned} 0 &\to \pi_* \mathcal{O}_\mathcal{C} \to \bigoplus_{V \in V(\Gamma)} \pi_* \mathcal{O}_{\mathcal{C}_V} \to \bigoplus_{E \in E(\Gamma)} \pi_* \mathcal{O}_{s_E(B)} \\ &\to R^1 \pi_* \mathcal{O}_\mathcal{C} \to \bigoplus_{V \in V(\Gamma)} R^1 \pi_* \mathcal{O}_{\mathcal{C}_V} \to 0. \end{aligned}$$

The kernel of

$$R^1 \pi_* \mathcal{O}_\mathcal{C} \to \bigoplus_{V \in V(\Gamma)} R^1 \pi_* \mathcal{O}_{\mathcal{C}_V}$$

is a free sheaf of rank $|E(\Gamma)| - |V(\Gamma)| + 1 = g_\Gamma$. We obtain the desired exact sequence by Serre duality.

Equivalently, if we choose g_Γ edges of Γ whose complement is a tree, we can understand the morphism

$$\pi_* \omega_\pi \to \mathcal{O}^{\oplus g_\Gamma}$$

as taking the residues at the corresponding g_Γ sections. □

Lemma 8 *Let B be a scheme over $\mathbb{C}$. Let $\pi \colon \mathcal{C} \to B$ be a prestable curve of arithmetic genus g over B. For every integer g' such that $0 \leqslant g' \leqslant g$, let $B_{g'}$ be the closed subset of B of points b such that the dual graph of the curve $\pi^{-1}(b)$ is of genus $\geqslant g'$. Then the lambda classes $\lambda_j \in H^{2j}(B, \mathbb{Q})$, defined by $\lambda_j = c_j(\pi_* \omega_\pi)$, satisfy*

$$\lambda_j|_{B_{g'}} = 0$$

in $H^{2j}(B_{g'}, \mathbb{Q})$ for all $j > g - g'$.

Proof Let $\tilde{B}_{g'}$ be the finite cover of $B_{g'}$ given by the possible choices of g' fully separating nodes, i.e. of nodes whose complement is of genus 0. Separating these g' fully separating nodes gives a way to write the pullback of $\mathcal{C}$ to $\tilde{B}_{g'}$ as the gluing of curves according to a dual graph Γ of genus g'. According to Lemma 7, the Hodge bundle of this family of curves has a trivial rank g' quotient. As $\tilde{B}_{g'}$ is finite over B'_g, it is enough to guarantee the desired vanishing in rational cohomology. □

4 Toric degeneration and decomposition formula

In Sect. 4.1, we review the natural link between log geometry and tropical geometry given by tropicalization. In Sect. 4.2, we start the proof of Theorem 1 by considering the Nishinou–Siebert toric degeneration. In Sect. 4.3, we apply the decomposition formula of Abramovich, Chen, Gross, Siebert [3] to this toric degeneration to write the log Gromov–Witten invariants $N_g^{\Delta,n}$ in terms of log Gromov–Witten invariants $N_g^{\Delta,h}$ indexed by parametrized tropical curves $h\colon \Gamma \to \mathbb{R}^2$. We use the vanishing result of Sect. 3 to restrict the tropical curves appearing.

4.1 Tropicalization

Log geometry is naturally related to tropical geometry. Every log scheme X admits a tropicalization $\Sigma(X)$.

Recall that a log scheme is a scheme X endowed with a sheaf of monoids $\mathcal{M}_X$ and a morphism of sheaves of monoids[8]

$$\alpha_X \colon \mathcal{M}_X \to \mathcal{O}_X,$$

where $\mathcal{O}_X$ is seen as a sheaf of multiplicative monoids, such that the restriction of α_X to $\alpha_X^{-1}(\mathcal{O}_X^*)$ is an isomorphism.

The ghost sheaf of a log scheme X is the sheaf of monoids

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / \alpha^{-1}(\mathcal{O}_X^*).$$

For the kind of log schemes that we are considering, fine and saturated, the ghost sheaf is of combinatorial nature. In this case, one can think of the log geometry of X as a combination of the geometry of the underlying scheme X and of the combinatorics of the ghost sheaf $\overline{\mathcal{M}}_X$. Non-trivial interactions between these two aspects of log geometry are encoded in the sequence

$$\mathcal{O}_X^* \to \mathcal{M}_X \to \overline{\mathcal{M}}_X.$$

8 All the monoids considered will be commutative and with an identity element.

A cone complex is an abstract gluing of convex rational cones along their faces. If X is a log scheme, the tropicalization $\Sigma(X)$ of X is the cone complex defined by gluing together the convex rational cones $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geqslant 0})$ for all $x \in X$ according to the natural specialization maps. Tropicalization is a functorial construction. For more details on tropicalization of log schemes, we refer to Appendix B of [24] and Section 2 of [3]. Tropicalization gives a pictorial way to describe the combinatorial part of log geometry contained in the ghost sheaf.

Examples

- Let X be a toric variety. We can view X as a log scheme for the toric divisorial log structure, i.e. the divisorial log stucture with respect to the toric boundary divisor ∂X. The sheaf $\mathcal{M}_X$ is the sheaf of functions non-vanishing outside ∂X and α_X is the natural inclusion of $\mathcal{M}_X$ in $\mathcal{O}_X$. The tropicalization $\Sigma(X)$ of X is naturally isomorphic as cone complex to the fan of X.
- Let $\overline{\mathcal{M}}$ be a monoid whose only invertible element is 0. Let X be the log scheme of underlying scheme the point $\mathrm{pt} = \mathrm{Spec}\ \mathbb{C}$, with $\mathcal{M}_X = \overline{\mathcal{M}} \oplus \mathbb{C}^*$ and

$$\begin{aligned} \alpha_X \colon \overline{\mathcal{M}} \oplus \mathbb{C}^* &\to \mathbb{C} \\ (m, a) &\mapsto a\delta_{m,0}. \end{aligned}$$

  We denote this log scheme as $\mathrm{pt}_{\overline{\mathcal{M}}}$ and such a log scheme is called a log point. By construction, we have $\overline{\mathcal{M}}_{\mathrm{pt}_{\overline{\mathcal{M}}}} = \overline{\mathcal{M}}$ and so the tropicalization $\Sigma(\mathrm{pt}_{\overline{\mathcal{M}}})$ is the cone $\mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$, i.e. the fan of the affine toric variety $\mathrm{Spec}\ \mathbb{C}[\overline{\mathcal{M}}]$.
- The log point $\mathrm{pt}_{\mathbb{N}}$ obtained for $\overline{\mathcal{M}} = \mathbb{N}$ is called the standard log point. Its tropicalization is simply $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geqslant 0}$, the fan of the affine line $\mathbb{A}^1$.
- The log point pt_0 obtained for $\overline{\mathcal{M}} = 0$ is called the trivial log point. Its tropicalization $\Sigma(\mathrm{pt}_0)$ is reduced to a point.
- A stable log map to some relative log scheme $X \to S$ determines a commutative diagram in the category of log schemes,

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \longrightarrow & S, \end{array}$$

  where $\mathrm{pt}_{\overline{\mathcal{M}}}$ is a log point and π is a log smooth proper integral curve. In particular, the scheme underlying C is a projective nodal curve with a natural set of smooth marked points. We can take the tropicalization of this diagram to obtain a commutative diagram of cone complexes

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X) \\ \downarrow{\scriptstyle \Sigma(\pi)} & & \downarrow \\ \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) & \longrightarrow & \Sigma(S). \end{array}$$

$\Sigma(C)$ is a family of graphs over the cone $\Sigma(\mathrm{pt}_{\overline{M}}) = \mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$: the fiber of $\Sigma(\pi)$ over a point in the interior of the cone is the dual graph of C. Fibers over faces of the cone are contractions of the dual graph. In particular, the fiber over the origin of the cone is obtained by fully contracting the dual graph of C to a graph with a unique vertex. If X is a toric variety with the toric divisorial log structure and S is the trivial log point, then $\Sigma(f)$ is a family of parametrized tropical curves in the fan of X. We refer to Section 2.5 of [3] for more details.

4.2 Toric degeneration

Let Δ be a balanced configuration of vectors, as in Sect. 2.1, and let n be a non-negative integer such that $g_{\Delta,n} \geqslant 0$. We fix $p = (p_1, \dots, p_n)$ a configuration of n points in $\mathbb{R}^2$ belonging to the open dense subset $U_{\Delta,n}$ of $(\mathbb{R}^2)^n$ given by Proposition 2. Let $T_{\Delta,p}$ be the set of n-pointed genus $g_{\Delta,n}$ parametrized tropical curves in $\mathbb{R}^2$ of type Δ passing through p. The set $T_{\Delta,p}$ is finite by Proposition 3. Proposition 2 shows that the elements of $T_{\Delta,p}$ are particularly nice parametrized tropical curves.

We can slightly modify p such that $p \in (\mathbb{Q}^2)^n \cap U_{\Delta,n}$ without changing the combinatorial type of the elements of $T_{\Delta,p}$ and so without changing the tropical counts $N_{\mathrm{trop}}^{\Delta,p}$ and $N_{\mathrm{trop}}^{\Delta,p}(q)$. In that case, for every parametrized tropical curve $h \colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$ and for every vertex V of Γ, we have $h(V) \in \mathbb{Q}^2$ and for every edge E of Γ, we have $\ell(E) \in \mathbb{Q}$. Indeed, the positions $h(V)$ of vertices in $\mathbb{R}^2$ and the lengths $\ell(E)$ of edges are natural parameters on the moduli space of genus $g_{\Delta,n}$ parametrized tropical curves of type Δ and this moduli space is a rational polyhedron in the space of these parameters. The set $T_{\Delta,p}$ is obtained as zero dimensional intersection of this rational polyhedron with the rational (because $p \in (\mathbb{Q}^2)^n$) linear space imposing to pass through p. It follows that the parameters $h(V)$ and $\ell(E)$ are rational for elements of $T_{\Delta,p}$.

We follow the toric degeneration approach introduced by Nishinou and Siebert [38] (see also Mandel and Ruddat [30]). According to [38] Proposition 3.9 and [30] Lemma 3.1, there exists a rational polyhedral decomposition $\mathcal{P}_{\Delta,p}$ of $\mathbb{R}^2$ such that

– The asymptotic fan of $\mathcal{P}_{\Delta,p}$ is the fan of X_Δ.

- For every parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, the images $h(V)$ of vertices V of Γ are vertices of $\mathcal{P}_{\Delta,p}$ and the images $h(E)$ of edges E of Γ are contained in union of edges of $\mathcal{P}_{\Delta,p}$

Remark that the points p_j in $\mathbb{R}^2$ are image of vertices of parametrized tropical curves in $T_{\Delta,p}$ and so are vertices of $\mathcal{P}_{\Delta,p}$.

Given a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, we construct a new parametrized tropical curve $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ by simply adding a genus zero bivalent unpointed vertex to Γ at each point $h^{-1}(V)$ for V a vertex of $\mathcal{P}_{\Delta,p}$ which is not the image by h of a vertex of Γ. The image $\tilde{h}(E)$ of each edge E of $\tilde{\Gamma}$ is now exactly an edge of $\mathcal{P}_{\Delta,p}$. The graph $\tilde{\Gamma}$ has three types of vertices:

- Trivalent unpointed vertices, coming from Γ.
- Bivalent pointed vertices, coming from Γ.
- Bivalent unpointed vertices, not coming from Γ.

Doing a global rescaling of $\mathbb{R}^2$ if necessary, we can assume that $\mathcal{P}_{\Delta,p}$ is an integral polyhedral decomposition, i.e. that all the vertices of $\mathcal{P}_{\Delta,p}$ are in $\mathbb{Z}^2$, and that all the lengths $\ell(E)$ of edges E of parametrized tropical curves $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$, coming from $h\colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, are integral.

Taking the cone over $\mathcal{P}_{\Delta,p} \times \{1\}$ in $\mathbb{R}^2 \times \mathbb{R}$, we obtain the fan of a three dimensional toric variety $X_{\mathcal{P}_{\Delta,p}}$ equipped with a morphism

$$\nu\colon X_{\mathcal{P}_{\Delta,p}} \to \mathbb{A}^1$$

coming from the projection $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ on the third $\mathbb{R}$ factor. We have $\nu^{-1}(t) \simeq X_\Delta$ for every $t \in \mathbb{A}^1 - \{0\}$. The special fiber $X_0 := \nu^{-1}(0)$ is a reducible surface whose irreducible components X_V are toric surfaces in one to one correspondence with the vertices V of $\mathcal{P}_{\Delta,p}$,

$$X_0 = \bigcup_V X_V .$$

In other words, $\nu\colon X_{\mathcal{P}_{\Delta,p}} \to \mathbb{A}^1$ is a toric degeneration of X_Δ.

We consider the toric varieties $\mathbb{A}^1$, $X_{\mathcal{P}_{\Delta,p}}$, X_Δ and X_V as log schemes with respect to the toric divisorial log structure. In particular, the toric morphism ν induces a log smooth morphism

$$\nu\colon X_{\mathcal{P}_{\Delta,n}} \to \mathbb{A}^1 .$$

Restricting to the special fiber gives a structure of log scheme on X_0 and a log smooth morphism to the standard log point

$$\nu_0\colon X_0 \to \mathrm{pt}_{\mathbb{N}} .$$

From now on, we will denote $\underline{X}_0$ the scheme underlying the log scheme X_0. Beware that the toric divisorial log structure that we consider on X_V is not the restriction of the log structure that we consider on X_0.

For every $j = 1, \dots, n$, the ray $\mathbb{R}_{\geqslant 0}(p_j, 1)$ in $\mathbb{R}^2 \times \mathbb{R}$ defines a one-parameter subgroup $\mathbb{C}^*_{p_j}$ of $(\mathbb{C}^*)^3 \subset X_{\mathcal{P}_{\Delta,n}}$. We choose a point $P_j \in (\mathbb{C}^*)^2$ and we write Z_{P_j} the affine line in $X_{\mathcal{P}_{\Delta,n}}$ defined as the closure of the orbit of $(P_j, 1)$ under the action of $\mathbb{C}^*_{p_j}$. We have

$$Z_{P_j} \cap \nu^{-1}(1) = Z_{P_j} \cap X_\Delta = P_j,$$

and

$$P_j^0 := Z_{P_j} \cap \nu^{-1}(0)$$

is a point in the dense torus $(\mathbb{C}^*)^2$ contained in the toric component of X_0 corresponding to the vertex p_j of $\mathcal{P}_{\Delta,p}$. In other words, Z_{P_j} is a section of ν degenerating $P_j \in X_\Delta$ to some $P_j^0 \in X_0$.

Recall from Sect. 2.2 that the log Gromov–Witten invariants $N_g^{\Delta,n}$ are defined using stable log maps of target X_Δ,

$$N_g^{\Delta,n} := \int_{[\overline{M}_{g,n,\Delta}]^{\text{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \prod_{j=1}^n \text{ev}_j^*(\text{pt}),$$

where $\overline{M}_{g,n,\Delta}$ is the moduli space of n-pointed stable log maps to X_Δ of genus g and of type Δ.

Let $\overline{M}_{g,n,\Delta}(X_0/\text{pt}_{\mathbb{N}})$ be the moduli space of n-pointed stable log maps to $\pi_0 \colon X_0 \to \text{pt}_{\mathbb{N}}$ of genus g and of type Δ. It is a proper Deligne-Mumford stack of virtual dimension

$$\text{vdim } \overline{M}_{g,n,\Delta}(X_0/\text{pt}_{\mathbb{N}}) = \text{vdim } \overline{M}_{g,n,\Delta} = g - g_{\Delta,n} + 2n$$

and it admits a virtual fundamental class

$$[\overline{M}_{g,n,\Delta}(X_0/\text{pt}_{\mathbb{N}})]^{\text{virt}} \in A_{g-g_{\Delta,n}+2n}(\overline{M}_{g,n,\Delta}(X_0/\text{pt}_{\mathbb{N}}), \mathbb{Q}).$$

Considering the evaluation morphism

$$\text{ev} \colon \overline{M}_{g,n,\Delta}(X_0/\text{pt}_{\mathbb{N}}) \to \underline{X}_0^n$$

and the inclusion

$$\iota_{P^0} \colon \left(P^0 := \left(P_1^0, \dots, P_n^0\right)\right) \hookrightarrow \underline{X}_0^n,$$

we can define the moduli space[9]

$$\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0) := \overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}) \times_{\underline{X}_0^n} P^0,$$

of stable log maps passing through P^0, and by the Gysin refined homomorphism (see Section 6.2 of [16]), a virtual fundamental class

$$[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}} := \iota_{P^0}^! [\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})]^{\mathrm{virt}}$$
$$\in A_{g-g_{\Delta,n}}(\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0), \mathbb{Q}).$$

Remark[10] that this definition is compatible with [3] because each P_j^0, seen as a log morphism $P_j^0 \colon \mathrm{pt}_{\mathbb{N}} \to X_0$, is strict. This follows from the fact that we have chosen P_j^0 in the dense torus $(\mathbb{C}^*)^2$ contained in the toric component of X_0 dual to the vertex p_j of $\mathcal{P}_{\Delta,p}$. If it were not the case,[11] then, following Section 6.3.2 of [3], the definition of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)$ should have been replaced by a fiber product in the category of fs log stacks and $[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}}$ should have been defined by some perfect obstruction theory directly on $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)$.

By deformation invariance of the virtual fundamental class on moduli spaces of stable log maps in log smooth families, we have

$$N_g^{\Delta,n} = \int_{[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}}.$$

4.3 Decomposition formula

As the toric degeneration breaks the toric surface X_Δ into many pieces, irreducible components of the special fiber X_0, one can similarly expect that it breaks the moduli space $\overline{M}_{g,n,\Delta}$ of stable log maps to X_Δ into many pieces, irreducible components of the moduli space $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ of stable log maps to X_0. Tropicalization gives a way to understand the combinatorics of this breaking into pieces.

As we recalled in Sect. 4.1, a n-pointed stable log map to $X_0/\mathrm{pt}_{\mathbb{N}}$ of type Δ gives a commutative diagram of log schemes

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9 As already mentioned in Sect. 2.2, we consider moduli spaces of stable log maps as stacks, not log stacks. In particular, the morphisms ev, ι_{P^0} and the fiber product defining $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)$ are defined in the category of stacks, not log stacks.

10 I thank the referee for stressing this point.

11 In Section 6.3.2 of [3], sections defining point constraints have to interact non-trivially with the log structure of the special fiber to produce something interesting because the degeneration considered there is a trivial product, whereas we are considering a non-trivial degeneration.

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \nu_0} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}}, \end{array}$$

which can be tropicalized in a commutative diagram of cone complexes

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \big\downarrow{\scriptstyle \Sigma(\pi)} & & \big\downarrow{\scriptstyle \Sigma(\nu_0)} \\ \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) & \xrightarrow{\Sigma(g)} & \Sigma(\mathrm{pt}_{\mathbb{N}}). \end{array}$$

We have $\Sigma(\mathrm{pt}_{\mathbb{N}}) \simeq \mathbb{R}_{\geqslant 0}$ and the fiber $\Sigma(\nu_0)^{-1}(1)$ is naturally identified with $\mathbb{R}^2$ equipped with the polyhedral decomposition $\mathcal{P}_{\Delta,p}$, whose asymptotic fan is the fan of X_Δ. So the above diagram gives a family over the polyhedron $\Sigma(g)^{-1}(1)$ of n-pointed parametrized tropical curves in $\mathbb{R}^2$ of type Δ

The moduli space $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$ of n-pointed genus g parametrized tropical curves in $\mathbb{R}^2$ of type Δ is a rational polyhedral complex. If $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$ were the tropicalization of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ (seen as a log stack over $\mathrm{pt}_{\mathbb{N}}$), then $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$ would be the dual intersection complex of $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$. In particular, irreducible components of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ would be in one to one correspondence with the 0-dimensional faces of $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$. As the polyhedral decomposition of $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$ is induced by the combinatorial type of tropical curves, the 0-dimensional faces of $\overline{M}^{\mathrm{trop}}_{g,n,\Delta}$ correspond to the rigid parametrized tropical curves, see Definition 4.3.1 of [3], i.e. to parametrized tropical curves which are not contained in a non-trivial family of parametrized tropical curves of the same combinatorial type.

According to the decomposition formula of Abramovich, Chen, Gross, Siebert [3], this heuristic description of the pieces of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ is correct at the virtual level: one can express $[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}}$ as a sum of contributions indexed by rigid tropical curves.

Let $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ be a n-pointed genus g rigid parametrized tropical curve to $\mathbb{R}^2$ of type Δ passing through p. For every V vertex of $\tilde{\Gamma}$, let Δ_V be the balanced collection of vectors $v_{V,E}$ for all edges E adjacent to V. Using the notations of Sect. 2.1 that we used all along for Δ but now for Δ_V, the toric surface X_{Δ_V} is the irreducible component of X_0 corresponding to the vertex $h(V)$ of the polyhedral decomposition $\mathcal{P}_{\Delta,p}$.

A n-pointed genus g stable log map to X^0 of type Δ passing through P^0 and marked by $\tilde{h}$ is the following data, see [3], Definition 4.4.1,[12]

[12] In [3], the marking includes also a choice of curve classes for the stable maps f_V. In our case, the curve classes are uniquely determined because a curve class in a toric variety is uniquely determined by its intersection numbers with the components of the toric boundary divisor.

- A n-pointed genus g stable log map $f: C/\mathrm{pt}_{\overline{\mathcal{M}}} \to X_0/\mathrm{pt}_{\mathbb{N}}$ of type Δ passing through P^0.
- For every vertex V of $\tilde{\Gamma}$, an ordinary stable map $f_V: C_V \to X_{\Delta_V}$ of class β_{Δ_V} with marked points x_v for every $v \in \Delta_V$, such that $f_V(x_v) \in D_v$, where D_v is the prime toric divisor of X_{Δ_V} dual to the ray $\mathbb{R}_{\geqslant 0}v$.

These data must satisfy the following compatibility conditions: the gluing of the curves C_V along the points corresponding to the edges of $\tilde{\Gamma}$ is isomorphic to the curve underlying the log curve C, and the corresponding gluing of the maps f_V is the map underlying the log map f.

By [3], the moduli space $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$ of n-pointed genus g stable log maps of type Δ passing through P^0 and marked by $\tilde{h}$ is a proper Deligne-Mumford stack, equipped with a natural virtual fundamental class $[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}$. Forgetting the marking by $\tilde{h}$ gives a morphism

$$i_{\tilde{h}}: \overline{M}_{g,n,\Delta}^{\tilde{h},P^0} \to \overline{M}_{g,n,\Delta}\left(X_0/\mathrm{pt}_{\mathbb{N}}, P^0\right).$$

According to the decomposition formula, [3] Theorem 6.3.9, we have

$$\left[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)\right]^{\mathrm{virt}} = \sum_{\tilde{h}} \frac{n_{\tilde{h}}}{|\mathrm{Aut}(\tilde{h})|} (i_{\tilde{h}})_* \left[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}\right]^{\mathrm{virt}},$$

where the sum is over the n-pointed genus g rigid parametrized tropical curves to $(\mathbb{R}^2, \mathcal{P}_{\Delta,p})$ of type Δ passing through p, $n_{\tilde{h}}$ is the smallest positive integer such that the scaling of $\tilde{h}$ by $n_{\tilde{h}}$ has integral vertices and integral lengths, and $|\mathrm{Aut}(\tilde{h})|$ is the order of the automorphism group of $\tilde{h}$.

Recall from Proposition 2 that a parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$ has a source graph Γ of genus $g_{\Delta,n}$ and that all vertices V of Γ are of genus zero: $g(V) = 0$. In Sect. 4.2, we explained that the polyhedral decomposition $\mathcal{P}_{\Delta,p}$ defines a new parametrized tropical $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$, for each $h: \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, by addition of unmarked genus zero bivalent vertices. Given such parametrized tropical curve $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$, one can construct genus g parametrized tropical curves by changing only the genus of vertices $g(V)$ so that

$$\sum_{V \in V(\Gamma)} g(V) = g - g_{\Delta,n}.$$

We denote $T_{\Delta,p}^g$ the set of genus g parametrized tropical curves obtained in this way.

Lemma 9 *Parametrized tropical curves* $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ *in* $T^g_{\Delta,p}$ *are rigid. Furthermore, for such* $\tilde{h}$*, we have* $n_{\tilde{h}} = 1$ *and* $|\mathrm{Aut}(\tilde{h})| = 1$.

Proof The rigidity of parametrized tropical curves in $T^g_{\Delta,p}$ follows from the rigidity of parametrized tropical curves in $T_{\Delta,p}$ because the genera attached to the vertices cannot change under a deformation preserving the combinatorial type, and added bivalent vertices to go from Γ to $\tilde{\Gamma}$ are mapped to vertices of $\mathcal{P}_{\Delta,p}$ and so cannot move without changing the combinatorial type.

We have $n_{\tilde{h}} = 1$ because in Sect. 4.2, we have chosen the polyhedral decomposition $\mathcal{P}_{\Delta,p}$ to be integral: vertices of $\tilde{h}$ map to integral points of $\mathbb{R}^2$ and edges E of $\tilde{\Gamma}$ have integral lengths $\ell(E)$. We have $|\mathrm{Aut}(\tilde{h})| = 1$ because $\tilde{h}$ is an immersion. The genus of vertices never enters in the above arguments. □

For every $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ parametrized tropical curve in $T^g_{\Delta,p}$, we define

$$N^{\Delta,n}_{g,\tilde{h}} := \int_{[\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}}.$$

Proposition 10 *For every* Δ*,* n *and* $g \geqslant g_{\Delta,n}$*, we have*

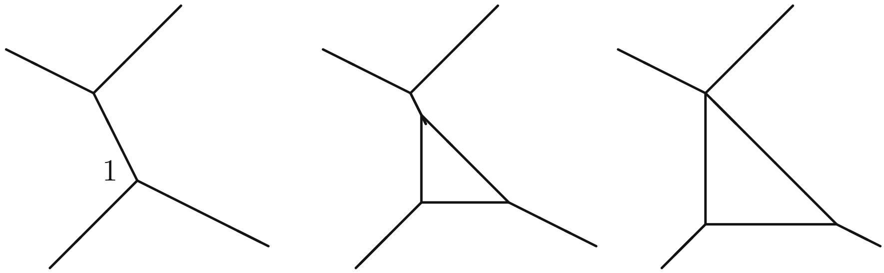
$$N^{\Delta,n}_g = \sum_{\tilde{h}\in T^g_{\Delta,p}} N^{\Delta,n}_{g,\tilde{h}}.$$

Proof This follows from the decomposition formula and from the vanishing property of lambda classes.

If $\tilde{h}$ is a rigid parametrized tropical curve of genus $g > g_{\Delta,n}$, then every point in $\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}$ is a stable log map whose tropicalization has genus $g > g_{\Delta,n}$. In particular, the dual intersection complex of the source curve has genus $g > g_{\Delta,n}$. By Lemma 8, $\lambda_{g-g_{\Delta,n}}$ is zero on restriction to such family of curves. □

Example The generic way to deform a parametrized tropical curve in $T^g_{\Delta,p}$ is to open $g(V)$ small cycles in place of a vertex of genus $g(V)$. When the cycles coming from various vertices grow and meet, we can obtain curves with vertices of valence strictly greater than three which can be rigid. Proposition 10 guarantees that such rigid curves do not contribute in the decomposition formula after integration of the lambda class.

Below is an illustration of a genus one vertex opening in one cycle and growing until forming a 4-valent vertex.



5 Non-torically transverse stable log maps in X_Δ

Let Δ be a balanced collections of vectors in $\mathbb{Z}^2$, as in Sect. 2.1. We consider the toric surface X_Δ with the toric divisorial log structure. In this Section, we prove some general properties of stable log maps of type Δ in X_Δ, using as tool the tropicalization procedure reviewed in Sect. 4.1.

We say that a stable log map $(f: C/\mathrm{pt}_{\overline{\mathcal{M}}} \to X_\Delta)$ to X_Δ is torically transverse[13] if its image does not contain any of the torus fixed points of X_Δ, i.e. if its image does not pass through the "corners" of the toric boundary divisor ∂X_Δ. The difficulty of log Gromov–Witten theory, with respect to relative Gromov–Witten theory for example, comes from the stable log maps which are not torically transverse: the "corners" of ∂X_Δ are the points where ∂X_Δ is not smooth and so are exactly the points where the log structure of X_Δ is locally more complicated that the divisorial log structure along a smooth divisor.

The following Proposition is a structure result for stable log maps of type Δ which are not torically transverse. Combined with vanishing properties of lambda classes reviewed in Sect. 3, this will give us in Sect. 7 a way to completely discard stable log maps which are not torically transverse.

Proposition 11 *Let $f: C/\mathrm{pt}_{\overline{\mathcal{M}}} \to X_\Delta$ be a stable log map to X_Δ of type Δ. Let $\Sigma(f): \Sigma(C)/\Sigma(\mathrm{pt}_{\mathbb{N}}) \to \Sigma(X_\Delta)$ be the family of tropical curves obtained as tropicalization of f. Assume that f is not torically transverse and that the unbounded edges of the fibers of $\Sigma(f)$ are mapped to rays of the fan of X_Δ. Then the dual graph of C has positive genus, i.e. C contains at least one non-separating node.*

Proof Recall that $\Sigma(f)$ is a family over the cone $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$ of parametrized tropical curves in $\mathbb{R}^2$. We assume that the unbounded edges of these parametrized tropical curves are mapped to rays of the fan of X_Δ.

13 We allow a torically transverse stable log map to have components contracted to points of ∂X_Δ which are not torus fixed points. In particular, we use a notion of torically transverse map which is slightly different from the one used by Nishinou and Siebert in [38].

We fix a point in the interior of the cone $\mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$ and we consider the corresponding parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ in $\mathbb{R}^2$. Combinatorially, Γ is the dual graph of C.

Lemma 12 *There exists a vertex V of Γ mapping away from the origin in $\mathbb{R}^2$ and a non-contracted edge E adjacent to V such that $h(E)$ is not included in a ray of the fan of X_Δ.*

Proof We are assuming that f is not torically transverse. This means that at least one component of C maps dominantly to a component of the toric boundary divisor ∂X_Δ or that at least one component of C is contracted to a torus fixed point of X_Δ.

If one component of C is contracted to a torus fixed point of X_Δ, then we are done because the corresponding vertex V of Γ is mapped away from the origin and from the rays of the fan of X_Δ, and any non-contracted edge of Γ adjacent to V is not mapped to a ray of the fan of X_Δ. Remark that there exists such non-contracted edge because if not, as Γ is connected, all the vertices of Γ would be mapped to $h(V)$ and so the curve C would be entirely contracted to a torus fixed point, contradicting $\beta_\Delta \neq 0$.

So we can assume that no component of C is contracted to a torus fixed point, i.e. that all the vertices of Γ are mapped either to the origin or to a point on a ray of the fan of X_Δ, and that at least one component of C maps dominantly to a component of ∂X_Δ. We argue by contradiction by assuming further that every edge of Γ is either contracted to a point or mapped inside a ray of the fan of X_Δ.

Let Γ_0 be the subgraph of Γ formed by vertices mapping to the origin and edges between them. For every ray ρ of the fan of X_Δ, let Δ_ρ be the set of $v \in \Delta$ such that $\mathbb{R}_{\geqslant 0} v = \rho$, and let Γ_ρ be the subgraph of Γ formed by vertices of Γ mapping to the ray ρ away from the origin and the edges between them.

By our assumption, there is no edge in Γ connecting Γ_ρ and $\Gamma_{\rho'}$ for two different rays ρ and ρ'. For every ray ρ, let $E(\Gamma_0, \Gamma_\rho)$ the set of edges of Γ connecting a vertex $V_0(E)$ of Γ_0 and a vertex $V_\rho(E)$ of Γ_ρ. It follows from the balancing condition that, for every ray ρ, we have

$$\sum_{E \in E(\Gamma_0, \Gamma_\rho)} v_{V_0(E), E} = \sum_{v \in \Delta_\rho} v.$$

Let C_0 be the curve obtained by taking the components of C intersecting properly the toric boundary divisor ∂X_Δ. The dual graph of C_0 is Γ_0 and the total intersection number of C_0 with the toric divisor D_ρ is

$$\sum_{E \in E(\Gamma_0, \Gamma_\rho)} |v_{V_0(E), E}|,$$

where $|v_{V_0(E),E}|$ is the divisibility of $v_{V_0(E),E}$ in $\mathbb{Z}^2$, i.e. the multiplicity of the corresponding intersection point of C_0 and D_ρ.

From the previous equality, we obtain that the intersection numbers of C_0 with the components of ∂X_Δ are equal to the intersection numbers of C with the components of ∂X_Δ so $[f(C_0)] = \beta_\Delta$. It follows that all the components of C not in C_0 are contracted, which contradicts the fact that at least one component of C maps dominantly to a component of ∂X_Δ. □

We continue the proof of Proposition 11. By Lemma 12, there exists a vertex V of Γ mapping away from the origin in $\mathbb{R}^2$ and a non-contracted edge E adjacent to V such that $h(E)$ is not included in a ray of the fan of X_Δ. We will use (V, E) as initial data for a recursive construction of a non-trivial cycle in Γ.

There exists a unique two-dimensional cone of the fan of X_Δ, containing $h(V) \in \mathbb{R}^2 - \{0\}$ and delimited by rays ρ_1 and ρ_2, such that the rays ρ_1, $\mathbb{R}_{\geqslant 0} h(V)$ and ρ_2 are ordered in the clockwise way and such that $h(V) \in \rho_1$ if $h(V)$ is on a ray. Let v_1 and v_2 be vectors in $\mathbb{R}^2 - \{0\}$ such that $\rho_1 = \mathbb{R}_{\geqslant 0} v_1$ and $\rho_2 = \mathbb{R}_{\geqslant 0} v_2$. The vectors v_1 and v_2 form a basis of $\mathbb{R}^2$ and for every $v \in \mathbb{R}^2$, we write (v, v_1) and (v, v_2) for the coordinates of v in this basis, i.e. the real numbers such that

$$v = (v, v_1)v_1 + (v, v_2)v_2.$$

By construction, we have $(h(V), v_1) > 0$ and $(h(V), v_2) \geqslant 0$. As $v_{V,E} \neq 0$, we have $(v_{V,E}, v_1) \neq 0$ or $(v_{V,E}, v_2) \neq 0$.

If $(v_{V,F}, v_2) = 0$ for every edge F adjacent to V, then $(v_{V,E}, v_1) \neq 0$ and $(h(V), v_2) > 0$. In particular, E is not an unbounded edge. By the balancing condition, up to replacing E by another edge adjacent to V, one can assume that $(v_{V,E}, v_1) > 0$. Then, the edge E is adjacent to another vertex V' with $(h(V'), v_1) > (h(V), v_1)$ and $(h(V'), v_2) = (h(V), v_2)$. By the balancing condition, there exists an edge E' adjacent to V' such that $(v_{V',E'}, v_1) > 0$. If $(v_{V,F'}, v_2) = 0$ for every edge F' adjacent to V', then in particular we have $(v_{V,E'}, v_2) = 0$ and so E' is adjacent to another vertex V'' with $(h(V''), v_1) > (h(V'), v_1)$ and $(h(V''), v_2) = (h(V'), v_2)$, and we can iterate the argument. Because Γ has finitely many vertices, this process has to stop: there exists a vertex $\tilde{V}$ in the cone generated by ρ_1 and ρ_2 and an edge $\tilde{E}$ adjacent to $\tilde{V}$ such that $(v_{\tilde{V},\tilde{E}}, v_2) \neq 0$.

The upshot of the previous paragraph is that, up to changing V and E, one can assume that $(v_{V,E}, v_2) \neq 0$. By the balancing condition, up to replacing E by another edge adjacent to V, one can assume that $(v_{V,E}, v_2) > 0$. The edge E is adjacent to another vertex V' with $(h(V'), v_2) > (h(V), v_2)$. By the balancing condition, one can find an edge E' adjacent to V' such that $(v_{V',E'}, v_2) > 0$. If $h(V')$ is in the interior of the cone generated by ρ_1 and ρ_2, then E' is not an unbounded edge and so is adjacent to another vertex V'' with $(h(V''), v_2) > (h(V'), v_2)$. Repeating this construction, we obtain a sequence of vertices of image in the cone generated by ρ_1 and ρ_2. Because Γ has finitely many vertices, this process has to terminate: there exists a vertex $\tilde{V}$ of Γ such that $h(\tilde{V}) \in \rho_2$ and connected to V by a path of edges mapping to the interior of the cone delimited by ρ_1 and ρ_2.

Repeating the argument starting from $\tilde{V}$, and so on, we construct a path of edges in Γ whose projection in $\mathbb{R}^2$ intersects successive rays in the clockwise order. Because the combinatorial type of Γ is finite, this path has to close eventually and so Γ contains a non-trivial closed cycle, i.e. Γ has positive genus. □

Remark It follows from Proposition 11 that the ad hoc genus zero invariants defined in terms of relative Gromov–Witten invariants of some open geometry used by Gross, Pandharipande, Siebert in [23] (Section 4.4), and Gross, Hacking, Keel in [22] (Section 3.1), coincide with log Gromov–Witten invariants.[14] In fact, our proof of Proposition 11 can be seen as a tropical analogue of the main properness argument of [23] (Proposition 4.2) which guarantees that the ad hoc invariants are well-defined.

6 Statement of the gluing formula

We continue the proof of Theorem 1 started in Sect. 4. In Sect. 6, we state a gluing formula, Corollary 16, expressing the invariants $N_{g,\tilde{h}}^{\Delta,n}$ attached to a parametrized tropical curve $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ in terms of invariants $N_{g,V}^{1,2}$ attached to the vertices V of Γ. This gluing formula is proved in Sect. 7, using the structure result of Sect. 5 and the vanishing result of Sect. 3 to reduce the argument to the locus of torically transverse stable log maps.

6.1 Preliminaries

We fix $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ a parametrized tropical curve in $T_{\Delta,p}^{g}$. The purpose of the gluing formula is to write the log Gromov–Witten invariant

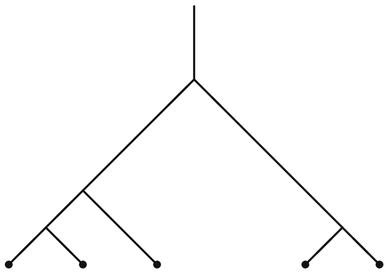
$$N_{g,\tilde{h}}^{\Delta,n} = \int_{[\overline{M}_{g,n,\Delta}^{\tilde{h},p^0}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}},$$

[14] This result was expected: see Remark 3.4 of [22] but it seems that no proof was published until now.

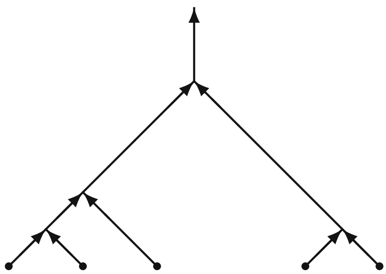
introduced in Sect. 4.3, in terms of log Gromov–Witten invariants of the toric surfaces X_{Δ_V} attached to the vertices V of $\tilde{\Gamma}$. Recall from Sect. 4.2 that $\tilde{\Gamma}$ has three types of vertices:

– Trivalent unpointed vertices, coming from Γ.
– Bivalent pointed vertices, coming from Γ.
– Bivalent unpointed vertices, not coming from Γ.

According to Lemma 4.20 of Mikhalkin [34], the connected components of the complement of the bivalent pointed vertices of $\tilde{\Gamma}$ are trees with exactly one unbounded edge.



In particular, we can fix an orientation of edges of $\tilde{\Gamma}$ consistently from the bivalent pointed vertices to the unbounded edges. Every trivalent vertex of $\tilde{\Gamma}$ has two ingoing and one outgoing edges with respect to this orientation. Every bivalent pointed vertex has two outgoing edges with respect to this orientation. Every bivalent unpointed vertex has one ingoing and one outgoing edges with respect to this orientation.



6.2 Contribution of trivalent vertices

Let V be a trivalent vertex of $\tilde{\Gamma}$. Let $\overline{M}_{g,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g and of type Δ_V. It has virtual dimension

$$\text{vdim}\ \overline{M}_{g,\Delta_V} = g + 2,$$

and admits a virtual fundamental class

$$[\overline{M}_{g,\Delta_V}]^{\text{virt}} \in A_{g+2}(\overline{M}_{g,\Delta_V}, \mathbb{Q}).$$

Let $E_V^{\mathrm{in},1}$ and $E_V^{\mathrm{in},2}$ be the two ingoing edges adjacent to V, and let E_V^{out} be the outgoing edge adjacent to V. Let $D_{E_V^{\mathrm{in},1}}$, $D_{E_V^{\mathrm{in},2}}$ and $D_{E_V^{\mathrm{out}}}$ be the corresponding toric divisors of X_{Δ_V}. We have evaluation morphisms

$$\left(\mathrm{ev}_V^{E_V^{\mathrm{in},1}}, \mathrm{ev}_V^{E_V^{\mathrm{in},2}}, \mathrm{ev}_V^{E_V^{\mathrm{out}}}\right) : \overline{M}_{g,\Delta_V} \to D_{E_V^{\mathrm{in},1}} \times D_{E_V^{\mathrm{in},2}} \times D_{E_V^{\mathrm{out}}}.$$

We define

$$N_{g,V}^{1,2} := \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g \left(\mathrm{ev}_V^{E_V^{\mathrm{in},1}}\right)^* \left(\mathrm{pt}_{E_V^{\mathrm{in},1}}\right) \left(\mathrm{ev}_V^{E_V^{\mathrm{in},2}}\right)^* \left(\mathrm{pt}_{E_V^{\mathrm{in},2}}\right),$$

where $\mathrm{pt}_{E_V^{\mathrm{in},1}} \in A^1(D_{E_V^{\mathrm{in},1}})$ and $\mathrm{pt}_{E_V^{\mathrm{in},2}} \in A^1(D_{E_V^{\mathrm{in},2}})$ are classes of a point on $D_{E_V^{\mathrm{in},1}}$ and $D_{E_V^{\mathrm{in},2}}$ respectively.

6.3 Contribution of bivalent pointed vertices

Let V be a bivalent pointed vertex of $\tilde{\Gamma}$. Let $\overline{M}_{g,\Delta_V}$ be the moduli space of 1-pointed[15] stable log maps to X_{Δ_V} of genus g and of type Δ_V. It has virtual dimension

$$\mathrm{vdim}\ \overline{M}_{g,\Delta_V} = g + 2,$$

and admits a virtual fundamental class

$$[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}} \in A_{g+2}(\overline{M}_{g,\Delta_V}, \mathbb{Q}).$$

We have the evaluation morphism at the extra marked point,

$$\mathrm{ev}\colon \overline{M}_{g,\Delta_V} \to X_{\Delta_V},$$

and we define

$$N_{g,V}^{1,2} := \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}^*(\mathrm{pt}),$$

where $\mathrm{pt} \in A^2(X_{\Delta_V})$ is the class of a point on X_{Δ_V}.

15 As in Sect. 2.2, 1-pointed means that the source curves are equipped with one marked point in addition to the marked points keeping track of the tangency conditions.

6.4 Contribution of bivalent unpointed vertices

Let V be a bivalent unpointed vertex of $\tilde{\Gamma}$. Let $\overline{M}_{g,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g and of type Δ_V. It has virtual dimension

$$\text{vdim } \overline{M}_{g,\Delta_V} = g + 1,$$

and admits a virtual fundamental class

$$[\overline{M}_{g,\Delta_V}]^{\text{virt}} \in A_{g+1}(\overline{M}_{g,\Delta_V}, \mathbb{Q}).$$

Let E_V^{in} be the ingoing edge adjacent to V and E_V^{out} the outgoing edge adjacent to V. Let $D_{E_V^{\text{in}}}$ and $D_{E_V^{\text{out}}}$ be the corresponding toric divisors of X_{Δ_V}. We have evaluation morphisms

$$\left(\text{ev}_V^{E_V^{\text{in}}}, \text{ev}_V^{E_V^{\text{out}}}\right) : \overline{M}_{g,\Delta_V} \to D_{E_V^{\text{in}}} \times D_{E_V^{\text{out}}}.$$

We define

$$N_{g,V}^{1,2} := \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \left(\text{ev}_V^{E_V^{\text{in}}}\right)^* \left(\text{pt}_{E_V^{\text{in}}}\right),$$

where $\text{pt}_{E_V^{\text{in}}} \in A^1(D_{E_V^{\text{in},1}})$ is the class of a point on $D_{E_V^{\text{in}}}$.

6.5 Statement of the gluing formula

The following gluing formula expresses the log Gromov–Witten invariant $N_{g,\tilde{h}}^{\Delta,n}$ attached to a parametrized tropical curve $\tilde{h} \colon \tilde{\Gamma} \to \mathbb{R}^2$ in terms of the log Gromov–Witten invariants $N_{g,V}^{1,2}$ attached to the vertices V of $\tilde{\Gamma}$ and of the weights $w(E)$ of the edges of $\tilde{\Gamma}$.

Proposition 13 *For every $\tilde{h} \colon \tilde{\Gamma} \to \mathbb{R}^2$ parametrized tropical curve in $T_{\Delta,p}^g$, we have*

$$N_{g,\tilde{h}}^{\Delta,n} = \left(\prod_{V \in V(\tilde{\Gamma})} N_{g(V),V}^{1,2}\right)\left(\prod_{E \in E_f(\tilde{\Gamma})} w(E)\right),$$

where the first product is over the vertices of $\tilde{\Gamma}$ and the second product is over the bounded edges of $\tilde{\Gamma}$.

The proof of Proposition 13 is given in Sect. 7.

In the following Lemmas, we compute the contributions $N_{g(V),V}^{1,2}$ of the bivalent vertices.

Lemma 14 *Let V be a bivalent pointed vertex of $\tilde{\Gamma}$. Then we have*

$$N_{g,V}^{1,2} = 0$$

for every $g > 0$, and

$$N_{0,V}^{1,2} = 1$$

for $g = 0$.

Proof Let w be the weight of the two edges of $\tilde{\Gamma}$ adjacent to V. We can take $X_{\Delta_V} = \mathbb{P}^1 \times \mathbb{P}^1$ and $\beta_{\Delta_V} = w([\mathbb{P}^1] \times [\mathrm{pt}])$. We have the evaluation map at the extra marked point

$$\mathrm{ev}\colon \overline{M}_{g,\Delta_V} \to \mathbb{P}^1 \times \mathbb{P}^1.$$

We fix a point $p = (p_1, p_2) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1 \times \mathbb{P}^1$ and we denote $\iota_p\colon p \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\iota_{p_1}\colon p \hookrightarrow \mathbb{P}^1 \times \{p_2\} \simeq \mathbb{P}^1$ the inclusion morphisms.

Let $\overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w)$ be the moduli space of genus g 1-pointed stable maps to $\mathbb{P}^1$, of degree w, relative to the divisor $\{0\} \cup \{\infty\}$, with intersection multiplicities w both along $\{0\}$ and $\{\infty\}$. We have an evaluation morphism at the extra marked point

$$\mathrm{ev}_1\colon \overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w) \to \mathbb{P}^1,$$

Because an element $(f\colon C \to \mathbb{P}^1 \times \mathbb{P}^1)$ of $\mathrm{ev}^{-1}(p)$ factors through $\mathbb{P}^1 \times \{p_2\} \simeq \mathbb{P}^1$, we have a natural identification of moduli spaces $\mathrm{ev}^{-1}(p) = \mathrm{ev}_1^{-1}(p)$, but the natural virtual fundamental classes are different. The class $\iota_p^![\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}$, defined by the refined Gysin homomorphism (see Section 6.2 of [16]), has degree g whereas the class $\iota_{p_1}^![\overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w)]^{\mathrm{virt}}$ is of degree

$$2g - 2 + 2w - (w-1) - (w-1) + (1-1) = 2g.$$

The two obstruction theories differ by the bundle whose fiber at

$$f\colon C \to \mathbb{P}^1$$

is $H^1(C, f^* N_{f(C)|\mathbb{P}^1 \times \mathbb{P}^1})$. Because $\beta_{\Delta_V}^2 = 0$, the normal bundle $N_{f(C)|\mathbb{P}^1 \times \mathbb{P}^1}$ is trivial of rank one, so the pullback $f^* N_{f(C)|\mathbb{P}^1 \times \mathbb{P}^1}$ is trivial of rank one and the two obstruction theories differ by the dual of the Hodge bundle. Therefore, we have

$$\iota_p^! \left[\overline{M}_{g,\Delta_V}\right]^{\text{virt}} = c_g(\mathbb{E}^*) \cap \iota_{p_1}^! \left[\overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w)\right]^{\text{virt}},$$

and so

$$N_{g,V}^1 = \int_{\iota_p^! [\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g = \int_{\iota_{p_1}^! [\overline{M}_{g,1}(\mathbb{P}^1/\{0\}\cup\{\infty\}, w; w, w)]^{\text{virt}}} \lambda_g^2.$$

But $\lambda_g^2 = 0$ for $g > 0$, as follows from Mumford's relation [36]

$$c(\mathbb{E})c(\mathbb{E}^*) = 1,$$

and so $N_{g,V}^1 = 0$ if $g > 0$.

If $g = 0$, we have $\lambda_0^2 = 1$, the moduli space is a point, given by the degree w map $\mathbb{P}^1 \to \mathbb{P}^1$ fully ramified over 0 and ∞, with trivial automorphism group (there is no non-trivial automorphism of $\mathbb{P}^1$ fixing 0, ∞ and the extra marked point), and so

$$N_{0,V}^{1,2} = 1.$$

□

Lemma 15 *Let V be a bivalent unpointed vertex of $\tilde{\Gamma}$ and $w(E_V)$ the common weight of the two edges adjacent to V. Then we have*

$$N_{g,V}^{1,2} = 0$$

for every $g > 0$, and

$$N_{0,V}^{1,2} = \frac{1}{w(E_V)}$$

for $g = 0$.

Proof The argument is parallel to the one used to prove Lemma 14. The only difference is that the vertex is no longer pointed and the invariant $N_{g,V}^{1,2}$ is defined using the evaluation map at one of the tangency point. The vanishing for $g > 0$ still follows from $\lambda_g^2 = 0$. For $g = 0$, the moduli space is a point, given by the degree $w(E_V)$ map $\mathbb{P}^1 \to \mathbb{P}^1$ fully ramified over 0 and ∞, but now with an automorphism group $\mathbb{Z}/w(E_V)$ (the extra marked point in Lemma 14 is no longer there to kill all non-trivial automorphisms). It follows that $N_{0,V}^{1,2} = \frac{1}{w(E_V)}$. □

Corollary 16 *Let $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ be a parametrized tropical curve in $T^g_{\Delta,p}$.*

- *If there exists one bivalent vertex V of $\tilde{\Gamma}$ with $g(V) \neq 0$, then*

$$N^{\Delta,n}_{g,\tilde{h}} = 0.$$

- *If $g(V) = 0$ for all the bivalent vertices V of $\tilde{\Gamma}$, then*

$$N^{\Delta,n}_{g,\tilde{h}} = \left(\prod_{V \in V^{(3)}(\tilde{\Gamma})} N^{1,2}_{g(V),V} \right) \left(\prod_{E \in E_f(\Gamma)} w(E) \right),$$

where the first product is over the trivalent vertices of Γ (or $\tilde{\Gamma}$), and the second product is over the bounded edges of Γ (not $\tilde{\Gamma}$).

Proof If $\tilde{\Gamma}$ has a bivalent vertex V with $g(V) > 0$, then, according to Lemmas 14 and 15, we have $N^{1,2}_{g(V),V} = 0$ and so $N^{\Delta,n}_{g,\tilde{h}} = 0$ by Proposition 13.

If $g(V) = 0$ for all the bivalent vertices V of $\tilde{\Gamma}$, then, according to Lemma 14, we have $N^{1,2}_{g(V),V} = 1$ for all the bivalent pointed vertices V of $\tilde{\Gamma}$ and according to Lemma 15, we have $N^{1,2}_{g(V),V} = \frac{1}{w(E_V)}$ for all the bivalent unpointed vertices V of $\tilde{\Gamma}$. It follows that Proposition 13 can be rewritten

$$N^{\Delta,n}_{g,\tilde{h}} = \left(\prod_{V \in V^{(3)}(\tilde{\Gamma})} N^{1,2}_{g(V),V} \right) \left(\prod_{V \in V^{(2up)}(\tilde{\Gamma})} \frac{1}{w(E_V)} \right) \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right),$$

where the first product is over the trivalent vertices of $\tilde{\Gamma}$ (which can be naturally identified with the trivalent vertices of Γ) and the second product is over the bivalent unpointed vertices of $\tilde{\Gamma}$. Recalling from Sect. 4.2 that the edges of $\tilde{\Gamma}$ are obtained as subdivision of the edges of Γ by adding the bivalent unpointed vertices, we have

$$\left(\prod_{V \in V^{(2up)}(\tilde{\Gamma})} \frac{1}{w(E_V)} \right) \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right) = \prod_{E \in E_f(\Gamma)} w(E).$$

□

7 Proof of the gluing formula

This Section is devoted to the proof of Proposition 13. Part of it is inspired the proof by Chen [13] of the degeneration formula for expanded stable log maps, and the proof by Kim, Lho and Ruddat [27] of the degeneration formula for stable log maps in degenerations along a smooth divisor. In Sect. 7.1, we define a cut morphism. Restricted to some open substack of torically transverse stable maps, we show in Sect. 7.2 that the cut morphism is étale, and in Sect. 7.3, that the cut morphism is compatible with the natural obstruction theories of the pieces. Using in addition Proposition 11 and the results of Sect. 3, we prove a gluing formula in Sect. 7.4. To finish the proof of Proposition 13, we explain in Sect. 7.5 how to organize the glued pieces.

7.1 Cutting

Let $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ be a parametrized tropical curve in $T^g_{\Delta,p}$. We denote $V^{(2p)}(\tilde{\Gamma})$ the set of bivalent pointed vertices of $\tilde{\Gamma}$ and $V^{(2up)}(\tilde{\Gamma})$ the set of bivalent unpointed vertices of $\tilde{\Gamma}$.

Evaluations $\mathrm{ev}_V^E\colon \overline{M}_{g(V),\Delta_V} \to D_E$ at the tangency points dual to the bounded edges of $\tilde{\Gamma}$ give a morphism

$$\mathrm{ev}^{(e)}\colon \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \to \prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2,$$

where D_E is the divisor of X_0 dual to an edge E of $\tilde{\Gamma}$.

Evaluations $\mathrm{ev}_V^{(p)}\colon \overline{M}_{g(V),\Delta_V} \to X_{\Delta_V}$ at the extra marked points corresponding to the bivalent pointed vertices give a morphism

$$\mathrm{ev}^{(p)}\colon \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \to \prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}.$$

Let

$$\delta\colon \prod_{E\in E_f(\tilde{\Gamma})} D_E \to \prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2$$

be the diagonal morphism. Let

$$\iota_{P^0}\colon \left(P^0 = (P_V^0)_{V\in V^{(2p)}(\tilde{\Gamma})}\right) \hookrightarrow \prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V},$$

be the inclusion morphism of P^0.

Using the fiber product diagram in the category of stacks

$$\begin{array}{ccc} \bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} & \xrightarrow{(\delta \times \iota_{P^0})_M} & \prod_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \\ \downarrow & & \downarrow \mathrm{ev}^{(e)} \times \mathrm{ev}^{(p)} \\ \left(\prod_{E \in E_f(\tilde{\Gamma})} D_E \right) \times P^0 & \xrightarrow{\delta \times \iota_{P^0}} & \prod_{E \in E_f(\tilde{\Gamma})} (D_E)^2 \times \prod_{V \in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}, \end{array}$$

we define the substack $\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}$ of $\prod_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}$ consisting of curves whose marked points keeping track of the tangency conditions match over the divisors D_E and whose extra marked points associated to the bivalent pointed vertices map to P^0.

Lemma 17 *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow \pi & & \downarrow \nu_0 \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}}, \end{array}$$

be a n-pointed genus g stable log map of type Δ passing through P^0 and marked by $\tilde{h} \colon \tilde{\Gamma} \to \mathbb{R}^2$, i.e. a point of $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$. Let

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \downarrow \Sigma(\pi) & & \downarrow \Sigma(\nu_0) \\ \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) & \xrightarrow{\Sigma(g)} & \Sigma(\mathrm{pt}_{\mathbb{N}}). \end{array}$$

be its tropicalization. For every $b \in \Sigma(g)^{-1}(1)$, let

$$\Sigma(f)_b \colon \Sigma(C)_b \to \Sigma(\nu_0)^{-1}(1) \simeq \mathbb{R}^2$$

be the fiber of $\Sigma(f)$ over b. Let E be an edge of Γ and let $E_{f,b}$ be the edge of $\Sigma(C)_b$ marked by E. Then $\Sigma(f)_b(E_{f,b}) \subset \tilde{h}(E)$.

Proof We recalled in Sect. 6 that the connected components of the complement of the bivalent pointed vertices of $\tilde{\Gamma}$ are trees with exactly one unbounded edge. We prove Lemma 17 by induction, starting with the edges connected to the bivalent pointed vertices and then we go through each tree following the orientation introduced in Sect. 6.

Let E be an edge of $\tilde{\Gamma}$ adjacent to a bivalent pointed vertex V of $\tilde{\Gamma}$. Let $P_V^0 \in X_{\Delta_V}$ be the corresponding marked point. As f is marked by $\tilde{h}$, we have an ordinary stable map $f_V : C_V \to X_{\Delta_V}$, a marked point x_E in C_V such that $f(x_E) \in D_E$ and $f_V(C_V)$ contains P_V^0. We can assume that $X_{\Delta_V} = \mathbb{P}^1 \times \mathbb{P}^1$, $D_E = \{0\} \times \mathbb{P}^1$, $\beta_{\Delta_V} = w(E)([\mathbb{P}^1] \times [\mathrm{pt}])$, and $P_V^0 = (P_{V,1}^0, P_{V,2}^0) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1 \times \mathbb{P}^1$. Then f_V factors through $\mathbb{P}^1 \times \{P_{V,2}^0\}$ and $x_E = (0, P_{V,2}^0)$. It follows that $\Sigma(f)_b(E_{f,b}) \subset \tilde{h}(E)$.

Let E be the outgoing edge of a trivalent vertex of $\tilde{\Gamma}$, of ingoing edges E^1 and E^2. By the induction hypothesis, we know that $\Sigma(f)_b(E_{f,b}^1) \subset \tilde{h}(E^1)$ and $\Sigma(f)_b(E_{f,b}^2) \subset \tilde{h}(E^2)$. We conclude that $\Sigma(f)_b(E_{f,b}) \subset \tilde{h}(E)$ by an application of the balancing condition, as in Proposition 30 (tropical Menelaus theorem) of Mikhalkin [35]. □

For a stable log map

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu_0} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} \end{array}$$

marked by $\tilde{h}$, we have nodes of C in correspondence with the bounded edges of $\tilde{\Gamma}$. Cutting C along these nodes, we obtain a morphism

$$\mathrm{cut} \colon \overline{M}_{g,n,\Delta}^{\tilde{h},P^0} \to \bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}.$$

Let us give a precise definition of the cut morphism.[16] By definition of the marking, for every vertex V of $\tilde{\Gamma}$, we have an ordinary stable map $f_V : C_V \to X_{\Delta_V}$, such that the underlying stable map to f is obtained by gluing together the maps f_V along nodes corresponding to the edges of $\tilde{\Gamma}$.

We have to give C_V the structure of a log curve, and enhance f_V to a log morphism. In particular, we need to construct a monoid $\overline{\mathcal{M}}_V$.

We fix a point b in the interior of $\Sigma(g)^{-1}(1)$. Let $\Sigma(f)_b : \Sigma(C)_b \to \mathbb{R}^2$ be the corresponding parametrized tropical curve. Let $\Sigma(C)_{V,b}$ be the subgraph of $\Sigma(C)_b$ obtained by taking the vertices of $\Sigma(C)_b$ dual to irreducible components of C_V, the edges between them, and considering the edges to other vertices of $\Sigma(C)_b$ as unbounded edges. Let $\Sigma(f)_{V,b}$ be the restriction of $\Sigma(f)_b$ to $\Sigma(C)_{V,b}$. It follows from Lemma 17 that one can view $\Sigma(f)_{V,b}$ as a parametrized tropical curve of type Δ_V to the fan of X_{Δ_V}.

We define $\overline{\mathcal{M}}_V$ as being the monoid whose dual is the monoid of integral points of the moduli space of deformations of $\Sigma(f)_{V,b}$ preserving its combi-

[16] We are considering a stable log map over a point. It is a notational exercise to extend the argument to a stable log map over a general base, which is required to really define a morphism between moduli spaces.

natorial type.[17] Let $i_{C_V} : C_V \to C$ and $i_{X_{\Delta_V}} : X_{\Delta_V} \to X_0$ be the inclusion morphisms of ordinary (not log) schemes. The parametrized tropical curves $\Sigma(f)_V$ encode a sheaf of monoids $\overline{\mathcal{M}}_{C_V}$ and a map $f_V^{-1}\overline{\mathcal{M}}_{X_{\Delta_V}} \to \overline{\mathcal{M}}_{C_V}$. We define a log structure on C_V by

$$\mathcal{M}_{C_V} = \overline{\mathcal{M}}_{C_V} \times_{i_{C_V}^{-1}\overline{\mathcal{M}}_C} i_{C_V}^{-1}\mathcal{M}_C.$$

The natural diagram

$$\begin{array}{ccc} f_V^{-1}\mathcal{M}_{X_{\Delta_V}} & & \mathcal{M}_{C_V} \\ \downarrow & & \downarrow \\ f_V^{-1} i_{X_{\Delta_V}}^{-1}\mathcal{M}_{X_0} & \longrightarrow & i_{C_V}^{-1}\mathcal{M}_C \end{array}$$

can be uniquely completed, by restriction, with a map

$$f_V^{-1}\mathcal{M}_{X_{\Delta_V}} \to \mathcal{M}_{C_V}$$

compatible with $f_V^{-1}\overline{\mathcal{M}}_{X_{\Delta_V}} \to \overline{\mathcal{M}}_{C_V}$. This defines a log enhancement of f_V and finishes the construction of the cut morphism.

Remark If one considers a general log smooth degeneration and if one applies the decomposition formula, it is in general impossible to write the contribution of a tropical curves in terms of log Gromov–Witten invariants attached to the vertices. This is already clear at the tropical level. The theory of punctured invariants developed by Abramovich, Chen, Gross, Siebert in [4] is the correct extension of log Gromov–Witten theory which is needed in order to be able to write down a general gluing formula. In our present case, the Nishinou–Siebert toric degeneration is extremely special because it has been constructed knowing a priori the relevant tropical curves. It follows from Lemma 17 that we always cut edges contained in an edge of the polyhedral decomposition, and so we don't have to consider punctured invariants.

7.2 Counting log structures

We say that a map to X_0 is torically transverse if its image does not contain any of the torus fixed points of the toric components X_{Δ_V}. In other words, its corestriction to each toric surface X_{Δ_V} is torically transverse in the sense of Sect. 5.

[17] The base monoid of a basic stable log map has always such description in terms of deformations of tropical curves. See Remark 1.18 and Remark 1.21 of [24] for more details.

Let $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}$ be the open locus of $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$ formed by the torically transverse stable log maps to X_0, and for every vertex V of $\tilde{\Gamma}$, let $\overline{M}_{g(V),\Delta_V}^{\circ}$ be the open locus of $\overline{M}_{g(V),\Delta_V}$ formed by the torically transverse stable log maps to X_{Δ_V}. The morphism cut restricts to a morphism

$$\mathrm{cut}^\circ \colon \overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ} \to \bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}.$$

Proposition 18 *The morphism*

$$\mathrm{cut}^\circ \colon \overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ} \to \bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}$$

is étale of degree

$$\prod_{E \in E_f(\tilde{\Gamma})} w(E),$$

where the product is over the bounded edges of $\tilde{\Gamma}$.

Proof Let $(f_V \colon C_V \to X_{\Delta_V})_V \in \times_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}$. We have to glue the stable log maps f_V together. Because we are assuming that the maps f_V are torically transverse, the image in X_0 by f_V of the curves C_V is away from the torus fixed points of the components X_{Δ_V}. The gluing operation corresponding to the bounded edge E of $\tilde{\Gamma}$ happens entirely along the torus $\mathbb{C}^*$ contained in the divisor D_E.

It follows that it is enough to study the following local model. Denote $\ell := \ell(E)w(E)$, where $\ell(E)$ is the length of E and $w(E)$ the weight of E. Let X_E be the toric variety Spec $\mathbb{C}[x, y, u^\pm, t]/(xy = t^\ell)$, equipped with a morphism $\nu_E \colon X_E \to \mathbb{C}$ given by the coordinate t. Using the natural toric divisorial log structures on X_E and $\mathbb{C}$, we define by restriction a log structure on the special fiber $X_{0,E} := \nu_E^{-1}(0)$ and a log smooth morphism to the standard log point $\nu_{0,E} \colon X_{0,E} \to \mathrm{pt}_{\mathbb{N}}$. The scheme underlying $X_{0,E}$ has two irreducible components, $X_{1,E} := \mathbb{C}_x \times \mathbb{C}_u^*$ and $X_{2,E} := \mathbb{C}_y \times \mathbb{C}_u^*$, glued along the smooth divisor $D_E^\circ := \mathbb{C}_u^*$. We endow $X_{1,E}$ and $X_{2,E}$ with their toric divisorial log structures.

Let $f_1 \colon C_1/\mathrm{pt}_{\overline{\mathcal{M}}_1} \to X_{1,E}$ be the restriction to $X_{1,E}$ of a torically transverse stable log map to some toric compactification of $X_{1,E}$, with one point p_1 of tangency order $w(E)$ along D_E, and let $f_2 \colon C_2/\mathrm{pt}_{\overline{\mathcal{M}}_2} \to X_{2,E}$ be the restriction to $X_{2,E}$ of a torically transverse stable log map to some toric compactification of $X_{2,E}$, with one point p_2 of tangency order $w(E)$ along D_E.

We assume that $f(p_1) = f(p_2)$ and so we can glue the underlying maps $\underline{f}_1 : \underline{C}_1 \to \underline{X}_{1,E}$ and $\underline{f}_2 : \underline{C}_2 \to \underline{X}_{2,E}$ to obtain a map $\underline{f} : \underline{C} \to \underline{X}_{0,E}$ where $\underline{C}$ is the curve obtained from $\underline{C}_1$ and $\underline{C}_2$ by identification of p_1 and p_2. We denote p the corresponding node of $\underline{C}$. We have to show that there are $w(E)$ ways to lift this map to a log map in a way compatible with the log maps f_1 and f_2 and with the basic condition. If C_1 and C_2 had no component contracted to $f(p) \in D_E^\circ$, this would follow from Proposition 7.1 of Nishinou, Siebert [38]. But we allow contracted components, so we have to present a variant of the proof of Proposition 7.1 of [38].

We first give a tropical description of the relevant objects. The tropicalization of $X_{0,E}$ is the cone $\Sigma(X_{0,E}) = \mathrm{Hom}(\overline{\mathcal{M}}_{X_{0,E},f(p)}, \mathbb{R}_{\geqslant 0})$. It is the fan of X_E, a two-dimensional cone generated by rays ρ_1 and ρ_2 dual to the divisors $X_{1,E}$ and $X_{2,E}$. The toric description $X_E = \mathrm{Spec}\ \mathbb{C}[x, y, u^{\pm}, t]/(xy = t^\ell)$ defines a natural chart for the log structure of $X_{0,E}$. Denote s_x, s_y, s_t the corresponding elements of $\mathcal{M}_{X_{0,E},f(p)}$ and $\overline{s}_x, \overline{s}_y, \overline{s}_t$ their projections in $\overline{\mathcal{M}}_{X_{0,E},f(p)}$. We have $s_x s_y = s_t^\ell$. Seeing elements of $\overline{\mathcal{M}}_{X_{0,E},f(p)}$ as functions on $\Sigma(X_{0,E})$, we have $\rho_1 = \overline{s}_y^{-1}(0)$, $\rho_2 = \overline{s}_x^{-1}(0)$ and $\overline{s}_t : \Sigma(X_{0,E}) \to \mathbb{R}_{\geqslant 0}$ is the tropicalization of the projection $X_{0,E} \to \mathrm{pt}_{\mathbb{N}}$. Level sets $\overline{s}_t^{-1}(c)$ are line segments $[P_1, P_2]$ in $\Sigma(X_{0,E})$, connecting a point P_1 of ρ_1 to a point P_2 of ρ_2, of length ℓc.

Denote $\underline{C}_{1,E}$ and $\underline{C}_{2,E}$ the irreducible components of $\underline{C}_1$ and $\underline{C}_2$ containing p_1 and p_2 respectively. We can see them as the two irreducible components of $\underline{C}$ meeting at the node p. Fix $j = 1$ or $j = 2$. The tropicalization of $C_j/\mathrm{pt}_{\overline{\mathcal{M}}_j}$ is a family $\Sigma(C_j)$ of tropical curves $\Sigma(C_j)_b$ parametrized by $b \in \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}_j}) = \mathrm{Hom}(\overline{\mathcal{M}}_j, \mathbb{R}_{\geqslant 0})$. Let $V_{j,E}$ be the vertex of these tropical curves dual to the irreducible component $\underline{C}_{j,E}$. The image $\Sigma(f_j)(V_{j,E})$ of $V_{j,E}$ by the tropicalization $\Sigma(f_j)$ of f_j is a point in the tropicalization $\Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$. This induces a map $\mathrm{Hom}(\overline{\mathcal{M}}_j, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ defined by an element $v_j \in \overline{\mathcal{M}}_j$. The component $\underline{C}_{j,E}$ is contracted by f_j onto $f_j(p_j)$ if and only if $v_j \neq 0$. In other words, v_j is the measure according to the log structures of "how" $\underline{C}_{j,E}$ is contracted by f_j. The marked point p_j on $C_{j,E}$ defines an unbounded edge E_j, of weight $w(E)$, whose image by $\Sigma(f_j)$ is the unbounded interval $[\Sigma(f_j)(V_{j,E}), +\infty) \subset \Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$.

We explain now the gluing at the tropical level. Let $j = 1$ or $j = 2$. Let $[0, \ell_j] \subset \Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$ be an interval. If c is a large enough positive real number, we denote $\varphi_c^j : [0, \ell_j] \hookrightarrow \overline{s}_t^{-1}(c) = [P_1, P_2]$ the linear inclusion such that $\varphi_c^j(0) = P_j$ and $\varphi_c^j([0, \ell_j])$ is a subinterval of $[P_1, P_2]$ of length ℓ_j. Let $b_j \in \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}_j})$. There exists ℓ_j large enough such that all images by $\Sigma(f_j)$ of vertices of $\Sigma(f_j)_{b_j}$ are contained in $[0, \ell_j] \subset \Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$.

For c large enough, the line segments $\varphi_c^1([0,\ell_1])$ and $\varphi_c^2([0,\ell_2])$ are disjoint. We have

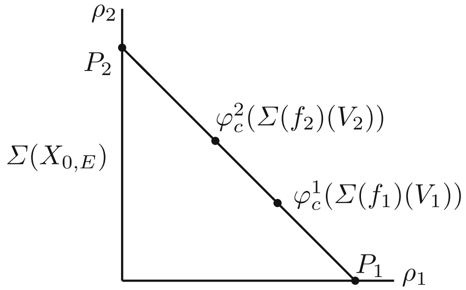
$$\begin{aligned}[P_1,P_2] = &\left[P_1, \varphi_c^1(\Sigma(f_1)(V_1))\right) \\ &\cup \left[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))\right] \\ &\cup \left(\varphi_c^2(\Sigma(f_2)(V_2)), P_2\right].\end{aligned}$$

We construct a new tropical curve $\Sigma_{b_1,b_2,c}$ by removing the unbounded edges E_1 and E_2 of $\Sigma(f_1)_{b_1}$ and $\Sigma(f_2)_{b_2}$, and gluing the remaining curves by an edge F connecting $V_{1,E}$ and $V_{2,E}$, of weight $w(E)$, and length $\frac{1}{w(E)}$ times the length of the line segment $[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))]$. We construct a tropical map $\Sigma_{b_1,b_2,c} \to \Sigma(X_{0,E})$ using $\Sigma(f_1)_{b_1}$, $\Sigma(f_2)_{b_2}$ and mapping the edge F to $[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))]$. We define $\overline{\mathcal{M}}$ as being the monoid whose dual is the monoid of integral points of the moduli space of deformations of these tropical maps.

We have $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$. The element $(0,0,1) \in \overline{\mathcal{M}}$ defines the function on the moduli space of tropical curves $\Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) = \mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$ given by the length of the gluing edge F. The function given by $\frac{1}{\ell}$ times the length of the line segment $[P_1,P_2]$ defines an element $\bar{s}_t^{\overline{\mathcal{M}}} \in \overline{\mathcal{M}}$. The morphism of monoids $\mathbb{N} \to \overline{\mathcal{M}}$, $1 \mapsto \bar{s}_t^{\overline{\mathcal{M}}}$, induces a map $g \colon \mathrm{pt}_{\overline{\mathcal{M}}} \to \mathrm{pt}_{\mathbb{N}}$. The decomposition of $[P_1,P_2]$ into the three intervals $[P_1, \varphi_c^1(\Sigma(f_1)(V_1)))$, $[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))]$ and $(\varphi_c^2(\Sigma(f_2)(V_2)), P_2]$, implies the relation

$$\ell\, \bar{s}_t^{\overline{\mathcal{M}}} = (v_1,0,0) + (0,0,w(E)) + (0,v_2,0)$$

in $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$.



From the tropical description of the gluing and from the fact that we want to obtain a basic log map, we find that there is a unique structure of log smooth curve $C/\mathrm{pt}_{\overline{\mathcal{M}}}$ compatible with the structures of log smooth curves on C_1 and C_2. As p is a node of C, we have for the ghost sheaf of C at p: $\overline{\mathcal{M}}_{C,p} =$

$\overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2$, with $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1,1)$, and $\mathbb{N} \to \overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$, $1 \mapsto \rho_p = (0,0,1)$.

It remains to lift $\underline{f} \colon \underline{C} \to \underline{X}_{0,E}$ to a log map $f \colon C \to X_{0,E}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X_{0,E} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu_{0,E}} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} \end{array}$$

commutes. The restriction of f to $C_j/\mathrm{pt}_{\overline{\mathcal{M}}_j}$ has to coincide with f_j, for $j=1$ and $j=2$. It follows from the explicit description of $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_C$ that such f exists and is unique away from the node p.

It follows from the tropical description of the gluing that at the ghost sheaves level, f at p is given by

$$\overline{f}^{\flat} \colon \overline{\mathcal{M}}_{X_{0,E},f(p)} \to \overline{\mathcal{M}}_{C,p} = \overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2 = (\overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}) \oplus_{\mathbb{N}} \mathbb{N}^2$$

$$\begin{aligned} \overline{s}_x &\mapsto ((v_1,0,0),(w(E),0)) \\ \overline{s}_y &\mapsto ((0,v_2,0),(0,w(E))) \\ \overline{s}_t &\mapsto \overline{\pi}^{\flat}(\overline{s}_t^{\overline{\mathcal{M}}}) = (\overline{s}_t^{\overline{\mathcal{M}}},(0,0)). \end{aligned}$$

The relation $\ell \, \overline{s}_t^{\overline{\mathcal{M}}} = (v_1, v_2, w(E))$ in $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$ implies that

$$\begin{aligned} \overline{f}^{\flat}(\overline{s}_x) + \overline{f}^{\flat}(\overline{s}_y) &= ((v_1,v_2,0),(w(E),w(E))) = ((v_1,v_2,w(E)),(0,0)) \\ &= \overline{f}^{\flat}(\ell \overline{s}_t^{\overline{\mathcal{M}}}), \end{aligned}$$

and so that this map is indeed well-defined.

The log maps $f_1 \colon C_1/\mathrm{pt}_{\overline{\mathcal{M}}_1} \to X_{1,E}$ and $f_2 \colon C_2/\mathrm{pt}_{\overline{\mathcal{M}}_2} \to X_{2,E}$ define morphisms

$$f_1^{\flat} \colon \mathcal{M}_{X_{1,E},f(p_1)} \to \mathcal{M}_{C_1,p_1},$$

and

$$f_2^{\flat} \colon \mathcal{M}_{X_{2,E},f(p_2)} \to \mathcal{M}_{C_2,p_2}.$$

For $j=1$ or $j=2$, let $\overline{\mathcal{M}}_j \oplus \mathbb{N} \to \mathcal{O}_{C_j,p_j}$ be a chart of the log structure of C_j at p_j. This realizes $\mathcal{M}_{C_j,p_j}$ as a quotient of $(\overline{\mathcal{M}}_j \oplus \mathbb{N}) \oplus \mathcal{O}_{C,p}^*$. Denote $s_{j,m} \in \mathcal{M}_{C_j,p_j}$ the image of $(m,1)$ for $m \in \overline{\mathcal{M}}_j \oplus \mathbb{N}$.

We fix a coordinate u on C_1 near p_1 such that

$$f_1^\flat(s_x) = s_{1,(v_1,0)} u^{w(E)}$$

and a coordinate v on C_2 near p_2 such that

$$f_2^\flat(s_y) = s_{2,(v_2,0)} v^{w(E)}.$$

We are trying to define some $f^\flat\colon \mathcal{M}_{X_{0,E},f(p)} \to \mathcal{M}_{C,p}$, lift of $\overline{f}^\flat$, compatible with $f_1^\flat$ and $f_2^\flat$. For every ζ a $w(E)$-th root of unity, the map

$$\begin{aligned} \overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2 &\to \mathcal{O}_{C,p} \\ (m,(a,b)) &\mapsto \begin{cases} \zeta^a u^a v^b & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases} \end{aligned}$$

defines a chart for the log structure of C at p. This realizes $\mathcal{M}_{C,p}$ as a quotient of $(\overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2) \oplus \mathcal{O}_{C,p}^*$. Denote $s_m^\zeta \in \mathcal{M}_{C,p}$ the image of $(m,1)$ for $m \in \overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2$. Remark that $s^\zeta_{((v_1,0,0),(0,0))}$, $s^\zeta_{((0,v_2,0),(0,0))}$ and $s^\zeta_{((0,0,0),(1,1))}$ are independent of ζ and we denote them simply as $s_{((v_1,0,0),(0,0))}$, $s_{((0,v_2,0),(0,0))}$ and $s_{((0,0,0),(1,1))}$.

Then

$$f^{\flat,\zeta}\colon \mathcal{M}_{X_{0,E},f(p)} \to \mathcal{M}_{C,p}$$

$$\begin{aligned} s_x &\mapsto s^\zeta_{((v_1,0,0),(w(E),0))} \\ s_y &\mapsto s^\zeta_{((0,v_2,0),(0,w(E)))} \\ s_t &\mapsto \pi^\flat((\overline{s}_t^{\overline{\mathcal{M}}},1)) \end{aligned}$$

is a lift of $\overline{f}^\flat$, compatible with $f_1^\flat$ and $f_2^\flat$.

Assume that $f^{\flat,\zeta} \simeq f^{\flat,\zeta'}$ for ζ and ζ' two $w(E)$-th roots of unity. It follows from the compatibility with $f_1^\flat$ and $f_2^\flat$ that there exists $\varphi_1 \in \mathcal{O}_{C,p}^*$ and $\varphi_2 \in \mathcal{O}_{C,p}^*$ such that $s^{\zeta'}_{((0,0,0),(1,0))} = \varphi_1 s^\zeta_{((0,0,0),(0,1))}$ and $s^{\zeta'}_{((0,0,0),(0,1))} = \varphi_2 s^\zeta_{((0,0,0),(0,1))}$. It follows from the definition of the charts that $\varphi_1 = \zeta' \zeta^{-1}$ in $\mathcal{O}_{C_1,p_1}$ and $\varphi_2 = 1$ in $\mathcal{O}_{C_2,p_2}$. Compatibility with $\mathrm{pt}_{\overline{\mathcal{M}}} \to \mathrm{pt}_{\mathbb{N}}$ implies that $\varphi_1\varphi_2 = 1$. This implies that $\varphi_1 = \varphi_2 = 1$ and $\zeta = \zeta'$.

It remains to show that any $f^\flat$, lift of $\overline{f}^\flat$ compatible with $f_1^\flat$ and $f_2^\flat$, is of the form $f^{\flat,\zeta}$ for some ζ a $w(E)$-th root of unity. For such $f^\flat$, there exists unique $s'_{(1,0)} \in \mathcal{M}_{C,p}$ and $s'_{(0,1)} \in \mathcal{M}_{C,p}$ such that $\alpha_C(s'_{(1,0)}) = u$,

$\alpha_C(s'_{(0,1)}) = v$, and $f^\flat(s_x) = s_{((v_1,0,0),(0,0))}(s'_{(1,0)})^{w(E)}$ and $f^\flat(s_y) = s_{((0,v_2,0),(0,0))}(s'_{(0,1)})^{w(E)}$. From $s_x s_y = s_t^\ell$, we get $(s'_{(1,0)}s'_{(0,1)})^{w(E)} = s^{w(E)}_{((0,0,0),(1,1))}$ and so $s'_{(1,0)}s'_{(0,1)} = \zeta^{-1}s_{((0,0,0),(1,1))}$ for some ζ a $w(E)$-th root of unity. It is now easy to check that $s'_{(1,0)} = \zeta^{-1}s^\zeta_{((0,0,0),(1,0))}$, $s'_{(0,1)} = s^\zeta_{((0,0,0),(0,1))}$ and $f^\flat = f^{\flat,\zeta}$. □

Remarks

- When $v_1 = v_2 = 0$, i.e. when the components $C_{1,E}$ and $C_{2,E}$ are not contracted, the above proof reduces to the proof of Proposition 7.1 of [38] (see also the proof of Proposition 4.23 of [21]). In general, log geometry remembers enough information about the contracted components, such as v_1 and v_2, to make possible a parallel argument.
- The gluing of stable log maps along a smooth divisor is discussed in Section 6 of [27], proving the degeneration formula along a smooth divisor. In the above proof, we only have to glue along one edge connecting two vertices. In Section 6 of [27], further work is required to deal with pair of vertices connected by several edges.

7.3 Comparing obstruction theories

As in the previous Sect. 7.2, let $\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}$ be the open locus of $\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}$ formed by the torically transverse stable log maps to X_0, and for every vertex V of $\tilde{\Gamma}$, let $\overline{M}^\circ_{g(V),\Delta_V}$ be the open locus of $\overline{M}_{g(V),\Delta_V}$ formed by the torically transverse stable log maps to X_{Δ_V}. The morphism cut restricts to a morphism

$$\mathrm{cut}^\circ \colon \overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} \to \bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}^\circ_{g(V),\Delta_V}.$$

The goal of the present Section is to use the morphism cut° to compare the virtual classes $[\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}]^{\mathrm{virt}}$ and $[\overline{M}^\circ_{g(V),\Delta_V}]^{\mathrm{virt}}$, which are obtained by restricting the virtual classes $[\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}]^{\mathrm{virt}}$ and $[\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}$ to the open loci of torically transverse stable log maps.

Recall that $X_0 = \nu^{-1}(0)$, where $\nu\colon X_{\mathcal{P}_{\Delta,n}} \to \mathbb{A}^1$. Following Section 4.1 of [3], we define $\mathcal{X}_0 := \mathcal{A}_X \times_{\mathcal{A}_{\mathbb{A}^1}} \{0\}$, where $\mathcal{A}_X$ and $\mathcal{A}_{\mathbb{A}^1}$ are Artin fans, see Section 2.2 of [3]. It is an algebraic log stack over $\mathrm{pt}_\mathbb{N}$. There is a natural morphism $X_0 \to \mathcal{X}_0$.

Following Section 4.5 of [3], let $\mathfrak{M}^{\tilde{h}}_{g,n,\Delta}$ be the stack of n-pointed genus g prestable basic log maps to $\mathcal{X}_0/\mathrm{pt}_\mathbb{N}$ marked by $\tilde{h}$ and of type Δ. There is a natural morphism of stacks $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0} \to \mathfrak{M}_{g,n,\Delta}^{\tilde{h}}$. Let $\pi : \mathcal{C} \to \overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$ be the universal curve and let $f : \mathcal{C} \to X_0/\mathrm{pt}_{\mathbb{N}}$ be the universal stable log map. According to Proposition 4.7.1 and Section 6.3.2 of [3], the virtual fundamental class $[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}$ is defined by $\mathbf{E}$, the cone of the morphism $(\mathrm{ev}^{(p)})^* L_{\iota_{P^0}}[-1] \to (R\pi_* f^* T_{X_0|\mathcal{X}_0})^\vee$, seen as a perfect obstruction theory relative to $\mathfrak{M}_{g,n,\Delta}^{\tilde{h}}$. Here, $T_{X_0|\mathcal{X}_0}$ is the relative log tangent bundle, and $L_{\iota_{P^0}} = \oplus_{V \in V^{(2p)}(\tilde{\Gamma})} (T_{X_{\Delta_V}}|_{P_V^0})^\vee[1]$ is the cotangent complex of ι_{P^0}. As $\mathcal{X}_0$ is log étale over $\mathrm{pt}_{\mathbb{N}}$, we have $T_{X_0|\mathcal{X}_0} = T_{X_0|\mathrm{pt}_{\mathbb{N}}}$. We denote $\mathbf{E}^\circ$ the restriction of $\mathbf{E}$ to the open locus $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}$ of torically transverse stable log maps.

For every vertex V of $\tilde{\Gamma}$, let $\pi_V : \mathcal{C}_V \to \overline{M}_{g(V),\Delta_V}$ be the universal curve and let $f_V : \mathcal{C}_V \to X_{\Delta_V}$ be the universal stable log map. Let $\mathcal{A}_{X_{\Delta_V}}$ be the Artin fan of X_{Δ_V} and let $\mathfrak{M}_{g(V),\Delta_V}$ be the stack of prestable basic log maps to $\mathcal{A}_{X_{\Delta_V}}$, of genus $g(V)$ and of type Δ_V. There is a natural morphism of stacks $\overline{M}_{g(V),\Delta_V} \to \mathfrak{M}_{g(V),\Delta_V}$. According to Section 6.1 of [6], the virtual fundamental class $[\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}$ is defined by $(R(\pi_V)_* f_V^* T_{X_{\Delta_V}})^\vee$, seen as a perfect obstruction theory relative to $\mathfrak{M}_{g(V),\Delta_V}$. Here, $T_{X_{\Delta_V}}$ is the log tangent bundle.

Recall that $\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}$ is defined by the fiber product diagram

$$\begin{array}{ccc}
\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} & \xrightarrow{(\delta \times \iota_{P^0})_M} & \prod_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \\
\Big\downarrow {\scriptstyle \mathrm{ev}^{(e)} \times \mathrm{ev}^{(p)}} & & \Big\downarrow {\scriptstyle \mathrm{ev}^{(e)} \times \mathrm{ev}^{(p)}} \\
\left(\prod_{E \in E_f(\tilde{\Gamma})} D_E \right) \times P^0 & \xrightarrow{\delta \times \iota_{P^0}} & \prod_{E \in E_f(\tilde{\Gamma})} (D_E)^2 \times \prod_{V \in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}.
\end{array}$$

We compare the deformation theory of the individual stable log maps f_V and the deformation theory of the stable log maps f_V constrained to match at the gluing nodes. Let $\mathbf{F}$ be the cone of the natural morphism

$$(\mathrm{ev}^{(e)} \times \mathrm{ev}^{(p)})^* L_{\delta \times \iota_{P^0}}[-1] \to (\delta \times \iota_{P^0})_M^* \left(\boxtimes_{V \in V(\tilde{\Gamma})} (R(\pi_V)_* f_V^* T_{X_{\Delta_V}})^\vee \right),$$

where $L_{\delta \times \iota_{P^0}}$ is the cotangent complex of the morphism $\delta \times \iota_{P^0}$. It defines a perfect obstruction theory on $\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}$ relative to $\prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}$, whose corresponding virtual fundamental class is, using Proposition 5.10 of [7],

$$(\delta \times \iota_{P^0})^! \prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}},$$

where $(\delta \times \iota_{P^0})^!$ is the refined Gysin homomorphism (see Section 6.2 of [16]). We denote $\mathbf{F}^\circ$ the restriction of $\mathbf{F}$ to the open locus $\times_{V \in V(\tilde{\Gamma})} \overline{M}^\circ_{g(V),\Delta_V}$ of torically transverse stable log maps.

The cut operation naturally extends to prestable log maps to $\mathcal{X}_0/\mathrm{pt}_{\mathbb{N}}$ marked by $\tilde{h}$, and so we have a commutative diagram

$$\begin{array}{ccc} \overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} & \xrightarrow{\mathrm{cut}^\circ} & \times_{V \in V(\tilde{\Gamma})} \overline{M}^\circ_{g(V),\Delta_V} \\ \downarrow{\scriptstyle \mu} & & \downarrow \\ \mathfrak{M}^{\tilde{h}}_{g,n,\Delta} & \xrightarrow{\mathrm{cut}_C} & \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}. \end{array}$$

By Proposition 18, the morphism cut° is étale and so $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ defines a perfect obstruction theory on $\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}$ relative to $\prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}$.

The maps $\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} \xrightarrow{\mu} \mathfrak{M}^{\tilde{h}}_{g,n,\Delta}(\mathcal{X}_0/\mathrm{pt}_{\mathbb{N}}) \xrightarrow{\mathrm{cut}_C} \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}$ define an exact triangle of cotangent complexes

$$L_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}} \to L_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} | \mathfrak{M}^{\tilde{h}}_{g,n,\Delta}} \to \mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}.$$

Adding the perfect obstruction theories $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ and $\mathbf{E}^\circ$, we get a diagram

$$\begin{array}{ccccc} (\mathrm{cut}^\circ)^*\mathbf{F}^\circ & & \mathbf{E}^\circ & & \\ \downarrow & & \downarrow & & \\ L_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}} & \to & L_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} | \mathfrak{M}^{\tilde{h}}_{g,n,\Delta}} & \to & \mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}. \end{array}$$

Proposition 19 *The above diagram can be completed into a morphism of exact triangles*

$$\begin{array}{ccccc} (\mathrm{cut}^\circ)^*\mathbf{F}^\circ & \longrightarrow & \mathbf{E}^\circ & \longrightarrow & \mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]} \\ \downarrow & & \downarrow & & \| \\ L_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}} & \to & L_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} | \mathfrak{M}^{\tilde{h}}_{g,n,\Delta}} & \to & \mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta} | \prod_{V \in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}. \end{array}$$

Proof Denote $X_0^\circ, X_{\Delta_V}^\circ, D_E^\circ$ the objects obtained from X_0, X_{Δ_V}, D_E by removing the torus fixed points of the toric surfaces X_{Δ_V}. Denote $\iota_{X_{\Delta_V}^\circ}$ the inclusion morphism of $X_{\Delta_V}^\circ$ in X_0°.

If E is a bounded edge of $\tilde{\Gamma}$, we denote V_E^1 and V_E^2 the two vertices of E. Let $\mathcal{F}$ be the sheaf on the universal curve $\mathcal{C}|_{\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}}$ defined as the kernel of

$$\bigoplus_{V \in V(\tilde{\Gamma})} f^*(\iota_{X_{\Delta_V}^\circ})_* T_{X_{\Delta_V}^\circ} \to \bigoplus_{E \in E_f(\tilde{\Gamma})} (\iota_E)_* (\mathrm{ev}^E)^* T_{D_E^\circ}$$
$$(s_V)_V \mapsto (s_{V_E^1}|_{D_E^\circ} - s_{V_E^2}|_{D_E^\circ})_E,$$

where ev^E is the evaluation at the node p_E dual to E, and ι_E the section of $\mathcal{C}$ given by p_E. It follows from the exact triangle obtained by applying $R\pi_*$ to the short exact sequence defining $\mathcal{F}$ and from $L_\delta = \oplus_{E \in E_f(\tilde{\Gamma})} T_{D_E}^\vee[1]$ that $(\mathrm{cut}^\circ)^* \mathbf{F}^\circ$ is given by the cone of the morphism $(\mathrm{ev}^{(p)})^* L_{\iota_{P^0}}[-1] \to (R\pi_* \mathcal{F})^\vee$. So in order to compare $\mathbf{E}^\circ$ and $(\mathrm{cut}^\circ)^* \mathbf{F}^\circ$, we have to compare $f^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}}$ and $\mathcal{F}$. The sheaf $f^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}}$ can be written as the kernel of

$$f^* \bigoplus_{V \in V(\tilde{\Gamma})} (\iota_{X_{\Delta_V}^\circ})_* (\iota_{X_{\Delta_V}^\circ})^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}} \to \bigoplus_{E \in E_f(\tilde{\Gamma})} (\iota_E)_* (\mathrm{ev}^E)^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}}$$
$$(s_V)_V \mapsto (s_{V_E^1}|_{D_E^\circ} - s_{V_E^2}|_{D_E^\circ})_E.$$

Remark that because X_0 is the special fiber of a toric degeneration, all the log tangent bundles $T_{X_0}, T_{X_{\Delta_V}}, T_{D_E}$ are free sheaves (see e.g. Section 7 of [38]). In particular, the restrictions $(\iota_{X_{\Delta_V}^\circ})^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}} \to T_{X_{\Delta_V}^\circ}$ are isomorphisms, the restriction

$$\bigoplus_{E \in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}} \to \bigoplus_{E \in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* T_{D_E^\circ}$$

has kernel $\bigoplus_{E \in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* \mathcal{O}_{D_E^\circ}$ and so there is an induced exact sequence

$$0 \to f^* T_{X_0^\circ|\mathrm{pt}_{\mathbb{N}}} \to \mathcal{F} \to \bigoplus_{E \in E_f(\tilde{\Gamma})} (\iota_E)_* (\mathrm{ev}^E)^* \mathcal{O}_{D_E^\circ} \to 0,$$

which induces an exact triangle on $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0\circ}$:

$$(\mathrm{cut}^\circ)^* \mathbf{F}^\circ \to \mathbf{E}^\circ \to \bigoplus_{E \in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* \mathcal{O}_{D_E^\circ}[1] \xrightarrow{[1]} .$$

It remains to check the compatibility of this exact triangle with the exact triangle of cotangent complexes. We have

$$\mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta}|\prod_{V\in V(\tilde{\Gamma})}\mathfrak{M}_{g(V),\Delta_V}} = \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)^*\mathcal{O}_{p_E}.$$

Indeed, restricted to the locus of torically transverse stable log maps, cut_C is smooth, and, given a torically transverse stable log map to $\mathcal{X}_0/\mathrm{pt}_{\mathbb{N}}$, a basis of first order infinitesimal deformations fixing its image by cut_C in $\prod_{V\in V(\tilde{\Gamma})}\mathfrak{M}_{g(V),\Delta_V}$ is indexed by the cutting nodes. The dual of the natural map

$$\bigoplus_{E\in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^*\mathcal{O}_{D_E^\circ} \to \mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta}|\prod_{V\in V(\tilde{\Gamma})}\mathfrak{M}_{g(V),\Delta_V}} = \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)^*\mathcal{O}_{p_E}$$

sends the canonical first order infinitesimal deformation indexed by the cutting node p_E to the canonical summand $\mathcal{O}_{D_E^\circ}$ in the normal bundle to the diagonal $\prod_{E\in E_f(\tilde{\Gamma})} D_E^\circ$ in $\prod_{E\in E_f(\tilde{\Gamma})}(D_E^\circ)^2$, and so is an isomorphism. This guarantees the compatibility with the exact triangle of cotangent complexes. □

Remark Restricted to the open locus of torically transverse stable maps, the discussion is essentially reduced to a collection of gluings along the smooth divisors D_E°. A comparison of the obstruction theories in the context of the degeneration formula along a smooth divisor is given with full details in Section 7 of [27].

Proposition 20 *We have*

$$(\mathrm{cut}^\circ)_*\left([\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}]^{\mathrm{virt}}\right) = \left(\prod_{E\in E_f(\tilde{\Gamma})} w(E)\right)\left((\delta\times\iota_{P^0})^!_M \prod_{V\in V(\tilde{\Gamma})}[\overline{M}^\circ_{g(V),\Delta_V}]^{\mathrm{virt}}\right).$$

Proof It follows from Proposition 19 and from Theorem 4.8 of [29] that the relative obstruction theories $\mathbf{E}^\circ$ and $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ define the same virtual fundamental class on $\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}$. By Proposition 18, cut° is étale, and so, by Proposition 7.2 of [7], the virtual fundamental class defined by $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ is the image by $(\mathrm{cut}^\circ)^*$ of the virtual fundamental class defined by $\mathbf{F}^\circ$. It follows that

$$[\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}]^{\mathrm{virt}} = (\mathrm{cut}^\circ)^*(\delta\times\iota_{P^0})^!_M \prod_{V\in V(\tilde{\Gamma})}[\overline{M}^\circ_{g(V),\Delta_V}]^{\mathrm{virt}}.$$

According to Proposition 18, the morphism cut° is étale of degree

$$\prod_{E \in E_f(\tilde{\Gamma})} w(E),$$

and so the result follows from the projection formula. □

7.4 Gluing

Recall that we have the morphism

$$(\delta \times \iota_{P^0})_M \colon \bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \to \prod_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}.$$

For every $V \in V(\tilde{\Gamma})$, we have a projection morphism

$$\mathrm{pr}_V \colon \prod_{V' \in V(\tilde{\Gamma})} \overline{M}_{g(V'),\Delta_{V'}} \to \overline{M}_{g(V),\Delta_V}.$$

On each moduli space $\overline{M}_{g(V),\Delta_V}$, we have the top lambda class $(-1)^{g(V)}\lambda_{g(V)}$.

Proposition 21 *We have*

$$N_{g,\tilde{h}}^{\Delta,n} = \int_{(\delta \times \iota_{P^0})^! \prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)} \right).$$

Proof By definition (see Sect. 4.3), we have

$$N_{g,\tilde{h}}^{\Delta,n} = \int_{[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}}.$$

Using the gluing properties of lambda classes given by Lemma 7, we obtain that

$$(-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} = (\mathrm{cut})^* (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)} \right).$$

It follows from the projection formula that

$$N_{g,\tilde{h}}^{\Delta,n} = \int_{(\mathrm{cut})_* [\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)} \right).$$

According to Proposition 20, the cycles

$$(\mathrm{cut})_* \left([\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}} \right)$$

and

$$\left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right) \left((\delta \times \iota_{P^0})^! \prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}} \right)$$

have the same restriction to the open substack

$$\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}$$

of

$$\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}.$$

It follows, by Proposition 1.8 of [16], that their difference is rationally equivalent to a cycle supported on the closed substack

$$Z := \left(\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \right) - \left(\bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ} \right).$$

If we have

$$(f_V \colon C_V \to X_{\Delta_V})_{V \in V(\tilde{\Gamma})} \in Z,$$

then at least one stable log map $f_V \colon C_V \to X_{\Delta_V}$ is not torically transverse. By Lemma 17, the unbounded edges of the tropicalization of f_V are contained in the rays of the fan of X_{Δ_V}. It follows that we can apply Proposition 11 to obtain that at least one of the source curves C_V contains a non-trivial cycle of components. By the vanishing result of Lemma 8, this implies that

$$\int_Z (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)} \right) = 0.$$

It follows that

$$\int_{(\mathrm{cut})_*[\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})^*_M \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right)$$
$$= \int_{(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})^*_M \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right).$$

This finishes the proof of Proposition 21. □

7.5 Identifying the pieces

Proposition 22 *We have*

$$\int_{(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})^*_M \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right)$$
$$= \prod_{V\in V(\tilde{\Gamma})} N^{1,2}_{g(V),V}.$$

Proof Using the definitions of δ and ι_{P^0}, we have

$$\int_{(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})^*_M \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right)$$
$$= \int_{\prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\mathrm{ev}^{(p)})^*([P^0])(\mathrm{ev}^{(e)})^*([\delta]) \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right),$$

where

$$[P^0] = \prod_{V\in V^{(2p)}(\tilde{\Gamma})} P^0_V \in A^*\left(\prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}\right)$$

is the class of P^0 and

$$[\delta] \in A^*\left(\prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2\right)$$

is the class of the diagonal $\prod_{E \in E_f(\tilde{\Gamma})} D_E$. As each D_E is a projective line, we have

$$[\delta] = \prod_{E \in E_f(\tilde{\Gamma})} (\mathrm{pt}_E \times 1 + 1 \times \mathrm{pt}_E),$$

where $\mathrm{pt}_E \in A^1(D_E)$ is the class of a point.

We fix an orientation of edges of $\tilde{\Gamma}$ as described in Sect. 6. In particular, every trivalent vertex has two ingoing and one outgoing adjacent edges, every bivalent pointed vertex has two outgoing adjacent edges, every bivalent unpointed vertex has one ingoing and one outgoing edges. For every bounded edge E of $\tilde{\Gamma}$, we denote V_E^s the source vertex of E and V_E^t the target vertex of E, as defined by the orientation. Furthermore, the connected components of the complement of the bivalent pointed vertices of $\tilde{\Gamma}$ are trees with exactly one unbounded edge.

We argue that the effect of the insertion $(\mathrm{ev}^{(p)})^*([P^0])(\mathrm{ev}^{(e)})^*([\delta])$ can be computed in terms of the combinatorics of ingoing and outgoing edges of $\tilde{\Gamma}$.[18] More precisely, we claim that the only term in

$$(\mathrm{ev}^{(e)})^*([\delta]) = \prod_{E \in E_f(\tilde{\Gamma})} \left((\mathrm{ev}^E_{V_E^s})^*(\mathrm{pt}_E) + (\mathrm{ev}^E_{V_E^t})^*(\mathrm{pt}_E) \right),$$

giving a non-zero contribution after multiplication by

$$\left(\prod_{V \in V^{(2p)}(\tilde{\Gamma})} (\mathrm{ev}_V^{(p)})^*(P_V^0) \right) \left(\prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)} \right) \right)$$

and integration over $\prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}$ is $\prod_{E \in E_f(\tilde{\Gamma})} (\mathrm{ev}^E_{V_E^t})^*(\mathrm{pt}_E)$.

We prove this claim by induction, starting at the bivalent pointed vertices, where things are constrained by the marked points P^0, and propagating these constraints following the flow on $\tilde{\Gamma}$ defined by the orientation of edges.

Let V be a bivalent pointed vertex, E an edge adjacent to V and V' the other vertex of E. The edge E is outgoing for V and ingoing for V', so $V' = V_E^t$. We have in $(\mathrm{ev}^{(p)})^*([P^0])(\mathrm{ev}^{(e)})^*([\delta])$ a corresponding factor

$$(\mathrm{ev}_V^{(p)})^*(P_V^0) \left((\mathrm{ev}_V^E)^*(\mathrm{pt}_E) + (\mathrm{ev}_{V'}^E)^*(\mathrm{pt}_E) \right).$$

[18] It is essentially a cohomological reformulation and generalization of the way the gluing is organized in Mikhalkin's proof of the tropical correspondence theorem, [34].

But $(\mathrm{ev}_V^{(p)})^*(P_V^0)(\mathrm{ev}_V^E)^*(\mathrm{pt}_E)(-1)^{g(V)}\lambda_{g(V)} = 0$ for dimension reasons (its insertion over $\overline{M}_{g(V),\Delta_V}$ defines an enumerative problem of virtual dimension -1) and so only the factor $(\mathrm{ev}_V^{(p)})^*(P_V^0)(\mathrm{ev}_{V'}^E)^*(\mathrm{pt}_E)$ survives, which proves the initial step of the induction.

Let E be an outgoing edge of a trivalent vertex V, of ingoing edges E^1 and E^2. Let V_E^t be the target vertex of E. By the induction hypothesis, every possibly non-vanishing term contains the insertion of $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_V^{E^2})^*(\mathrm{pt}_{E^2})$. But $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_V^{E^2})^*(\mathrm{pt}_{E^2})(\mathrm{ev}_V^E)^*(\mathrm{pt}_E)(-1)^{g(V)}\lambda_{g(V)} = 0$ for dimension reasons (its insertion over $\overline{M}_{g(V),\Delta_V}$ defines an enumerative problem of virtual dimension -1) and so only the factor $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_E^1)(\mathrm{ev}_V^{E^2})^*(\mathrm{pt}_E^2)$ $(\mathrm{ev}_{V_E^t}^E)^*(\mathrm{pt}_E)$ survives.

Let E be an outgoing edge of a bivalent unpointed vertex V, of ingoing edges E^1. Let V_E^t the target vertex of E. By the induction hypothesis, every possibly non-vanishing term contains the insertion of $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})$. But $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_V^E)^*(\mathrm{pt}_E)(-1)^{g(V)}\lambda_{g(V)} = 0$ for dimension reasons (its insertion over $\overline{M}_{g(V),\Delta_V}$ defines an enumerative problem of virtual dimension -1) and so only the factor $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_{V_E^t}^E)^*(\mathrm{pt}_E)$ survives. This finishes the proof by induction of the claim.

Using the notations introduced in Sect. 6, we can rewrite

$$\prod_{E\in E_f\left(\tilde{\Gamma}\right)} \left(\mathrm{ev}_{V_E^t}^E\right)^* (\mathrm{pt}_E)$$

as

$$\left(\prod_{V\in V^{(3)}(\tilde{\Gamma})} (\mathrm{ev}_V^{E_V^{\mathrm{in},1}})^*(\mathrm{pt}_{E_V^{\mathrm{in},1}})(\mathrm{ev}_V^{E_V^{\mathrm{in},2}})^*(\mathrm{pt}_{E_V^{\mathrm{in},2}})\right)\left(\prod_{V\in V^{(2up)}(\tilde{\Gamma})} (\mathrm{ev}_V^{E_V^{\mathrm{in}}})^*(\mathrm{pt}_{E_V^{\mathrm{in}}})\right)$$

and so we proved

$$\int_{(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})}[\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta\times\iota_{P^0})_M^* \prod_{V\in V(\tilde{\Gamma})} \mathrm{pr}_V^*\left((-1)^{g(V)}\lambda_{g(V)}\right)$$
$$= \left(\prod_{V\in V^{(3)}(\tilde{\Gamma})} N_{g(V),V}^{1,2}\right)\left(\prod_{V\in V^{(2p)}(\tilde{\Gamma})} N_{g(V),V}^{1,2}\right)\left(\prod_{V\in V^{(2up)}(\tilde{\Gamma})} N_{g(V),V}^{1,2}\right).$$

This finishes the proof of Proposition 22. □

7.6 End of the proof of the gluing formula

The gluing identity given by Proposition 13 follows from the combination of Proposition 21 and Proposition 22.

8 Vertex contribution

In this Section, we evaluate the invariants $N_{g,V}^{1,2}$ attached to the vertices V of Γ and appearing in the gluing formula of Corollary 16. The first step, carried out in Sect. 8.1 is to rewrite these invariants in terms of more symmetric invariants $N_{g,V}$ depending only on the multiplicity of the vertex V. In Sect. 8.2, we use the consistency of the gluing formula to deduce non-trivial relations between these invariants and to reduce the question to the computation of the invariants attached to vertices of multiplicity one and two. Invariants attached to vertices of multiplicity one and two are explicitly computed in Sect. 8.3 and this finishes the proof of Theorem 1. Modifications needed to prove Theorem 6 are discussed at the end of Sect. 8.4.

8.1 Reduction to a function of the multiplicity

The gluing formula of the previous Section, Corollary 16, expresses the log Gromov–Witten invariant $N_{g,h}^{\Delta,n}$ attached to a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ as a product of log Gromov–Witten $N_{g(V),V}^{1,2}$ attached to the trivalent vertices V of Γ, and of the weights $w(E)$ of the edges E of Γ. The definition of $N_{g(V),V}^{1,2}$ given in Sect. 6 depends on a specific choice of orientation on the edges of Γ. In particular, the definition of $N_{g(V),V}^{1,2}$ does not treat the three edges adjacent to V in a symmetric way.

Let $E_V^{\text{in},1}$ and $E_V^{\text{in},2}$ be the two ingoing edges adjacent to V, and let E_V^{out} be the outgoing edge adjacent to V. Let $D_{E_V^{\text{in},1}}$, $D_{E_V^{\text{in},2}}$ and $D_{E_V^{\text{out}}}$ be the corresponding toric divisors of X_{Δ_V}. We have evaluation morphisms

$$\text{ev} = (\text{ev}_1, \text{ev}_2, \text{ev}_{\text{out}})\colon \overline{M}_{g,\Delta_V} \to D_{E_V^{\text{in},1}} \times D_{E_V^{\text{in},2}} \times D_{E_V^{\text{out}}}.$$

In Sect. 6, we defined

$$N_{g,V}^{1,2} = \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_1^*(\text{pt}_1)\text{ev}_2^*(\text{pt}_2),$$

where $\text{pt}_1 \in A^1(D_{E_V^{\text{in},1}})$ and $\text{pt}_2 \in A^1(D_{E_V^{\text{in},2}})$ are classes of a point on $D_{E_V^{\text{in},1}}$ and $D_{E_V^{\text{in},2}}$ respectively.

But one could similarly define

$$N_{g,V}^{2,\text{out}} := \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_2^*(\text{pt}_2)\text{ev}_{\text{out}}^*(\text{pt}_{\text{out}}),$$

and

$$N_{g,V}^{\text{out},1} := \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_{\text{out}}^*(\text{pt}_{\text{out}})\text{ev}_1^*(\text{pt}_1),$$

where $\text{pt}_{\text{out}} \in A^*(D_{E_V^{\text{out}}})$ is the class of a point on E_V^{out}. The following Lemma gives a relation between these various invariants.

Lemma 23 *We have*

$$N_{g,V}^{1,2} w(E_V^{\text{in},1}) w(E_V^{\text{in},2}) = N_{g,V}^{2,\text{out}} w(E_V^{\text{in},2}) w(E_V^{\text{out}}) = N_{g,V}^{\text{out},1} w(E_V^{\text{out}}) w(E_V^{\text{in},1})$$

and we denote by $N_{g,V}$ this number.

Proof Let Γ_V be the trivalent tropical curve given by V and its three edges $E_V^{\text{in},1}$, $E_V^{\text{in},2}$ and E_V^{out}. Let $\Gamma_{V'}$ be the trivalent tropical curve with a unique vertex V' and edges $E_{V'}^{\text{in},1}$, $E_{V'}^{\text{in},2}$ and $E_{V'}^{\text{out}}$, such that

$$w(E_V^{\text{in},1}) = w(E_{V'}^{\text{in},1}), w(E_V^{\text{in},2}) = w(E_{V'}^{\text{in},2}), w(E_V^{\text{out}}) = w(E_{V'}^{\text{out}}),$$

and

$$v_{V,E_V^{\text{in},1}} = -v_{V',E_{V'}^{\text{in},1}}, v_{V,E_V^{\text{in},2}} = -v_{V',E_{V'}^{\text{in},2}}, v_{V,E_V^{\text{out}}} = -v_{V',E_{V'}^{\text{out}}}.$$

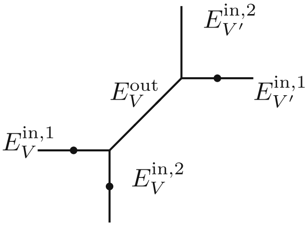
Let $\Gamma_{V,V'}$ be the tropical curve obtained by gluing E_V^{out} and $E_{V'}^{\text{out}}$ together.

Taking

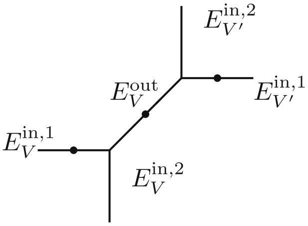
$$\Delta = \left\{ v_{V,E_V^{\text{in},1}}, -v_{V',E_{V'}^{\text{in},1}}, v_{V,E_V^{\text{in},2}}, -v_{V',E_{V'}^{\text{in},2}} \right\}$$

and $n = 3$, we have $g_{\Delta,n} = 0$ and $T_{\Delta,p}$ consists of a unique tropical curve $\Gamma_{V,V'}^p$, obtained from $\Gamma_{V,V'}$ by adding three bivalent vertices corresponding to the three point p_1, p_2 and p_3 in $\mathbb{R}^2$.

Choosing differently $p = (p_1, p_2, p_3)$, the tropical curve $\Gamma_{V,V'}^{p}$ can look like



or like



But the log Gromov–Witten invariants $N_g^{\Delta,3}$ are independent of the choice of p and so can be computed for any choice of p. For each of the two above choices of p, the gluing formula of Corollary 16 gives an expression for $N_g^{\Delta,3}$. These two expressions have to be equal. Writing

$$F(u) = \sum_{g \geqslant 0} N_g u^{2g+1}$$

we obtain[19]

$$\begin{aligned} &F_V^{1,2}(u) F_{V'}^{1,\text{out}}(u) w(E_V^{\text{in},1}) w(E_V^{\text{in},2}) w(E_V^{\text{out}}) w(E_{V'}^{\text{in},1}) \\ &\quad = F_V^{1,\text{out}}(u) F_{V'}^{1,\text{out}}(u) w(E_V^{\text{in},1}) w(E_V^{\text{out}}) w(E_V^{\text{out}}) w(E_{V'}^{\text{in},1}), \end{aligned}$$

and so after simplification

$$F_V^{1,2}(u) F_{V'}^{1,\text{out}}(u) w(E_V^{\text{in},2}) = F_V^{1,\text{out}}(u) F_{V'}^{1,\text{out}}(u) w(E_V^{\text{out}}).$$

By $GL_2(\mathbb{Z})$ invariance, we have $F_V^{1,2}(u) = F_{V'}^{1,2}(u)$ and $F_V^{1,\text{out}}(u) = F_{V'}^{1,\text{out}}(u)$. By the unrefined correspondence theorem, we know that $F_V^{1,\text{out}}(u) \neq 0$, so we obtain

$$F_V^{1,2}(u) w(E_V^{\text{in},2}) = F_V^{1,\text{out}}(u) w(E_V^{\text{out}}),$$

[19] Recall that we are considering marked points as bivalent vertices and that this affects the notion of bounded edge. According to the gluing formula of Corollary 16, we need to include one weight factor for each bounded edge.

which finishes the proof of Lemma 23. □

We define the contribution $F_V(u) \in \mathbb{Q}[\![u]\!]$ of a trivalent vertex V of Γ as being the power series

$$F_V(u) = \sum_{g \geqslant 0} N_{g,V} u^{2g+1}.$$

Proposition 24 *For every Δ and n such that $g_{\Delta,n} \geqslant 0$, and for every $p \in U_{\Delta,n}$, we have*

$$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = \sum_{(h\colon \Gamma \to \mathbb{R}^2) \in T_{\Delta,p}} \prod_{V \in V^{(3)}(\Gamma)} F_V(u)$$

where the product is over the trivalent vertices of Γ.

Proof This follows from the decomposition formula, Proposition 10, from the gluing formula, Corollary 16, and from Lemma 23. Indeed, every bounded edge of Γ is an ingoing edge for exactly one trivalent vertex of Γ and every trivalent vertex of Γ has exactly two ingoing edges. Combining the invariant $N_{g(V),V}^{1,2}$ of a trivalent vertex V with the weights of its two ingoing edges, one can rewrite the double product of Corollary 16 as a single product in terms of the invariants defined by Lemma 23. □

Proposition 25 *The contribution $F_V(u)$ of a vertex V only depends on the multiplicity $m(V)$ of V.*

In particular, for every m positive integer, one can define the contribution $F_m(u) \in \mathbb{Q}[\![u]\!]$ as the contribution $F_V(u)$ of a vertex V of multiplicity m.

Proof We follow closely Brett Parker, [42] (Section 3).

For $v_1, v_2 \in \mathbb{Z}^2 - \{0\}$, let us denote by $F_{v_1,v_2}(u)$ the contribution $F_V(u)$ of a vertex V of adjacent edges E_1, E_2 and E_3 such that $v_{V,E_1} = v_1$ and $v_{V,E_2} = v_2$. The contribution $F_{v_1,v_2}(u)$ depends on (v_1, v_2) only up to linear action of $GL_2(\mathbb{Z})$ on $\mathbb{Z}^2$. In particular, we can change the sign of v_1 and/or v_2 without changing $F_{v_1,v_2}(u)$.

By the balancing condition, we have $v_{V,E_3} = -v_{V,E_1} - v_{V,E_2}$ and so

$$F_{v_1,v_2}(u) = F_{-v_1,v_2}(u) = F_{v_1-v_2,v_2}(u).$$

By $GL_2(\mathbb{Z})$ invariance, we can assume $v_1 = (|v_1|, 0)$ and $v_2 = (v_{2x}, *)$ with $v_{2x} \geqslant 0$. If $|v_1|$ divides v_{2x}, $v_{2x} = a|v_1|$, then replacing v_2 by $v_2 - av_1$, which does not change F_{v_1,v_2}, we can assume that $v_1 = (|v_1|, 0)$ and $v_2 = (0, *)$. If not, we do the Euclidean division of v_{2x} by $|v_1|$, $v_{2x} = a|v_1| + b$, $0 \leqslant b < |v_1|$, and we replace v_2 by $v_2 - av_1$ to obtain $v_2 = (b, *)$. Exchanging the roles of v_1 and v_2, we can assume by $GL_2(\mathbb{Z})$ invariance that $v_1 = (|v_1|, 0)$, for some $|v_1| \leqslant b$ and $v_2 = (v_{2x}, *)$ for some $v_{2x} \geqslant 0$, and we repeat the above procedure. By the Euclidean algorithm, this process terminates and at the end we have $v_1 = (|v_1|, 0)$ and $v_2 = (0, |v_2|)$. In particular, for every $v_1, v_2 \in \mathbb{Z}^2 - \{0\}$, the contribution F_{v_1,v_2} only depends on $\gcd(|v_1|, |v_2|)$ and on the multiplicity $|\det(v_1, v_2)|$.

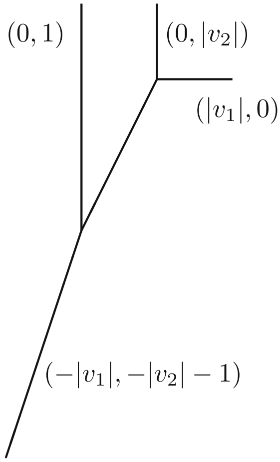
By the previous paragraph, we can assume that $v_1 = (|v_1|, 0)$ and $v_2 = (0, |v_2|)$.

Taking

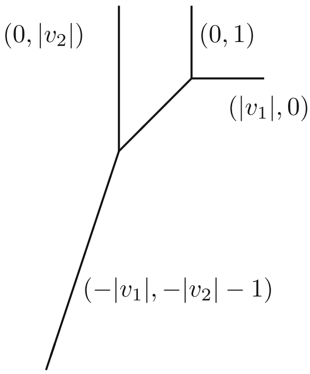
$$\Delta = \{(|v_1|, 0), (0, |v_2|), (0, 1), (-|v_1|, -|v_2| - 1)\},$$

and $n = 3$, we have $g_{\Delta,n} = 0$ and $T_{\Delta,p}$ contains a unique tropical curve Γ^p.

Choosing differently $p = (p_1, p_2, p_3)$, the tropical curve Γ^p can look like



or like



But the log Gromov–Witten invariants $N_g^{\Delta,3}$ are independent of the choice of p and so can be computed for any choice of p. For each of the two above choices of p, the gluing formula of Proposition 24 gives an expression for

$N_g^{\Delta,3}$. These two expressions have to be equal and we obtain

$$F_{(|v_1|,0),(0,|v_2|)}(u)F_{(0,1),(-|v_1|,-|v_2|-1)}(u) = F_{(|v_1|,0),(0,1)}(u)F_{(0,|v_2|),(-|v_1|,-|v_2|-1)}(u).$$

For both pairs of vectors $(|v_1|, 0)$, $(0, 1)$ and $(0, 1)$, $(-|v_1|, -|v_2|-1)$, the gcd of the divisibilities is equal to one and the absolute value of the determinant is equal to $|v_1|$, so we have

$$F_{(0,1),(-|v_1|,-|v_2|-1)}(u) = F_{(|v_1|,0),(0,1)}(u).$$

As this quantity is non-zero by the unrefined correspondence theorem, we can simplify it from the previous equality to obtain

$$F_{(|v_1|,0),(0,|v_2|)}(u) = F_{(0,|v_2|),(-|v_1|,-|v_2|-1)}(u).$$

As

$$\gcd(|(0, |v_2|)|, |(-|v_1|, -|v_2| - 1)|) = 1,$$

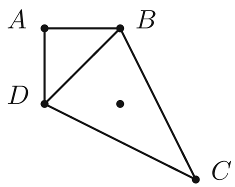
we obtain the desired result. □

8.2 Reduction to vertices of multiplicity 1 and 2

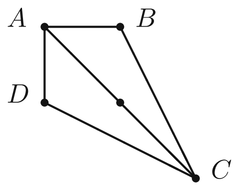
We start reviewing the key step in the argument of Itenberg and Mikhalkin [25] proving the tropical deformation invariance of Block-Göttsche invariants. We consider a tropical curve with a 4-valent vertex V. Let Q be the quadrilateral dual to V. We assume that Q has no pair of parallel sides. In that case, there exists a unique parallelogram P having two sides in common with Q and being contained in Q. Let A,B,C and D denote the four vertices of Q, such that A,B and D are vertices of P. Let E be the fourth vertex of P, contained in the interior of Q. There are three combinatorially distinct ways to deform this tropical curve into a simple one, corresponding to the three ways to decompose Q into triangles or parallelograms:

1. We can decompose Q into the triangles ABD and BCD.
2. We can decompose Q into the triangles ABC and ACD.
3. We can decompose Q into the triangles BCE, DEC and the parallelogram P.

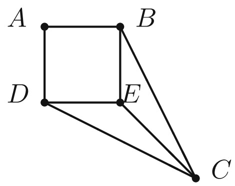
Case (1):



Case (2):



Case (3):



The deformation invariance result then follows from the identity

$$\begin{aligned}&(q^{|ACD|}-q^{-|ACD|})(q^{|ABC|}-q^{-|ABC|})\\&\quad=(q^{|BCD|}-q^{-|BCD|})(q^{|ABD|}-q^{-|ABD|})\\&\qquad+(q^{|BCE|}-q^{-|BCE|})(q^{|DEC|}-q^{-|DEC|}),\end{aligned}$$

where $|-|$ denotes the area. This identity can be proved by elementary geometry considerations.

The following result goes in the opposite direction and shows that the constraints imposed by tropical deformation invariance are quite strong. The generating series of log Gromov–Witten invariants $F_m(u)$ will satisfy these constraints. Indeed, they are defined independently of any tropical limit, so applications of the gluing formula to different degenerations have to give the same result.

Proposition 26 *Let $F : \mathbb{Z}_{>0} \to R$ be a function of positive integers valued in a commutative ring R, such that, for any quadrilateral Q as above, we have*[20]

$$\begin{aligned}F(2|BCD|)F(2|ABD|) = {} & F(2|ACD|)F(2|ABC|)\\&+F(2|BCE|)F(2|DEC|).\end{aligned}$$

[20] All the relevant areas are half-integers and so their doubles are indeed integers.

Then for every integer n $\geqslant 2$*, we have*

$$F(n)^2 = F(2n-1)F(1) + F(n-1)^2$$

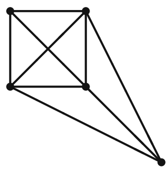
and for every integer n $\geqslant 3$*, we have*

$$F(n)^2 = F(2n-2)F(2) + F(n-2)^2.$$

In particular, if $F(1)$ *and* $F(2)$ *are invertible in* R*, then the function* F *is completely determined by its values* $F(1)$ *and* $F(2)$*.*

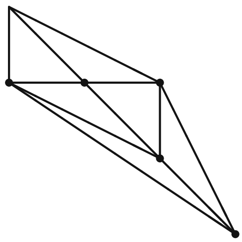
Proof The first equality is obtained by taking Q to be the quadrilateral of vertices $(-1,0)$, $(-1,1)$, $(0,1)$, $(n-1, -(n-1))$.

Picture of Q for $n = 2$:



The second equality is obtained by taking Q to be the quadrilateral of vertices $(-1,0)$, $(-1,1)$, $(1,0)$, $(n-1, -(n-1))$.

Picture of Q for $n = 3$:



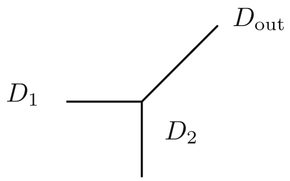
□

8.3 Contribution of vertices of multiplicity 1 and 2

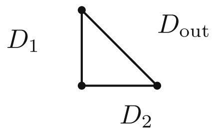
8.3.1 Vertex of multiplicity one

We now evaluate the contribution $F_1(u)$ of a vertex of multiplicity 1 by direct computation.

We consider $\Delta = \{(-1,0), (0,-1), (1,1)\}$. The corresponding toric surface X_Δ is simply $\mathbb{P}^2$, of fan



and of dual polygon



Let D_1, D_2 and D_{out} be the toric boundary divisors of $\mathbb{P}^2$. The class β_Δ is simply the class of a curve of degree one, i.e. of a line, on $\mathbb{P}^2$. Let $\overline{M}_{g,\Delta}$ be the moduli space of genus g stable log maps of type Δ. We have evaluation maps

$$(\text{ev}_1, \text{ev}_2) \colon \overline{M}_{g,\Delta} \to D_1 \times D_2,$$

and in Sect. 6, we defined

$$N_{g,\Delta}^{1,2} = \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_1^*(\text{pt}_1)\text{ev}_2^*(\text{pt}_2),$$

where $\text{pt}_1 \in A^*(D_1)$ and $\text{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

By definition (see Sect. 8.1), we have

$$F_1(u) = \sum_{g \geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}.$$

Proposition 27 *The contribution of a vertex of multiplicity one is given by*

$$F_1(u) = 2\sin\left(\frac{u}{2}\right) = -i\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)$$

where $q = e^{iu}$.

Proof Let P_1 and P_2 be points on D_1 and D_2 respectively, away from the torus fixed points. Let S be the surface obtained by blowing-up $\mathbb{P}^2$ at P_1 and P_2. Denote by D the strict transform of the class of a line in $\mathbb{P}^2$ and by E_1, E_2 the exceptional divisors. Denote ∂S the strict transform of the toric boundary $\partial \mathbb{P}^2$ of $\mathbb{P}^2$. We endow S with the divisorial log structure with respect to ∂S. Let $\overline{M}_g(S)$ be the moduli space of genus g stable log maps to S of class $D - E_1 - E_2$ with tangency condition to intersect ∂S in one point with multiplicity one. It has virtual dimension g and we define

$$N_g^S := \int_{[\overline{M}_g(S)]^{\mathrm{virt}}} (-1)^g \lambda_g.$$

The strict transform C of the line L in $\mathbb{P}^2$ passing through P_1 and P_2 is the unique genus zero curve satisfying these conditions and has normal bundle $N_{C|S} = \mathcal{O}_{\mathbb{P}^1}(-1)$ in S. All the higher genus maps factor through C, and as C is away from the preimage of the torus fixed points of $\mathbb{P}^2$, log invariants coincide with relative invariants [5]. More precisely, we can consider the moduli space $\overline{M}_g(\mathbb{P}^1/\infty, 1, 1)$ genus g stable maps to $\mathbb{P}^1$, of degree one, and relative to a point $\infty \in \mathbb{P}^1$. If $\pi : \mathcal{C} \to \overline{M}_g(\mathbb{P}^1/\infty, 1, 1)$ is the universal curve and $f : \mathcal{C} \to \mathbb{P}^1 \simeq C$ is the universal map, the difference in obstruction theories between stable maps to S and stable maps to $\mathbb{P}^1$ comes from $R^1\pi_* f^* N_{C|S} = R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)$. So we obtain

$$N_g^S = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} (-1)^g \lambda_g \, e\left(R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)\right),$$

where $e(-)$ is the Euler class. Rewriting

$$(-1)^g \lambda_g = e(R^1\pi_* \mathcal{O}_{\mathcal{C}}) = e(R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}),$$

we get

$$N_g^S = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} e\left(R^1\pi_* f^* (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right).$$

These integrals have been computed by Bryan and Pandharipande[12], (see the proof of the Theorem 5.1), and the result is

$$\sum_{g \geqslant 0} N_g^S u^{2g-1} = \frac{1}{2\sin\left(\frac{u}{2}\right)}.$$

As in [23], we will work with the non-compact varieties $(\mathbb{P}^2)^\circ$, D_1°, D_2°, S° obtained by removing the torus fixed points of $\mathbb{P}^2$ and their preimages in S.

Denote $\mathbb{P}_1^\circ$ the projectivized normal bundle to D_1° in $(\mathbb{P}^2)^\circ$, coming with two natural sections $(D_1^\circ)_0$ and $(D_1^\circ)_\infty$. Denote $\tilde{\mathbb{P}}_1^\circ$ the blow-up of $\mathbb{P}_1^\circ$ at the point $P_1 \in (D_1^\circ)_\infty$, $\tilde{E}_1$ the corresponding exceptional divisor and C_1 the strict transform of the fiber of $\mathbb{P}_1^\circ$ passing through P_1. In particular, $\tilde{E}_1$ and C_1 are both projective lines with degree -1 normal bundle in $(\tilde{\mathbb{P}}_1)^\circ$. Furthermore, $\tilde{E}_1$

and C_1 intersect in one point. Similarly, denote $\mathbb{P}_2^\circ$ the projectivized normal bundle to D_2° in $(\mathbb{P}^2)^\circ$, coming with two natural sections $(D_2^\circ)_0$ and $(D_2^\circ)_\infty$. Denote $\tilde{\mathbb{P}}_2^\circ$ the blow-up of $\mathbb{P}_2^\circ$ at the point $P_2 \in (D_2^\circ)_\infty$, $\tilde{E}_2$ the corresponding exceptional divisor and C_2 the strict transform of the fiber of $\mathbb{P}_2^\circ$ passing through P_2. In particular, $\tilde{E}_2$ and C_2 are both projective lines with degree -1 normal bundle in $(\tilde{\mathbb{P}}_2)^\circ$. Furthermore, $\tilde{E}_2$ and C_2 intersect in one point.

We degenerate S° as in Section 5.3 of [23]. We first degenerate $(\mathbb{P}^2)^\circ$ to the normal cone of $D_1^\circ \cup D_2^\circ$, i.e. we blow-up $(D_1^\circ \cup D_2^\circ) \times \{0\}$ in $(\mathbb{P}^2)^\circ \times \mathbb{C}$. The fiber over $0 \in \mathbb{C}$ has three irreducible components: $(\mathbb{P}^2)^\circ$, $\mathbb{P}_1^\circ$, $\mathbb{P}_2^\circ$, with $\mathbb{P}_1^\circ$ and $\mathbb{P}_2^\circ$ glued along $(D_1^\circ)_0$ and $(D_2^\circ)_0$ to D_1° and D_2° in $(\mathbb{P}^2)^\circ$. We then blow-up the strict transforms of the sections $P_1 \times \mathbb{C}$ and $P_2 \times \mathbb{C}$. The fiber of the resulting family away from $0 \in \mathbb{C}$ is isomorphic to S°. The fiber over zero has three irreducible components: $(\mathbb{P}^2)^\circ$, $\tilde{\mathbb{P}}_1^\circ$, $\tilde{\mathbb{P}}_2^\circ$.

We would like to apply a degeneration formula to this family in order to compute N_g^S. As discussed above, all the maps in $\overline{M}_g(S)$ factor through C and so N_g^S can be seen as a relative Gromov–Witten invariant of the non-compact surface S°, relatively to the strict transforms of D_1° and D_2°.

The key point is that for homological degree reasons, the degenerating relative stable maps do not leave the non-compact geometries we are considering. More precisely, any limiting relative stable map has to factor through $C_1 \cup L \cup C_2$, with degree one over each of the components C_1, L and C_2. So, even if the target geometry is non-compact, all the relevant moduli spaces of relative stable maps are compact. It follows that we can apply the ordinary degeneration formula in relative Gromov–Witten theory [28].

We obtain

$$\sum_{g \geqslant 0} N_g^S u^{2g-1} = \left(\sum_{g \geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1} \right) \left(\sum_{g \geqslant 0} N_g^{C_1} u^{2g-1} \right) \left(\sum_{g \geqslant 0} N_g^{C_2} u^{2g-1} \right).$$

The invariants $N_g^{C_1}$ and $N_g^{C_2}$, coming from curves factoring through C_1 and C_2, which are (-1)-curves in $\tilde{\mathbb{P}}_1^\circ$ and $\tilde{\mathbb{P}}_2^\circ$ respectively, can be written as relative invariants of $\mathbb{P}^1$:

$$N_g^{C_1} = N_g^{C_2} = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} e\left(R^1 \pi_* f^* (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right),$$

which is exactly the formula giving N_g^S, and so

$$\sum_{g \geqslant 0} N_g^{C_1} u^{2g-1} = \sum_{g \geqslant 0} N_g^{C_2} u^{2g-1} = \frac{1}{2 \sin\left(\frac{u}{2}\right)}.$$

Remark that this equality is a higher genus version of Proposition 5.2 of [23]. Combining the previous equalities, we obtain

$$\frac{1}{2\sin\left(\frac{u}{2}\right)} = \left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right)\left(\frac{1}{2\sin\left(\frac{u}{2}\right)}\right)^2,$$

and so

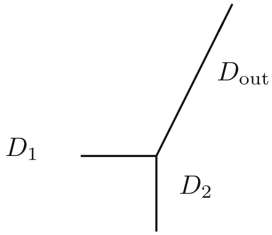
$$\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1} = 2\sin\left(\frac{u}{2}\right).$$

□

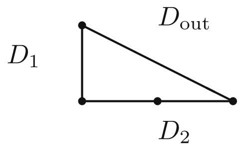
8.3.2 Vertex of multiplicity 2

We now evaluate the contribution $F_2(u)$ of a vertex of multiplicity 2 by direct computation.

We consider $\Delta = \{(-1,0),(0,-2),(1,2)\}$. The corresponding toric surface X_Δ is simply the weighted projective plane $\mathbb{P}^{1,1,2}$, of fan



and of dual polygon



Let D_1, D_2 and D_{out} be the toric boundary divisors of $\mathbb{P}^{1,1,2}$. We have the following numerical properties:

$$2D_1 = D_2 = 2D_{\text{out}},$$
$$D_1.D_2 = 1,\ D_1.D_{\text{out}} = \frac{1}{2},\ D_2.D_{\text{out}} = 1,$$
$$D_1^2 = \frac{1}{2},\ D_2^2 = 2,\ D_{\text{out}}^2 = \frac{1}{2}.$$

The class β_Δ satisfies $\beta_\Delta.D_1 = 1$, $\beta_\Delta.D_2 = 2$, $\beta_\Delta.D_{\text{out}} = 1$ and so

$$\beta_\Delta = 2D_1 = D_2 = 2D_{\text{out}}.$$

Let $\overline{M}_{g,\Delta}$ be the moduli space of genus g stable log maps of type Δ. We have evaluation maps

$$(\text{ev}_1, \text{ev}_2) \colon \overline{M}_{g,\Delta} \to D_1 \times D_2,$$

and in Sect. 6, we defined

$$N_{g,\Delta}^{1,2} = \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_1^*(\text{pt}_1)\text{ev}_2^*(\text{pt}_2),$$

where $\text{pt}_1 \in A^*(D_1)$ and $\text{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

By definition (see Sect. 8.1), we have

$$F_1(u) = 2\left(\sum_{g \geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right).$$

Proposition 28 *The contribution of a vertex of multiplicity two is given by*

$$F_2(u) = 2\sin(u) = (-i)\left(q - q^{-1}\right)$$

where $q = e^{iu}$.

Proof We have to prove that

$$\sum_{g \geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1} = \sin(u).$$

Let P_2 be a point on D_2 away from the torus fixed points. Let S be the surface obtained by blowing-up $\mathbb{P}^{1,1,2}$ at P_2. Still denote β_Δ the strict transform of the class β_Δ and by E_2 the exceptional divisor. Denote ∂S the strict transform of the toric boundary $\partial\mathbb{P}^{1,1,2}$ of $\mathbb{P}^{1,1,2}$. We endow S with the divisorial log structure with respect to ∂S. Let $\overline{M}_g(S)$ be the moduli space of genus g stable log maps to S of class $\beta_\Delta - 2E_2$ with tangency condition to intersect D_1 in one point with multiplicity one and D_{out} in one point with multiplicity one. It has virtual dimension g and we have an evaluation map

$$\text{ev}_1 \colon \overline{M}_{g,S} \to D_1$$

We define

$$N_g^S := \int_{[\overline{M}_g(S)]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1),$$

where $\mathrm{pt}_1 \in A^1(D_1)$ is the class of a point on D_1.

In fact, because a curve in the linear system $\beta_\Delta - 2E_2$ is of arithmetic genus g_a given by

$$\begin{aligned} 2g_a - 2 &= (\beta_\Delta - 2E_2) \cdot (\beta_\Delta - 2E_2 + K_S) \\ &= (2D_1 - 2E_2) \cdot (2D_1 - 4D_1 - E_2) \\ &= -4D_1^2 + 2E_2^2 \\ &= -4, \end{aligned}$$

i.e. $g_a = -1 < 0$, all the moduli spaces $\overline{M}_g(S)$ are empty and so

$$\sum_{g \geqslant 0} N_g^S u^{2g-1} = 0.$$

We write $\tilde{\Delta} = \{(-1,0), (0,-1), (0,-1), (1,2)\}$ and $\overline{M}_{g,\tilde{\Delta}}$ the moduli space of genus g stable log maps of type $\tilde{\Delta}$. We have evaluation maps

$$(\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_{2'}) \colon \overline{M}_{g,\tilde{\Delta}} \to D_1 \times D_2 \times D_2,$$

and we define

$$N_{g,\tilde{\Delta}}^{1,2,2'} := \int_{[\overline{M}_{g,\tilde{\Delta}}]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1) \mathrm{ev}_2^*(\mathrm{pt}_2) \mathrm{ev}_{2'}^*(\mathrm{pt}_2),$$

where $\mathrm{pt}_1 \in A^*(D_1)$ and $\mathrm{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

As in [23], we will work with the non-compact varieties $(\mathbb{P}^{1,1,2})^\circ$, D_1°, D_2°, S° obtained by removing the torus fixed points of $\mathbb{P}^{1,1,2}$ and their preimages in S. Denote $\mathbb{P}_2^\circ$ the projectivized normal bundle to D_2° in $(\mathbb{P}^2)^\circ$, coming with two natural sections $(D_2^\circ)_0$ and $(D_2^\circ)_\infty$. Denote $\tilde{\mathbb{P}}_2^\circ$ the blow-up of $\mathbb{P}_2^\circ$ at the point $P_2 \in (D_2^\circ)_\infty$, $\tilde{E}_2$ the corresponding exceptional divisor and C_2 the strict transform of the fiber of $\mathbb{P}_2^\circ$ passing through P_2. In particular, $\tilde{E}_2$ and C_2 are both projective lines with degree -1 normal bundle in $(\tilde{\mathbb{P}}_2)^\circ$. Furthermore, $\tilde{E}_2$ and C_2 intersect in one point.

We degenerate S° as in Section 5.3 of [23]. We first degenerate $(\mathbb{P}^{1,1,2})^\circ$ to the normal cone of D_2°, i.e. we blow-up $D_2^\circ \times \{0\}$ in $(\mathbb{P}^{1,1,2})^\circ \times \mathbb{C}$. The fiber over $0 \in \mathbb{C}$ has two components: $(\mathbb{P}^{1,1,2})^\circ$ and $\mathbb{P}_2^\circ$, with $\mathbb{P}_2^\circ$ glued along $(D_2^\circ)_0$ to D_2° in $(\mathbb{P}^{1,1,2})^\circ$. We then blow-up the strict transform of the section $P_2 \times \mathbb{C}$. The fiber of the resulting family away from $0 \in \mathbb{C}$ is isomorphic to S°. The fiber over zero has two components: $(\mathbb{P}^{1,1,2})^\circ$ and $\tilde{\mathbb{P}}_2^\circ$.

We would like to apply a degeneration formula to this family in order to compute N_g^S. The key point is that for homological degree reasons, the relevant degenerating relative stable maps do not leave the non-compact geometries we are considering. More precisely, after fixing a point $P_1 \in D_1^\circ$, realizing the insertion $\mathrm{ev}_1^*(\mathrm{pt}_1)$, any limiting relative stable map has to factor through $L \cup C_2$, with degree one over L and degree two over C_2, where L is the unique curve in $\mathbb{P}^{1,1,2}$, of class β_Δ, passing through P_1 and through P_2 with tangency order two along D_2°. So, even if the target geometry is non-compact, all the relevant moduli spaces of relative stable maps are compact. It follows that we can apply the ordinary degeneration formula in relative Gromov–Witten theory [28].

The application of the degeneration formula gives two terms, corresponding to the two partitions $2 = 1 + 1$ and $2 = 2$ of the intersection number

$$(\beta_\Delta - 2E_2).E_2 = 2.$$

For the first term, the invariants on the side of $\mathbb{P}^{1,1,2}$ are $N_{g,\tilde{\Delta}}^{1,2,2'}$, whereas on the side of $\tilde{\mathbb{P}}_2$, we have disconnected invariants, corresponding to two degree one maps to C_2. As in the proof of Proposition 27, the relevant connected degree one invariants of C_2 are given by

$$N_g^{C_2} = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} e\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right),$$

satisfying

$$\sum_{g \geqslant 0} N_g^{C_2} u^{2g-1} = \frac{1}{2\sin\left(\frac{u}{2}\right)}.$$

For the second term, the invariants on the side of $\mathbb{P}^{1,1,2}$ are $N_{g,\Delta}^{1,2}$, whereas on the side of $\tilde{\mathbb{P}}_2$, we have connected invariants, corresponding to one degree two map to C_2. More precisely, the relevant connected degree two invariants of C_2 are given by

$$N_g^{2C_2} = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,2,2)]^{\mathrm{virt}}} e\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right),$$

where $\overline{M}_g(\mathbb{P}^1/\infty, 2, 2)$ is the moduli space of genus g stable maps to $\mathbb{P}^1$, of degree two, and relative to a point $\infty \in \mathbb{P}^1$ with maximal tangency order two.

According to [12] (see the proof of Theorem 5.1), we have

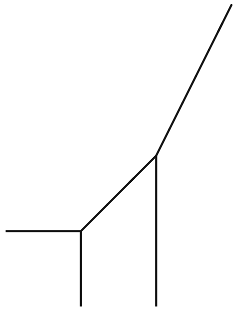
$$\sum_{g\geqslant 0} N_g^{2C_2} u^{2g-1} = -\frac{1}{2}\frac{1}{2\sin(u)}.$$

It follows that the degeneration formula takes the form

$$\sum_{g\geqslant 0} N_g^S u^{2g-1} = \frac{1}{2}\left(\sum_{g\geqslant 0} N_{g,\tilde{\Delta}}^{1,2,2'} u^{2g+2}\right)\left(\frac{1}{2\sin\left(\frac{u}{2}\right)}\right)^2 +2\left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right)\frac{(-1)}{2}\frac{1}{2\sin(u)}.$$

The factor $\frac{1}{2}$ in front of the fist term is a symmetry factor and the factor 2 in front of the second term is a multiplicity.

There exists a unique tropical curve of type $\tilde{\Delta}$, which looks like



This tropical curve has two vertices of multiplicity one, so using the gluing formula of Proposition 24 and Proposition 27, we find

$$\sum_{g\geqslant 0} N_{g,\tilde{\Delta}}^{1,2,2'} u^{2g+2} = (F_1(u))^2 = \left(2\sin\left(\frac{u}{2}\right)\right)^2.$$

Combining the previous results, we obtain

$$0 = \frac{1}{2} - \frac{1}{2\sin(u)}\left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right),$$

and so the desired formula. □

Remark The proofs of Propositions 27 and 28 rely on the fact that the involved curves have low degree. More precisely, in each case, the key point is that the dual polygon does not contain any interior integral point, i.e. a generic curve in the corresponding linear system on the surface has genus zero. This implies that, after imposing tangency constraints, all the higher genus stable maps factor through some rigid genus zero curve in the surface. This guarantees the compactness result needed to work as we did with relative Gromov–Witten theory of non-compact geometries. The higher genus generalization of the most general case of the degeneration argument of Section 5.3 of [23] cannot be dealt with in the same way. This generalization will be treated and applied in [9], using techniques similar to those used to prove the gluing formula in Sect. 7.

8.4 Contribution of a general vertex

Proposition 29 *The contribution of a vertex of multiplicity m is given by*

$$F_m(u) = (-i)\left(q^{\frac{m}{2}} - q^{-\frac{m}{2}}\right).$$

Proof By Proposition 27, the result is true for $m = 1$ and by Proposition 28, the result is true for $m = 2$. By consistency of the gluing formula of Proposition 24, the function $F(m) := F_m$ valued in the ring $R := \mathbb{Q}[\![u]\!]$ satisfies the hypotheses of Proposition 26. The result follows by induction on m using Proposition 26. □

The proof of Theorem 1 (Theorem 5 in Sect. 2.5) follows from the combination of Propositions 24, 25 and 29.

To prove Theorem 6, generalizing Theorem 1 by allowing to fix the positions of some of the intersection points with the toric boundary, we only have to organize the gluing procedure slightly differently. The connected components of the complement of the bivalent vertices of Γ, as at the beginning of Sect. 6, are trees with one unfixed unbounded edge and possibly several fixed unbounded edges. We fix an orientation of the edges such that edges adjacent to bivalent pointed vertices go out of the bivalent pointed vertices, such that the fixed unbounded edges are ingoing and such that the unfixed unbounded edge is outgoing. With respect to this orientation, every trivalent vertex has two ingoing and one outgoing edges, and so, without any modification, we obtain the analogue of the gluing formula of Corollary 16:

$$N_{g,h}^{\Delta,n} = \left(\prod_{V \in V^{(3)}(\Gamma)} N_{g(V),V}^{1,2}\right)\left(\prod_{E \in E_f(\Gamma)} w(E)\right).$$

In Lemma 23, we defined $N_{g,V} := N^{1,2}_{g(V),V} w(E_V^{\text{in},1}) w(E_V^{\text{in},2})$, where $E_V^{\text{in},1}$ and $E_V^{\text{in},1}$ are the ingoing edges adjacent to V. Every bounded edge is an ingoing edge to some vertex but some ingoing edges are fixed unbounded edges and so

$$N_{g,h}^{\Delta,n} = \left(\prod_{E_\infty^F \in E_\infty^F(\Gamma)} \frac{1}{w(E_\infty^F)} \right) \left(\prod_{V \in V^{(3)}(\Gamma)} N_{g(V),V} \right),$$

where the first product is over the fixed unbounded edges of Γ. Theorem 6 then follows from Proposition 29.

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A An explicit example

In this appendix, we check by a direct computation one of the consequences of Theorem 1. Let us consider the problem of counting rational cubic curves in $\mathbb{P}^2$ passing through 8 points in general position. To match the notations of the Introduction, we choose Δ containing three times the vector $(1, 0)$, three times the vector $(0, 1)$ and three times the vector $(-1, -1)$.

The toric surface X_Δ is then $\mathbb{P}^2$ and the curve class β_Δ is the class of a cubic curve in $\mathbb{P}^2$. We have $|\Delta| = 9$, $n = 8$, $g_{\Delta,n} = 0$. Let us write N_g for $N_g^{\Delta,n}$. We have $N_0 = 12$ and the corresponding Block-Göttsche invariant is $q + 10 + q^{-1}$ (see Example 1.3 of [37] for pictures of tropical curves). From the point of view of Göttsche–Shende [19], the relevant relative Hilbert scheme to consider happens to be the pencil of cubics passing through the 8 given points, i.e. $\mathbb{P}^2$ blown-up in 9 points, whose Hirzebruch genus is indeed $1 + 10q + q^2$.

According to Theorem 5, we have

$$\begin{aligned}\sum_{g\geqslant 0} N_g u^{2g-2+9} &= i(q+10+q^{-1})(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^7\\ &= i(q^{\frac{9}{2}}+3q^{\frac{7}{2}}-48q^{\frac{5}{2}}+168q^{\frac{3}{2}}-294q^{\frac{1}{2}}+294q^{-\frac{1}{2}}\\ &\quad -168q^{-\frac{3}{2}}+48q^{-\frac{5}{2}}-3q^{-\frac{7}{2}}-q^{-\frac{9}{2}})\\ &= 12u^7-\frac{9}{2}u^9+\frac{137}{160}u^{11}-\frac{1253}{11520}u^{13}+\cdots\end{aligned}$$

We will check directly that $N_1=-\frac{9}{2}$. Remark that a Block-Göttsche invariant equal to 12 rather than to $q+10+q^{-1}$ would lead to $N_1=-\frac{7}{2}$. In particular, the value of N_1 is already sensitive to the choice of the correct refinement.

We have[21]

$$N_1=\int_{[\overline{M}_{1,8}(\mathbb{P}^2,3)]^{\mathrm{virt}}}(-1)^1\lambda_1\prod_{j=1}^{8}\mathrm{ev}_j^*(\mathrm{pt}),$$

where $\mathrm{pt}\in A^2(\mathbb{P}^2)$ is the class of a point. Introducing an extra marked point and using the divisor equation, one can write

$$N_1=\frac{1}{3}\int_{[\overline{M}_{1,8+1}(\mathbb{P}^2,3)]^{\mathrm{virt}}}(-1)^1\lambda_1\left(\prod_{j=1}^{8}\mathrm{ev}_j^*(\mathrm{pt})\right)\mathrm{ev}_9^*(h),$$

where $h\in A^1(\mathbb{P}^2)$ is the class of a line. On $\overline{M}_{1,1}$, we have

$$\lambda_1=\frac{1}{12}\delta_0,$$

where δ_0 is the class of a point. Taking for representative of δ_0 the point corresponding to the nodal genus one curve, with j-invariant $i\infty$, and resolving the node, we can write

$$N_1=-\frac{1}{12}\cdot\frac{1}{2}\cdot\frac{1}{3}\int_{[\overline{M}_{0,8+1+2}(\mathbb{P}^2,3)]^{\mathrm{virt}}}\left(\prod_{j=1}^{8}\mathrm{ev}_j^*(\mathrm{pt})\right)\mathrm{ev}_9^*(h)(\mathrm{ev}_{10}^*\times\mathrm{ev}_{11}^*)(D),$$

[21] A general choice of representative for λ_1 cuts out a locus in the moduli space made entirely of torically transverse stable maps. In particular, we do not have to worry about the difference between log and usual stable maps. A general form of this argument is used in the proof of the gluing formula in Sect. 7.

where the factor $\frac{1}{2}$ comes from the two ways of labeling the two points resolving the node, and D is the class of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. We have

$$D = 1 \times \mathrm{pt} + \mathrm{pt} \times 1 + h \times h.$$

The first two terms do not contribute to N_1 for dimension reasons so

$$N_1 = -\frac{1}{12} \cdot \frac{1}{2} \cdot \frac{1}{3} \int_{[\overline{M}_{0,8+1+2}(\mathbb{P}^2,3)]^{\mathrm{virt}}} \left(\prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}) \right) \mathrm{ev}_9^*(h) \mathrm{ev}_{10}^*(h) \mathrm{ev}_{11}^*(h).$$

Using the divisor equation, we obtain

$$N_1 = -\frac{1}{12} \cdot \frac{1}{2} \cdot 3 \cdot 3 \int_{[\overline{M}_{0,8}(\mathbb{P}^2,3)]^{\mathrm{virt}}} \left(\prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}) \right) = -\frac{9}{24} N_0 = -\frac{9}{2},$$

as expected.

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