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TRUDINGER TYPE INEQUALITIES IN $\mathbf{R}^N$ AND THEIR BEST EXPONENTS

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ABSTRACT. We study Trudinger type inequalities in $\mathbf{R}^N$ and their best exponents α_N. We show for $\alpha \in (0, \alpha_N)$, $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ (ω_{N-1} is the surface area of the unit sphere in $\mathbf{R}^N$), there exists a constant $C_\alpha > 0$ such that

$$(*) \qquad \int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_{L^N(\mathbf{R}^N)}}\right)^{\frac{N}{N-1}}\right) dx \le C_\alpha \frac{\|u\|^N_{L^N(\mathbf{R}^N)}}{\|\nabla u\|^N_{L^N(\mathbf{R}^N)}}$$

for all $u \in W^{1,N}(\mathbf{R}^N) \setminus \{0\}$. Here $\Phi_N(\xi)$ is defined by

$$\Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2} \frac{1}{j!}\xi^j.$$

It is also shown that $(*)$ with $\alpha \ge \alpha_N$ is false, which is different from the usual Trudinger's inequalities in bounded domains.

0. INTRODUCTION

In this note, we study the limit case of Sobolev's inequalities; suppose $N \ge 2$ and let $D \subset \mathbf{R}^N$ be an open set. We denote by $W_0^{1,N}(D)$ the usual Sobolev space, that is, the completion of $C_0^\infty(D)$ with the norm $\|u\|_{W_0^{1,p}(D)} = \|\nabla u\|_p + \|u\|_p$. Here

$$\|u\|_p = \left(\int_D |u|^p \, dx\right)^{1/p}.$$

It is well-known that

$$W_0^{1,p}(D) \subset L^{\frac{pN}{N-p}}(D) \quad \text{if } 1 \le p < N,$$
$$W_0^{1,p}(D) \subset L^\infty(D) \qquad \text{if } N < p.$$

The case $p = N$ is the limit case of these imbeddings and it is known that

$$W_0^{1,N}(D) \subset L^q(D) \quad \text{for } N \le q < \infty,$$
$$W_0^{1,N}(D) \not\subset L^\infty(D).$$

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This case is studied by Trudinger [14] more precisely and he showed for bounded domains $D \subset \mathbf{R}^N$

$$\int_D \exp\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right)dx \le C\,|D| \tag{0.1}$$

for $u \in W_0^{1,N}(D)\setminus\{0\}$, where the constants α, C are independent of u and D.

Trudinger's result is extended into two directions; the first one is to find the best exponents in (0.1). Moser [8] proved that (0.1) holds for $\alpha \le \alpha_N$ but not for $\alpha > \alpha_N$, where

$$\alpha_N = N\omega_{N-1}^{1/(N-1)} \tag{0.2}$$

and ω_{N-1} is the surface area of the unit sphere in $\mathbf{R}^N$. See also Adams [1]. We also refer to [3], [5], [7], [13] for the attainability of

$$\sup\left\{\int_D \exp\left(\alpha_N\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right)dx;\ u \in W_0^{1,N}(D)\setminus\{0\}\right\}.$$

The second direction is to extend Trudinger's result for unbounded domains and for Sobolev spaces of higher order and fractional order. We refer to D. R. Adams [1], R. A. Adams [2], Ogawa [9], Ogawa-Ozawa [10], Ozawa [11], Strichartz [12].

In this paper, we study a version of Trudinger inequalities in $\mathbf{R}^N$ and their best exponents; we show

(0.3)

$$\int_{\mathbf{R}^N} \exp\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) - \sum_{j=0}^{N-2}\frac{1}{j!}\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right)^j dx \le C\frac{\|u\|_N^N}{\|\nabla u\|_N^N}$$

for $u \in W^{1,N}(\mathbf{R}^N)\setminus\{0\}$, where α, $C > 0$ are independent of u, and we also find the best exponents α for (0.3).

In [9], [11], [2], (0.3) and related inequalities are obtained without studying their best exponents; Ogawa [9] obtained (0.3) for $N = 2$ and Ozawa [11] extended it for functions in the Sobolev space $H^{N/p,p}(\mathbf{R}^N) = (1-\Delta)^{-N/2p}L^p(\mathbf{R}^N)$ of fractional order. See also [10]. Adams [2] studied a different version of (0.3); however the dependence in u of the right-hand side is not given explicitly.

The main purpose of this paper is to study the best exponents α in (0.3) as well as to give a simplified proof of (0.3).

To simplify notation, we use

$$\Phi_N(\xi) = \exp(\xi) - \sum_{j=0}^{N-2}\frac{1}{j!}\xi^j. \tag{0.4}$$

With this notation, (0.3) becomes

$$\int_{\mathbf{R}^N}\Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right)dx \le C\frac{\|u\|_N^N}{\|\nabla u\|_N^N}. \tag{0.5}$$

One of the virtues of the inequality (0.5) is its scale-invariance; for $u \in W^{1,N}(\mathbf{R}^N)$ and $\lambda > 0$, we set

$$u_\lambda(x) = u(\lambda x). \tag{0.6}$$

We can easily see that $\|\nabla u_\lambda\|_N = \|\nabla u\|_N$ and

$$\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u_\lambda(x)|}{\|\nabla u_\lambda\|_N}\right)^{\frac{N}{N-1}}\right) dx = \lambda^{-N}\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) dx, \tag{0.7}$$

$$\|u_\lambda\|_N^N = \lambda^{-N}\|u\|_N^N. \tag{0.8}$$

Thus (0.5) is invariant under the scaling (0.6) and we believe the best exponents α in (0.5) are of interest.

Our main result is the following.

Theorem 0.1. *Suppose $N \geq 2$. Then for any $\alpha \in (0, \alpha_N)$ (α_N is given in (0.2)), there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) dx \leq C_\alpha \frac{\|u\|_N^N}{\|\nabla u\|_N^N} \qquad \text{for } u \in W^{1,N}(\mathbf{R}^N)\setminus\{0\}.$$

We remark that the restriction $\alpha < \alpha_N$ is optimal. The limit exponent α_N is excluded for (0.5). It is quite different from Moser's result for (0.1).

Theorem 0.2. *For $\alpha \geq \alpha_N$, there exists a sequence $(u_k(x))_{k=1}^\infty \subset W^{1,N}(\mathbf{R}^N)$ such that $\|\nabla u_k\|_N = 1$ and*

$$\begin{aligned}\frac{1}{\|u_k\|_N^N}\int_{\mathbf{R}^N} \Phi_N\left(\alpha\left(\frac{|u_k(x)|}{\|\nabla u_k\|_N}\right)^{\frac{N}{N-1}}\right) dx \\ \geq \frac{1}{\|u_k\|_N^N}\int_{\mathbf{R}^N} \Phi_N\left(\alpha_N\left(\frac{|u_k(x)|}{\|\nabla u_k\|_N}\right)^{\frac{N}{N-1}}\right) dx \to \infty\end{aligned} \tag{0.9}$$

as $k \to \infty$.

Remark 0.3. Even if we consider (0.5) in a bounded domain D, i.e.,

$$\int_D \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_N}\right)^{\frac{N}{N-1}}\right) dx \leq C_\alpha \frac{\|u\|_N^N}{\|\nabla u\|_N^N} \qquad \text{for } u \in W^{1,N}(D)\setminus\{0\}, \tag{0.10}$$

the limit exponent α_N is still excluded. It is because of the scale-invariance (0.7)–(0.8). See Remark 2.1 below.

As to the proof of the inequality (0.5), following the original idea of Trudinger, [9], [10], [11] made use of a combination of the power series expansion of the exponential function and sharp multiplicative inequalities:

$$\|u\|_q \leq C(N,q)\|u\|_N^{N/q}\|\nabla u\|_N^{1-N/q}. \tag{0.11}$$

For multiplicative inequalities of type (0.11) and their applications, we refer to Edmunds-Ilyin [4] and Kozono-Ogawa-Sohr [6]. We also remark that in Ozawa [11] multiplicative inequalities for functions $H^{N/p,p}(\mathbf{R}^N)$ are given and they are applied to obtain Brezis-Gallouet-Wainger type inequalities.

We give proofs of Theorems 0.1 and 0.2 in the following sections. We take a different approach from [9], [10], [11], we use Moser's idea; we take symmetrization of functions and we reduce (0.5) to one-dimensional inequality.

1. Proof of Theorem 0.1

To prove Theorem 0.1, we use an idea of Moser [8]. By means of symmetrization, it suffices to show the desired inequality (0.5) for functions $u(x) = u(|x|)$, which are non-negative, compactly supported, radially symmetric, and $u(|x|) : [0, \infty) \to \mathbf{R}$ are decreasing.

Following Moser's argument, we set

$$w(t) = N^{\frac{N-1}{N}} \omega_{N-1}^{\frac{1}{N}} u\left(e^{-\frac{t}{N}}\right), \qquad |x|^N = e^{-t}. \tag{1.1}$$

Then $w(t)$ is defined on $(-\infty, \infty)$ and satisfies

$$w(t) \geq 0 \quad \text{for } t \in \mathbf{R}, \tag{1.2}$$

$$\dot{w}(t) \geq 0 \quad \text{for } t \in \mathbf{R}, \tag{1.3}$$

$$w(t_0) = 0 \quad \text{for some } t_0 \in \mathbf{R}\,. \tag{1.4}$$

Moreover we have

$$\int_{\mathbf{R}^N} |\nabla u|^N \, dx = \int_{-\infty}^{\infty} |\dot{w}(t)|^N \, dt, \tag{1.5}$$

$$\int_{\mathbf{R}^N} \Phi_N \left(\alpha u^{\frac{N}{N-1}}\right) dx = \frac{\omega_{N-1}}{N} \int_{-\infty}^{\infty} \Phi_N \left(\frac{\alpha}{\alpha_N} w(t)^{\frac{N}{N-1}}\right) e^{-t} \, dt, \tag{1.6}$$

$$\int_{\mathbf{R}^N} |u(x)|^N \, dx = \frac{1}{N^N} \int_{-\infty}^{\infty} |w(t)|^N e^{-t} \, dt. \tag{1.7}$$

Thus, to prove Theorem 0.1, it suffices to show that for any $\beta \in (0, 1)$ there exists a constant $C_\beta > 0$ such that

$$\int_{-\infty}^{\infty} \Phi_N \left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t} \, dt \leq C_\beta \int_{-\infty}^{\infty} |w(t)|^N e^{-t} \, dt \tag{1.8}$$

for all functions $w(t)$ satisfying (1.2)–(1.4) and

$$\int_{-\infty}^{\infty} |\dot{w}(t)|^N \, dt = 1. \tag{1.9}$$

Proof of Theorem 0.1. Let $w(t)$ be a function satisfying (1.2)–(1.4) and (1.9). We set

$$T_0 = \sup\{t \in \mathbf{R}; w(t) \leq 1\} \in (-\infty, \infty].$$

We decompose the integral on the left-hand side of (1.8) according to the decomposition $(-\infty, \infty) = (-\infty, T_0] \cup [T_0, \infty)$.

For $t \in (-\infty, T_0]$, we have $w(t) \in [0, 1]$. We can find a constant $m_N > 0$ such that

$$\Phi_N(\xi) \leq m_N \xi^{N-1} \quad \text{for } \xi \in [0, 1].$$

Thus we have

$$\int_{-\infty}^{T_0} \Phi_N \left(\beta w(t)^{\frac{N}{N-1}}\right) e^{-t} \, dt \leq m_N \int_{-\infty}^{T_0} w(t)^N e^{-t} \, dt. \tag{1.10}$$

Next we consider the integral over $[T_0, \infty)$. Since $w(T_0) = 1$, we have for $t \geq T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^{t} \dot{w}(\tau) d\tau \\ &\leq w(T_0) + (t - T_0)^{\frac{N-1}{N}} \left(\int_{T_0}^{\infty} \dot{w}(\tau)^N d\tau \right)^{\frac{1}{N}} \\ &\leq 1 + (t - T_0)^{\frac{N-1}{N}}. \end{aligned}$$

We remark that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$1 + s^{\frac{N-1}{N}} \leq ((1+\varepsilon)s + C_\varepsilon)^{\frac{N-1}{N}} \quad \text{for all } s \geq 0.$$

Thus, we have

$$|w(t)|^{\frac{N}{N-1}} \leq (1+\varepsilon)(t - T_0) + C_\varepsilon \quad \text{for } t \geq T_0.$$

Since $\beta \in (0,1)$, we can choose $\varepsilon > 0$ small so that $\beta(1+\varepsilon) < 1$. Thus we have

$$\begin{aligned} \int_{T_0}^{\infty} \Phi_N \left(\beta w(t)^{\frac{N}{N-1}} \right) e^{-t}\, dt &\leq \int_{T_0}^{\infty} \exp \left(\beta w(t)^{\frac{N}{N-1}} - t \right) dt \\ &\leq \int_{T_0}^{\infty} \exp \left((\beta(1+\varepsilon) - 1)(t - T_0) + \beta C_\varepsilon - T_0 \right) dt \\ &= \frac{1}{1 - \beta(1+\varepsilon)} e^{\beta C_\varepsilon} e^{-T_0}. \end{aligned} \tag{1.11}$$

On the other hand,

$$\int_{T_0}^{\infty} |w(t)|^N e^{-t}\, dt \geq \int_{T_0}^{\infty} e^{-t}\, dt = e^{-T_0}. \tag{1.12}$$

Therefore it follows from (1.11) and (1.12) that

$$\int_{T_0}^{\infty} \Phi_N \left(\beta w(t)^{\frac{N}{N-1}} \right) e^{-t}\, dt \leq \frac{e^{\beta C_\varepsilon}}{1 - \beta(1+\varepsilon)} \int_{T_0}^{\infty} |w(t)|^N e^{-t}\, dt. \tag{1.13}$$

Thus, setting $C_\beta = \max\{m_N, \dfrac{e^{\beta C_\varepsilon}}{1 - \beta(1+\varepsilon)}\}$, we obtain (1.8). $\square$

2. Proof of Theorem 0.2

It suffices to show Theorem 0.2 for $\alpha = \alpha_N$. We use the idea of Moser again. Repeating the argument of the previous section, it suffices to find a sequence of functions $w_k(t) : \mathbf{R} \to \mathbf{R}$ which satisfies (1.1)–(1.4), (1.9) and

$$\int_{-\infty}^{\infty} |w_k(t)|^N e^{-t}\, dt \to 0 \quad \text{as } k \to \infty, \tag{2.1}$$

$$\int_{-\infty}^{\infty} \Phi_N \left(w_k(t)^{\frac{N}{N-1}} \right) e^{-t}\, dt \geq \frac{1}{2} \quad \text{for large } k. \tag{2.2}$$

If we define a sequence of functions $(u_k(x))_{k=1}^{\infty} \subset W^{1,N}(\mathbf{R}^N)$ from $(w_k(t))_{k=1}^{\infty}$ through the relation (1.1), it follows from (1.5)–(1.7), (1.9), (2.1) and (2.2) that $\|\nabla u_k\|_N = 1$ and (0.9). Thus $(u_k)_{k=1}^{\infty}$ has a desired property in Theorem 0.2.

Here we give an example of $(w_k(t))_{k=1}^\infty$ explicitly. We set

$$w_k(t) = \begin{cases} 0 & \text{for } t \le 0, \\ k^{\frac{N-1}{N}} \dfrac{t}{k} & \text{for } 0 \le t \le k, \\ k^{\frac{N-1}{N}} & \text{for } k \le t. \end{cases}$$

Such functions appeared in [8] to show that the integral on the left-hand side of (0.1) can be made arbitrarily large for $\alpha > \alpha_N$. It is easily seen that $w_k(t)$ satisfies (1.2)–(1.4) and (1.9).

First we verify (2.1).

$$\begin{aligned} \int_{-\infty}^\infty |w_k(t)|^N e^{-t}\,dt &= \int_0^k \left(k^{\frac{N-1}{N}} \frac{t}{k}\right)^N e^{-t}\,dt + \int_k^\infty k^{N-1} e^{-t}\,dt \\ &\le \frac{1}{k}\int_0^\infty t^N e^{-t}\,dt + k^{N-1}e^{-k} \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Next we deal with (2.2).

$$\begin{aligned} &\int_{-\infty}^\infty \Phi_N\left(w_k(t)^{\frac{N}{N-1}}\right) e^{-t}\,dt \\ &\qquad = \int_0^k \Phi_N \left(k \left(\frac{t}{k}\right)^{\frac{N}{N-1}}\right) e^{-t}\,dt + \int_k^\infty \Phi_N(k) e^{-t}\,dt \\ &= \int_0^k \left(\exp\left(k\left(\frac{t}{k}\right)^{\frac{N}{N-1}}\right) - \sum_{j=0}^{N-2} \frac{1}{j!} \left(k \left(\frac{t}{k}\right)^{\frac{N}{N-1}}\right)^j \right) e^{-t}\,dt \\ &\qquad + \Phi_N(k) e^{-k} \\ &= \int_0^k \exp\left(k\left(\frac{t}{k}\right)^{\frac{N}{N-1}} - t\right) dt - \sum_{j=0}^{N-2} \frac{1}{j!} k^{-\frac{j}{N-1}} \int_0^k t^{\frac{N}{N-1}j} e^{-t}\,dt \\ &\qquad + \left(e^k - \sum_{j=0}^{N-2} \frac{1}{j!} k^j \right) e^{-k} \\ &\ge \int_0^k e^{-t}\,dt - \sum_{j=0}^{N-2} \frac{1}{j!} k^{-\frac{j}{N-1}} \int_0^k t^{\frac{N}{N-1}j} e^{-t}\,dt \\ &\qquad + \left(e^k - \sum_{j=0}^{N-2} \frac{1}{j!} k^j \right) e^{-k} \\ &\to 1 - 1 + 1 = 1 \quad \text{as } k \to \infty. \end{aligned}$$

Thus we obtain (2.1) and (2.2). This completes the proof of Theorem 0.2. □

Remark 2.1. The function $u_k(x)$ corresponding $w_k(t)$ has a compact support, i.e., $supp\, u_k(x) \subset \{x \in \mathbf{R}^N; |x| \le 1\}$. Thus (0.10) with $\alpha = \alpha_N$ is false for $D = \{x \in \mathbf{R}^N; |x| < 1\}$. If we set for $a > 0$

$$w_{a,k}(t) = w_k(t + N \log a),$$

then corresponding $u_{a,k}(x)$ has a compact support, i.e., $supp\, u_{a,k}(x) \subset \{x \in \mathbf{R}^N; |x| \leq a\}$ and satisfies $\|\nabla u_{a,k}\|_N = 1$ and (0.9). Since we can choose $a > 0$ arbitrarily small, (0.10) with $\alpha = \alpha_N$ is false for any domain D.

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