

TRUE MANY-PARTICLE SCATTERING IN THE OSCILLATOR REPRESENTATION

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Scattering theory in the oscillator representation is generalized to true many-particle scattering.

In fond memory of N. A. Sveshnikov

1. Introduction

We suggest an approach to many-particle scattering theory in the oscillator representation (OR) and treat the simplest case, in which the wave function in the asymptotic domain has the form of a spherical wave in a multidimensional space. This corresponds to the so-called true many-particle scattering (TMS) [1, 2], i.e., to inclusion of only those states for which “democracy” is observed in the system and no single pair or group of particles is selected in the sense of forming bound states or scattering on an energy surface.

Constructing a TMS wave function is part of the problem of finding a wave function for a many-body system [1]. However, investigating the TMS approximation is independently worthwhile from the physical standpoint because this approximation can adequately describe many processes in the disintegration of light nuclei into several fragments (see [3, 4]). Moreover, it ceases to be an approximation and becomes, in fact, an exact theory for the “democratic decay” processes [4], i.e., three-particle decay processes $A \rightarrow A_1 + A_2 + A_3$ in which no binary subsystem $(A_i A_j)$ has a bound state, four-particle decay processes $A \rightarrow A_1 + A_2 + A_3 + A_4$ in which no binary subsystem $(A_i A_j)$ or ternary subsystem $(A_i A_j A_k)$ has a bound state, etc. Democratic decay processes have recently become very important because radioactive beams have begun to be used to systematically investigate exotic short-lived neutron-excessive nuclei, many of which can only decay over “democratic channels” (e.g., $^{11}\text{Li} \rightarrow ^9\text{Li} + n + n$, $^6\text{He} \rightarrow ^4\text{He} + n + n$, $^{14}\text{Be} \rightarrow ^{12}\text{Be} + n + n$, and the like) under low excitation energies that are of great interest at the present. The excited states of such nuclei have recently been theoretically investigated precisely in the framework of the TMS method (e.g., [5–10]). Some other currently central problems (e.g., resonances in two- and three-neutron systems [11]) are also studied in the TMS approximation.

Furthermore, as is shown below, the TMS approximation can also be used to find the S -matrix for a many-particle system, and calculation of its poles not only can uniquely determine the parameters of resonance states but also can refine the bonding energy of the system. This refinement proves essential for weakly developed systems [8, 10]. The TMS approximation in this approach can probably be very exact for studying the ground state and other bound states of a many-particle system.

It is natural to use the K -harmonics method, related to expanding the system wave function with respect to the hyperspherical basis, to calculate TMS characteristics (see [2–4] and references therein). We suggest that the expansion of the wave function with respect to the eigenfunctions of a $(3A - 3)$ -dimensional harmonic oscillator (A is the number of particles in the system) in the hyperspherical coordinates ρ and Ω be

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used to solve the equations in the K -harmonics method (including the domain of the continuous spectrum). As a result, we study the Schrödinger equation in the hyperspherical OR: the problem reduces to analyzing an infinite system of linear algebraic equations instead of solving a system of coupled differential equation of the K -harmonics method. The OR method [12–16] is very efficient in scattering theory and the theory of reactions with binary channels [17–21]. Our approach is a natural generalization of this method to the TMS case.

We published some of the results in this paper in [22], and they were used in [23, 8, 10] for concrete calculations of nuclear systems. The calculation results show that the suggested approach is very effective; its efficiency can be further improved using the analytic results in the present paper that were not published earlier.

2. The Schrödinger equation in the hyperspherical OR

Assuming that the center-of-mass motion has already been separated off, we write the wave function of an A -particle system characterized by the total energy E and other quantum numbers denoted by the multi-index β as the expansion

$$|E\beta\rangle = \sum_{nK\gamma} \langle nK\gamma|E\beta\rangle |nK\gamma\rangle, \quad (1)$$

where the basis functions $|nK\gamma\rangle$ are the eigenfunctions of a $(3A - 3)$ -dimensional harmonic oscillator with the frequency $\hbar\omega$ in the hyperspherical coordinates,

$$|nK\gamma\rangle = \Phi_{nK}(\rho)\mathfrak{Y}_{K\gamma}(\Omega). \quad (2)$$

Here, K is the hypermomentum, and γ is the set of the other quantum numbers characterizing the functions $\mathfrak{Y}_{K\gamma}(\Omega)$, which are the hyperspherical harmonics $Y_{K\gamma}(\Omega)$ of the angular variables Ω for a system of spinless particles. If the spin and isospin degrees of freedom of the particles are taken into account, then the $\mathfrak{Y}_{K\gamma}(\Omega)$ are products of hyperspherical harmonics and spin–isospin functions (antisymmetrized or symmetrized combinations of these products if the system involves identical fermions or bosons respectively); in this case, γ includes the spin–isospin quantum numbers as well. The variable ρ in (2) is the hyperradius,

$$\rho = \sqrt{\sum_{i=1}^A (\mathbf{r}_i - \mathbf{R})^2},$$

where \mathbf{r}_i is the coordinate of an individual particle, \mathbf{R} is the center-of-mass coordinate, and n is the principal quantum number characterizing the radial function

$$\Phi_{nK} \equiv \Phi_n^{\mathcal{L}}(\rho) = \rho^{-(3A-4)/2} \varphi_{nK}(\rho), \quad (3)$$

$$\varphi_{nK}(\rho) \equiv \varphi_n^{\mathcal{L}}(\rho) = (-1)^n \sqrt{\frac{2\lambda n!}{\Gamma(n + \mathcal{L} + 3/2)}} (\lambda\rho)^{\mathcal{L}+1} e^{-\frac{\lambda^2 \rho^2}{2}} L_n^{\mathcal{L}+\frac{1}{2}}(\lambda^2 \rho^2). \quad (4)$$

Here,

$$\mathcal{L} = K + \frac{3A - 6}{2},$$

$L_n^\alpha(x)$ is the associated Laguerre polynomial [24], and $\lambda^{-1} = \sqrt{\hbar/(m\omega)}$ plays the part of the “oscillator radius” and is a free parameter whose value is chosen to ensure a sufficiently fast convergence of the computational scheme. Basis (2) is orthonormalized,

$$\langle nK\gamma|n'K'\gamma'\rangle = \delta_{nn'}\delta_{\Gamma\Gamma'}, \quad (5)$$

i.e.,

$$\int \varphi_{nK}^*(\rho) \varphi_{n'K}(\rho) d\rho = \delta_{nn'}$$

(the symbol Γ in (5) denotes the set of the quantum numbers K and γ).

For a two-particle system ($A = 2$), the theory presented below becomes the ordinary two-particle scattering theory. Two-particle scattering is a particular TMS case in which $K = \mathcal{L} = L$, where L is the orbital angular momentum, the hyperspherical functions $Y_{K\gamma}(\Omega)$ become the ordinary spherical functions $Y_{Lm}(\vartheta, \varphi)$, and (3) and (4) become the radial functions of a three-dimensional spherically symmetrical oscillator. The indices Γ play the same formal role as those distinguishing between the channels in the two-particle scattering theory. However, we are only interested in the situation in which the threshold energy Γ is the same for all the "channels." In the two-particle case, this corresponds to considering the general theory of scattering on a noncentral potential that mixes states with different momenta L or at least with different spin projections σ . The theory can be easily extended to the general case that includes different physical channels corresponding to different types of excitation of the internal degrees of freedom of the particles in question and characterized by different threshold energies. Such a generalization can also be interesting for systems with $A \geq 3$.

The Schrödinger equation becomes

$$\sum_{n'K'\gamma'} \langle nK\gamma | H - E | n'K'\gamma' \rangle \langle n'K'\gamma' | E\beta \rangle = 0 \quad (6)$$

in the hyperspherical OR, where $H = T + V$. The kinetic-energy matrix T is diagonal with respect to the indices Γ and Γ' and tridiagonal with respect to n and n' ,

$$\begin{aligned} \langle nK\gamma | T | n'K'\gamma' \rangle = \frac{\hbar\omega}{2} \delta_{\Gamma\Gamma'} \left[-\sqrt{(n+1)\left(n+\mathcal{L}+\frac{3}{2}\right)} \delta_{n+1,n'} + \right. \\ \left. + \left(2n+\mathcal{L}+\frac{3}{2}\right) \delta_{nn'} - \sqrt{n\left(n+\mathcal{L}+\frac{1}{2}\right)} \delta_{n-1,n'} \right]. \end{aligned} \quad (7)$$

In contrast to the kinetic-energy matrix, the potential-energy matrix, which we write as the sum of pairwise interaction potentials

$$V = \sum_{i < j} V(\mathbf{r}_i - \mathbf{r}_j),$$

is not diagonal with respect to Γ and Γ' (except for the case $A = 2$ with central forces).

One of the advantages of the oscillator basis in the TMS problem is that it admits an easy transition from one set of Jacobi coordinates to another (substantially facilitating the calculation of the potential-energy matrix elements [2, 19, 25]). System (6) consists of infinitely many linear algebraic equations for the coefficients $\langle nK\gamma | E\beta \rangle \equiv \langle n\Gamma | E\beta \rangle$ in expansion (1). However, the situation is simplified because for sufficiently large n , the elements of the potential-energy matrix $\langle n\Gamma | V | n'\Gamma' \rangle$ are small compared with those of the kinetic-energy matrix $\langle n\Gamma | T | n'\Gamma' \rangle$ (as is seen from (7), the latter grow linearly with increasing n for $n \gg 1$). Therefore, we can discard the matrix elements of the potential energy V for large values of n and retain those of the kinetic energy for $n \gg 1$, i.e., set

$$\langle n\Gamma | H | n'\Gamma' \rangle = \langle n\Gamma | T | n'\Gamma' \rangle \quad \text{if } n > n_\Gamma \text{ or } n' > n_{\Gamma'}. \quad (8)$$

The specific values of n_Γ chosen for each partial wave Γ must be so large that the calculation results do not substantially change under their increase.

In approximation (8), system of equations (6) splits into two parts that correspond to an “interior” domain with sufficiently small values of $n \leq n_\Gamma$ and an “exterior” domain with $n > n_\Gamma$, where only the kinetic-energy matrix elements are taken into account and the interaction between particles is neglected. Because the kinetic energy is diagonal with respect to the indices Γ and Γ' , the equations for the coefficients $\langle n\Gamma|E\beta\rangle$ corresponding to different partial waves Γ are separated in the exterior domain and can be solved independently. The coefficients $\langle n\Gamma|E\beta\rangle$ satisfy the three-term recurrence relation (TRR)

$$\begin{aligned} \sqrt{(n+1)\left(n+\mathcal{L}+\frac{1}{2}\right)}\langle n-1,\Gamma|E\beta\rangle - \left(2n+\mathcal{L}+\frac{3}{2}-q^2\right)\langle n\Gamma|E\beta\rangle + \\ + \sqrt{n\left(n+\mathcal{L}+\frac{3}{2}\right)}\langle n+1,\Gamma|E\beta\rangle = 0 \end{aligned} \quad (9)$$

in this domain, where $q = \kappa/\lambda = \sqrt{2E/(\hbar\omega)}$. Relation (9) should be regarded as a second-order difference equation, which has two linearly independent solutions in general. The solution to Eq. (9) can be chosen uniquely by imposing the corresponding asymptotic conditions in each partial wave $\Gamma = K, \gamma$. The matrix elements of the Hamiltonian are diagonal with respect to the indices Γ and Γ' in the thoroughly studied case of ordinary potential scattering ($A = 2$) with central interaction [12–16] (in this case, K coincides with the conserved angular momentum L), and the solution can be constructed independently in each partial wave. However, in the more general TMS case ($A \geq 3$), the matrix elements of the Hamiltonian H are no longer diagonal with respect to Γ and Γ' . Therefore, the solutions for different partial waves $\Gamma = K, \gamma$ in the interior domain are coupled. As a result, the TMS problem becomes similar to the coupled-channel problem. We note that in the case of a structureless fragment, the TMS problem is only formally a coupled-channel problem because every “channel” under consideration, being associated with the same physical process, corresponds to the same physical channel. Nevertheless, we speak of a partial wave $\Gamma = K, \gamma$ as a channel Γ and introduce various quantities used in multichannel scattering theory, e.g., the S -matrix relating to the solutions in different channels, and so on. Different physical channels can also be taken into account using our approach.

3. Solution of the Schrödinger equation in the asymptotic domain

3.1. Qualitative analysis of solutions of difference equation (9). In the exterior domain (i.e., for $n > n_\Gamma$), the expansion coefficients $\langle n\Gamma|E\beta\rangle$ in (1) satisfy second-order difference equation (9) in each channel Γ . Because the interaction between particles is neglected in this domain, i.e., (9) is the free Schrödinger equation in the hyperspherical OR, exact solutions can be found for it. However, before deriving them in explicit form, we discuss the qualitative character of the expected solutions. For this, we approximate difference equation (9) with a second-order differential equation. The method for approximating second-order difference equations by differential equations was developed in [26, 27], where we obtained all the necessary relations, investigated their accuracy and field of applicability, etc. For the justification of the approach applied below, see those papers. We are only interested in a qualitative analysis of solutions of Eq. (9) in this section; we therefore use the simplest version of the approximation of difference equation (9) by differential equations without taking the effect of the additional potential (see [27]) into account and only include terms of a comparatively low order of smallness. When necessary, the relations obtained below can be easily refined by the methods in [27].

We introduce the expression

$$z = 2\sqrt{n + \frac{\mathcal{L}}{2} + \frac{3}{4}},$$

which is regarded as a continuous variable. The function $\chi(z) = z^{1/2}\langle n\Gamma|E\beta\rangle$ is used instead of the expansion coefficients in (1). Expanding the left-hand side of Eq. (9) as a series in powers of $1/z$ and

retaining only the terms up to and including the second order with respect to $1/z$, we obtain the differential equation

$$\chi'' - \frac{\mathcal{L}(\mathcal{L} + 1)}{z^2} + q^2\chi = 0, \quad (10)$$

which approximates TRR (9) sufficiently accurately for $z \gg 1$.

The solutions of one-dimensional Schrödinger equation (10) with a centrifugal potential corresponding to the “momentum” \mathcal{L} (we recall that the “momentum” \mathcal{L} is integer for an even number of particles A and half-integer for an odd A) can be expressed via linear combinations of the Bessel and Neumann functions $J_{\mathcal{L}+1/2}(qz)$ and $N_{\mathcal{L}+1/2}(qz)$, which we define in accordance with [28]. As a result, we find that the expansion coefficients in (1) have the asymptotic behavior for $n \gg 1$

$$\begin{aligned} \langle n\Gamma|E\beta\rangle &\xrightarrow{n \rightarrow \infty} \alpha\sqrt{qz}J_{\mathcal{L}+\frac{1}{2}}(qz) + \beta\sqrt{qz}N_{\mathcal{L}+\frac{1}{2}}(qz) \xrightarrow{n \rightarrow \infty} \\ &\xrightarrow{n \rightarrow \infty} \sqrt{\frac{2}{\pi}} \left[\alpha \sin\left(qz - \frac{\pi\mathcal{L}}{2}\right) - \beta \cos\left(qz - \frac{\pi\mathcal{L}}{2}\right) \right], \end{aligned} \quad (11)$$

where α and β are constants not depending on n . The numerical values of α and β are determined by the behavior of $\langle n\Gamma|E\beta\rangle$ at small values of n , for which asymptotic equation (9) can no longer be used and the potential energy of particle interaction should be taken into account (see Sec. 4).

The quasi-classical considerations in [14] show that for large values of n , the oscillator basis functions behave like the delta function at the classical turning point $\rho_{\text{turn}} = z/\lambda$ (for more details, see Appendix A) and lead to the conclusion (see expression (A.2)) that

$$\langle n\Gamma|E\beta\rangle \sim \Psi_{E\beta}^{\mathcal{L}}\left(\frac{z}{\lambda}\right), \quad (12)$$

where $\Psi_{E\beta}^{\mathcal{L}}(\rho)$ is the wave function of the system in the coordinate representation. It follows that in the coordinate representation, the functions $\langle n\Gamma|E\beta\rangle$ with asymptotic behavior (11) correspond to the radial functions with the asymptotic expression

$$\begin{aligned} \Psi_{E\beta}^{\mathcal{L}}(\rho) &= \sum_n \langle n\Gamma|E\beta\rangle \varphi_n^{\mathcal{L}}(\rho) \xrightarrow{n \rightarrow \infty} G\sqrt{\varkappa\rho} \left[\alpha J_{\mathcal{L}+\frac{1}{2}}(\varkappa\rho) + \beta N_{\mathcal{L}+\frac{1}{2}}(\varkappa\rho) \right] \xrightarrow{n \rightarrow \infty} \\ &\xrightarrow{n \rightarrow \infty} G \sqrt{\frac{2}{\pi}} \left[\alpha \sin\left(\varkappa\rho - \frac{\pi\mathcal{L}}{2}\right) - \beta \cos\left(\varkappa\rho - \frac{\pi\mathcal{L}}{2}\right) \right], \end{aligned} \quad (13)$$

where G is a normalization constant and $\varkappa = q\lambda$ is the wave number. Thus, $\varphi = \tan^{-1}(\beta/\alpha)$ represents the TMS phase in the given partial wave Γ . If one of the coefficients α or β is imaginary, then (13) corresponds to converging or diverging waves in the partial wave Γ . As can be seen from (11)–(13), in order to obtain the solution of TRR (9) leading to the asymptotic expression $\sqrt{\varkappa\rho}J_{\mathcal{L}+1/2}(\varkappa\rho)$ for the coordinate wave function, we should require that

$$\langle n\Gamma|E\beta\rangle \xrightarrow{n \rightarrow \infty} \sqrt{qz}J_{\mathcal{L}+\frac{1}{2}}(qz);$$

in order to obtain the solution with the coordinate asymptotic expression $\sqrt{\varkappa\rho}N_{\mathcal{L}+1/2}(\varkappa\rho)$, we should require that

$$\langle n\Gamma|E\beta\rangle \xrightarrow{n \rightarrow \infty} \sqrt{qz}N_{\mathcal{L}+\frac{1}{2}}(qz).$$

We now derive explicit solutions of TRR (9) with this asymptotic behavior (and also solutions of the type of converging and diverging waves $\sqrt{\varkappa\rho}H_{\mathcal{L}+1/2}^{(1,2)}(qz)$, where $H_{\nu}^{(1,2)}(z)$ are the Hankel functions [24, 28]).

3.2. Fundamental systems of solutions for the free Schrödinger equation in the hyper-spherical OR. We consider the functions $S_n^\mathcal{L}(q)$, $C_n^\mathcal{L}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$ given by the expressions

$$S_n^\mathcal{L}(q) = \sqrt{\frac{2n!}{\lambda\Gamma(n+\mathcal{L}+3/2)}} q^{\mathcal{L}+1} e^{-\frac{q^2}{2}} L_n^{\mathcal{L}+\frac{1}{2}}(q^2), \quad (14)$$

$$C_n^\mathcal{L}(q) = -\frac{2q}{\pi S_0^\mathcal{L}(q)} \mathcal{P} \int_0^\infty \frac{S_0^\mathcal{L}(q') S_n^\mathcal{L}(q')}{q^2 - q'^2} dq', \quad (15)$$

$$C_n^{\mathcal{L}(+) } (q) = -\frac{2q}{\pi S_0^\mathcal{L}(q)} \int_0^\infty \frac{S_0^\mathcal{L}(q') S_n^\mathcal{L}(q')}{q^2 - q'^2 + i\epsilon} dq', \quad (16)$$

$$C_n^{\mathcal{L}(-) } (q) = -\frac{2q}{\pi S_0^\mathcal{L}(q)} \int_0^\infty \frac{S_0^\mathcal{L}(q') S_n^\mathcal{L}(q')}{q^2 - q'^2 - i\epsilon} dq'. \quad (17)$$

The additional term $\pm i\epsilon$ in (16) and (17) indicates the direction in which the contours going around the poles are described; \mathcal{P} means that the integral must be calculated in the sense of the principal value. We show that $S_n^\mathcal{L}(q)$, $C_n^\mathcal{L}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$ are solutions to difference equation (9).

That $S_n^\mathcal{L}(q)$ satisfies (9) follows directly from the recurrence relation for the Laguerre polynomials [24]. Using this, we substitute (15)–(17) directly in (9) to ensure that $C_n^\mathcal{L}(q)$, $C_n^{\mathcal{L}(+) } (q)$, and $C_n^{\mathcal{L}(-) } (q)$ also satisfy Eq. (9) for $n > 0$ or, more precisely, the inhomogeneous second-order difference equation

$$\sqrt{(n+1)\left(n+\mathcal{L}+\frac{1}{2}\right)} A_{n-1}^\mathcal{L} - \left(2n+\mathcal{L}+\frac{3}{2}-q^2\right) A_n^\mathcal{L} + \sqrt{n\left(n+\mathcal{L}+\frac{3}{2}\right)} A_{n+1}^\mathcal{L} = -\frac{2q}{\lambda\pi S_0^\mathcal{L}(q)} \delta_{n0}, \quad (18)$$

where $n \geq 0$ and $A_n^\mathcal{L}$ denotes $C_n^\mathcal{L}(q)$, $C_n^{\mathcal{L}(+) } (q)$, or $C_n^{\mathcal{L}(-) } (q)$.

Using the Sochozki formula

$$\frac{1}{x \pm i\epsilon} = \mp i\pi\delta(x) + \mathcal{P}\frac{1}{x}, \quad (19)$$

we find the relationship between solutions (15)–(17),

$$C_n^{\mathcal{L}(\pm) } (q) = C_n^\mathcal{L}(q) \pm iS_n^\mathcal{L}(q), \quad (20)$$

i.e.,

$$C_n^\mathcal{L}(q) = \frac{1}{2} \left(C_n^{\mathcal{L}(+) } (q) + C_n^{\mathcal{L}(-) } (q) \right). \quad (21)$$

Any two of solutions (14)–(17) to Eq. (9) are linearly independent and form a fundamental system of solutions. To show this, it suffices to calculate the so-called Casorati determinant [29] playing the part of the Wronskian in the theory of difference equations and show that it is not identically zero for any n . The Casorati determinant of two solutions u_n and v_n of a second-order difference equation is defined by the relation

$$\mathcal{C}_n(u, v) = u_{n+1}v_n - u_nv_{n+1}. \quad (22)$$

To calculate $\mathcal{C}_n(C^\mathcal{L}, S^\mathcal{L})$, the Christoffel–Darboux sum for the Laguerre polynomials [24],

$$\frac{(n+1)!}{\Gamma(n+\alpha+1)(x-y)} \left[L_n^\alpha(x)L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x)L_n^\alpha(y) \right] = \sum_{m=0}^n \frac{m!}{\Gamma(m+\alpha+1)} L_m^\alpha(x)L_m^\alpha(y), \quad (23)$$

and the orthogonality property of $S_n^\mathcal{L}(q)$ expressed by the relation

$$\int_0^\infty S_n^\mathcal{L}(q) S_{n'}^\mathcal{L}(q) dq = \lambda^{-1} \delta_{nn'} \quad (24)$$

should be used. This results in

$$c_n(C^\mathcal{L}, S^\mathcal{L}) = -\frac{2q}{\pi\lambda\sqrt{(n+2)(n+\mathcal{L}+3/2)}}. \quad (25)$$

Using (20), we can now easily calculate the Casorati determinant for the other pairs of solutions as well, namely,

$$c_n(C^{\mathcal{L}(\pm)}, S^\mathcal{L}) = \mp i c_n(C^{\mathcal{L}(\pm)}, C^\mathcal{L}) = \frac{i}{2} c_n(C^{\mathcal{L}(+)}, C^{\mathcal{L}(-)}) = c_n(C^\mathcal{L}, S^\mathcal{L}). \quad (26)$$

Solutions (14)–(17) are treated in detail in Appendix B: various properties are considered, their expressions in terms of special functions are found, the asymptotic behavior as $n \rightarrow \infty$ is studied, the analytic continuation is investigated, and problems in numerically calculating $S_n^\mathcal{L}(q)$, $C_n^\mathcal{L}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$ (which is important for applications) are discussed. We now proceed to the asymptotic representations corresponding to (14)–(17) in the coordinate space.

3.2.1. The solution $S_n^\mathcal{L}(q)$. Difference equation (9), for which $S_n^\mathcal{L}(q)$ is a solution, plays the part of the free Schrödinger equation in the OR for the partial wave with hypermomentum K . The free radial Schrödinger equation has the form

$$\left[-\frac{d^2}{d\rho^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} - \varkappa^2 \right] \chi_{K\gamma}^{\text{free}}(\rho) = 0 \quad (27)$$

in the coordinate representation. The regular (normalized to the delta function of \varkappa) solution of Eq. (27) is $\chi_{K\gamma}^{\text{free}}(\rho) = \sqrt{\varkappa\rho} J_{\mathcal{L}+1/2}(\varkappa\rho)$. The oscillator functions in the momentum representation $S_n^\mathcal{L}(q)$ are the expansion coefficients for $\chi_{K\gamma}^{\text{free}}(\rho)$ with respect to radial oscillator function (4), i.e.,

$$\sum_{n=0}^{\infty} S_n^\mathcal{L}(q) \varphi_n^\mathcal{L}(\rho) = \chi_{K\gamma}^{\text{free}}(\rho) = \sqrt{\varkappa\rho} J_{\mathcal{L}+1/2}(\varkappa\rho) \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2}{\pi}} \sin\left(\varkappa\rho - \frac{\pi\mathcal{L}}{2}\right). \quad (28)$$

To prove (28), it suffices to calculate the integral

$$S_n^\mathcal{L}(q) = \int_0^\infty \sqrt{\varkappa\rho} J_{\mathcal{L}+1/2}(\varkappa\rho) \varphi_n^\mathcal{L}(\rho) d\rho, \quad (29)$$

which can be easily done using formulas in standard reference books (e.g., [30, 31]) and verifying that the resulting expressions coincide with (14).

3.2.2. The solutions $C_n^\mathcal{L}(q)$ and $C_n^{\mathcal{L}(\pm)}(q)$. The functions $C_n^\mathcal{L}(q)$ and $C_n^{\mathcal{L}(\pm)}(q)$ satisfy inhomogeneous second-order difference equation (18). It is easy to show that the inhomogeneous Schrödinger equation

$$\left[-\frac{d^2}{d\rho^2} + \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} - \varkappa^2 \right] \chi_{K\gamma}(\rho) = \frac{2\varkappa}{\pi S_0^\mathcal{L}(q)} \varphi_0^\mathcal{L}(\rho) \quad (30)$$

in the coordinate representation corresponds to (18). We use Green's functions to solve Eq. (30). Because the functions $\chi_{K\gamma}^{\text{free}}(\rho)$ defined by (28) form a complete system of solutions for free Schrödinger equation (27) and are normalized to the delta function of \varkappa , the Green's functions can be written as

$$G_E^{(\pm)}(\rho, \rho') = -\int_0^\infty \frac{t \sqrt{\rho\rho'} J_{\mathcal{L}+1/2}(t\rho) J_{\mathcal{L}+1/2}(t\rho')}{\varkappa^2 - t^2 \pm i\varepsilon} dt. \quad (31)$$

Computing the integral in (31) with standard formulas (see [30, 31]), we find

$$G_E^{(+)}(\rho, \rho') = \frac{i\pi}{2} \sqrt{\rho\rho'} J_{\mathcal{L}+1/2}(\varkappa\rho_<) H_{\mathcal{L}+1/2}^{(1)}(\varkappa\rho_>), \quad (32)$$

$$G_E^{(-)}(\rho, \rho') = -\frac{i\pi}{2} \sqrt{\rho\rho'} J_{\mathcal{L}+1/2}(\varkappa\rho_<) H_{\mathcal{L}+1/2}^{(2)}(\varkappa\rho_>), \quad (33)$$

where $\rho_>$ ($\rho_<$) is the greatest (smallest) of the parameters ρ and ρ' .

We consider the partial solutions of Eq. (30)

$$\chi_{K\gamma}^{(\pm)}(\rho) = \frac{2\kappa}{\pi S_0^{\mathcal{L}}(q)} \int_0^\infty G_E^{(\pm)}(\rho, \rho') \varphi_0^{\mathcal{L}}(\rho') d\rho'. \quad (34)$$

Using (29), (32), and (33), we can easily calculate the asymptotic expressions for (34)

$$\chi_{K\gamma}^{(+)}(\rho) \xrightarrow{\rho \rightarrow \infty} i\sqrt{\kappa\rho} H_{\mathcal{L}+\frac{1}{2}}^{(1)}(\kappa\rho) \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2}{\pi}} e^{i(\kappa\rho - \frac{\pi\mathcal{L}}{2})}, \quad (35)$$

$$\chi_{K\gamma}^{(-)}(\rho) \xrightarrow{\rho \rightarrow \infty} -i\sqrt{\kappa\rho} H_{\mathcal{L}+\frac{1}{2}}^{(2)}(\kappa\rho) \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2}{\pi}} e^{-i(\kappa\rho - \frac{\pi\mathcal{L}}{2})}. \quad (36)$$

We now find the coefficients $B_n^{(\pm)}(q)$ for $\chi_{K\gamma}^{(\pm)}(\rho)$ in the expansion

$$\chi_{K\gamma}^{(\pm)}(\rho) = \sum_{n=0}^{\infty} B_n^{(\pm)}(q) \varphi_n^{\mathcal{L}}(\rho). \quad (37)$$

For this, the integral

$$B_n^{(\pm)}(q) = \int_0^\infty \chi_{K\gamma}^{(\pm)}(\rho) \varphi_n^{\mathcal{L}}(\rho) d\rho \quad (38)$$

should be calculated. Substituting (34) in (38) and taking (31) and (29) into account, we obtain expressions for $B_n^{(\pm)}(q)$ that exactly coincide with (16) and (17).

Hence, the physical meaning of $C_n^{\mathcal{L}(\pm)}(q)$ is that the series

$$\sum_n C_n^{\mathcal{L}(\pm)}(q) \varphi_n^{\mathcal{L}}(\rho)$$

converges to functions with an asymptotic representation of the type of converging or diverging waves for $\rho \rightarrow \infty$, i.e.,

$$\sum_{n=0}^{\infty} C_n^{\mathcal{L}(\pm)}(q) \varphi_n^{\mathcal{L}}(\rho) \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2}{\pi}} e^{\pm i(\kappa\rho - \frac{\pi\mathcal{L}}{2})}. \quad (39)$$

Using the Green's function

$$G_E^{\mathcal{P}}(\rho, \rho') = \frac{1}{2} \left(G_E^{(+)}(\rho, \rho') + G_E^{(-)}(\rho, \rho') \right), \quad (40)$$

we can similarly show that

$$\sum_{n=0}^{\infty} C_n^{\mathcal{L}}(q) \varphi_n^{\mathcal{L}}(\rho) \xrightarrow{\rho \rightarrow \infty} \sqrt{\frac{2}{\pi}} \cos\left(\kappa\rho - \frac{\pi\mathcal{L}}{2}\right). \quad (41)$$

3.2.3. Bound states and resonances. Up to now, we have investigated solutions in the exterior domain for states with positive energies E . In the case of bound states $E < 0$, we should study Eq. (9) in the exterior domain with $q^2 < 0$, i.e., q should be assumed to be purely imaginary. To investigate resonance states of an A -particle system, the solutions of the Schrödinger equation must be sought for complex values of energy E and momentum q .

We consider the analytic continuation of wave functions obtained in the suggested formalism to the complex plane with respect to the argument q . The asymptotic expressions in the coordinate space for wave functions as $\rho \rightarrow \infty$ are determined by series of the form

$$\sum_n S_n^{\mathcal{L}}(q) \varphi_n^{\mathcal{L}}(\rho), \quad \sum_n C_n^{\mathcal{L}}(q) \varphi_n^{\mathcal{L}}(\rho), \quad \sum_n C_n^{\mathcal{L}(\pm)}(q) \varphi_n^{\mathcal{L}}(\rho).$$

Studying the analytic continuation of wave functions to the complex plane with respect to the argument q reduces to studying that of the functions $S_n^{\mathcal{L}}(q)$, $C_n^{\mathcal{L}}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$, which is done in Appendix B. It follows from the results in Appendix B that for bound states with $q = i|q|$, only the series

$$\sum_n C_n^{\mathcal{L}^{(+)}}(i|q|) \varphi_n^{\mathcal{L}}(\rho)$$

contains no exponentially growing terms for $\rho \rightarrow \infty$. Precisely the solutions $C_n^{\mathcal{L}^{(+)}}(i|q|)$ should be “sewn” to the solutions in the interior domain for bound states of the system. (See (B.13) for an explicit formula for their calculation.) For resonance states, it is most natural to write asymptotic expressions for the wave function as a linear combination of the series

$$\sum_n C_n^{\mathcal{L}^{(+)}}(q) \varphi_n^{\mathcal{L}}(\rho) \quad \text{and} \quad \sum_n C_n^{\mathcal{L}^{(-)}}(q) \varphi_n^{\mathcal{L}}(\rho),$$

which results in asymptotic representations of the type of diverging and converging spherical waves for $\rho \rightarrow \infty$. Hence, in this case, the solutions in the interior domain should be “sewn” to the solutions $C_n^{\mathcal{L}^{(+)}}(q)$ and $C_n^{\mathcal{L}^{(-)}}(q)$ in the exterior domain.

4. Constructing the complete TMS wave function

4.1. “Sewing” conditions for solutions in the interior and exterior domains. We have constructed the fundamental system of solutions in the exterior domain, i.e. for $n > n_\Gamma$. We should now consider the relationship between these solutions and those in the interior domain. For this, we generalize the results in [12, 32] to the TMS case where the solutions in the interior and exterior domains for the multichannel case are “sewn” together.

The Schrödinger equation in the hyperspherical OR reduces to system of linear algebraic equations (6). Taking (7) and (8) into account, we can write those equations in this system for which $n \leq n_\Gamma$ in the form

$$\sum_{\Gamma', n \leq n_{\Gamma'}} \langle n\Gamma | H - E | n'\Gamma' \rangle \langle n'\Gamma' | E\beta \rangle = -\langle n_\Gamma\Gamma | T | n_\Gamma + 1, \Gamma \rangle \langle n_\Gamma + 1, \Gamma | E\beta \rangle \delta_{nn_\Gamma}. \quad (42)$$

We assume that system of homogeneous equations (42) has already been solved, i.e., the eigenvalues E_λ and the eigenvectors $\langle n_\Gamma\Gamma | \lambda \rangle$ of the truncated matrix of the Hamiltonian $\langle n\Gamma | H | n'\Gamma' \rangle$, $n \leq n_\Gamma$, $n' \leq n_{\Gamma'}$, have been found. The coefficients $\langle n\Gamma | E\beta \rangle$ for the interior domain can then be easily expressed using the set of the coefficients $\langle n_\Gamma + 1, \Gamma | E\beta \rangle$, namely,

$$\langle n\Gamma | E\beta \rangle = \sum_{\Gamma'} \langle n\Gamma | P | n_{\Gamma'} + 1, \Gamma' \rangle \langle n_{\Gamma'} + 1, \Gamma' | E\beta \rangle, \quad (43)$$

where

$$\langle n\Gamma | P | n_{\Gamma'} + 1, \Gamma' \rangle = \langle n\Gamma | \mathfrak{P} | n_{\Gamma'}\Gamma' \rangle \langle n_{\Gamma'}\Gamma' | T | n_{\Gamma'} + 1, \Gamma' \rangle. \quad (44)$$

It is convenient to calculate the matrix elements $\langle n\Gamma | \mathfrak{P} | n'\Gamma' \rangle$ ($n \leq n_\Gamma$, $n' \leq n_{\Gamma'}$) of $\mathfrak{P} \equiv (H - E)^{-1}$ in the truncated oscillator space using the expressions

$$\langle n\Gamma | \mathfrak{P} | n'\Gamma' \rangle = \sum_\lambda \frac{\langle n\Gamma | \lambda \rangle \langle \lambda | n'\Gamma' \rangle}{E - E_\lambda}. \quad (45)$$

Formulas (43)–(45) permit the wave function in the interior domain (for $n \leq n_\Gamma$) to be calculated if the solution in the exterior domain is known. On the other hand, the coefficient $\langle n_\Gamma\Gamma | E\beta \rangle$ is related to all the coefficients $\langle n\Gamma | E\beta \rangle$ in the exterior domain (for $n > n_\Gamma$) by (9) and can consequently be similarly expressed via the fundamental system of solutions for Eq. (9). Therefore, setting $n = n_\Gamma$ in (43), we obtain a system of equations for solutions in the exterior domain. In Secs. 4.2 and 4.3, we more thoroughly investigate the procedure for constructing the wave functions separately in the discrete and the continuous spectra.

4.2. Discrete spectrum. The wave function for discrete-spectrum states must decay exponentially as $\rho \rightarrow \infty$. Therefore, the expansion coefficients $\langle n\Gamma|E\beta\rangle$ in (1) are proportional to $C_n^{\mathcal{L}^{(+)}}(i|q|)$ for each channel in the exterior domain (see (B.13)). Letting $-[\mathbf{S}]_{\Gamma\Gamma}/\sqrt{\varkappa}$ denote the proportionality factor in the channel Γ , we obtain

$$\langle n\Gamma|E\beta\rangle = -\frac{1}{\sqrt{\varkappa}}[\mathbf{S}]_{\Gamma\Gamma}C_n^{\mathcal{L}^{(+)}}(i|q|) \quad (46)$$

for $n \geq n_\Gamma$. Substituting (46) in (43), where we set $n = n_\Gamma$ and assume for simplicity that $\varkappa = |q|\lambda$ is the same in all the channels, results in

$$[\mathbf{S}]_{\Gamma\Gamma}C_{n_\Gamma}^{\mathcal{L}^{(+)}}(i|q|) = \sum_{\Gamma'} \langle n_\Gamma\Gamma|P|n_{\Gamma'} + 1, \Gamma'\rangle [\mathbf{S}]_{\Gamma'\Gamma'} C_{n_{\Gamma'}+1}^{\mathcal{L}^{(+)}}(i|q|). \quad (47)$$

Relations (47) form a system of linear algebraic equations for the coefficients $[\mathbf{S}]_{\Gamma\Gamma}$. The solvability condition for this system has the form

$$\det \mathbf{A} = 0, \quad (48)$$

where the matrix elements of \mathbf{A} are given by

$$[\mathbf{A}]_{\Gamma\Gamma'} = \langle n_\Gamma\Gamma|P|n_{\Gamma'} + 1, \Gamma'\rangle C_{n_{\Gamma'}+1}^{\mathcal{L}^{(+)}}(i|q|) - \delta_{\Gamma\Gamma'} C_{n_\Gamma}^{\mathcal{L}^{(+)}}(i|q|). \quad (49)$$

Condition (48) should be regarded as an equation for determining the energy levels of the system (not only the expressions $C_n^{\mathcal{L}^{(+)}}(i|q|)$ but also the matrix elements $\langle n_\Gamma\Gamma|P|n_{\Gamma'} + 1, \Gamma'\rangle$ depend on the energy; see (44) and (45)). The set of discrete-spectrum energy levels found by solving Eq. (48) refines the values of E_λ obtained for the bound states by diagonalizing the truncated matrix of the Hamiltonian.

Solving system of equations (47) for each energy level in the discrete spectrum, we find the coefficients $[\mathbf{S}]_{\Gamma\Gamma}$ and then use (46) and (43) to calculate the wave function. Because the set of the coefficients $[\mathbf{S}]_{\Gamma\Gamma}$ can only be determined to within a common factor, the wave function should be normalized.

4.3. Continuous spectrum. We construct the solution for the continuous spectrum in the exterior domain in the form of a superposition of diverging and converging waves (i.e., solutions to (16) and (17) respectively). Let a converging wave be present only in the channel Γ' , and let all the other channels contain diverging waves. Then we have

$$\langle n\Gamma|E\beta\rangle = \frac{1}{2\sqrt{\varkappa}} \left(\delta_{\Gamma\Gamma'} C_n^{\mathcal{L}^{(-)}}(q) - C_n^{\mathcal{L}^{(+)}}(q) [\mathbf{S}]_{\Gamma\Gamma'} \right) \quad (50)$$

for $n \geq n_\Gamma$, where the meaning of $[\mathbf{S}]_{\Gamma\Gamma'}$ is similar to that of the elements of the S -matrix. The equation for determining the S -matrix is derived in the same way as (47) to obtain

$$\mathbf{A}^s \mathbf{S} = \mathbf{B}^s, \quad (51)$$

where the matrices \mathbf{A}^s and \mathbf{B}^s have the elements

$$[\mathbf{A}^s]_{\Gamma'\Gamma} = \langle n_{\Gamma'}\Gamma'|P|n_\Gamma + 1, \Gamma\rangle C_{n_\Gamma+1}^{\mathcal{L}^{(+)}}(q) - \delta_{\Gamma'\Gamma} C_{n_{\Gamma'}}^{\mathcal{L}^{(+)}}(q), \quad (52)$$

$$[\mathbf{B}^s]_{\Gamma'\Gamma} = \langle n_{\Gamma'}\Gamma'|P|n_\Gamma + 1, \Gamma\rangle C_{n_\Gamma+1}^{\mathcal{L}^{(-)}}(q) - \delta_{\Gamma'\Gamma} C_{n_{\Gamma'}}^{\mathcal{L}^{(-)}}(q). \quad (53)$$

We thus find

$$\mathbf{S} = (\mathbf{A}^s)^{-1} \mathbf{B}^s. \quad (54)$$

Here, as in (47), we have assumed that the momenta \varkappa are the same in all the channels. Calculating the S -matrix for a given energy E using (54), we use solutions (16) and (17) to construct the wave function with the required asymptotic behavior first in the exterior domain with (50) and then in the interior domain

with (43). In this case, we recall that solutions (16) and (17) correspond to the normalization of the wave function to the delta function of the momentum $\kappa = q\lambda$.

From the standpoint of numerical calculations, it is more efficient to first determine the so-called K -matrix instead of directly calculating the S -matrix (which involves complex matrices (52) and (53)) using formula (54). The K -matrix is real; it corresponds to representing the solution for the channel Γ in the form

$$\langle n\Gamma|E\beta\rangle = \frac{1}{\sqrt{\varkappa}} e^{i\delta_\Gamma} \left(\delta_{\Gamma\Gamma'} C_n^\mathcal{L}(q) - S_n^\mathcal{L}(q) [\mathbf{K}]_{\Gamma\Gamma'} \right) \quad (55)$$

instead of (50), where the TMS phase δ_Γ is given by the expression $e^{2i\delta_\Gamma} = [\mathbf{S}]_{\Gamma\Gamma}$. Introducing the matrices

$$[\mathbf{A}^K]_{\Gamma'\Gamma} = \langle n_{\Gamma'}\Gamma'|P|n_\Gamma + 1, \Gamma\rangle S_{n_\Gamma+1}^\mathcal{L}(q) - \delta_{\Gamma\Gamma'} S_{n_\Gamma}^\mathcal{L}(q), \quad (56)$$

$$[\mathbf{B}^K]_{\Gamma'\Gamma} = \langle n_{\Gamma'}\Gamma'|P|n_\Gamma + 1, \Gamma\rangle C_{n_\Gamma+1}^\mathcal{L}(q) - \delta_{\Gamma\Gamma'} C_{n_\Gamma}^\mathcal{L}(q), \quad (57)$$

we obtain

$$\mathbf{K} = (\mathbf{A}^K)^{-1} \mathbf{B}^K. \quad (58)$$

We now use the relation

$$\mathbf{S} = (\mathbf{I} + i\mathbf{K})(i\mathbf{K} - \mathbf{I})^{-1} \quad (59)$$

to calculate the S -matrix, where \mathbf{I} is the identity matrix. We can separate the real and imaginary parts in (59),

$$\text{Re } \mathbf{S} = \mathbf{I} - 2(\mathbf{I} + \mathbf{K}^2)^{-1}, \quad (60)$$

$$\text{Im } \mathbf{S} = -2\mathbf{K}(\mathbf{I} + \mathbf{K}^2)^{-1}. \quad (61)$$

We note that (54) implies that the poles of the S -matrix in the complex plane can be found by solving the equations

$$\det \mathbf{A}^s = 0, \quad (62)$$

where \mathbf{A}^s is determined by (52). In this situation, Eq. (48) defining the energy of the bound states of the system is obviously a special case of (62) for the S -matrix poles on the imaginary axis.

It is possible to construct the wave function in the suggested TMS approximation approach and include a great many components with different values of the hypermomentum K . However, because of the high centrifugal barrier

$$\mathcal{L}(\mathcal{L} + 1) \frac{1}{\rho^2} = \left(K + \frac{3A - 6}{2} \right) \left(K + \frac{3A - 4}{2} \right) \frac{1}{\rho^2},$$

including channels with large values of K in the exterior domain is probably insignificant for limited system energies E . Therefore, as a simple “minimal” approximation, we can take only one channel with $K = K_{\min}$ in the exterior domain into account (in this connection, see, for instance, [23], where monopole excitations of a ^{12}C nucleus were investigated with regard to its decomposition into three alpha particles). The minimal approximation includes coupling between the open channel with $K = K_{\min}$ and the artificially closed channels with $K > K_{\min}$ if components with $K > K_{\min}$ are included in the expansion of the wave function in the interior domain. The calculation of many parameters is simplified in this case. In particular, the expansion coefficient $\langle n\Gamma|E\beta\rangle$ in (1) for the exterior domain can be found using the expression

$$\langle n\Gamma|E\beta\rangle = \left[\cos \delta_0 S_n^{\mathcal{L}_0}(q) + \sin \delta_0 C_n^{\mathcal{L}_0}(q) \right] \delta_{K K_{\min}}, \quad (63)$$

where

$$\mathcal{L}_0 = K_{\min} + \frac{3A - 6}{2}$$

and δ_0 is the TMS phase in the channel corresponding to $K = K_{\min}$. In this case, the “sewing” conditions yield the following relation for calculating the phase δ_0 :

$$\tan \delta_0 = -\frac{S_{\nu}^{\mathcal{L}_0}(q) - S_{\nu+1}^{\mathcal{L}_0}(q) P_0}{C_{\nu}^{\mathcal{L}_0}(q) - C_{\nu+1}^{\mathcal{L}_0}(q) P_0}, \quad (64)$$

where $\nu = n_{\Gamma_0}$, $P_0 = \langle \nu \Gamma_0 | P | \nu + 1, \Gamma_0 \rangle$, and $\Gamma_0 = \{K_{\min}, \gamma\}$.

Instead of one channel in the exterior domain, two or more channels corresponding to minimal values of the hypermomentum K can be taken into account. In this case, a much greater number of components corresponding to different values of K can be left in the exterior domain. Physically, this means that only significant components of the wave function are included and those with large K , which are suppressed at large distances by the centrifugal barrier, are discarded; this does not, however, spoil the description of the wave function at small distances, where the account for components with large K is important for describing short-range correlations. This flexibility makes it possible to elaborate highly efficient and exact approaches for describing many-particle systems.

5. Conclusion

We have presented all the expressions required for constructing the wave function of a many-particle system for the discrete or the continuous spectrum in the TMS approximation. An important advantage of the suggested approach is that the most laborious part—constructing and diagonalizing the truncated matrix of the Hamiltonian—must be performed only once, after which the wave functions can be calculated for any given energy. The energy dependence of the S -matrix, the wave functions, and the other observables is only determined by the energy difference $E - E_{\lambda}$ in the denominator in Eq. (45) and by the dependence of the free solutions in the OR on the energy, i.e., of the functions $S_n^{\mathcal{L}}(q)$, $C_n^{\mathcal{L}}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$, which have analytic expressions. This makes it possible to perform detailed calculations in a broad energy range with small energy steps, to use the extension to the complex energy plane to calculate the poles of the many-particle S -matrix, etc.

Undoubtedly, an advantage of the formalism for applications in nuclear physics is the representation of the wave function as a superposition of oscillator functions, which permits the resulting solution to be clearly interpreted in terms of the shell model.

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Appendix A: Delta function as the limit of the oscillator function for large values of the principal quantum number

The assertion that the oscillator function $\varphi_n^{\mathcal{L}}(\rho)$ behaves like the Dirac delta function for large values of the principal quantum number seems to first appear in [14]. However, so far as we know, no explicit expression for the delta function as the limit of $\varphi_n^{\mathcal{L}}(\rho)$ for $n \rightarrow \infty$ has been published. Because this expression is very useful for analyzing asymptotic representations of wave functions in the OR, we present it here.

For the Schrödinger equation of a harmonic oscillator, the eigenstate with the characteristic energy $E_n = \hbar\omega(2n + \mathcal{L} + 3/2)$ has the classical turning point

$$\begin{aligned} \rho_{\text{turn}} &= \frac{1}{\lambda} \left[2n + \mathcal{L} + \frac{3}{2} + \sqrt{\left(2n + \mathcal{L} + \frac{3}{2}\right)^2 - \mathcal{L}(\mathcal{L} + 1)} \right]^{\frac{1}{2}} \approx \\ &\approx \frac{1}{\lambda} \sqrt{2\left(2n + \mathcal{L} + \frac{3}{2}\right)}. \end{aligned} \quad (\text{A.1})$$

Formula (A.1) is accurate to second-order terms with respect to $1/n$. The function $\varphi_n^{\mathcal{L}}(\rho)$ oscillates sufficiently fast for $n \gg 1$ on the interval $0 < \rho < \rho_{\text{turn}}$. Hence, the integration interval $0 < \rho < \rho_{\text{turn}}$ does not contribute to an integral of the type

$$\int f(\rho) \varphi_n^{\mathcal{L}}(\rho) d\rho$$

with sufficiently slowly varying functions $f(\rho)$. At the same time, the integration interval $\rho > \rho_{\text{turn}}$ does not contribute to the integral either, because the function $\varphi_n^{\mathcal{L}}(\rho)$ rapidly decreases with the damping factor $e^{-\lambda^2 \rho^2/2}$ in (4). Therefore, only a sufficiently small neighbourhood of the turning point contributes to the integral, i.e.,

$$\int f(\rho) \varphi_n^{\mathcal{L}}(\rho) d\rho \sim f(\rho_{\text{turn}}),$$

or, in other words, $\varphi_n^{\mathcal{L}}(\rho) = g \delta(\rho - \rho_{\text{turn}})$ for $n \gg 1$, where the factor g generally depends on n .

To calculate the factor g , we use (29) with the free OR wave function $S_n^{\mathcal{L}}(q)$ in the left-hand side replaced by its asymptotic expression (B.18). This results in

$$\varphi_n^{\mathcal{L}}(\rho) \xrightarrow{n \rightarrow \infty} \lambda^{-\frac{1}{2}} \left(n + \frac{\mathcal{L}}{2} + \frac{3}{4} \right)^{-\frac{1}{4}} \delta \left(\rho - \frac{2}{\lambda} \sqrt{n + \frac{\mathcal{L}}{2} + \frac{3}{4}} \right). \quad (\text{A.2})$$

Appendix B: Properties of the functions $S_n^{\mathcal{L}}(q)$, $C_n^{\mathcal{L}}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$

B.1. Expressions for the functions $C_n^{\mathcal{L}}(q)$ and $C_n^{\mathcal{L}(\pm)}(q)$ in terms of special functions. We find the expression for the solutions $C_n^{\mathcal{L}(\pm)}(q)$ via the known special functions. For this, we modify the integral in (16) somewhat by introducing the notation $s = q^2$ and $\alpha = \mathcal{L} + 1/2$ and using the Rodrigues formula for the Laguerre polynomials [33],

$$L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}). \quad (\text{B.1})$$

Then, n -fold integration by parts leads to

$$C_n^{\mathcal{L}(+)}(q) = \frac{1}{\pi} \left(\frac{2n!}{\lambda \Gamma(\mathcal{L} + n + 3/2)} \right)^{\frac{1}{2}} e^{\frac{q^2}{2}} q^{-\mathcal{L}} \mathfrak{S}_n^{\alpha(+)}(q), \quad (\text{B.2})$$

where

$$\mathfrak{S}_n^{\alpha(+)}(q) = \int_0^{\infty} \frac{e^{-s} s^{\alpha+n}}{(s - (q^2 + i\epsilon))^{n+1}} ds. \quad (\text{B.3})$$

To calculate the integral in (B.3) for $q^2 > 0$, we define the variable $z = -(q^2 + i\epsilon)$ with the condition that $\arg z = \phi = -\pi + \beta$, $\beta \rightarrow +0$ (i.e., $\phi > -\pi$). After the change of variable $t = s/z$ in (B.3), we obtain

$$\mathfrak{S}_n^{\alpha(+)}(q) = z^{\alpha} \int_0^{\infty e^{i\varphi}} e^{-tz} t^{\alpha+n} (t+1)^{-n-1} dt, \quad (\text{B.4})$$

where $\varphi = -\phi < \pi$. We compare (B.4) with the integral representation

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\varphi}} e^{-xt} t^{a-1} (1+t)^{c-a-1} dt, \quad (\text{B.5})$$

$$\text{Re } a > 0, \quad -\pi < \varphi < \pi, \quad -\frac{\pi}{2} < \varphi + \arg x < \frac{\pi}{2},$$

for the Tricomi confluent hypergeometric function $\Psi(a, c; x)$ [34], where the principal branches of the functions t^{a-1} and $(1+t)^{c-a-1}$ are taken. As a result, we clearly see that the integral $\Im_n^{\mathcal{L}^{(\pm)}}(q)$ can be expressed in terms of the principal value of the Tricomi function $\Psi(a, c; x)$ on the lower edge of the cut along the negative real axis, i.e., at the point $x = e^{-i\pi}q^2$, where $a = n + \mathcal{L} + 3/2$ and $c = \mathcal{L} + 3/2$. A similar argument can also be used for $C_n^{\mathcal{L}^{(-)}}(q)$. Thus, the final expression for $C_n^{\mathcal{L}^{(\pm)}}(q)$ becomes

$$C_n^{\mathcal{L}^{(\pm)}}(q) = \frac{1}{\pi\sqrt{\lambda}} \sqrt{2n!\Gamma\left(n + \mathcal{L} + \frac{3}{2}\right)} e^{\frac{q^2}{2}} q^{\mathcal{L}+1} e^{\mp i\pi(\mathcal{L}+\frac{1}{2})} \Psi\left(n + \mathcal{L} + \frac{3}{2}, \mathcal{L} + \frac{3}{2}; e^{\mp i\pi}q^2\right). \quad (\text{B.6})$$

The function $C_n^{\mathcal{L}}(q)$ can be expressed via the Laguerre function of the second kind $Q_n^\alpha(z)$ [33, 35]:

$$C_n^{\mathcal{L}}(q) = \frac{1}{\pi} \sqrt{\frac{2n!}{\lambda\Gamma\left(n + \mathcal{L} + \frac{3}{2}\right)}} q^{\mathcal{L}+1} e^{-\frac{q^2}{2}} Q_n^{\mathcal{L}+\frac{1}{2}}(q^2). \quad (\text{B.7})$$

Formula (B.7) follows from comparing (15) with the integral representation for $Q_n^\alpha(z)$ [33, 35]

$$Q_n^\alpha(z) = \frac{1}{z^\alpha e^{-z}} \mathcal{P} \int_0^\infty \frac{t^\alpha e^{-t} L_n^\alpha(t)}{t-z} dt. \quad (\text{B.8})$$

Furthermore, (21) and (B.6) can be applied to express $C_n^{\mathcal{L}}(q)$ as the sum of the Tricomi functions $\Psi(n + \mathcal{L} + 3/2, \mathcal{L} + 3/2; -q^2)$ on the upper and lower edges of the cut along the negative real axis.

All the above expressions can be used for a system of arbitrarily many particles. We note that \mathcal{L} is an integer if the number of particles A is even, and the parameters of the Tricomi functions $(n + \mathcal{L} + 3/2)$ and $(\mathcal{L} + 3/2)$ are accordingly half-integer. In this situation, the expressions for the solutions $C_n^{\mathcal{L}}(q)$ and $C_n^{\mathcal{L}^{(\pm)}}(q)$ can be simplified because in this case, the function $\Psi(a, c; -x)$ can be expressed via the confluent hypergeometric functions ${}_1F_1(c-a, c; x)$ and ${}_1F_1(1-a, 2-c; x)$ (e.g., see [34]); moreover, ${}_1F_1(c-a, c; x)$ can be written in terms of the associated Laguerre polynomials because $c-a = -n$. As a result, the desired expressions become

$$C_n^{\mathcal{L}}(q) = \left(\frac{2n!}{\lambda\Gamma\left(\mathcal{L} + n + \frac{3}{2}\right)}\right)^{\frac{1}{2}} \frac{\Gamma\left(\mathcal{L} + \frac{1}{2}\right)}{\pi} e^{-\frac{q^2}{2}} q^{-\mathcal{L}} {}_1F_1\left(-n - \mathcal{L} - \frac{1}{2}, -\mathcal{L} + \frac{1}{2}; q^2\right), \quad (\text{B.9})$$

$$C_n^{\mathcal{L}^{(\pm)}}(q) = \left(\frac{2n!}{\lambda\Gamma\left(\mathcal{L} + n + \frac{3}{2}\right)}\right)^{\frac{1}{2}} e^{-\frac{q^2}{2}} \left[\frac{\Gamma\left(\mathcal{L} + \frac{1}{2}\right)}{\pi} q^{-\mathcal{L}} {}_1F_1\left(-n - \mathcal{L} - \frac{1}{2}, -\mathcal{L} + \frac{1}{2}; q^2\right) \pm \right. \\ \left. \pm iq^{\mathcal{L}+1} L_n^{\mathcal{L}+\frac{1}{2}}(q^2) \right]. \quad (\text{B.10})$$

Formula (B.9) for ordinary two-particle scattering ($A = 2$) was derived in [15].

B.2. Analytic continuation of the functions $C_n^{\mathcal{L}}(q)$ and $C_n^{\mathcal{L}^{(\pm)}}(q)$ to the complex plane with respect to the argument q . The analytic continuation of the OR oscillator function $S_n^{\mathcal{L}}(q)$ (which can be expressed in terms of the Laguerre polynomial) to the complex plane with respect to the argument q is trivial (see (14)). Therefore, we only discuss the analytic continuation of $C_n^{\mathcal{L}^{(\pm)}}(q)$ and $C_n^{\mathcal{L}}(q)$.

We note that to derive (B.6) for positive real values of q^2 , we used the change of variable $-(q^2 \pm i\varepsilon) \rightarrow e^{\mp i\pi}(q^2 \pm i\varepsilon)$, which made it possible to express $C_n^{\mathcal{L}^{(\pm)}}(q)$ via the principal branch of the Tricomi function $\Psi(a, c; x)$. Formula (B.6) can be regarded as the definition of $C_n^{\mathcal{L}^{(+)}}(q)$ ($C_n^{\mathcal{L}^{(-)}}(q)$) in the upper (lower) complex half-plane for the momenta q , where $0 < \arg q \leq \pi$ ($-\pi < \arg q \leq 0$). The function $C_n^{\mathcal{L}^{(+)}}(q)$ ($C_n^{\mathcal{L}^{(-)}}(q)$) can be analytically continued to the lower (upper) half-plane q using the formulas for the analytic continuation of the function $\Psi(a, c; x)$ (e.g., see [34]). The resulting expressions are

$$C_n^{\mathcal{L}^{(+)}}(q) = C_n^{\mathcal{L}^{(-)}}(q) + 2iS_n^{\mathcal{L}}(q) \quad (\text{B.11})$$

and

$$C_n^{\mathcal{L}(-)}(q) = C_n^{\mathcal{L}(+)}(q) - 2iS_n^{\mathcal{L}}(q). \quad (\text{B.12})$$

In view of (21), defining the functions $C_n^{\mathcal{L}(\pm)}(q)$ in the entire complex plane q with analytic continuation formulas (B.11) and (B.12), we thus define $C_n^{\mathcal{L}}(q)$ throughout this plane.

Analytic continuation formulas (B.11) and (B.12) are of practical interest for calculating the S -matrix poles. In the resonance case, we are first interested in the functions $C_n^{\mathcal{L}(\pm)}(q)$ in the lower complex half-plane q ; for bound states, the function $C_n^{\mathcal{L}(\pm)}(q)$ of the purely imaginary argument $q = i|q|$ is most interesting. Because this is a very important case, we present the integral representation of $C_n^{\mathcal{L}(\pm)}(i|q|)$ in the explicit form involving no singularities

$$C_n^{\mathcal{L}(\pm)}(i|q|) = \frac{2}{\pi} \sqrt{\frac{2n!}{\lambda\Gamma(n+\mathcal{L}+\frac{3}{2})}} |q|^{\mathcal{L}} e^{-\frac{|q|^2}{2}} \int_0^\infty \frac{t^{2\mathcal{L}+2} e^{-t^2} L_n^{\mathcal{L}+\frac{1}{2}}(t^2)}{|q|^2 + t^2} dt. \quad (\text{B.13})$$

B.3. Asymptotic formulas for the functions $S_n^{\mathcal{L}}(q)$, $C_n^{\mathcal{L}}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$ as $n \rightarrow \infty$. We find the asymptotic expression for $C_n^{\mathcal{L}(\pm)}(q)$ (see (16) and (17)) at large values of the index n (and bounded q). For this, we use the asymptotic representation of the Tricomi function $\Psi(a, c; x)$ in [36], which can be written in the following form with an obvious renaming of the variables:

$$\Psi\left(\frac{1}{4}u^2 + \frac{1}{2}b, b; z^2\right) \xrightarrow{u \rightarrow \infty} e^{\frac{z^2}{2}} z^{1-b} \frac{2^{2-b} u^{b-1}}{\Gamma(\frac{1}{4}u^2 + \frac{1}{2}b)} K_{b-1}(uz), \quad (\text{B.14})$$

where b and z^2 are assumed to be bounded and $K_\nu(x)$ is the modified Hankel function. Passing to the functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$, we obtain the asymptotic expression for $C_n^{\mathcal{L}(\pm)}(q)$,

$$\begin{aligned} C_n^{\mathcal{L}(+)}(q) &\xrightarrow{n \rightarrow \infty} +i\sqrt{\frac{2q}{\lambda}} H_{\mathcal{L}+\frac{1}{2}}^{(1)}\left(2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q\right) \xrightarrow{n \rightarrow \infty} \\ &\xrightarrow{n \rightarrow \infty} \sqrt{\frac{2}{\pi\lambda}} \left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{-\frac{1}{4}} \exp\left(i\left[2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q - \frac{\pi\mathcal{L}}{2}\right]\right), \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} C_n^{\mathcal{L}(-)}(q) &\xrightarrow{n \rightarrow \infty} -i\sqrt{\frac{2q}{\lambda}} H_{\mathcal{L}+\frac{1}{2}}^{(2)}\left(2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q\right) \xrightarrow{n \rightarrow \infty} \\ &\xrightarrow{n \rightarrow \infty} \sqrt{\frac{2}{\pi\lambda}} \left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{-\frac{1}{4}} \exp\left(-i\left[2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q - \frac{\pi\mathcal{L}}{2}\right]\right). \end{aligned} \quad (\text{B.16})$$

Thus, resulting expressions (B.6) for the solutions $C_n^{\mathcal{L}(\pm)}(q)$ of TRR (9) have an asymptotic representation in the form of diverging and converging waves in the OR.

The asymptotic expression for $C_n^{\mathcal{L}}(q)$ can be easily found from (21) and asymptotic formulas (B.15) and (B.16), namely,

$$\begin{aligned} C_n^{\mathcal{L}}(q) &\xrightarrow{n \rightarrow \infty} -\sqrt{\frac{2q}{\lambda}} N_{\mathcal{L}+\frac{1}{2}}\left(2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q\right) \xrightarrow{n \rightarrow \infty} \\ &\xrightarrow{n \rightarrow \infty} \sqrt{\frac{2}{\pi\lambda}} \left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{-\frac{1}{4}} \cos\left(2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q\right). \end{aligned} \quad (\text{B.17})$$

Finally, the asymptotic representation for $S_n^{\mathcal{L}}(q)$ can be readily found from (20) and asymptotic relations (B.15)–(B.17) or directly from asymptotic expansions for the Laguerre polynomials (e.g., see [24]),

$$\begin{aligned} S_n^{\mathcal{L}}(q) &\xrightarrow{n \rightarrow \infty} \sqrt{\frac{2q}{\lambda}} J_{\mathcal{L}+\frac{1}{2}}\left(2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q\right) \xrightarrow{n \rightarrow \infty} \\ &\xrightarrow{n \rightarrow \infty} \sqrt{\frac{2}{\pi\lambda}} \left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{-\frac{1}{4}} \sin\left(2\left(n+\frac{\mathcal{L}}{2}+\frac{3}{4}\right)^{\frac{1}{2}}q\right). \end{aligned} \quad (\text{B.19})$$

B.4. Finding the numerical values of $S_n^{\mathcal{L}}(q)$, $C_n^{\mathcal{L}}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$. The most efficient method for computing the numerical values of solutions (14)–(17) for arbitrary n is the direct application of recurrence relations (9) (for $S_n^{\mathcal{L}}(q)$) and (18) (for (15)–(17)). In this case, the recursion can be performed both from top to bottom (from large values to small of n) and from bottom to top (from small to large n). In the first case, we can choose some $m \gg 1$ and calculate a pair of seed values $A_m^{\mathcal{L}}(q)$ and $A_{m+1}^{\mathcal{L}}(q)$ ($A_n^{\mathcal{L}}(q)$ is understood as one of the functions $S_n^{\mathcal{L}}(q)$, $C_n^{\mathcal{L}}(q)$, and $C_n^{\mathcal{L}(\pm)}(q)$) that are needed for starting calculations according to recurrence relation (9) or (18) with the help of asymptotic expressions (B.15)–(B.18). Realization of this algorithm is very simple but can, in some conditions, lead to numerical instability for small values of n .

The bottom-to-top recursion is more stable. To determine the initial values of the functions in this algorithm, the relations

$$L_{-1}^{\alpha}(x) = 0, \quad L_0^{\alpha}(x) = 1 \quad (\text{B.19})$$

should be used. This results in $S_{-1}^{\mathcal{L}}(q) = C_{-1}^{\mathcal{L}}(q) = C_{-1}^{\mathcal{L}(\pm)}(q) = 0$. The value of $S_0^{\mathcal{L}}(q)$ is obtained directly from (14) using (B.19); to find $C_0^{\mathcal{L}}(q)$ and $C_0^{\mathcal{L}(\pm)}(q)$, the integrals entering (15)–(17) should first be calculated taking (B.19) into account. We note that integral (15) for $n = 0$ can be easily expressed via the exponential integral for odd A and via the error function for even A (the corresponding expressions for $A = 2$ are presented in [15]).

Solution (B.13) for bound states and also the functions $C_n^{\mathcal{L}(\pm)}(q)$ for complex values of q can be calculated similarly.

Instead of the numerical calculation of integrals to find seed values $C_0^{\mathcal{L}(\pm)}(q)$ and $C_0^{\mathcal{L}}(q)$, it is possible to use analytic expressions (B.6) and (B.7) or (B.9) and (B.10) for these functions via the Tricomi function or the confluent hypergeometric functions and also various relations for these special functions. In particular, it is possible to use continued fractions, which provide one of the most stable and exact methods for calculating various hypergeometric functions [37]. However, in applying different relations for the Tricomi functions, caution is required when dealing with an odd number of particles A (a half-integer \mathcal{L}): in this situation, the functions

$$\Psi\left(n + \mathcal{L} + \frac{3}{2}, \mathcal{L} + \frac{3}{2}; e^{\mp i\pi} q^2\right)$$

relate to the so-called special logarithmic case, and they should be calculated using only expressions that are valid for the case.

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