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TRUNCATION OF THE BECHHOFER-KIEFER-SOBEL SEQUENTIAL PROCEDURE FOR SELECTING THE MULTINOMIAL EVENT WHICH HAS THE LARGEST PROBABILITY (II): EXTENDED TABLES AND AN IMPROVED PROCEDURE

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ABSTRACT

In an earlier article the authors studied the performance of a truncated version of a vector-at-a-time sequential sampling procedure $(\mathcal{P}_B^*)$ proposed by Bechhofer, Kiefer and Sobel for selecting the multinomial event which has the largest probability. Among the performance characteristics studied for both the original (open) basic procedure $(\mathcal{P}_B^*)$ and the truncated (closed) procedure $(\mathcal{P}_{B_T}^*)$ were the achieved probability of a correct selection, and the expected number of vector-observations (n) to terminate sampling when the event probabilities are in the so-called least-favorable (LF) and equal-parameter (EP) configurations. These were compared with the same quantities for a competing (closed) procedure of Ramey and Alam (R-A). All three procedures guarantee the same requirement on the probability of a correct selection. The truncated procedure was shown to be greatly

superior to the untruncated version both in terms of $E\{n\}$ and $Var\{n\}$, and also to be superior to the R-A procedure.

A limited set of truncation numbers (n_0) necessary to implement $\mathcal{P}^*_{B_T}$, and associated performance characteristics of $\mathcal{P}^*_{B_T}$, were given in our earlier article. In the present article we supply a greatly expanded set of n_0-values and associated performance characteristics. The tables presented herein are sufficiently detailed to enable implementation of $\mathcal{P}^*_{B_T}$ over a broad range of specifications, and should be sufficient for all practical purposes. A method of interpolating in the tables is given, thereby extending their domain of usefulness.

We also describe an improved version of the truncated procedure $(\mathcal{P}^*_{B_{T(I)}})$ and give its performance characteristics. The distribution of n for $\mathcal{P}^*_{B_{T(I)}}$ is stochastically smaller (never larger) than that for $\mathcal{P}^*_{B_T}$ for almost all (for any) values of the specified constants. As a consequence, $E\{n\}$ is decreased, uniformly in the unknown event probabilities, although in most instances the decrease is rather small.

1. INTRODUCTION AND SUMMARY

This article presents an expansion of results reported on earlier in Bechhofer and Goldsman (1985b) (B-G(b)) concerning the effect of truncation on the Bechhofer-Kiefer-Sobel (1968) (B-K-S) open sequential procedure for selecting the multinomial event which has the largest probability. In that article we studied in great detail the expected number of vector-observations (n) to terminate sampling for the basic untruncated (open) procedure $\mathcal{P}^*_B$ and the corresponding truncated (closed) procedure $\mathcal{P}^*_{B_T}$ when both guarantee the same probability requirement. $\mathcal{P}^*_{B_T}$ was shown to be superior to $\mathcal{P}^*_B$, not only in terms of $E\{n\}$ but also in terms of Var{n}; $\mathcal{P}^*_{B_T}$ was also shown to be superior to a competing procedure of Ramey and Alam (1979) which the authors had claimed to be the best to that date for the selection problem under study. The reader is referred to Bechhofer and Goldsman (1985a,1985b) for a history of the problem, and the detailed earlier results.

Our article (B-G(b)) was exploratory in nature in the sense that, although we knew that $\mathcal{P}^*_{B_T}$ would be superior to $\mathcal{P}^*_{B}$ in terms of E{n} and Var{n}, we had no knowledge as to the magnitude of the improvement that would be obtained. Since calculations were quite costly, particularly for θ^* and/or P^* (the specified constants) close to unity, a course grid of the pairs (θ^*,P^*) was chosen at which to conduct our experiments. Specifically, in B-G(b) the performance of $\mathcal{P}^*_{B_T}$ was studied for $k = 2(1)5, 10$ at $\theta^* = 1.6, 2.0, 2.4, 3.0$ for $P^* = 0.75$ and 0.90, and at $\theta^* = 2.0, 2.4, 3.0$ for $P^* = 0.95$, yielding a total of 55 $(k;\theta^*,P^*)$ combinations. In the present article the performance of $\mathcal{P}^*_{B_T}$ is studied for $k = 2(1)6$ at $\theta^* = 1.2(0.2)3.0$ for $P^* = 0.75, 0.90, 0.95$, yielding a total of 150 combinations; 44 of these 150 were given in B-G(b) but are included in the present article for completeness. For many of the repeated combinations we have been able to obtain better results either via MC estimates based on larger numbers of replications or (in one case) by exact calculations replacing MC estimates. These cases are noted in Section 6. All of the E{n}-values for $\mathcal{P}^*_{B_T}$ in the present tables can be compared directly with the corresponding values for the improved procedure $\mathcal{P}^*_{B_{T(I)}}$ of Section 4. When we made such comparisons for the same $(k;\theta^*,P^*)$ and $n_0 = n_0(k;\theta^*,P^*)$ combinations with $(p_1,p_2,\ldots,p_k)$ fixed, $\mathcal{P}^*_{B_{T(I)}}$ proved to have smaller values of E{n} for 142 out of the 150 $(k;\theta^*,P^*)$ combinations. The exceptions were for $k = 2$ at

$\theta^* = 1.8(0.2)3.0$ with $P^* = 0.75$ and at $\theta^* = 3.0$ with $P^* = 0.90$; for these 8 combinations, the $E\{n\}$-values were <u>equal</u> for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$. However, for most for the 142 combinations the decrease in $E\{n\}$ was rather small.

<u>2. STATEMENT OF THE PROBLEM</u>

In order to make the present article self-contained we state here the problem being considered, and the assumptions being made. We also introduce the notation which we will employ.

We consider a single k-variate multinomial population Π with probability vector $\underline{p} = (p_1, p_2, \ldots, p_k)$ where p_i is the probability of the event E_i $(1 \leq i \leq k)$. Let $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$ denote the ordered values of the p_i $(1 \leq i \leq k)$. It is assumed that the values of the p_i and of the $p_{[j]}$ $(1 \leq i,j \leq k)$ are unknown, and it is further assumed that the pairing of the E_i with the $p_{[j]}$ $(1 \leq i,j \leq k)$ is completely unknown.

The <u>goal</u> of the experiment is to select the event associated with $p_{[k]}$. When a procedure selects this event then we say that a correct selection (CS) has been made. Consideration is limited to procedures which guarantee the following indifference-zone probability requirement:

$$P\{CS\} \geq P^* \quad \text{whenever} \quad p_{[k]} \geq \theta^* p_{[k-1]} \tag{2.1}$$

where $\{\theta^*, P^*\}$ with $1 < \theta^* < \infty$, $1/k < P^* < 1$ are specified by the experimentor prior to the start of experimentation. It was proved in B-K-S that the basic sequential selection procedure $\mathcal{P}^*_B$ described in B-G(b) guarantees (2.1). It was also proved in B-K-S that the configuration of the p_i $(1 \leq i \leq k)$ for which the probability in (2.1) is minimized subject to $p_{[k]} \geq \theta^* p_{[k-1]}$ is $p_{[1]} = p_{[k-1]} = p_{[k]}/\theta^*$, which is referred to as the <u>least-favorable</u> (LF) configuration. When $p_{[1]} = p_{[k]}$, the p_i $(1 \leq i \leq k)$ are referred to as being in the <u>equal-parameter</u> (EP) configuration.

The truncated B-K-S sequential selection procedure $\mathcal{P}^*_{B_T}$ for the multinomial introduced in B-G(b) and described in Section 3 also guarantees (2.1) as does the <u>improved</u> truncated B-K-S sequential selection procedure $\mathcal{P}^*_{B_{T(I)}}$ described in Section 4.

We introduce the same notation as was employed in B-G(b). Let $\underset{\sim}{x}_j = (x_{1j}, x_{2j}, \ldots, x_{kj})$ $(j = 1,2,\ldots)$ denote independent vector-observations from Π. Also let $y_{im} = \Sigma_{j=1}^{m} x_{ij}$ $(1 \leq i \leq k,$ $m = 1,2,\ldots)$, and let $y_{[1]m} \leq y_{[2]m} \leq \cdots \leq y_{[k]m}$ denote the ordered values of the y_{im} $(1 \leq i \leq k)$ after the mth vector-observation has been taken. In Section 3 we present $\mathcal{P}^*_{B_T}$, and in Section 4 we show how this procedure can be modified to obtain $\mathcal{P}^*_{B_{T(I)}}$.

3. THE TRUNCATED B-K-S PROCEDURE $(P^*_{B_T})$ FOR SELECTING THE EVENT ASSOCIATED WITH $p_{[k]}$

<u>Sampling rule</u>: Take vector-observations $(x_{1j}, \ldots, x_{kj})$ $(j = 1,2,\ldots)$ one-at-a-time from Π. (3.1a)

<u>Stopping rule</u>: After the mth vector-observation, compute

$$z_m = \sum_{i=1}^{k-1} (1/\theta^*)^{(y_{[k]m} - y_{[i]m})}.$$

Stop sampling when, for the first time, <u>either</u>

$$z_n \leq (1 - P^*)/P^*$$

<u>or</u> (3.1b)

$$n = n_0,$$

whichever occurs first; here n (a random variable) is the value of m at termination.

<u>Terminal decision rule</u>: After stopping, select the event associated with $y_{[k]n}$. If two or more events yield y_{in}-values equal to $y_{[k]n}$, then select one of them at random. (3.1c)

The stopping constant $n_0 = n_0(k;\theta^*,P^*)$ employed in (3.1b) is predetermined as the smallest integer that will guarantee (2.1) when $\mathcal{P}^*_{B_T}$ is used.

4. THE IMPROVED TRUNCATED B-K-S PROCEDURE ($\mathcal{P}^*_{B_{T(I)}}$) FOR SELECTING THE EVENT ASSOCIATED WITH $p_{[k]}$

The improved truncated procedure $\mathcal{P}^*_{B_{T(I)}}$ can be obtained from the original truncated procedure $\mathcal{P}^*_{B_T}$ simply be replacing the stopping rule (3.1b) of the latter by

Stopping rule for $\mathcal{P}^*_{B_{T(I)}}$: After the mth vector-observation compute

$$z_m = \sum_{i=1}^{k-1} (1/\theta^*)^{(y_{[k]m} - y_{[i]m})}.$$

Stop sampling when, for the first time, either

$$z_n \leqq (1 - P^*)/P^*,$$

or

$$n = n_0 \tag{4.1}$$

or

$$y_{[k]n} - y_{[k-1]n} \geqq n_0 - n,$$

whichever occurs first; here n (a random variable) is the value of m at termination.

Note: The LF-configuration of the p_i $(1 \leqq i \leqq k)$ is the same for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ as for $\mathcal{P}^*_B$. Our studies in Section 6 were conducted for the LF- and EP- configurations.

The rationale for adding the third inequality $y_{[k]n} - y_{[k-1]n} \geqq n_0 - n$ in (4.1) to the two original inequalities in (3.1b) is the following: If the third inequality holds, then the category currently associated with $y_{[k-1]n}$ can, in the remaining $n_0 - n$ possible vector-observations, at best <u>tie</u> the category currently associated with $y_{[k]n}$. We thus are employing so-called "strong" curtailment (as used in Bechhofer and Kulkarni [1982] for the Bernoulli). It is easy to show (see Remark 2.4 in Jennison [1983]) that $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ achieve the <u>same</u> probability of a correct selection $(P\{CS|\underset{\sim}{p}\})$ <u>uniformly</u> in $\underset{\sim}{p} = (p_1,p_2,\ldots,p_k)$.

5. DETERMINATION OF THE TRUNCATION NUMBERS (n_0) FOR $\mathcal{P}^*_{B_T}$

In Section 4 of B-G(b) we described in detail how the truncation numbers (n_0) for $\mathcal{P}^*_{B_T}$ were determined: Specifically, for small k, and θ^* and P^* not "close" to unity, the required n_0-values were obtained by trial and error after enumerating sample paths. Using these values of n_0, associated <u>exact</u> values of $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ were calculated. For larger k, and θ^* and/or P^* closer to unity, recursion formulae were used to calculate the required n_0-values and associated <u>exact</u> values of $P\{CS\}$ and $E\{n\}$; for still larger values of k and θ^* and/or P^* even closer to unity, Monte Carlo (MC) sampling was used to <u>estimate</u> the required n_0-values and the associated values of $P\{CS\}$ and $E\{n\}$, the latter quantities being estimated with very high precision. The same techniques were used in the present article; however, more efficient and faster computing routines enabled us to obtain results for $\theta^* = 1.2$ and 1.4 (for all P^*) which are smaller values of θ^* than any for which computations were feasible at the time of B-G(b).

6. TABLES OF n_0, P{CS} AND E{n} FOR $\mathcal{P}^*_{B_T}$ AND $\mathcal{P}^*_{B_{T(I)}}$

In this section we present a comprehensive set of tables of n_0 needed to implement $\mathcal{P}^*_{B_T}$ (or equivalently $\mathcal{P}^*_{B_{T(I)}}$) for $k = 2(1)6$, $\theta^* = 1.2(0.2)3.0$ and $P^* = 0.75, 0.90, 95$. Also given for each $(k;\theta^*,P^*)$ and $n_0 = n_0(k;\theta^*,P^*)$ are corresponding performance characteristics, namely, $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ for both $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$. (As pointed out in Section 4, P{CS} is the same for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$, uniformly in $(p_1,p_2,\ldots,p_k)$.)

All of the results for $k = 2$ (Table I) are _exact_ having been calculated using recursion formulae. Most of the results for $k = 3$ (Table II) are _exact_, the remaining results being _estimated_ by Monte Carlo sampling. (The computer program employed for Monte Carlo sampling was checked carefully to insure that the MC results agreed with the exact results when the latter were available.) In Tables II-V, all MC sampling results are given along with their estimated standard error recorded below them in parentheses, while the figure to the left of each pair gives the number of independent replications (in thousands) on which the estimate and its standard error are based. Thus, for example, in Table II $(k = 3)$ the estimate of $E\{n|LF\}$ for $P^* = 0.95$, $\theta^* = 1.4$ is 98.98 with an estimated standard error of 0.33, the latter two figures being based on 30,000 independent replications. When _estimated_ values of $P\{CS|LF\}$ are given, these were obtained by a method which uses a statistic $\bar{W}_{[k]n}$ described on p. 290 of B-G(b); this method yields very precise unbiased estimates of $P\{CS|LF\}$.

It is clear that the stopping rule (4.1) _sometimes_ permits sampling to terminate _earlier_ (but _never later_) than does the stopping rule (3.1b). However, the resulting decrease in E{n} for any given $(k,\theta^*,P^*,n_0,(p_1,\ldots,p_k))$ is usually modest; values of this decrease can be obtained by comparing corresponding $E\{n|LF\}$ or $E\{n|EP\}$ entries in Tables I-V.

In order to obtain MC estimates of $E\{n|LF\}$ for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ which are directly comparable, the same random number sequences were used for each particular replication, the sequences differing from replication to replication; we thus were able to estimate the small $E\{n|LF\}$ differences between the two procedures; of course, the estimates obtained in this way are highly positively correlated but a sufficient number of independent replications were carried out so that the standard errors of each estimate are small. The same procedure was followed for the MC estimates of $E\{n|EP\}$ with $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$.

Remark 6.1: It would have been possible to calculate directly from the differences $n(\mathcal{P}^*_{B_T}) - n(\mathcal{P}^*_{B_{T(I)}})$ obtained in each replication, an estimate of $\text{Var}\{n(\mathcal{P}^*_{B_T}) - n(\mathcal{P}^*_{B_{T(I)}})\}$ and hence an estimate of $\text{Var}\{\bar{n}(\mathcal{P}^*_{B_T}) - \bar{n}(\mathcal{P}^*_{B_{T(I)}})\}$; because the differences between the n-values obtained in each replication were highly positively correlated the estimate of the variance of the difference between the two means would have been very, very small. However, our focus was on the estimates of $E\{n\}$ separately for each procedure rather on the estimate of their difference, and hence we did not perform the calculations just described.

Remark 6.2: We point out that if the Bechhofer-Kulkarni Bernoulli one-observation-at-a-time sequential selection procedure (which employs strong curtailment) is used, then the E{Total number of scalar observations} can be reduced substantially over the same procedure employing weak curtailment. However, for the multinomial, the reduction in E{Number of vector observations} is typically very small when $\mathcal{P}^*_{B_{T(I)}}$ (which employs strong curtailment) is used in place of $\mathcal{P}^*_{B_T}$.

As mentioned at the end of Section 1, for several $(k;\theta^*,P^*)$ combinations we were able to obtain better results for $\mathcal{P}^*_{B_T}$ than those reported in B-G(b). Specifically, smaller values of $n_0 = n_0(k;\theta^*,P^*)$ with resulting decreases in $E\{n\}$, are now given for $k = 4$ with $\{P^*,\theta^*\} = \{0.90,1.6\}$, $\{0.95,2.0\}$ and $k = 5$ with $\{P^*,\theta^*\} = \{0.75,2.0\}$, $\{0.90,1.6\}$. For $k = 2$ with $\{P^*,\theta^*\} = \{0.90,3.0\}$ we now have $n_0 = \infty$, i.e., the untruncated procedure $\mathcal{P}^*_B$ achieves exactly $P^* = 0.90$ when the p_i $(1 \leqq i \leqq 2)$ are in the LF-configurations. For $k = 4$ with $(P^*,\theta^*) = (0.75,1.6)$ the $E\{n|EP\}$- value is now calculated exactly rather than being estimated by MC sampling. Finally, many estimates of $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ are now much more precise, the estimates being based on larger numbers of replications.

7. INTERPOLATION IN THE TABLES

When we studied the behavior of n_0, $E\{n|LF\}$ and $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$ as a function of θ^* for fixed k and P^*, we noticed that $\ln n_0$, $\ln [E\{n|LF\}]$ and $\ln [E\{n|EP\}]$ increase almost quadratically in $1/\theta^*$, at least for P^* "close to" unity. This fact suggested a method of interpolating in the tables of Section 6. The method which we illustrate below by an example proved to yield excellent results.

Example 7.1: We wish to estimate n_0 for $\mathcal{P}^*_{B_T}$ and the corresponding value of $E\{n|LF\}$ and $E\{n|EP\}$ for the case $k = 3$, $P^* = 0.95$, $\theta^* = 1.65$. From Table II we find the n_0-, $E\{n|LF\}$- and $E\{n|EP\}$- values associated with the three θ^*-values which are closest to 1.65, namely,

P^*	θ^*	n_0	$E\{n\|LF\}$	$E\{n\|EP\}$
0.95	1.8	71	32.804	51.228
0.95	1.6	125	50.432	82.184
0.95	1.4	266	98.98	166.75

(7.1)

Fitting quadratic equations of the form

$$\ln n_0 = \beta_{01} + \beta_{11}/\theta^* + \beta_{21}/(\theta^*)^2 \qquad (7.2a)$$

$$\ln E\{n|LF\} = \beta_{02} + \beta_{12}/\theta^* + \beta_{22}/(\theta^*)^2 \qquad (7.2b)$$

$$\ln E\{n|EP\} = \beta_{03} + \beta_{13}/\theta^* + \beta_{23}/(\theta^*)^2 \qquad (7.2c)$$

to the data of (7.1) we obtain

$$\ln n_0 = 0.42212 + 5.8178/\theta^* + 1.9714/(\theta^*)^2 \qquad (7.3a)$$

$$\ln E\{n|LF\} = 3.0208 - 3.9083/\theta^* + 8.5567/(\theta^*)^2 \qquad (7.3b)$$

$$\ln E\{n|EP\} = 2.6005 - 1.5085/\theta^* + 7.0433/(\theta^*)^2. \qquad (7.3c)$$

Using (7.3a) for $\theta^* = 1.65$ yields $\ln n_0 = 4.6722$, i.e., $n_0 = 106.93$ or 107. The <u>exact</u> answer (obtained in the same way as were the entries in Table II for $\theta^* = 1.4$, 1.6 and 1.8) is 106.

Using (7.3b) for $\theta^* = 1.65$ yields $\ln E\{n|LF\} = 3.7951$, i.e., $E\{n|LF\} = 44.483$. The <u>exact</u> answer for $n_0 = 106$ is 44.538 and for $n_0 = 107$ is 44.577.

Using (7.3c) for $\theta^* = 1.65$ yields $\ln E\{n|LF\} = 4.2733$, i.e., $E\{n|EP\} = 71.760$. The <u>exact</u> answer for $n_0 = 106$ is 71.963 and for $n_0 = 107$ is 72.285.

Based on results such as these we recommend the use of this method for interpolation, particularly for $k = 4, 5, 6$ and P^* close to unity where the determination of n_0-values and associated $E\{n|LF\}$- and $E\{n|EP\}$-values is extremely costly. However, we caution that the method be used only for <u>interpolation</u> in the tables and <u>not</u> for <u>extrapolation</u> to θ^*-values outside the range of the tables.

8. CONCLUDING REMARKS

We were greatly impressed by the performance of $\mathcal{P}^*_{B_T}$ relative to that of $\mathcal{P}^*_B$ as reported in B-G(b), and in particular by the very substantial decrease in $E\{n|\underset{\sim}{p}\}$ and $Var\{n|\underset{\sim}{p}\}$ achieved by $\mathcal{P}^*_{B_T}$. To the best of our knowledge $P^*_{B_T}$ is superior, over a broad range of the practical $(k;\theta^*,P^*)$-values, to all procedures proposed thus far for the multinomial selection problem under consideration. Thus we decided to produce a definitive set of tables of n_0-values necessary to implement $\mathcal{P}^*_{B_T}$, and to provide some of its important performance characteristics. These are contained in the present article. However, when we undertook this project we had little appreciation for the substantial cost that would be incurred to obtain certain of the table entries. For example, CPU requirements per table entry ranged from a few seconds (for large θ^* and small P^*) to approximately three hours (for $k = 6$ and θ^* and P^* close to unity). For these computations all computer runs were conducted in FORTRAN on an IBM 4381 machine at Georgia Tech; an optimized version of the FORTVS compiler was used. In retrospect it is felt that the effort and time expended to prepare these extended tables was justified.

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Table I

Exact P{CS|LF}, E{n|LF} and E{n|EP}
for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ with associated truncation number n_0
and selected {P*,θ*} when k = 2

P*	θ*	n_0	$\mathcal{P}^*_{B_T}$			$\mathcal{P}^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Exact P{CS|LF}	Exact E{n|LF}	Exact E{n|EP}	Exact E{n|LF}	Exact E{n|EP}
0.75	3.0	1	0.7500	1.000	1.000	1.000	1.000
	2.8	3	0.8287	2.388	2.500	2.388	2.500
	2.6	3	0.8114	2.401	2.500	2.401	2.500
	2.4	3	0.7914	2.415	2.500	2.415	2.500
	2.2	3	0.7681	2.430	2.500	2.430	2.500
	2.0	5	0.7737	3.086	3.250	3.086	3.250
	1.8	7	0.7524	3.437	3.625	3.437	3.625
	1.6	9	0.7559	6.144	6.469	5.956	6.258
	1.4	19	0.7555	11.582	12.326	11.354	12.056
	1.2	67	0.7514	37.151	39.775	36.748	39.283
0.90	3.0	∞	0.9000	3.200	4.000	3.200	4.000
	2.8	7	0.9132	4.840	5.625	4.633	5.344
	2.6	9	0.9168	5.364	6.469	5.234	6.258
	2.4	11	0.9113	5.806	7.102	5.718	6.943
	2.2	15	0.9037	6.366	7.932	6.326	7.843
	2.0	15	0.9033	9.128	10.957	8.899	10.587
	1.8	27	0.9021	11.101	14.050	11.040	13.907
	1.6	41	0.9006	17.098	21.705	17.001	21.482
	1.4	79	0.9000+	33.079	42.198	32.924	41.835
	1.2	267	0.9002	112.609	144.281	112.282	143.502
0.95	3.0	11	0.9522	5.314	7.102	5.251	6.943
	2.8	15	0.9515	5.674	7.932	5.650	7.843
	2.6	13	0.9536	7.756	10.092	7.541	9.658
	2.4	17	0.9548	8.586	11.696	8.465	11.380
	2.2	23	0.9505	9.484	13.323	9.426	13.127
	2.0	27	0.9537	13.256	18.349	13.091	17.898
	1.8	35	0.9508	18.295	24.955	18.034	24.301
	1.6	59	0.9502	26.755	37.697	26.559	37.094
	1.4	151	0.9501	48.376	72.716	48.312	72.356
	1.2	455	0.9500+	166.779	246.357	166.544	245.305

Table II

Exact[+] P{CS|LF}, E{n|LF} and E{n|EP}

for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ with associated truncation number n_0

and selected {P*,θ*} when k = 3

P*	θ*	n_0	$\mathcal{P}^*_{B_T}$			$\mathcal{P}^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Exact P{CS|LF}	Exact E{n|LF}	Exact E{n|EP}	Exact E{n|LF}	Exact E{n|EP}
0.75	3.0	5	0.7574	3.482	3.852	3.242	3.482
	2.8	6	0.7623	3.795	4.247	3.699	4.148
	2.6	7	0.7533	4.096	4.609	3.937	4.379
	2.4	8	0.7602	5.586	6.226	5.403	5.938
	2.2	10	0.7517	6.226	7.012	6.003	6.680
	2.0	13	0.7512	8.181	9.273	7.966	8.934
	1.8	18	0.7503	11.631	13.153	11.338	12.736
	1.6	32	0.7517	17.802	20.592	17.597	20.254
	1.4	71	0.7502	34.178	40.139	34.015	39.837
	1.2	285	0.7509	118.04	141.20	117.89	140.85
			(24)	(24)	(12)	(24)	(12)
			(0.0003)	(0.46)	(0.74)	(0.46)	(0.74)
0.90	3.0	12	0.9029	7.206	9.394	6.969	8.933
	2.8	15	0.9053	7.926	10.734	7.771	10.364
	2.6	16	0.9013	9.374	12.328	9.171	11.827
	2.4	22	0.9021	10.528	14.577	10.429	14.247
	2.2	25	0.9008	13.473	18.275	13.297	17.772
	2.0	34	0.9016	17.341	23.801	17.165	23.296
	1.8	50	0.9004	23.854	33.390	23.707	32.887
	1.6	83	0.9003	37.398	53.146	37.261	52.614
	1.4	170	0.9004	73.58	106.07	73.42	105.39
			(40)	(40)	(15)	(40)	(15)
			(0.0003)	(0.21)	(0.42)	(0.21)	(0.41)
	1.2	670	0.9003	255.00	370.60	254.85	369.78
			(48)	(48)	(12)	(48)	(12)
			(0.0002)	(0.70)	(1.81)	(0.70)	(1.80)
0.95	3.0	20	0.9505	8.970	13.932	8.901	13.573
	2.8	22	0.9518	10.574	16.273	10.481	15.793
	2.6	25	0.9514	12.427	18.881	12.267	18.268
	2.4	31	0.9516	14.600	22.653	14.479	22.086
	2.2	41	0.9509	17.641	28.270	17.559	27.760

Table II (continued) (k = 3)

0.95	2.0	52	0.9508	23.159	36.625	23.032	35.972
	1.8	71	0.9503	32.805	51.228	32.629	50.407
	1.6	125	0.9502	50.432	82.184	50.321	81.434
	1.4	266	0.9504 (30) (0.0002)	98.98 (30) (0.33)	166.75 (15) (0.66)	98.88 (30) (0.33)	165.90 (15) (0.65)
	1.2	960	0.9503 (24) (0.0002)	346.57 (24) (1.31)	579.26 (12) (2.62)	346.42 (24) (1.31)	577.82 (12) (2.61)

+All E{n} and P{CS} results are exact except where indicated by standard errors for {P*,θ*} = {0.75,1.2}, {0.90,1.4}, {0.90,1.2}, {0.95,1.4}, {0.95,1.2}, these exceptions and their standard errors being estimated by Monte Carlo sampling.

Table III

Exact+ P{CS|LF}, E{n|LF} and E{n|EP}

for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ with associated truncation number n_0

and selected {P*,θ*} when k = 4

P*	θ*	n_0	$\mathcal{P}^*_{B_T}$			$\mathcal{P}^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Exact P{CS|LF}	Exact E{n|LF}	Exact E{n|EP}	Exact E{n|LF}	Exact E{n|EP}
0.75	3.0	9	0.7517	5.029	5.973	4.907	5.747
	2.8	9	0.7508	6.170	7.083	5.997	6.789
	2.6	11	0.7548	7.275	8.496	7.051	8.162
	2.4	15	0.7569	8.416	10.150	8.286	9.911
	2.2	17	0.7503	10.640	12.605	10.439	12.264
	2.0	24	0.7541	13.944	16.760	13.781	16.449
	1.8	35	0.7530	19.592	23.671	19.423	23.344
	1.6	57	0.7512	31.295	38.031	31.109	37.649
	1.4	124	0.7515 (24) (0.0005)	62.48 (24) (0.21)	76.42 (24) (0.24)	62.31 (24) (0.21)	76.01 (24) (0.23)
	1.2	495	0.7508 (48) (0.0003)	219.87 (48) (0.56)	271.37 (12) (1.29)	219.69 (48) (0.56)	270.89 (12) (1.28)

Table III (continued) (k = 4)

	3.0	19	0.9036	9.988	14.273	9.844	13.852
	2.8	22	0.9042	11.420	16.351	11.294	15.941
	2.6	26	0.9033	13.320	19.309	13.195	18.873
	2.4	31	0.9022	16.053	23.231	15.927	22.767
	2.2	39	0.9009	19.929	28.899	19.792	28.389
	2.0	53	0.9000+	25.840	37.818	25.706	37.305
0.90	1.8	75	0.9012 (24) (0.0005)	37.09 (24) (0.12)	54.39 (24) (0.14)	36.94 (24) (0.12)	53.77 (24) (0.14)
	1.6	126	0.9006 (48) (0.0003)	58.84 (48) (0.14)	87.51 (12) (0.33)	58.69 (48) (0.14)	86.83 (12) (0.33)
	1.4	274	0.9009 (24) (0.0004)	117.02 (24) (0.42)	177.29 (12) (0.74)	116.89 (24) (0.42)	176.56 (12) (0.73)
	1.2	1050	0.9005 (24) (0.0003)	413.83 (24) (1.51)	628.71 (6) (4.02)	413.68 (24) (1.50)	627.68 (6) (4.00)
	3.0	26	0.9513	13.070	20.903	12.968	20.341
	2.8	30	0.9502	14.849	23.898	14.739	23.292
	2.6	36	0.9505	17.318	27.984	17.192	27.357
	2.4	44	0.9506	20.789	34.027	20.679	33.383
	2.2	56	0.9503	25.851	42.846	25.751	42.164
	2.0	74	0.9502 (72) (0.0002)	33.98 (72) (0.07)	56.23 (24) (0.13)	33.86 (72) (0.07)	55.47 (24) (0.13)
0.95	1.8	106	0.9506 (24) (0.0004)	47.94 (24) (0.17)	79.94 (12) (0.27)	47.80 (24) (0.16)	79.05 (12) (0.26)
	1.6	180	0.9502 (72) (0.0002)	76.17 (72) (0.15)	128.70 (12) (0.47)	76.06 (72) (0.15)	127.76 (12) (0.46)
	1.4	380	0.9503 (72) (0.0002)	152.82 (72) (0.31)	265.43 (12) (1.00)	152.72 (72) (0.31)	264.25 (12) (0.99)
	1.2	1500	0.9508 (12) (0.0003)	537.19 (12) (2.72)	964.12 (6) (5.69)	537.10 (12) (2.71)	962.45 (6) (5.67)

+ All $E\{n\}$ and $P\{CS\}$ results are exact except where indicated by standard errors, these exceptions and their standard errors being estimated by Monte Carlo sampling.

Table IV

Estimated+ P{CS|LF}, E{n|LF} and E{n|EP}

for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ with associated truncation number n_0

and selected {P*,θ*} when k = 5

P*	θ*	n_0	$\mathcal{P}^*_{B_T}$			$\mathcal{P}^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Estd. P{CS|LF}	Estd. E{n|LF}	Estd. E{n|EP}	Estd. E{n|LF}	Estd. E{n|EP}
0.75	3.0	12	0.7612 (24) (0.0009)	7.59 (24) (0.02)	9.20 (24) (0.02)	7.44 (24) (0.02)	8.95 (24) (0.02)
	2.8	13	0.7509 (24) (0.0010)	8.59 (24) (0.02)	10.30 (12) (0.03)	8.39 (24) (0.02)	9.96 (12) (0.03)
	2.6	17	0.7559 (24) (0.0008)	9.94 (24) (0.03)	12.21 (12) (0.05)	9.80 (24) (0.03)	11.93 (12) (0.04)
	2.4	20	0.7527 (24) (0.0008)	12.07 (24) (0.04)	14.86 (24) (0.04)	11.91 (24) (0.03)	14.55 (24) (0.03)
	2.2	25	0.7504 (72) (0.0005)	15.15 (72) (0.03)	18.53 (24) (0.04)	14.99 (72) (0.03)	18.20 (24) (0.04)
	2.0	34	0.7508 (72) (0.0004)	19.98 (72) (0.04)	24.71 (24) (0.06)	19.81 (72) (0.03)	24.35 (24) (0.06)
	1.8	50	0.7511 (24) (0.0007)	28.62 (24) (0.09)	35.27 (12) (0.13)	28.44 (24) (0.09)	34.88 (12) (0.13)
	1.6	86	0.7510 (36) (0.0005)	45.84 (36) (0.13)	57.53 (24) (0.16)	45.68 (36) (0.12)	57.14 (24) (0.16)
	1.4	184	0.7510 (24) (0.0005)	92.85 (24) (0.31)	118.10 (12) (0.49)	92.68 (24) (0.31)	117.62 (12) (0.49)
	1.2	730	0.7511 (12) (0.0005)	329.55 (12) (1.62)	422.05 (12) (1.90)	329.36 (12) (1.62)	421.47 (12) (1.90)

Table IV (continued) (k = 5)

P^*	θ^*	n_0	$P^*_{B_T}$			$P^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Estd. P{CS|LF}	Estd. E{n|LF}	Estd. E{n|EP}	Estd. E{n|LF}	Estd. E{n|EP}
0.90	3.0	24	0.9025 (24) (0.0007)	13.29 (24) (0.04)	19.25 (24) (0.04)	13.13 (24) (0.04)	18.76 (24) (0.04)
	2.8	28	0.9023 (24) (0.0007)	15.15 (24) (0.05)	22.36 (12) (0.07)	15.01 (24) (0.05)	21.87 (12) (0.06)
	2.6	34	0.9017 (24) (0.0006)	17.55 (24) (0.06)	26.15 (12) (0.08)	17.43 (24) (0.06)	25.69 (12) (0.08)
	2.4	42	0.9027 (36) (0.0005)	21.21 (36) (0.06)	32.16 (24) (0.07)	21.17 (36) (0.06)	31.66 (24) (0.07)
	2.2	52	0.9004 (48) (0.0004)	26.76 (48) (0.06)	39.93 (12) (0.13)	26.63 (48) (0.06)	39.37 (12) (0.13)
	2.0	71	0.9011 (36) (0.0004)	35.29 (36) (0.10)	53.34 (24) (0.13)	35.16 (36) (0.10)	52.75 (24) (0.12)
	1.8	104	0.9012 (24) (0.0005)	50.47 (24) (0.17)	76.33 (12) (0.27)	50.34 (24) (0.17)	75.69 (12) (0.26)
	1.6	172	0.9004 (48) (0.0003)	81.07 (48) (0.19)	124.72 (12) (0.44)	80.93 (48) (0.19)	123.96 (12) (0.44)
	1.4	374	0.9006 (48) (0.0003)	163.68 (48) (0.40)	253.35 (12) (0.99)	163.55 (48) (0.40)	252.55 (12) (0.98)
	1.2	1460	0.9009 (24) (0.0003)	585.13 (24) (2.04)	924.77 (6) (5.43)	585.00 (24) (2.04)	923.61 (6) (5.42)

Table IV (continued) (k = 5)

P^*	θ^*	n_0	$P^*_{B_T}$ LF Estd. P{CS|LF}	$P^*_{B_T}$ LF Estd. E{n|LF}	$P^*_{B_T}$ EP Estd. E{n|EP}	$P^*_{B_{T(I)}}$ LF Estd. E{n|LF}	$P^*_{B_{T(I)}}$ EP Estd. E{n|EP}
0.95	3.0	34	0.9512 (36) (0.0004)	16.51 (36) (0.04)	27.79 (24) (0.06)	16.42 (36) (0.04)	27.19 (24) (0.05)
	2.8	39	0.9513 (24) (0.0005)	19.30 (24) (0.06)	32.16 (12) (0.09)	19.19 (24) (0.06)	31.50 (12) (0.08)
	2.6	46	0.9509 (24) (0.0005)	22.68 (24) (0.07)	37.57 (12) (0.11)	22.55 (24) (0.07)	36.87 (12) (0.10)
	2.4	58	0.9513 (24) (0.0004)	27.14 (24) (0.09)	46.05 (12) (0.14)	27.04 (24) (0.09)	45.37 (12) (0.14)
	2.2	74	0.9505 (48) (0.0003)	34.15 (48) (0.08)	58.55 (12) (0.18)	34.05 (48) (0.08)	57.80 (12) (0.17)
	2.0	98	0.9502 (72) (0.0002)	45.12 (72) (0.09)	77.11 (24) (0.17)	45.01 (72) (0.09)	76.28 (24) (0.16)
	1.8	142	0.9504 (72) (0.0002)	64.49 (72) (0.13)	110.02 (12) (0.35)	64.37 (72) (0.12)	109.09 (12) (0.34)
	1.6	240	0.9503 (72) (0.0002)	103.64 (72) (0.20)	181.09 (12) (0.61)	103.53 (72) (0.20)	180.02 (12) (0.60)
	1.4	510	0.9503 (48) (0.0002)	209.74 (48) (0.50)	366.73 (12) (1.32)	209.64 (48) (0.50)	365.49 (12) (1.31)
	1.2	2000	0.9507 (12) (0.0003)	741.81 (12) (3.60)	1352.66 (3) (10.49)	741.70 (12) (3.60)	1350.73 (3) (10.45)

+ All E{n} and P{CS} results and their standard errors are estimated by Monte Carlo sampling.

Table V

Estimated[+] $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$
for $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_{B_{T(I)}}$ with associated truncation number n_0
and selected $\{P^*, \theta^*\}$ when $k = 6$

P^*	θ^*	n_0	$\mathcal{P}^*_{B_T}$			$\mathcal{P}^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Estd. $P\{CS\|LF\}$	Estd. $E\{n\|LF\}$	Estd. $E\{n\|EP\}$	Estd. $E\{n\|LF\}$	Estd. $E\{n\|EP\}$
0.75	3.0	16	0.7587 (12) (0.0012)	9.40 (12) (0.04)	11.92 (12) (0.04)	9.29 (12) (0.04)	11.66 (12) (0.04)
	2.8	18	0.7527 (12) (0.0012)	10.89 (12) (0.05)	13.61 (12) (0.05)	10.75 (12) (0.04)	13.32 (12) (0.04)
	2.6	21	0.7528 (12) (0.0013)	13.10 (12) (0.05)	16.19 (12) (0.05)	12.95 (12) (0.05)	15.89 (12) (0.05)
	2.4	26	0.7522 (12) (0.0012)	15.89 (12) (0.07)	19.87 (12) (0.06)	15.73 (12) (0.06)	19.55 (12) (0.06)
	2.2	34	0.7510 (24) (0.0008)	19.95 (24) (0.06)	25.14 (12) (0.09)	19.80 (24) (0.06)	24.79 (12) (0.09)
	2.0	46	0.7524 (24) (0.0008)	26.55 (24) (0.08)	33.68 (12) (0.12)	26.39 (24) (0.08)	33.32 (12) (0.12)
	1.8	67	0.7506 (48) (0.0005)	38.03 (48) (0.08)	48.29 (12) (0.17)	37.86 (48) (0.08)	47.89 (12) (0.17)
	1.6	116	0.7512 (24) (0.0006)	62.04 (24) (0.20)	79.80 (12) (0.30)	61.87 (24) (0.20)	79.37 (12) (0.30)
	1.4	250	0.7507 (24) (0.0005)	126.88 (24) (0.42)	163.35 (12) (0.65)	126.71 (24) (0.42)	162.86 (12) (0.65)
	1.2	990	0.7503 (48) (0.0003)	453.41 (48) (1.09)	592.33 (6) (3.63)	453.22 (48) (1.09)	591.72 (6) (3.62)

Table V (continued) (k = 6)

P^*	θ^*	n_0	$\mathcal{P}^*_{B_T}$			$\mathcal{P}^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Estd. P{CS|LF}	Estd. E{n|LF}	Estd. E{n|EP}	Estd. E{n|LF}	Estd. E{n|EP}
0.90	3.0	31	0.9052 (12) (0.0009)	16.19 (12) (0.08)	24.71 (12) (0.08)	16.08 (12) (0.07)	24.25 (12) (0.07)
	2.8	36	0.9039 (24) (0.0006)	18.71 (24) (0.06)	28.52 (12) (0.09)	18.59 (24) (0.06)	28.04 (12) (0.09)
	2.6	42	0.9009 (24) (0.0006)	22.14 (24) (0.07)	33.56 (12) (0.10)	22.02 (24) (0.07)	33.03 (12) (0.10)
	2.4	52	0.9015 (48) (0.0004)	26.74 (48) (0.06)	41.05 (12) (0.13)	26.63 (48) (0.06)	40.52 (12) (0.12)
	2.2	66	0.9005 (72) (0.0004)	33.88 (72) (0.06)	51.93 (12) (0.16)	33.76 (72) (0.06)	51.33 (12) (0.16)
	2.0	90	0.9012 (24) (0.0005)	45.22 (24) (0.15)	69.41 (12) (0.22)	45.09 (24) (0.15)	68.78 (12) (0.22)
	1.8	134	0.9024 (24) (0.0005)	64.14 (24) (0.21)	101.08 (12) (0.34)	64.02 (24) (0.21)	100.38 (12) (0.34)
	1.6	224	0.9007 (24) (0.0005)	104.66 (24) (0.35)	165.02 (12) (0.57)	104.54 (24) (0.35)	164.22 (12) (0.57)
	1.4	475	0.9005 (72) (0.0002)	213.06 (72) (0.41)	336.42 (12) (1.23)	212.93 (72) (0.41)	335.45 (12) (1.22)
	1.2	1890	0.9007 (24) (0.0003)	769.62 (24) (2.61)	1231.11 (3) (9.90)	769.48 (24) (2.61)	1229.83 (3) (9.87)

Table V (continued) (k = 6)

P^*	θ^*	n_0	$P^*_{B_T}$			$P^*_{B_{T(I)}}$	
			LF		EP	LF	EP
			Estd. P{CS|LF}	Estd. E{n|LF}	Estd. E{n|EP}	Estd. E{n|LF}	Estd. E{n|EP}
0.95	3.0	42	0.9514 (12) (0.0007)	20.49 (12) (0.09)	34.96 (12) (0.09)	20.40 (12) (0.09)	34.34 (12) (0.09)
	2.8	48	0.9505 (48) (0.0003)	23.67 (48) (0.05)	40.22 (12) (0.11)	23.57 (48) (0.05)	39.54 (12) (0.10)
	2.6	57	0.9509 (48) (0.0004)	27.93 (48) (0.06)	47.32 (12) (0.13)	27.81 (48) (0.06)	46.61 (12) (0.13)
	2.4	72	0.9511 (24) (0.0004)	33.82 (24) (0.11)	58.90 (12) (0.17)	33.73 (24) (0.11)	58.17 (12) (0.16)
	2.2	92	0.9517 (24) (0.0004)	42.60 (24) (0.14)	74.58 (12) (0.21)	42.51 (24) (0.14)	73.79 (12) (0.21)
	2.0	124	0.9512 (24) (0.0004)	56.68 (24) (0.19)	99.57 (12) (0.29)	56.58 (24) (0.19)	98.70 (12) (0.29)
	1.8	178	0.9511 (24) (0.0004)	81.19 (24) (0.27)	141.86 (12) (0.43)	81.07 (24) (0.27)	140.87 (12) (0.42)
	1.6	305	0.9512 (24) (0.0003)	131.63 (24) (0.44)	233.82 (12) (0.75)	131.53 (24) (0.44)	232.72 (12) (0.75)
	1.4	635	0.9505 (24) (0.0003)	268.50 (24) (0.89)	473.78 (12) (1.60)	268.39 (24) (0.88)	472.38 (12) (1.59)
	1.2	2540	0.9504 (24) (0.0002)	966.65 (24) (3.24)	1760.25 (3) (13.26)	966.55 (24) (3.24)	1758.36 (3) (13.23)

+ All E{n} and P{CS} results and their standard errors are estimated by Monte Carlo sampling.