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TRUNCATION OF THE BECHHOFER-KIEFER-SOBEL SEQUENTIAL PROCEDURE FOR SELECTING THE MULTINOMIAL EVENT WHICH HAS THE LARGEST PROBABILITY

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ABSTRACT

In this article we study the effect of truncation on the performance of an open vector-at-a-time sequential sampling procedure $(\mathcal{P}_B^*)$, proposed by Bechhofer, Kiefer and Sobel, for selecting the multinomial event which has the largest probability. The performance of the truncated version $(\mathcal{P}_{B_T}^*)$ is compared to that of the original basic procedure $(\mathcal{P}_B^*)$. The performance characteristics studied include the probability of a correct selection, the expected number of vector-observations (n) to terminate sampling, and the variance of n. Both procedures guarantee the specified probability of a correct selection. Exact results and Monte Carlo sampling results are obtained. It is shown that $\mathcal{P}_{B_T}^*$ is far superior to $\mathcal{P}_B^*$ in terms of $E\{n\}$ and $Var\{n\}$, particularly when the event probabilities are equal.

The performance of $\mathcal{P}_{B_T}^*$ is also compared to that of a closed vector-at-a-time sequential sampling procedure proposed for the

same problem by Ramey and Alam; this procedure has heretofore been claimed to be the best one for this problem. It is shown that $P^*_{B_T}$ is superior to the Ramey-Alam procedure for most of the specifications of practical interest.

1. INTRODUCTION AND SUMMARY

Bechhofer, Kiefer and Sobel (1968) (B-K-S) proposed an open vector-at-a-time sequential sampling procedure (P^*_B) for ranking $k \geq 2$ Koopman-Darmois populations. As a special case, this procedure can be applied to a single k-category multinomial population for selecting that one of the $k \geq 2$ multinomial events which has the largest probability. This multinomial selection problem was first posed by Bechhofer, Elmaghraby and Morse (1959) (B-E-M) who proposed a single-stage selection procedure for solving it. It was proved in B-K-S that P^*_B guaranteed a certain indifference-zone probability requirement; for the multinomial this requirement reduces to the same one that was adopted by B-E-M. It has been proved that both procedures have the same so-called least-favorable (LF-) configuration. Since both procedures guarantee the same probability requirement and have the same LF-configuration, they therefore can be regarded as comparable and in direct competition. Their relative merits must then be judged based on their performance characteristics.

Heretofore no quantitative assessment has been made of the important performance characteristics of P^*_B as applied to the multinomial. In particular, there has been no study of the achieved probability of a correct selection ($P\{CS|LF\}$) when the probabilities p_i ($1 \leq i \leq k$) are in the LF-configuration, or of the distribution of the number of vector-observations (n) to terminate experimentation, as a function of $\underset{\sim}{p} = (p_1, p_2, \ldots, p_k)$. One purpose of the present article is to report on the results of our study of such performance characteristics.

We have found that the achieved $P\{CS|LF\}$ of $\mathcal{P}_B^*$ always exceeds its specified lower bound P^* by a substantial amount. In addition, the distribution of n is highly skewed to the right (as might be expected for an open procedure) resulting in a sizable proportion of experiments which terminate with excessively large (from a practical point of view) values of n; this latter phenomenon results in large values of $E\{n|\underset{\sim}{p}\}$ and $Var\{n|\underset{\sim}{p}\}$, particularly when all of the p_i $(1 \leq i \leq k)$ are equal (or almost equal). Although most of these undesirable characteristics are direct consequences of the fact that the procedure is open, these effects are magnified because of the $P\{CS\}$ "overprotection."

It was suggested in B-K-S that it would be possible to truncate $\mathcal{P}_B^*$ in such a way as to maintain $P\{CS|LF\} \geq P^*$. The reason for truncating would be to decrease $P\{CS|LF\}$ and at the same time eliminate the excessively large n-values; $E\{n|\underset{\sim}{p}\}$ and $Var\{n|\underset{\sim}{p}\}$ would thereby be reduced, not only in the LF-configuration but also uniformly in $\underset{\sim}{p}$. Our studies show that the concept of truncating $\mathcal{P}_B^*$ has considerable merit. The main purpose of the present article is to report on the $E\{n|\underset{\sim}{p}\}$ results that we obtained using that device.

We have carried out our studies in such a way that the performance of the truncated procedure $(\mathcal{P}_{B_T}^*)$ can be compared directly with that of the original untruncated procedure $(\mathcal{P}_B^*)$. Our findings concerning the effects of truncation were startling, showing $\mathcal{P}_{B_T}^*$ to be far superior relative to $\mathcal{P}_B^*$ in terms of $E\{n|\underset{\sim}{p}\}$ and $Var\{n|\underset{\sim}{p}\}$, particularly when the p_i $(1 \leq i \leq k)$ are equal and k is large. (See Section 5.)

Some of our results were obtained by direct calculation; most were obtained by Monte Carlo (MC) simulation. The techniques that we used in these studies are described in detail in the Appendix.

We have also compared the results reported herein with those obtained with another competing multinomial selection procedure due to Ramey and Alam (1979) (R-A) for which they claimed a certain optimality property. The properties of their procedure as well as a history of the multinomial selection problem are described in great detail in Bechhofer and Goldsman (1985) (B-G) which can be regarded as a companion paper to the present one, and should be read for background. In the present paper we show that in several important respects $\mathcal{P}^*_{B_T}$ is superior to the R-A procedure.

2. THE BASIC BECHHOFER-KIEFER-SOBEL SEQUENTIAL RANKING PROCEDURE ($\mathcal{P}^*_B$) FOR THE MULTINOMIAL

It is shown in B-K-S, Section 5.3.3, that their open sequential ranking procedure ($\mathcal{P}^*_B$) can be used for selecting the multinomial event which has the largest single-trial event probability when the indifference-zone approach is adopted with the "distances" between these event probabilities p_i $(1 \leq i \leq k)$ being measured in terms of the ratios of the probabilities. We first describe their formulation of this selection problem, and give the procedure that they proposed for solving it. Then we discuss certain properties of the procedure.

In the sequel we denote the ordered values of the p_i $(1 \leq i \leq k)$ by $p_{[1]} \leq \cdots \leq p_{[k]}$. The goal of the experiment is to select the event associated with $p_{[k]}$. If that event is selected, we say that a correct selection has been made. Prior to the start of experimentation the experimenter specifies two constants $\{\theta^*, P^*\}$ with $1 < \theta^* < \infty$, $1/k < P^* < 1$. These constants are incorporated into the following indifference-zone probability requirement:

$$P\{\text{Correct selection}\} \geq P^* \quad \text{whenever} \quad p_{[k]} \geq p_{[k-1]}\theta^*. \tag{2.1}$$

Consideration is restricted to procedures which guarantee (2.1). The single-stage procedure of Bechhofer, Elmaghraby and Morse (1959) who were the first to study this selection problem has been shown to guarantee (2.1).

THE B-K-S PROCEDURE (P_B^*) FOR SELECTING THE EVENT ASSOCIATED WITH $p_{[k]}$.

The open basic sequential ranking procedure (P_B^*) for the multinomial distribution (Π), described in Sections 5.1.3 and 5.3.3 of B-K-S, employs the following sampling rule, stopping rule and terminal decision rule:

Sampling rule: Take observations $(x_{1j},\ldots,x_{kj})$ $(j = 1,2,\ldots)$ one-at-a-time from Π. (2.2)

Stopping rule: After the mth observation $(m = 1,2,\ldots)$ compute

$$z_m = \sum_{i=1}^{k-1} (1/\theta^*)^{(y_{[k]m}-y_{[i]m})} \tag{2.3}$$

where $y_{im} = \sum_{j=1}^{m} x_{ij}$ $(1 \leq i \leq k)$ and $y_{[1]m} \leq \cdots \leq y_{[k]m}$ are the ordered values of the y_{im}. Stop sampling when, for the first time,

$$z_n \leq (1-P^*)/P^*. \tag{2.4}$$

Here n (a random variable) is the value of m at termination.

Terminal decision rule: After stopping, select the event associated with $y_{[k]n}$. If two or more events yield y_{in}-values equal to $y_{[k]n}$, then select one of them at random. (2.5)

Remark 2.1: It is proved in B-K-S, Section 6.1.1, (see also Levin [1984]), that P_B^* guarantees (2.1). It is also proved in Section 6.1.1 (see in addition, Section 12.5) that the P{CS} of P_B^* is minimized subject to (2.1) when

$$p_{[1]} = p_{[k-1]} = p_{[k]}/\theta^*, \tag{2.6}$$

the so-called least-favorable configurations of the p_i $(1 \leq i \leq k)$ for P_B^*. Kesten and Morse (1959) had proved earlier that (2.6) is also the LF-configuration for the single-stage multinomial selection procedure of B-E-M.

Remark 2.2: It is proved in B-K-S, Corollary 3.2.2, that when the p_i $(1 \leq i \leq k)$ are in the LF-configuration, then

$$E\{W_{[k]n}\} = P\{CS|LF\} \tag{2.7}$$

where $W_{[k]n} = 1/(1+Z_n)$, and the r.v. Z_n is the value of (2.3) for $\mathcal{P}_B^*$ at termination. This result makes it possible to estimate $P\{CS|LF\}$ with high precision using Monte Carlo (MC) sampling. (See B-K-S, Section 12.6.4, and Sections 4.2 and 5 of the present paper.)

Because of the discreteness of the Z_m of (2.3), the procedure $\mathcal{P}_B^*$ usually terminates with a strict inequality in (2.4) resulting in so-called "undershoot." When (2.1) holds, the undershoot results in a larger achieved $P\{CS|LF\}$ than the specified P^*; this overprotection is purchased at the cost of a larger-than-necessary $E\{n|LF\}$. Not only does $E\{n|LF\}$ itself increase but also the distribution of n increases stochastically with the amount of overprotection, resulting in occasional excessively large (from a practical point of view) values of n, particularly for $p_{[k]}-p_{[1]}$ close to zero. ($E\{n|\underset{\sim}{p}\}$ assumes its maximum when $p_{[1]} = p_{[k]}$.)

To illustrate the extreme skewness of the distribution of n, we give in Table I two typical empirical distributions of n obtained by MC sampling, each being based on 10,000 independent replications. Both are for $k = 4$, $\theta^* = 1.6$, $P^* = 0.75$; in the first the p_i $(1 \leq i \leq 4)$ are in the least favorable (LF-) configuration $p_{[1]} = p_{[3]} = p_{[4]}/\theta^*$ while in the second they are in the equal-parameter (EP-) configuration $p_{[1]} = p_{[4]}$. The n-values are grouped in class intervals of width 10.

We point out that the single-stage procedure of B-E-M requires exactly 46 observations to guarantee (2.1) when $k = 4$, $P^* = 0.75$, $\theta^* = 1.6$. (See Table 4.1, p. 445 of Gibbons, Olkin and Sobel (1977).) For the ungrouped data on which Table I is based, 21.36 percent of the n-values exceeded $n = 46$ in the LF-configuration while 38.25 percent exceeded $n = 46$ in the EP-configuration; in fact, 5.02 percent of the n-values exceeded 75 and 5.05 percent exceeded 110 in the LF- and EP-configurations, respectively. It

Table I

Empiric Distribution of n for P_B^*

obtained by Monte Carlo sampling

for k = 4, $P^* = 0.75$, $\theta^* = 1.6$; LF- and EP-configurations

(Each distribution is based on 10,000 replications.)

(a,b)	Configuration (LF) Proportion of experiments terminating with $\{a < n \leq b\}$	Configuration (EP) Proportion of experiments terminating with $\{a < n \leq b\}$	(a,b)	Configuration (LF) Proportion of experiments terminating with $\{a < n \leq b\}$	Configuration (EP) Proportion of experiments terminating with $\{a < n \leq b\}$
(1,10)	0.0645	0.0382	(181,190)	0.0000	0.0017
(11,20)	0.2516	0.1660	(191,200)	0.0001	0.0012
(21,30)	0.2196	0.1774	(201,210)	--	0.0011
(31,40)	0.1723	0.1540	(211,220)	--	0.0005
(41,50)	0.1061	0.1102	(221,230)	--	0.0005
(51,60)	0.0720	0.0917	(231,240)	--	0.0001
(61,70)	0.0433	0.0716	(241,250)	--	0.0001
(71,80)	0.0305	0.0519	(251,260)	--	0.0001
(81,90)	0.0156	0.0353	(261,270)	--	0.0002
(91,100)	0.0109	0.0314	(271,280)	--	0.0000
(101,110)	0.0064	0.0204	(281,290)	--	0.0004
(111,120)	0.0024	0.0153	(291,300)	--	0.0000
(121,130)	0.0017	0.0098	(301,310)	--	0.0000
(131,140)	0.0012	0.0076	(311,320)	--	0.0000
(141,150)	0.0013	0.0044	(321,330)	--	0.0000
(151,160)	0.0003	0.0045	(331,340)	--	0.0000
(161,170)	0.0001	0.0025	(341,350)	--	0.0000
(171,180)	0.0001	0.0017	(351,360)	--	0.0002

is thus seen that P_B^* can, with sizable probability, yield n-values which are excessively large.

In Table II we provide estimates of $E\{n|LF\}$, $Var\{n|LF\}$, $E\{n|EP\}$, $Var\{n|EP\}$ and $P\{CS|LF\}$, all of them calculated from the ungrouped data on which Table I is based. The $E\{n\}$ and $P\{CS|LF\}$ estimates have very small standard errors since each is based on 10,000 replications. The first estimate of $P\{CS|LF\}$ is based on the observed proportion of correct selections in the 10,000 replications while the second estimate (see Remark 2.2) is based on $\overline{W}_{[k]n}$ = (the observed average of the $W_{[k]n}$ in the same 10,000 replications). Both are unbiased estimates of $P\{CS|LF\}$; however, the second estimate has a much smaller standard error than the first. We note that $P\{CS|LF\}$ is estimated as being approximately 0.785 which is much greater than the specified $P^* = 0.75$.

The results cited above are typical of those that would be obtained for P_B^* employed with "modest" k, and θ^* and P^* not too close to unity. We point out that since the skewness of the distribution of n increases as k increases for fixed $\{\theta^*,P^*\}$ and/or as θ^* and/or P^* approaches unity for fixed k, all of the undesirable effects noted above would then be greatly magnified. It is these reasons that prompted us to study the effect of truncating P_B^*. This strategem is pursued in Section 3. Corresponding results obtained with the untruncated and truncated procedures are compared in Tables IIIA-VIIB.

Remark 2.3: We have pointed out that the single-stage procedure of B-E-M requires exactly 46 observations to guarantee (2.1) when $k = 4$, $\theta^* = 1.6$, $P^* = 0.75$. We have also seen from Table II that $E\{n|LF\}$ and $E\{n|EP\}$ for P_B^* are approximately 34 and 47, respectively, for this same specification. These results might suggest to the reader that little is to be gained by using P_B^* in place of the single-stage procedure of B-E-M. Such a conclusion would, however, be unwarranted, the issues being more complicated than they appear on the surface.

Table II

MC Estimates[a] of $E\{n|LF\}$, $Var\{n|LF\}$, $E\{n|EP\}$, $Var\{n|EP\}$ and $P\{CS|LF\}$ for P_B^* when $k = 4$, $P^* = 0.75$, $\theta^* = 1.6$

(Each estimate is based on 10,000 replications.)

Least-favorable configuration		Equal-parameter configuration	
Estimate of $E\{n\|LF\}$	33.84 (0.22)[b]	Estimate of $E\{n\|EP\}$	46.98 (0.34)[b]
Estimate of $Var\{n\|LF\}$	473.58	Estimate of $Var\{n\|EP\}$	1127.41
Estimate of $P\{CS\|LF\}$ based on observed proportion of CS		0.7816 (0.0041)[b]	
Estimate of $P\{CS\|LF\}$ based on $\overline{W}_{[k]n}$ of (2.7)		0.7848 (0.0002)[b]	

[a] All MC estimates are based on the ungrouped data which yielded the empiric distribution of n given in Table I.

[b] The numbers in parentheses are the estimated standard errors of the averages above them.

If the configuration of the p_i $(1 \leq i \leq k)$ is <u>very favorable</u> to the experimenter, then $\mathcal{P}_B^*$ <u>always</u> should be used. In fact, in the most extreme configuration wherein $p_{[k]} = 1$ it is easy to see that $\mathcal{P}_B^*$ will require <u>exactly</u>

$$n_{min} = \left[\left(\log\left\{\frac{(k-1)P^*}{1-P^*}\right\}\right) \Big/ \log \theta^*\right]^+ \qquad (2.8)$$

observations to terminate sampling; here $[x]^+$ denotes the smallest integer equal to or greater than x. For $k = 4$, $\theta^* = 1.6$, $P^* = 0.75$ we find that $n_{min} = 5$. As the configuration of the p_i $(1 \leq i \leq k)$ becomes less and less favorable to the experimenter, e.g., if $p_{[1]} = p_{[k-1]} = p_{[k]}/\theta$ $(1 \leq \theta < \infty)$ with $\theta \to 1$, then $E\{n|\underset{\sim}{p}\}$ increases from n_{min} to $E\{n|EP\}$ <u>and</u> at the same time $Var\{n|\underset{\sim}{p}\}$ <u>increases from zero to</u> $Var\{n|EP\}$ with the distribution of n becoming more and more skewed to the right. Of course, the experimenter does not know the true configuration of the p_i $(1 \leq i \leq k)$. However, regardless of what the unknown values of the p_i $(1 \leq i \leq k)$ happen to be, truncation of $\mathcal{P}_B^*$ will always improve the performance of the procedure.

3. THE TRUNCATED BECHHOFER-KIEFER-SOBEL SEQUENTIAL RANKING PROCEDURE ($\mathcal{P}_{B_T}^*$) FOR THE MULTINOMIAL

We now propose to modify $\mathcal{P}_B^*$ by <u>truncation</u>, and to study the performance characteristics of the resulting <u>truncated</u> procedure which we shall refer to as $\mathcal{P}_{B_T}^*$. For $\mathcal{P}_{B_T}^*$ we employ the same sampling and terminal decision rules, (2.2) and (2.5), respectively, as for $\mathcal{P}_B^*$ but replace (2.4) of the stopping rule of $\mathcal{P}_B^*$ by the following:

<u>Stopping rule for</u> $\mathcal{P}_{B_T}^*$: After the mth observation $(m = 1,2,\ldots)$ compute z_m of (2.3). Stop sampling when, for the first time, <u>either</u>

$$z_n \leq (1-P^*)/P^*$$

or (3.1)

$$n = n_0,$$

whichever occurs first; here n (a random variable) is the value of m at termination, and $n_0(k;\theta^*,P^*) = n_0$ (say) is predetermined as the smallest integer that will guarantee (2.1) when $\mathcal{P}^*_{B_T}$ is used.

The possibility of modifying $\mathcal{P}^*_B$ by truncation was suggested in B-K-S, Section 12.6.4.

4. DETERMINATION OF THE TRUNCATION INTEGER (n_0) FOR $\mathcal{P}^*_{B_T}$

4.1 Exact Determination of n_0

It is mentioned in B-K-S, Section 3.7(e), that sampling for the multinomial using $\mathcal{P}^*_B$ can be truncated at some predetermined stage (if sampling has not terminated before that stage), and that the exact $P\{CS|\underset{\sim}{p}\}$ of the truncated procedure can be evaluated for any $\underset{\sim}{p}$ by the enumeration of termination sequences. The $P\{CS|\underset{\sim}{p}\}$ of $\mathcal{P}^*_{B_T}$ is an increasing function of the truncation integer for fixed k, $\{\theta^*,P^*\}$ and given $\underset{\sim}{p}$. Thus for $\underset{\sim}{p}$ satisfying (2.6) it is possible to determine n_0 by trial and error. We did this for $(k;\theta^*,P^*)$-values for which the resulting value of n_0 is "small." However, the enumeration process quickly gets out of hand as k increases and/or as θ^* and/or P^* approaches unity. We next calculated $P\{CS|\underset{\sim}{p}\}$ exactly using recursion formulae (see the Appendix), and in this way we were able to determine n_0-values for larger values of k and/or for θ^* and/or P^* closer to unity; the results obtained using the recursion formulae were then checked against those obtained for cases in which we used complete enumeration, and were found to be in complete agreement.

4.2 Estimation of n_0 Using MC Sampling

For larger values of k and/or for θ^* and/or P^* even closer to unity we found it necessary to employ Monte Carlo (MC) simulation to estimate $P\{CS|LF\}$ and thereby n_0; a sufficient number of replications were carried out in order that the estimates of $P\{CS|LF\}$ would have very small standard errors. These estimates were then checked against the calculated exact results obtained using the recursion formulae, and the two were found to be in very close agreement (within the sampling error).

When using MC sampling to estimate the $P\{CS|LF\}$ achieved by $\mathcal{P}^*_{B_T}$ we employed the estimate obtained by averaging the values of the $W_{[k]n}$ since for any given number of replications this estimate has a much smaller standard error than does the estimate based on the observed proportion of correct selections (as can be seen, e.g., in Table II). (In B-K-S, Section 14.1.1.1 and also Section 18.3, this same technique was used for the normal means selection problem when the populations have a common known variance.) The resulting value of n_0 (obtained by exact calculations, or by MC sampling when exact calculations were not feasible) was then employed with $\mathcal{P}^*_{B_T}$ for the p_i $(1 \leq i \leq k)$ in the LF-configuration given by (2.6) and also for the p_i in the EP-configuration given by $p_{[1]} = p_{[k]}$. The outcomes obtained are summarized and discussed in Section 5.

5. THE PERFORMANCE OF $\mathcal{P}^*_{B_T}$ COMPARED TO THAT OF $\mathcal{P}^*_B$.

In this section we report on the results of the studies which we carried out to compare the performance of $\mathcal{P}^*_B$ and $\mathcal{P}^*_{B_T}$. The performance characteristics studied for $\mathcal{P}^*_B$ were $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$; in addition $\max\{n|LF\}$ and $\max\{n|EP\}$ for $t = 4000$ replications were recorded. The characteristics studied for $\mathcal{P}^*_{B_T}$ employed with predetermined truncation integers n_0 were $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$. Results were obtained for both procedures with $k = 2,3,4,5$ and 10 (this being the range of most practical interest), and all combinations of $P^* = 0.75, 0.90, 0.95$ with $\theta^* = 3.0, 2.4, 2.0, 1.6$ except $P^* = 0.95$, $\theta^* = 1.6$; the latter was omitted because of cost considerations. Ideally we would have liked to have obtained results for P^* and θ^* closer to unity but the cost of obtaining such information would have been prohibitive. However, the results that we did obtain were overwhelmingly convincing, and enabled us to draw definitive conclusions. These results are summarized in Tables IIIA-B, IVA-B, VA-B, VIA-B and VIIA-B for $k = 2,3,4,5$ and 10, respectively.

For every k, all results for $\mathcal{P}_B^*$ were obtained by MC sampling. Unless noted otherwise, all MC sampling data for $\mathcal{P}_B^*$ are based on 4,000 replications each; the only exceptions here are for $k = 10$, EP-configuration, where the results for $(P^*,\theta^*) = (0.75,1.6)$, $(0.90,2.0)$, $(0.90,1.6)$, $(0.95,2.0)$ are based on 2,000 replications each. For $k = 2$ and 3, all data for $\mathcal{P}_{B_T}^*$ were obtained by exact calculations as were some of those for $k = 4$ (the exceptions being noted in Table VB); all of the remaining data for $\mathcal{P}_{B_T}^*$ were obtained by MC sampling. In our tables the symbol (x) placed to the left of a MC estimate means that the estimate is based on x,000 replications.

We point out that the results given in Tables VA-B for $P^* = 0.75$, $\theta^* = 1.6$ are in close agreement with those given in Tables I and II; the first are based on 4,000 independent replications while the second are based on 10,000 independent replications, the second being independent of the first.

The corresponding results for $\mathcal{P}_B^*$ and $\mathcal{P}_{B_T}^*$ in each table demonstrate the dramatic decreases in $E\{n|EP\}$ that can be achieved by truncation, these improvements being particularly substantial for P^* and/or θ^* approaching unity and k "large." If we consider the difference $E_{\mathcal{P}_B^*}\{n|EP\} - E_{\mathcal{P}_{B_T}^*}\{n|EP\}$ as a function of any one of the arguments (k,P^*,θ^*), the other two arguments remaining fixed, we find that the difference increases with increasing k or P^*, and with decreasing θ^*. For example, for $k = 10$ (Tables VIIA-B) and $P^* = 0.90$ we have for $\theta^* = 3.0$, 2.4, 2.0 and 1.6 with $\mathcal{P}_{B_T}^*$ employing $n_0 = 55$, 98, 174 and 424, respectively, that the corresponding $E\{n\}$ differences are 38.25, 48.84, 72.16 and 135.19. In the latter case a decrease in $E\{n|EP\}$ of approximately 29 percent was achieved by truncation. Truncation appears to offer only modest improvements in $E\{n|LF\}$. However, truncation forces $\max\{n|\underset{\sim}{p}\} \leq n_0$ for all $\underset{\sim}{p}$. The practical gains here can be substantial. For example, for $k = 10$, $P^* = 0.90$, $\theta^* = 1.6$ we see that $\max\{n|LF\}$ and $\max\{n|EP\}$ for $\mathcal{P}_B^*$ were 887 and 1777 based on 4,000 and 2,000 replications, respectively, both n-values being considerably larger than $n_0 = 424$. These large n-values are, of course, consequences of the extreme skewness of the distributions of n (already noted for $k = 4$, $P^* = 0.75$, $\theta^* = 1.6$ in Table I). The evidence presented in Tables IIIA-VIIB establishes the substantial superiority of $\mathcal{P}_{B_T}^*$ over $\mathcal{P}_B^*$.

Note: The particular pairs $\{\theta^*, P^*\}$ considered in the present study were chosen to match those used by Ramey and Alam (1979). In this way results obtained with $\mathcal{P}_{B_T}^*$ could be compared directly with the R-A results; this is done in Section 6.

6. THE PERFORMANCE OF $\mathcal{P}_{B_T}^*$ COMPARED TO THAT OF THE RAMEY-ALAM PROCEDURE

Ramey and Alam (1979) (R-A) proposed a closed sequential procedure for selecting that one of $k \geq 2$ multinomial events which has the largest probability. They provided tables of constants ((r,N)-values) necessary to implement their procedure in order that it would guarantee (2.1) and at the same time minimize $E\{n|LF\}$. Based on the constants that they provided, Ramey and Alam concluded that their procedure was "uniformly better" than all of the known competing procedures in terms of minimizing $E\{n|LF\}$. The reader is referred to B-G (1985) for a description of the R-A and competing procedures.

In the course of studying the performance of $\mathcal{P}_{B_T}^*$ we had occasion to check the accuracy of the R-A constants and found quite a few of them to be incorrect. In B-G (1985) we provided corrected sets of constants for their procedure. Use of these new

Table IIIA

Estimated $P\{CS|LF\}$, $E\{n|LF\}$, $E\{n|EP\}$ and $\max\{n|LF\}$, $\max\{n|EP\}$ based on 4000 replications for P_B^* with selected $\{P^*,\theta^*\}$ when $k = 2$

P^*	θ^*	Untruncated procedure (P_B^*)				
		LF-configuration			EP-configuration	
		$\max\{n\|LF\}$ t = 4000	Estd. $P\{CS\|LF\}$	Estd. $E\{n\|LF\}$	$\max\{n\|EP\}$ t = 4000	Estd. $E\{n\|EP\}$
0.75	3.0	1	0.7500 (0)	1.00 (0)	1	1.00 (0)
	2.4	24	0.8521 (0)	3.47 (0.04)	38	4.06 (0.05)
	2.0	24	0.8000 (0)	3.60 (0.04)	30	4.04 (0.05)
	1.6	45	0.8038 (0)	7.88 (0.09)	55	9.08 (0.11)
0.90	3.0	22	0.9000 (0)	3.21 (0.03)	24	3.97 (0.04)
	2.4	51	0.9325 (0)	6.35 (0.07)	73	8.90 (0.11)
	2.0	62	0.9412 (0)	10.63 (0.11)	138	16.02 (0.21)
	1.6	143	0.9129 (0)	17.79 (0.21)	137	24.89 (0.30)
0.95	3.0	37	0.9643 (0)	5.55 (0.05)	59	8.84 (0.11)
	2.4	84	0.9707 (0)	9.17 (0.10)	118	15.98 (0.20)
	2.0	101	0.9697 (0)	14.14 (0.15)	179	25.08 (0.31)

Table IIIB

Exact $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ for $P^*_{B_T}$ with associated truncation number n_0 and selected $\{P^*,\theta^*\}$ when $k = 2$

P^*	θ^*	Truncated procedure ($P^*_{B_T}$)			
		Truncation number (n_0)	LF-configuration		EP-configuration
			Exact $P\{CS\|LF\}$	Exact $E\{n\|LF\}$	Exact $E\{n\|EP\}$
0.75	3.0	1	0.7500	1.000	1.000
	2.4	3	0.7914	2.415	2.500
	2.0	5	0.7737	3.086	3.250
	1.6	9	0.7559	6.145	6.469
0.90	3.0	23	0.9000	3.200	3.999
	2.4	11	0.9113	5.806	7.102
	2.0	15	0.9032	9.128	10.957
	1.6	41	0.9006	17.098	21.705
0.95	3.0	11	0.9522	5.314	7.102
	2.4	17	0.9548	8.586	11.696
	2.0	27	0.9536	13.256	18.348

Table IVA

Estimated $P\{CS|LF\}$, $E\{n|LF\}$, $E\{n|EP\}$
and $\max\{n|LF\}$, $\max\{n|EP\}$ based on 4000 replications
for $\mathcal{P}_B^*$ with selected $\{P^*,\theta^*\}$ when $k = 3$

P^*	θ^*	Untruncated procedure ($\mathcal{P}_B^*$)				
		LF-configuration			EP-configuration	
		$\max\{n\|LF\}$ t = 4000	Estd. $P\{CS\|LF\}$	Estd. $E\{n\|LF\}$	$\max\{n\|EP\}$ t = 4000	Estd. $E\{n\|EP\}$
0.75	3.0	31	0.8421 (0.0004)	4.52 (0.05)	42	6.23 (0.08)
	2.4	45	0.8295 (0.0005)	6.70 (0.07)	58	9.00 (0.10)
	2.0	72	0.8003 (0.0005)	9.47 (0.10)	93	12.33 (0.14)
	1.6	102	0.7866 (0.0004)	19.40 (0.22)	225	25.84 (0.32)
0.90	3.0	51	0.9459 (0.0002)	8.22 (0.09)	93	15.70 (0.20)
	2.4	61	0.9213 (0.0002)	11.00 (0.11)	125	18.98 (0.23)
	2.0	92	0.9265 (0.0003)	18.44 (0.19)	222	33.16 (0.40)
	1.6	247	0.9150 (0.0001)	38.59 (0.39)	451	67.41 (0.80)
0.95	3.0	54	0.9615 (0.0001)	9.20 (0.09)	126	18.57 (0.22)
	2.4	70	0.9655 (0.0001)	14.96 (0.14)	201	33.06 (0.40)
	2.0	160	0.9625 (0.0001)	23.72 (0.24)	318	50.76 (0.63)

Table IVB

Exact $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$ with associated truncation number n_0 and selected $\{P^*,\theta^*\}$ when $k = 3$

P^*	θ^*	Truncated procedure ($\mathcal{P}^*_{B_T}$)			
		Truncation number (n_0)	LF-configuration		EP-configuration
			Exact $P\{CS\|LF\}$	Exact $E\{n\|LF\}$	Exact $E\{n\|EP\}$
0.75	3.0	5	0.7574	3.482	3.852
	2.4	8	0.7602	5.586	6.226
	2.0	13	0.7512	8.181	9.273
	1.6	32	0.7517	17.802	20.592
0.90	3.0	12	0.9029	7.206	9.394
	2.4	22	0.9021	10.528	14.577
	2.0	34	0.9016	17.341	23.801
	1.6	83	0.9003	37.398	53.146
0.95	3.0	20	0.9505	8.970	13.932
	2.4	31	0.9516	14.600	22.653
	2.0	52	0.9508	23.159	36.625

Table VA

Estimated $P\{CS|LF\}$, $E\{n|LF\}$, $E\{n|EP\}$ and $\max\{n|LF\}$, $\max\{n|EP\}$ based on 4000 replications for P_B^* with selected $\{P^*,\theta^*\}$ when $k = 4$

P^*	θ^*	Untruncated procedure (P_B^*)				
		LF-configuration			EP-configuration	
		$\max\{n\|LF\}$ $t = 4000$	Estd. $P\{CS\|LF\}$	Estd. $E\{n\|LF\}$	$\max\{n\|EP\}$ $t = 4000$	Estd. $E\{n\|EP\}$
0.75	3.0	38	0.8034 (0.0008)	5.81 (0.07)	53	8.55 (0.11)
	2.4	53	0.7986 (0.0006)	9.22 (0.10)	94	13.30 (0.16)
	2.0	107	0.7966 (0.0005)	15.44 (0.16)	127	21.91 (0.25)
	1.6	163	0.7850 (0.0004)	34.35 (0.36)	324	48.03 (0.54)
0.90	3.0	62	0.9342 (0.0004)	10.75 (0.11)	131	22.43 (0.28)
	2.4	82	0.9269 (0.0003)	16.84 (0.16)	231	33.26 (0.38)
	2.0	137	0.9214 (0.0002)	26.58 (0.27)	311	50.66 (0.60)
	1.6	255	0.9162 (0.0001)	59.74 (0.56)	646	113.50 (1.30)
0.95	3.0	61	0.9672 (0.0001)	13.52 (0.12)	209	33.51 (0.39)
	2.4	107	0.9644 (0.0001)	21.63 (0.20)	322	51.09 (0.61)
	2.0	181	0.9625 (0.0001)	34.84 (0.33)	451	82.31 (0.93)

Table VB

Exact† $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$
for $P^*_{B_T}$ with associated truncation number n_0
and selected $\{P^*,\theta^*\}$ when $k = 4$

P^*	θ^*	Truncated procedure ($P^*_{B_T}$)			
		Truncation number (n_0)	LF-configuration		EP-configuration
			Exact† $P\{CS\|LF\}$	Exact† $E\{n\|LF\}$	Exact† $E\{n\|EP\}$
0.75	3.0	9	0.7517	5.029	5.973
	2.4	15	0.7569	8.416	10.150
	2.0	24	0.7541	13.944	16.760
	1.6	57	0.7522	31.499	(24) 38.020 (0.107)
0.90	3.0	19	0.9036	9.988	14.273
	2.4	31	0.9022	16.053	23.231
	2.0	53	0.9001	25.840	37.818
	1.6	128	(24) 0.9007 (0.0005)	59.11 (0.20)	(12) 88.41 (0.34)
0.95	3.0	26	0.9513	13.070	20.90
	2.4	44	0.9507	20.789	34.03
	2.0	75	(36) 0.9505 (0.0003)	34.07 (0.10)	(24) 56.67 (0.13)

†All results are exact except where indicated by standard errors for $(P^*,\theta^*) = (0.75,1.6)$, $(0.90,1.6)$ and $(0.95,2,0)$, these exceptions being estimated.

Table VIA

Estimated $P\{CS|LF\}$, $E\{n|LF\}$, $E\{n|EP\}$
and $\max\{n|LF\}$, $\max\{n|EP\}$ based on 4000 replications
for P_B^* with selected $\{P^*,\theta^*\}$ when k = 5

P*	θ*	Untruncated procedure (P_B^*)				
		LF-configuration			EP-configuration	
		$\max\{n\|LF\}$ t = 4000	Estd. $P\{CS\|LF\}$	Estd. $E\{n\|LF\}$	$\max\{n\|EP\}$ t = 4000	Estd. $E\{n\|EP\}$
0.75	3.0	50	0.8295 (0.0006)	8.78 (0.09)	86	13.82 (0.16)
	2.4	85	0.8072 (0.0006)	13.32 (0.13)	142	21.10 (0.24)
	2.0	118	0.7961 (0.0005)	22.23 (0.23)	213	33.47 (0.38)
	1.6	243	0.7811 (0.0004)	48.11 (0.48)	541	72.10 (0.81)
0.90	3.0	85	0.9385 (0.0003)	14.63 (0.15)	217	33.17 (0.41)
	2.4	105	0.9252 (0.0003)	22.61 (0.22)	247	47.51 (0.54)
	2.0	181	0.9220 (0.0002)	37.41 (0.36)	540	74.25 (0.84)
	1.6	452	0.9156 (0.0001)	84.69 (0.79)	927	165.97 (1.86)
0.95	3.0	90	0.9654 (0.0001)	17.23 0.16	293	46.57 (0.54)
	2.4	130	0.9642 (0.0001)	27.82 0.25	536	73.87 (0.88)
	2.0	242	0.9620 (0.0001)	47.07 (0.43)	657	115.07 1.32

Table VIB

Estimated $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$ with associated truncation number n_0 and selected $\{P^*,\theta^*\}$ when $k = 5$

P^*	θ^*	Truncated procedure ($\mathcal{P}^*_{B_T}$) Truncation number (n_0)	LF-configuration Estd. $P\{CS\|LF\}$	LF-configuration Estd. $E\{n\|LF\}$	EP-configuration Estd. $E\{n\|EP\}$
0.75	3.0	12	(12) 0.7621 (0.0013)	7.62 (0.03)	(12) 9.14 (0.03)
	2.4	20	(12) 0.7520 (0.0012)	12.11 (0.05)	(12) 14.93 (0.05)
	2.0	35	(24) 0.7540 (0.0008)	20.13 (0.06)	(12) 24.88 (0.09)
	1.6	86	(24) 0.7512 (0.0006)	45.68 (0.15)	(12) 57.16 (0.23)
0.90	3.0	24	(12) 0.9030 (0.0010)	13.29 (0.06)	(12) 19.23 (0.06)
	2.4	42	(24) 0.9025 (0.0006)	21.41 (0.07)	(12) 32.39 (0.10)
	2.0	71	(24) 0.9008 (0.0005)	35.31 (0.12)	(12) 53.47 (0.18)
	1.6	176	(12) 0.9021 (0.0006)	81.46 (0.39)	(12) 125.28 (0.46)
0.95	3.0	34	(24) 0.9507 (0.0005)	16.46 (0.05)	(12) 27.78 (0.08)
	2.4	58	(12) 0.9508 (0.0006)	27.19 (0.13)	(12) 46.13 (0.13)
	2.0	98	(24) 0.9506 (0.0004)	45.07 (0.15)	(12) 77.31 (0.24)

Table VIIA

Estimated $P\{CS|LF\}$, $E\{n|LF\}$, $E\{n|EP\}$ and $\max\{n|LF\}$, $\max\{n|EP\}$ based on 4000 replications† for P_B^* with selected $\{P^*,\theta^*\}$ when $k = 10$

P^*	θ^*	Untruncated procedure (P_B^*)				
		LF-configuration			EP-configuration	
		$\max\{n\|LF\}$ t = 4000	Estd. $P\{CS\|LF\}$	Estd. $E\{n\|LF\}$	$\max\{n\|EP\}$ t = 4000	Estd. $E\{n\|EP\}$
0.75	3.0	112	0.8169 (0.0007)	20.54 (0.21)	258	38.32 (0.44)
	2.4	175	0.8064 (0.0006)	35.38 (0.33)	284	61.49 (0.67)
	2.0	253	0.7957 (0.0005)	59.39 (0.57)	555	100.06 (1.08)
	1.6	604	0.7815 (0.0003)	138.39 (1.24)	(2)1263	225.75 (3.31)
0.90	3.0	161	0.9341 (0.0003)	30.82 (0.29)	487	84.51 (0.98)
	2.4	244	0.9252 (0.0002)	52.38 (0.48)	771	129.91 (1.42)
	2.0	463	0.9220 (0.0002)	89.38 (0.81)	(2) 993	211.40 (3.11)
	1.6	887	0.9151 (0.0002)	212.25 (1.83)	(2)1777	468.06 (6.63)
0.95	3.0	178	0.9662 (0.0002)	36.84 (0.32)	659	121.04 (1.35)
	2.4	253	0.9633 (0.0001)	64.73 (0.56)	1239	192.61 (2.10)
	2.0	451	0.9614 (0.0001)	109.38 (0.95)	(2)1804	305.88 (4.71)

†All results are based on $t = 4000$ replications except those for the EP-configuration for $(P^*,\theta^*) = (0.75,1.6)$, $(0.90,2.0)$, $(0.90,1.6)$ and $(0.95,2.0)$ which are based on 2,000 replications.

Table VIIB

Estimated $P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$ for $P^*_{B_T}$ with associated truncation number n_0 and selected $\{P^*,\theta^*\}$ when $k = 10$

P^*	θ^*	Truncated procedure ($P^*_{B_T}$): Truncation number (n_0)	LF-configuration: Estd. $P\{CS\|LF\}$	LF-configuration: Estd. $E\{n\|LF\}$	EP-configuration: Estd. $E\{n\|EP\}$
0.75	3.0	31	(12) 0.7539 (0.0014)	18.73 (0.08)	(4) 24.46 (0.13)
	2.4	54	(12) 0.7535 (0.0012)	32.03 (0.13)	(4) 42.36 (0.23)
	2.0	96	(12) 0.7520 (0.0011)	55.16 (0.24)	(4) 72.91 (0.41)
	1.6	244	(20) 0.7510 (0.0007)	132.38 (0.45)	(4) 175.45 (1.07)
0.90	3.0	55	(12) 0.9023 (0.0010)	29.42 (0.13)	(4) 46.26 (0.21)
	2.4	98	(16) 0.9013 (0.0008)	50.63 (0.20)	(4) 81.07 (0.38)
	2.0	174	(16) 0.9020 (0.0007)	86.68 (0.34)	(4) 139.24 (0.71)
	1.6	424	(32) 0.9004 (0.0004)	206.19 (0.56)	(4) 332.87 (1.74)
0.95	3.0	72	(16) 0.9517 (0.0006)	36.19 (0.14)	(4) 64.03 (0.27)
	2.4	130	(16) 0.9509 (0.0005)	61.98 (0.24)	(4) 110.11 (0.49)
	2.0	228	(16) 0.9507 (0.0005)	105.76 (0.41)	(4) 191.84 (0.85)

sets of constants does not change the basic conclusions of Ramey and Alam; in several respects their procedure appears to be the best one proposed to that date for minimizing $E\{n|LF\}$.

This corrected set of (r,N)-values has permitted us to make fairer comparisons between the performance of $\mathcal{P}^*_{B_T}$ and that of R-A. In the present section we show that in several important respects $\mathcal{P}^*_{B_T}$ is superior to the R-A procedure.

Tables VIII and IX which are for $k = 5$ and $k = 10$, respectively, give comparative data for results obtained with $\mathcal{P}^*_{B_T}$ and the R-A procedure. The R-A results are abstracted from Tables IVA and VA of B-G (1985) (these being based on the corrected R-A (r,N)-values) while the $\mathcal{P}^*_{B_T}$ results are abstracted from Tables VIB and VIIB of the present article. Here, for each k, $E\{n|LF\}$ for $\mathcal{P}^*_{B_T}$ is to be compared with $E\{n|LF\}$ for R-A as a function of $\{\theta^*,P^*\}$, and similarly, $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$ with $E\{n|EP\}$ for R-A. The corresponding values of $P\{CS|LF\}$ for the two procedures also contain relevant information.

It is to be noted in Table VIII that in eight of the eleven $E\{n|LF\}$ comparisons, and in eight of the eleven $E\{n|EP\}$ comparisons, $\mathcal{P}^*_{B_T}$ yields smaller $E\{n\}$-values than does R-A. Similarly, in Table IX, in eight of the eleven $E\{n|LF\}$ comparisons and in ten of the eleven $E\{n|EP\}$ comparisons, $\mathcal{P}^*_{B_T}$ yields smaller $E\{n\}$-values than does R-A. We conjecture that $\mathcal{P}^*_{B_T}$ will improve relative to R-A in terms of $E\{n\}$ for both the LF- and EP- configurations as k increases and/or as P^* and/or θ^* approaches unity.

Remark 6.1: We point out that the $E\{n\}$ results for $\mathcal{P}^*_{B_T}$ and R-A are not completely comparable because they are not based on exactly the same achieved $P\{CS|LF\}$. It is possible to decrease $P\{CS|LF\}$ for $\mathcal{P}^*_{B_T}$ almost continuously by decreasing n_0 (especially when the desired $n_0(k;\theta^*,P^*)$ is large); however, $P\{CS|LF\}$ for R-A decreases in more discrete steps as r or N of

Table VIII

$P\{CS|LF\}$, $E\{n|LF\}$ and $E\{n|EP\}$
for $P^*_{B_T}$ with associated n_0, and for R-A with associated (r,N) for selected $\{P^*,\theta^*\}$ when $k = 5$

P^*	θ^*	n_0 for $P^*_{B_T}$	(r,N) for R-A	LF-configuration				EP-configuration	
				$P\{CS\|LF\}$		$E\{n\|LF\}$		$E\{n\|EP\}$	
				$P^*_{B_T}$	R-A	$P^*_{B_T}$	R-A	$P^*_{B_T}$	R-A
0.75	3.0	12	(2,5)	0.7621	0.7544	7.62	6.66	9.14	8.76
	2.4	20	(3,6)	0.7520	0.7683	12.11	12.65	14.93	15.58
	2.0	35	(3,11)	0.7540	0.7504	20.13	18.85	24.88	24.29
	1.6	86	(5,19)	0.7512	0.7533	45.68	48.60	57.16	58.56
0.90	3.0	24	(3,8)	0.9030	0.9046	13.29	12.58	19.23	19.95
	2.4	42	(4,11)	0.9025	0.9046	21.41	21.87	32.39	32.46
	2.0	71	(5,18)	0.9008	0.9093	35.31	37.76	53.47	56.94
	1.6	176	(7,36)	0.9021	0.9016	81.46	84.33	125.28	122.00
0.95	3.0	34	(4,10)	0.9507	0.9573	16.46	17.37	27.78	29.80
	2.4	58	(5,15)	0.9508	0.9577	27.19	29.05	46.13	48.16
	2.0	98	(6,23)	0.9506	0.9518	45.07	47.27	77.31	78.04

Table IX

P{CS|LF}, E{n|LF} and E{n|EP}

for $P^*_{B_T}$ with associated n_0, and for R-A with associated (r,N) for selected {P*,θ*} when k = 10

P*	θ*	n_0 for $P^*_{B_T}$	(r,N) for R-A	Lf-configuration				EP-configuration	
				P{CS|LF}		E{n|LF}		E{n|EP}	
				$P^*_{B_T}$	R-A	$P^*_{B_T}$	R-A	$P^*_{B_T}$	R-A
0.75	3.0	31	(3,6)	0.7539	0.7803	18.73	20.04	24.46	27.21
	2.4	54	(3,10)	0.7535	0.7605	32.03	32.22	42.36	44.66
	2.0	96	(4,14)	0.7520	0.7626	55.16	58.62	72.91	76.17
	1.6	244	(5,33)	0.7510	0.7572	132.38	129.56	175.45	174.73
0.90	3.0	55	(4,9)	0.9023	0.9047	29.42	31.33	46.26	48.23
	2.4	98	(4,16)	0.9013	0.9060	50.63	50.36	81.07	85.87
	2.0	174	(5,25)	0.9020	0.9030	86.68	87.61	139.24	145.87
	1.6	424	(7,58)	0.9004	0.9069	206.19	206.67	332.87	350.84
0.95	3.0	72	(4,13)	0.9517	0.9566	36.19	36.17	64.03	71.70
	2.4	130	(5,19)	0.9509	0.9529	61.98	63.26	110.11	116.16
	2.0	228	(6,31)	0.9507	0.9538	105.76	107.39	191.84	196.93

(r,N) is decreased. Thus we see in Tables VIII and IX that it is almost always the case that $P\{CS|LF\}$ for R-A is greater than $P\{CS|LF\}$ for $\mathcal{P}^*_{B_T}$. This fact, of course, militates against R-A. However, we still conjecture that even if $P\{CS|LF\}$ were the same for both $\mathcal{P}^*_{B_T}$ and R-A, the former would yield uniformly smaller values of $E\{n|LF\}$ and $E\{n|EP\}$ for large k and/or for P^* and/or θ^* close to unity. (For $k = 2$, the R-A procedure can never terminate after $\mathcal{P}^*_{B_T}$.)

Remark 6.2: Use of $\mathcal{P}^*_{B_T}$ forces $\max\{n|\underset{\sim}{p}\} \le n_0$. For the R-A procedure, $\max\{n|\underset{\sim}{p}\} \le k(N-1)+1$. Typically $n_0 << k(N-1)+1$ which is an additional virtue of $\mathcal{P}^*_{B_T}$ relative to R-A.

Remark 6.3: Based on limited calculations we have determined that $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$ is greater than $E\{n|EP\}$ for the Bechhofer-Kulkarni (1984) curtailed sampling procedure when k is sufficiently large and/or P^* and/or θ^* is sufficiently close to unity. Thus we cannot claim that $\mathcal{P}^*_{B_T}$ has smaller $E\{n|\underset{\sim}{p}\}$ uniformly in $\underset{\sim}{p}$ for all competing procedures when the above-noted conditions on (k,P^*,θ^*) obtain.

7. CONCLUDING REMARKS

We have demonstrated conclusively that $\mathcal{P}^*_{B_T}$ is greatly superior to $\mathcal{P}^*_B$ in terms of $E\{n|\underset{\sim}{p}\}$ and $Var\{n|\underset{\sim}{p}\}$ uniformly in $\underset{\sim}{p}$, and that the improvement is greatest when $p_{[1]} = p_{[k]}$. Both $\mathcal{P}^*_{B_T}$ and $\mathcal{P}^*_B$ lead to very early termination for $p_{[k]} \to 1$. Since $\mathcal{P}^*_{B_T}$ is a closed procedure it has great practical advantages over $\mathcal{P}^*_B$. Our results would appear to justify the calculation of extended tables of $n_0(k,\theta^*,P^*)$ in order that $\mathcal{P}^*_{B_T}$ can be implemented over a broader range of values of (k,θ^*,P^*). It is recognized, of course, that such calculations would be quite costly, particularly for θ^* and/or P^* close to unity.

We have also shown that $\mathcal{P}^*_{B_T}$ is superior to the R-A procedure over a broad range of the practical (k,θ^*,P^*)-values. We conjecture that $\mathcal{P}^*_{B_T}$ will dominate the R-A procedure uniformly in $\underset{\sim}{p}$ in terms of $E\{n|\underset{\sim}{p}\}$ for k moderately large and P^* or θ^* close to unity.

APPENDIX

ITERATIVE METHOD OF CALCULATING EXACT VALUES OF $P\{CS|LF\}$, $E\{n|LF\}$ AND $E\{n|EP\}$ FOR $\mathcal{P}^*_{B_T}$

For given (k,P^*,θ^*,n_0), we can calculate $P\{CS|LF\}$, $E\{n|LF\}$, and $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$; the LF-configuration is given by (2.6). We wish to find the smallest value of n_0 such that $P\{CS|LF\} \geq P^*$.

Consider the counts $y_{1m},y_{2m},\ldots,y_{km}$ at stage m of sampling $(m = 1,2,\ldots)$. $\mathcal{P}^*_{B_T}$ terminates sampling when either $\sum_{i=1}^{k-1} (1/\theta^*)^{y_{[k]m}-y_{[i]m}} \leq (1-P^*)/P^*$ or $m = n_0$.

Define $\mathcal{T}(k,P^*,\theta^*,n_0) \equiv \{(\ell_1,\ldots,\ell_k)$: For fixed (k,P^*,θ^*,n_0), $P\{\mathcal{P}^*_{B_T}$ terminates exactly when $\underset{\sim}{y}_m = (\ell_1,\ldots,\ell_k)\} > 0\}$, i.e., procedure termination occurs at the first m such that $\underset{\sim}{y}_m \in \mathcal{T}(k,P^*,\theta^*,n_0)$.

Example: Suppose that $k = 2$, $P^* = 0.75$, $\theta^* = 2$, $n_0 = 5$. Then $\mathcal{P}^*_{B_T}$ terminates sampling when $(1/2)^{y_{[2]m}-y_{[1]m}} \leq 1/3$ or $m = 5$. Thus $\mathcal{P}^*_{B_T}$ terminates when $(y_{[2]m},y_{[1]m}) = (2,0)$ or $(3,1)$ or $(3,2)$. Hence, $\mathcal{T}(2,0.75,2,3) = \{(0,2),(1,3),(2,0),(2,3),(3,1),(3,2)\}$. Note that $(3,0) \notin \mathcal{T}(\cdot)$ since termination would have occurred at $m = 2$.

Define $\#(\ell_1,\ldots,\ell_k)$ to be the number of distinct paths of the sampling process $\{\underset{\sim}{y}_m,\ m = 1,2,\ldots\}$ which lead to procedure termination exactly when $\underset{\sim}{y}_m = (\ell_1,\ldots,\ell_k)$.

Example: Again suppose that $k = 2$, $P^* = 0.75$, $\theta^* = 2$, $n_0 = 5$. Then $\#(2,0) = 1$ since only one path of the sampling process leads to termination exactly when $(y_{1m},y_{2m}) = (2,0)$; viz., $\underset{\sim}{y}_1 = (1,0)$, $\underset{\sim}{y}_2 = (2,0)$.

It is obvious that

$$\#(\ell_1,\ldots,\ell_k) = \left[\begin{array}{c}\text{total number of}\\ \text{paths to}\\ (\ell_1,\ldots,\ell_k)\end{array}\right] - \left[\begin{array}{l}\text{number of paths to}\\ (\ell_1,\ldots,\ell_k) \text{ for which}\\ P^*_{B_T} \text{ terminates \underline{en} \underline{route}}\end{array}\right]$$

Example: Again, suppose that $k = 2$, $P^* = 0.75$, $\theta^* = 2$, $n_0 = 5$. We calculate $\#(3,1)$. Noting that $P^*_{B_T}$ terminates [en route to $\underset{\sim}{y}_4 = (3,1)$] if $\underset{\sim}{y}_2 = (2,0)$, we have:

$$\#(3,1) = \binom{4}{1} - \left[\begin{array}{c}\text{number of paths}\\ \text{from } (2,0) \text{ to } (3,1)\end{array}\right]\#(2,0)$$

$$= 4 - \binom{2}{1}1 = 2.$$

Remark: $\#(\ell_1,\ldots,\ell_k) = 0, \ \forall \ (\ell_1,\ldots,\ell_k) \notin T(\cdot)$.

It thus is clear that:

$$\#(\ell_1,\ldots,\ell_k) = \binom{\sum_{i=1}^k \ell_i}{\ell_1,\ell_2,\ldots,\ell_k} - \underset{(j_1,\ldots,j_k) \in T(\cdot)}{\sum_{j_1=0}^{\ell_1} \sum_{j_2=0}^{\ell_2} \cdots \sum_{j_k=0}^{\ell_k}} \cdot \binom{\sum_{i=1}^k (\ell_i - j_i)}{\ell_1-j_1,\ell_2-j_2,\ldots,\ell_k-j_k} \#(j_1,\ldots,j_k) \qquad (1)$$

Remark: By symmetry, $\#(\ell_1,\ldots,\ell_k) = \#(\text{any permutation of } \ell_1,\ldots,\ell_k)$. Hence, we need only to calculate explicitly those $\#(\ell_1,\ldots,\ell_k)$'s such that $(\ell_1,\ldots,\ell_k) \in T(\cdot)$ and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$.

Remark: The $\#(\ell_1,\ldots,\ell_k)$'s are to be calculated in an iterative manner:

A. Initialize all $\#(\cdot)$'s to equal zero.

B. Using (1), left-lexicographically calculate those $\#(\cdot)$'s for which $(\ell_1,\ldots,\ell_k) \in T(\cdot)$ and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$. [By the above Remarks, we obtain (with no further calculations) all other $\#(\cdot)$'s]; i.e.,

Calculate $\#(1,0,0,\ldots,0)$ if $(1,0,0,\ldots,0) \in T(\cdot)$
Calculate $\#(1,1,0,\ldots,0)$ if $(1,1,0,\ldots,0) \in T(\cdot)$

$\vdots$

Calculate $\#(1,1,1,\ldots,1)$ if $(1,1,1,\ldots,1) \in T(\cdot)$
Calculate $\#(2,0,0,\ldots,0)$ if $(2,0,0,\ldots,0) \in T(\cdot)$
Calculate $\#(2,1,0,\ldots,0)$ if $(2,1,0,\ldots,0) \in T(\cdot)$

$\vdots$

Calculate $\#(2,1,1,\ldots,1)$ if $(2,1,1,\ldots,1) \in T(\cdot)$
Calculate $\#(2,2,0,\ldots,0)$ if $(2,2,0,\ldots,0) \in T(\cdot)$

$\vdots$

Calculate $\#(n_0,0,0,\ldots,0)$ if $(n_0,0,0,\ldots,0) \in T(\cdot)$.

This lexicographic order of calculation is necessary since the computation of $\#(\ell_1,\ldots,\ell_k)$ involves all of the previous $\#(\cdot)$'s. "On the fly" storage of the values of these previous $\#(\cdot)$'s avoids recursive re-computation in (1).

Suppose that $\underset{\sim}{p}$ is in the LF-configuration. Without loss of generality, assume for the remainder of this section that $p_1 = \theta^* p_i$, $i = 2,\ldots,k$; i.e., $p_1 = p_{[k]}$. For all $(\ell_1,\ldots,\ell_k) \in T(\cdot)$, define

$$R(\ell_1,\ldots,\ell_k) \equiv \begin{cases} \left|\{\ell_i : \ell_1 = \ell_i,\ i = 1,\ldots,k\}\right|^{-1} & \text{if } \ell_1 \geq \max\{\ell_2,\ldots,\ell_k\} \\ 0 & \text{otherwise} \end{cases}$$

$R(\ell_1,\ldots,\ell_k)$ is simply the $P\{CS \mid \underset{\sim}{y}_m = (\ell_1,\ldots,\ell_k) \in T(\cdot)\}$.

<u>Example</u>: $R(3,1) = 1$, $R(1,3) = 0$, $R(5,1,5,5) = 1/3$.

Letting $p_2 = \ldots = p_k = p$ (say) and denoting θ^* by θ, we have:

$$P\{CS|LF\} = \sum_{\substack{j_1}}\sum_{j_2}\cdots\sum_{\substack{j_k \\ (j_1,\ldots,j_k)\in T(\cdot) \text{ and} \\ j_1 \geq \max\{j_2,\ldots,j_k\}}} R(j_1,\ldots,j_k)\#(j_1,\ldots,j_k)\left(\prod_{i=1}^{k} p_i^{j_i}\right)$$

$$= \sum_{\substack{j_1}}\sum_{j_2}\cdots\sum_{\substack{j_k \\ (j_1,\ldots,j_k)\in T(\cdot) \text{ and} \\ j_1 \geq \max\{j_2,\ldots,j_k\}}} R(j_1,\ldots,j_k)\#(j_1,\ldots,j_k)\theta^{j_1} p^{\sum_{i=1}^{k} j_i}) \quad (2)$$

and

$$E\{n|LF\} = \sum_{j_1}\sum_{j_2}\cdots\sum_{\substack{j_k \\ (j_1,\ldots,j_k)\in T(\cdot)}} \left(\sum_{i=1}^{k} j_i\right)\#(j_1,\ldots,j_k)\left(\prod_{i=1}^{k} p_i^{j_i}\right)$$

$$= \sum_{j_1}\sum_{j_2}\cdots\sum_{\substack{j_k \\ (j_1,\ldots,j_k)\in T(\cdot)}} \left(\sum_{i=1}^{k} j_i\right)\#(j_1,\ldots,j_k)\theta^{j_1} p^{\sum_{i=1}^{k} j_i}. \quad (3)$$

$E\{n|EP\}$ is obtained by setting $\theta = 1$ and $p = 1/k$ in equation (3).

<u>Example</u>: Let $k = 3$, $P^* = .75$, $\theta^* = 3$, $n_0 = 5$. Then $\mathcal{P}_{B_T}^*$ terminates sampling if

$$(1/3)^{y_{[3]m}-y_{[2]m}} + (1/3)^{y_{[3]m}-y_{[1]m}} \leq 1/3 \quad \text{or } m = 5.$$

Hence, $\mathcal{P}_{B_T}^*$ terminates if $y_{[3]m}-y_{[2]m} \geq 2$ or $m = 5$. This yields:

$T(3,0.75,3,5) = \{(\ell_1,\ell_2,\ell_3)$: (ℓ_1,ℓ_2,ℓ_3) is a permutation of $(2,0,0)$ or $(2,2,1)$ or $(3,1,0)$ or $(3,1,1)$ or $(3,2,0)\}$.

Using (1), we calculate (lexicographically):

$\#(2,0,0) = 1,$

$\#(2,2,1) = \binom{5}{2,2,1} - \binom{3}{0,2,1}\#(2,0,0) - \binom{3}{2,0,1}\#(0,2,0) = 24,$

$$\#(3,1,0) = \binom{4}{3,1,0} - \binom{2}{1,1,0}\#(2,0,0) = 2,$$

$$\#(3,1,1) = \binom{5}{3,1,1} - \binom{1}{0,0,1}\#(3,1,0) - \binom{1}{0,1,0}\#(3,0,1) - \binom{3}{1,1,1}\#(2,0,0) = 10$$

$$\#(3,2,0) = \binom{5}{3,2,0} - \binom{1}{0,1,0}\#(3,1,0) - \binom{3}{1,2,0}\#(2,0,0) - \binom{3}{3,0,0}\#(0,2,0) = 4$$

Assuming $p_1 = 3/5$ and $p \equiv p_2 = p_3 = 1/5$ then (2) yields

$$\begin{aligned} P\{CS|LF\} = {} & R(2,0,0)\#(2,0,0)\theta^2p^2 + R(2,1,2)\#(2,1,2)\theta^2p^5 \\ & + R(2,2,1)\#(2,2,1)\theta^2p^5 + R(3,0,1)\#(3,0,1)\theta^3p^4 \\ & + R(3,1,0)\#(3,1,0)\theta^3p^4 + R(3,1,1)\#(3,1,1)\theta^3p^5 \\ & + R(3,0,2)\#(3,0,2)\theta^3p^5 + R(3,2,0)\#(3,2,0)\theta^3p^5 \\ = {} & 0.7574. \end{aligned}$$

Also, (3) yields:

$$\begin{aligned} E\{n|LF\} = {} & [2\#(0,0,2)p^2 + 2\#(0,2,0)p^2 + 2\#(2,0,0)\theta^2p^2] \\ & + [5\#(1,2,2)\theta p^5 + 5\#(2,1,2)\theta^2p^5 + 5\#(2,2,1)\theta^2p^5] \\ & + [4\#(0,1,3)p^4 + 4\#(0,3,1)p^4 + 4\#(1,0,3)\theta p^4 \\ & \quad + 4\#(1,3,0)\theta p^4 + 4\#(3,0,1)\theta^3p^4 + 4\#(3,1,0)\theta^3p^4] \\ & + [5\#(1,1,3)\theta p^5 + 5\#(1,3,1)\theta p^5 + 5\#(3,1,1)\theta^3p^5] \\ & + [5\#(0,2,3)p^5 + 5\#(0,3,2)p^5 + 5\#(2,0,3)\theta^2p^5 \\ & \quad + 5\#(2,3,0)\theta^2p^5 + 5\#(3,0,2)\theta^3p^5 + 5\#(3,2,0)\theta^5p^5] \\ = {} & 3.4816. \end{aligned}$$

Substituting $p = 1/k = 1/3$ and $\theta = 1$ in the above expression, we obtain $E\{n|EP\} = 3.8519$.

Remarks concerning the computer work

Exact results: For many values of (k,P^*,θ^*,n_0), it was possible to use certain efficient versions of (1), (2), and (3) in order to calculate exact values of $P\{CS|LF\}$, $E\{n|LF\}$, and $E\{n|EP\}$ for $\mathcal{P}^*_{B_T}$. These calculations were carried out on Purdue University's VAX network and on IBM 3081 and 4341 computers at Cornell.

Monte Carlo results: This work was carried out on Cornell's IBM machines and on CDC 6500 and 6600 computers at Purdue. The random variates we used were generated from the IMSL (1982) subroutines GGUBS or GGMTN.

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