Algebra Univers. (2021) 82:27
1420-8911/21/020001-19
published online April 6, 2021
https://doi.org/10.1007/s00012-021-00717-6

Algebra Universalis

Truncations of ordered abelian groups

Paola D'Aquino, Jamshid Derakhshan and Angus Macintyre

Abstract. We give axioms for a class of ordered structures, called truncated ordered abelian groups (TOAG's) carrying an addition. TOAG's come naturally from ordered abelian groups with a 0 and a +, but the addition of a TOAG is not necessarily even a cancellative semigroup. The main examples are initial segments $[0,\tau]$ of an ordered abelian group, with a truncation of the addition. We prove that any model of these axioms (i.e. a truncated ordered abelian group) is an initial segment of an ordered abelian group. We define Presburger TOAG's, and give a criterion for a TOAG to be a Presburger TOAG, and for two Presburger TOAG's to be elementarily equivalent, proving analogues of classical results on Presburger arithmetic. Their main interest for us comes from the model theory of certain local rings which are quotients of valuation rings valued in a truncation $[0,a]$ of the ordered group $\mathbb{Z}$ or more general ordered abelian groups, via a study of these truncations without reference to the ambient ordered abelian group. The results are used essentially in a forthcoming paper (D'Aquino and Macintyre, The model theory of residue rings of models of Peano Arithmetic: The prime power case, 2021, arXiv:2102.00295) in the solution of a problem of Zilber about the logical complexity of quotient rings, by principal ideals, of nonstandard models of Peano arithmetic.

Mathematics Subject Classification. 20F60, 03C64, 12J12, 03C60, 06F05, 12J10.

Keywords. Ordered abelian groups and their truncations, Model theory of local rings, Quotients of valuation rings and truncated valuations, Semi-groups.

Presented by A. Dow.
The third author was supported by a Leverhulme Emeritus Fellowship.

1. Introduction

The model theory of ordered abelian groups is well understood, and highly relevant for the model theory of Henselian valued fields (and, less directly, for nonstandard models of arithmetic). The ring of p-adic integers is easier to understand logically than the theory of the class of all its finite quotients. The latter is interpretable in the former, but some vital issues of definability get obscured. The finite quotients are local rings, valued in a truncation $[0, a]$ of the ordered group $\mathbb{Z}$.

For a direct study of these local rings (and for some associated semilocal rings) a direct study of the truncations is needed, not merely in $\mathbb{Z}$ but in more general ordered abelian groups.

This paper is a component of two pieces of research, one by D'Aquino and Macintyre [1,2], and one by Derakhshan and Macintyre [4]. The common theme is the model theory of local rings $V/(\alpha)$, where V is a Henselian valuation domain, and α is a nonzero nonunit of V. If $v : V \to P$ is the valuation of V, with P the semigroup of non-negative elements of the value group Γ of the fraction field K of V, v induces a "truncated valuation" from $V/(\alpha)$ onto the segment $[0, v(\alpha)]$ of P defined by

$$v(x + (\alpha)) = \begin{cases} v(x) & v(x) < v(\alpha), \\ v(\alpha) & v(x) \geq v(\alpha). \end{cases}$$

The segment $[0, v(\alpha)]$ inherits from Γ an ordering $\leq$, with 0 as least element and $v(\alpha)$ as greatest element. From our assumption on α, $v(\alpha) \neq 0$.

Next, $[0, v(\alpha)]$ gets a truncated semi-group structure as follows. Let $\oplus$ be the addition on Γ. Define, for $\gamma_1, \gamma_2 \in [0, v(\alpha)]$

$$\gamma_1 + \gamma_2 = \min(\gamma_1 \oplus \gamma_2, v(\alpha)).$$

The basic laws are

$$v(x + y) \geq \min(v(x), v(y)),$$
$$v(xy) = \min(v(\alpha), v(x) + v(y)).$$

Ideas connected to this, but much more sophisticated, appear in the work of Hiranouchi [7,8].

Forgetting the valuation in the preceding, we have an ordered abelian group Γ, with order $\leq$, addition $\oplus$, subtraction $\ominus$, zero 0, and a distinguished element $\tau > 0$ (where $\tau = v(\alpha)$ in the preceding). By the above, we can define a "truncated addition" $+$ on $[0, \tau]$. This can be done for any ordered abelian group Γ.

We give a natural first-order set of axioms in the language $\{\leq, 0, \tau, +\}$ for a class of linear orders, that we call truncated ordered abelian groups (TOAG's), as well as analogues of classical results on Presburger arithmetic, namely on Presburger TOAG's. The main examples are the truncations $[0, a]$ arising from ordered abelian groups Γ as above, equipped with a truncated addition and semi-group structure.

We prove that any TOAG is an initial segment of an ordered abelian group. We give a criterion for a TOAG to be a Presburger TOAG, and for two Presburger TOAG's to be elementarily equivalent.

These results play an essential role in [2] in the solution of a problem of Zilber on the logical complexity of quotient rings, by principal ideals, of models of Peano arithmetic (PA). The problem asks about the interpretability of arithmetic in $\mathcal{M}/k\mathcal{M}$ where $k \in \mathcal{M}$ and $\mathcal{M}$ a nonstandard model of PA. This itself can be solved easily in the negative, but in [2] a much more substantial analysis is given of definability in the structures $\mathcal{M}/k\mathcal{M}$.

Our work naturally applies to the study of definability in quotient rings of valuation rings and their value semi-groups which are TOAG's, without reference to the ambient ordered abelian group (which is not necessarily assumed to be $\mathbb{Z}$). A natural example is the class of all finite quotients of the ring of p-adic integers and related local and semi-local rings.

2. Truncated ordered abelian groups and their algebra

In this section we give the intended axioms in the language $\{\leq, 0, \tau, +\}$. We shall verify that they hold in the structures $[0, \tau]$ with the truncated addition $+$ defined as above that are got from abelian groups Γ with order $\leq$, addition $\oplus$, subtraction $\ominus$, zero 0, and a distinguished element $\tau > 0$.

Recall that the addition $+$ is defined by

$$\gamma_1 + \gamma_2 = \min(\gamma_1 \oplus \gamma_2, \tau)$$

for $\gamma_1, \gamma_2 \in [0, \tau]$. Let P denote the semigroup of non-negative elements of Γ.

2.1. Obvious axioms

The following are obvious, via immediate calculations in P.

Axiom 1. Addition $+$ is commutative.

Axiom 2. $x + 0 = x$.

Axiom 3. $x + \tau = \tau$.

Axiom 4. If $x_1 \leq y_1$ and $x_2 \leq y_2$ then $x_1 + x_2 \leq y_1 + y_2$.

2.2. Less obvious axioms

Axiom 5. Addition $+$ is associative.

Verification. Suppose x, y, z in $[0, \tau]$.

Case 1 $x \oplus y \oplus z \leq \tau$. Then

$$\begin{aligned} x + (y + z) &= x + (y \oplus z) = x \oplus (y \oplus z) \\ &= (x \oplus y) \oplus z = (x + y) + z. \end{aligned}$$

Case 2 $x \oplus y \oplus z > \tau$.

Subcase 1 $y \oplus z \geq \tau$ and $x \oplus y \geq \tau$.

Then

$$x + (y + z) = x + \tau = \tau,$$

and

$$(x + y) + z = \tau + z = \tau.$$

Subcase 2 $y \oplus z < \tau$ and $x \oplus y \geq \tau$.

Then

$$\begin{aligned} x + (y + z) &= x + (y \oplus z) = \min(x \oplus (y \oplus z), \tau) \\ &= \min((x \oplus y) \oplus z, \tau) = \tau, \end{aligned}$$

and

$$(x + y) + z = \tau + z = \tau.$$

Subcase 3 $y \oplus z \geq \tau$ and $x \oplus y < \tau$.

Then

$$x + (y + z) = x + \tau = \tau,$$

and

$$(x + y) + z = (x \oplus y) + z = \min((x \oplus y) \oplus z, \tau) = \min(x \oplus (y \oplus z), \tau) = \tau.$$

Subcase 4 $y \oplus z < \tau$ and $x \oplus y < \tau$.

Now

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$

and so

$$x \oplus (y + z) = (x + y) \oplus z,$$

hence

$$\min(x \oplus (y + z), \tau) = \min((x + y) \oplus z, \tau),$$

thus

$$x + (y + z) = (x + y) + z.$$

2.3. Axioms concerning cancellation

Axiom 6. If $x + y = x + z < \tau$, then $y = z$.

Verification. $x \oplus y = x + y$ and $x \oplus z = x + z$ in this case, so use cancellation in P.

Axiom 7. If $x \leq y < \tau$, then there is a unique z with $x + z = y$.

Verification. Immediate from definition and the fact that P is the non-negative part of Γ.

Notation. We write $y \dot{-} x$ for the z in the above.

Axiom 8. There are (in general, many) z in $[0, \tau]$, for x in $[0, \tau]$, so that $x + z = \tau$, and there is a minimal one to be denoted $\tau \dot{-} x$.

Verification. Obvious by working in P and taking $\tau \dot{-} x = \tau - x$.

Note that $\tau \dot{-} \tau = 0$.

Axiom 9. $\tau \dot{-} (\tau \dot{-} x) = x$.

Verification. For $x < \tau$,

$$\tau \dot{-} (\tau \dot{-} x) = \tau - (\tau - x).$$

For $x = \tau$,

$$\tau \dot{-} (\tau \dot{-} x) = \tau \dot{-} 0 = \tau = x.$$

2.4. Crucial axioms

There now follows a series of axioms which are basic in what follows. It is not clear to us what are the dependencies between these axioms over the preceding nine.

Axiom 10. Suppose $0 \leq x,\ y < \tau$ and $x + y = \tau$. Then

$$y \dot{-} (\tau \dot{-} x) = x \dot{-} (\tau \dot{-} y).$$

Verification. Both are equal to $(x \oplus y) \ominus \tau$.

Before getting to the remaining axioms, we prove some useful lemmas without using Axiom 10.

Lemma 2.1. *Suppose* $0 \leq y \leq z < \tau$. *Then* $\tau \dot{-} z \leq \tau \dot{-} y$.

Proof. We have that

$$y + (\tau \dot{-} y) = \tau$$

and

$$z + (\tau \dot{-} z) = \tau.$$

If $(\tau \dot{-} z) > (\tau \dot{-} y)$, then (Axiom 8)

$$z + (\tau \dot{-} y) < \tau,$$

so

$$y + (\tau \dot{-} y) < \tau,$$

contradiction. □

Lemma 2.2. *Suppose* $x, y < \tau$. *Then* $\tau \dot{-} x = \tau \dot{-} y$ *implies* $x = y$.

Proof. Assume $\tau \dot{-} x = \tau \dot{-} y$. Then

$$\tau \dot{-} (\tau \dot{-} x) = \tau \dot{-} (\tau \dot{-} y),$$

so (Axiom 9) $x = y$. □

Corollary 2.3. *If* $x < y < \tau$. *Then* $\tau \dot{-} y < \tau \dot{-} x$.

Proof. By Lemma 2.1, $\tau \dot{-} y \leq \tau \dot{-} x$. If $\tau \dot{-} y = \tau \dot{-} x$, then applying Lemma 2.2 we get $x = y$. □

2.5. A miscellany of other axioms

In the course of proving Associativity in Theorem 3.2 we need various axioms about $+$, $\dot{-}$, and τ. Each of these axioms is true (and with a trivial proof) in the $[0,\tau]$ coming from the ordered abelian group Γ with $\oplus$, so it is natural to use them. One may hope to deduce them from the axioms listed already, but we have not succeeded in doing so. Thus we settle for quite a long list of "ad hoc" axioms, which we now consider in the order in which they occur in the proof of Associativity in Subsection 3.5.

Axiom 11. Assume $y+z<\tau$, $x+(y+z)=\tau$, and $y+x<\tau$. Then

$$x \dot{-} (\tau \dot{-} (y+z)) = z \dot{-} (\tau \dot{-} (x+y)).$$

Note. By Axiom 5 we may safely write $x+y+z$ for $x+(y+z)$ and $(x+y)+z$ and will do so henceforward.

Verification. In all that follows, we construe x,y,z,τ as in the $[0,\tau]$ in

$$(P, \leq_P, \oplus)$$

the non-negative part of an ordered abelian group Γ, where $\leq$ is identified with $\leq_P$ (the restriction of $\leq$ to P), and $+$ with the truncation of $\oplus$. Then

$$x \oplus y \oplus z = \tau \oplus \epsilon,$$

for some ϵ in P.

Now $y \oplus z < \tau$ and $y \oplus x < \tau$, so

$$x \oplus y \oplus z < 2\tau,$$

so $\epsilon < \tau$.

Let $\mu = \tau - (y+z)$ and $\delta = \tau - (x+y)$. Then $x = \mu + \epsilon$, hence

$$x \dot{-} (\tau \dot{-} (y+z)) = \epsilon,$$

and $z = \delta + \epsilon$, so

$$z \dot{-} (\tau \dot{-} (x+y)) = \epsilon,$$

giving the verification.

The subsequent verifications are at the same level of difficulty.

Axiom 12. Assume $y+z<\tau$, $x+y+z=\tau$, $y+x=\tau$, and

$$z + (y \dot{-} (\tau \dot{-} x)) < \tau.$$

Then

$$z + (y \dot{-} (\tau \dot{-} x)) = x \dot{-} (\tau \dot{-} (y+z)).$$

Verification. (Convention: Any time we write $A \dot{-} B$ we assume $A \geq B$). We have

$$x \oplus y \oplus z = \tau \oplus \epsilon$$

for some $\epsilon \in P$, and

$$y \oplus x = \tau \oplus \gamma$$

for some $\gamma \in P$. Now

$$x \oplus y \oplus z < 2\tau,$$

so $\epsilon < \tau$, and

$$y \oplus z < 2\tau,$$

so $\gamma < \tau$.

Let $\mu = \tau - (y + x)$. Then $x = \mu \oplus \epsilon$, so

$$x \dot{-} (\tau \dot{-} (y + z)) = \epsilon,$$

whereas

$$z + (y \dot{-} (\tau \dot{-} x)) = z + (y - (\tau - x)) = x + y + z - \tau = \epsilon.$$

Axiom 13. Assume $y + x = \tau$ and $y + z < \tau$. Then $z + (y \dot{-} (\tau \dot{-} x)) < \tau$.

Verification. Let $y \oplus x = \tau \oplus \epsilon$, where $0 \leq \epsilon < \tau$. Then

$$y \dot{-} (\tau \dot{-} x) = \epsilon.$$

and

$$z \oplus \epsilon = z \oplus ((y \oplus x) \ominus \tau) = (x \oplus y \oplus z) \ominus \tau < \tau,$$

since $x < \tau$ and $y + z < \tau$.

Axiom 14. Assume $y + z = y + x = \tau$ and $z + (y \dot{-} (\tau \dot{-} x)) < \tau$. Then

$$x + (y \dot{-} (\tau \dot{-} z)) = (x \dot{-} (\tau \dot{-} y)) + z.$$

Verification. Let $y \oplus z = \tau \oplus \epsilon$ and $y \oplus x = \tau \oplus \delta$, with $0 \leq \epsilon, \delta < \tau$. Then

$$y \dot{-} (\tau \dot{-} z)) = \epsilon,$$
$$x \dot{-} (\tau \dot{-} y) = \delta,$$
$$y \oplus z \oplus \delta = \tau \oplus \epsilon \oplus \delta,$$

and

$$y \oplus x \oplus \epsilon = \tau \oplus \epsilon \oplus \delta,$$

so $x + \epsilon = z + \delta$ as required.

Axiom 15. Assume $y + z = \tau$, $y + x = \tau$, and $x + (y \dot{-} (\tau \dot{-} z)) = \tau$. Then $z + (y \dot{-} (\tau \dot{-} x)) = \tau$.

Verification. Let $y \oplus z = \tau \oplus \delta$ and $y \oplus x = \tau \oplus \epsilon$ as before. Then

$$y \dot{-} (\tau \dot{-} z)) = \delta,$$

and

$$y \dot{-} (\tau \dot{-} x)) = \epsilon.$$

We have

$$x \oplus (y \ominus (\tau \ominus z)) = (x \oplus y \oplus z) \ominus \tau,$$

since

$$x + (y \dot{-} (\tau \dot{-} z)) = \tau,$$

and

$$x \oplus y \oplus z \geq 2\tau.$$

Now

$$z \oplus (y \ominus (\tau \ominus x)) = (x \oplus y \oplus z) \ominus \tau,$$

so

$$z + (y \dot{-} (\tau \dot{-} x)) = \tau,$$

since $x \oplus y \oplus z \geq 2\tau$.

Axiom 16. Assume

$$y + z = y + x = x + (y \dot{-} (\tau \dot{-} z)) = \tau.$$

Then

$$(y \dot{-} (\tau \dot{-} x)) \dot{-} (\tau \dot{-} z) = (y \dot{-} (\tau \dot{-} z)) \dot{-} (\tau \dot{-} x).$$

Verification.

$$\begin{aligned}(y \dot{-} (\tau \dot{-} x)) \dot{-} (\tau \dot{-} z) &= ((y \oplus x) \ominus \tau) \ominus (\tau \ominus z)\\ &= (x \oplus y \oplus z) \ominus (2\tau) = ((y \oplus z) \ominus \tau) \ominus (\tau \ominus x)\\ &= (y \dot{-} (\tau \dot{-} z)) \dot{-} (\tau \dot{-} x).\end{aligned}$$

3. Truncated ordered abelian groups and ordered abelian groups

Definition 3.1. A truncated ordered abelian group (TOAG) is a linear order which is an $\mathcal{L} = \{\leq, 0, \tau, +\}$-structure $[0, \tau]$ with an operation $+$ satisfying Axioms 1–16.

Theorem 3.2. *Let $[0, \tau]$ be a truncated ordered abelian group with operation $+$ and order $\leq$. Then there is an ordered abelian group Γ, under an operation $\oplus$ and an order $\leq_\Gamma$, with P the semigroup of non-negative elements, and an element τ_P of P so that $[0, \tau]$ with $+$ and $\leq$ is isomorphic to $[0, \tau_P]$ with the addition and order induced by $\oplus$ and $\leq_\Gamma$ on $[0, \tau_P]$.*

Proof. We begin by constructing P, and then we laboriously verify that it has the required properties.

3.1. Construction

P is $\omega \times [0, \tau)$, where ω is the set of finite ordinals k under ordinal addition $(+)$ and order $(\leq)$. $[0, \tau)$ has the order induced from $[0, \tau]$. We denote the elements of P by $\langle a, b \rangle$, where $a \in \omega$, $b \in [0, \tau)$.

P is lexicographically ordered with respect to the two orderings just specified. Let $\leq_P$ be the lexicographic order.

Let $0 = \langle 0, 0 \rangle \in P$, the least element of P. Let $\tau_P = \langle 1, 0 \rangle \in P$. We have $[0, \tau_P) = \{0\} \times [0, \tau)$, giving natural order isomorphisms $[0, \tau_P) \cong [0, \tau)$ and $[0, \tau_P] \cong [0, \tau]$.

Now we define $\oplus$ on P.

3.2. The Case 1/Case 2 distinction for $\langle y, z\rangle \in [0, \tau)^2$

This comes up all the time in what follows, and the axioms from Axiom 10 on relate to it.

Case 1 $y + z < \tau$

Case 2 $y + z = \tau$

Note that both are symmetric in y and z and are complements of each other.

The main point is that Case 2 is equivalent to both $y \geq \tau \dot{-} z$ and $z \geq \tau \dot{-} y$.

Moreover Axiom 10 implies that in Case 2

$$y \dot{-} (\tau \dot{-} z) = z \dot{-} (\tau \dot{-} y).$$

3.3. Defining $\oplus$

We define $\langle k, y\rangle \oplus \langle l, z\rangle$ to be

$$\langle k + l, y + z\rangle$$

if $\langle y, z\rangle$ in Case 1, and

$$\langle k + l + 1, y \dot{-} (\tau \dot{-} z)\rangle$$

if $\langle y, z\rangle$ in Case 2.

Lemma 3.3. *$\oplus$ is commutative, with 0 as neutral element.*

Proof. Let $\langle k, y\rangle, \langle l, z\rangle \in P$.

Case 1. If $\langle y, z\rangle$ in Case 1 (and so then is $\langle z, y\rangle$). Then

$$\begin{aligned}\langle k, y\rangle \oplus \langle l, z\rangle &= \langle k + l, y + z\rangle \\ &= \langle l + k, z + y\rangle = \langle l, z\rangle \oplus \langle k, y\rangle.\end{aligned}$$

Case 2. If $\langle y, z\rangle$ in Case 2 (again symmetric). Then

$$\begin{aligned}\langle k, y\rangle \oplus \langle l, z\rangle &= \langle k + l + 1, y \dot{-} (\tau \dot{-} z)\rangle \\ &= \langle l + k + 1, z \dot{-} (\tau \dot{-} y)\rangle,\end{aligned}$$

by Axiom 10.

If $y = 0$ we are in Case 1 so

$$\langle k, y\rangle \oplus \langle l, z\rangle = \langle k + l, z\rangle,$$

so if $k = 0$, this is equal to $\langle 0, z\rangle$.

This completes the proof in all the cases. □

3.4. $\leq_P$ and $\oplus$

Lemma 3.4. *If* $\langle k,x\rangle \leq_P \langle l,y\rangle$ *and* $\langle m,z\rangle \leq_P \langle n,w\rangle$*, then*
$$\langle k,x\rangle \oplus \langle m,z\rangle \ \leq\ \langle l,y\rangle \oplus \langle n,w\rangle.$$

Proof. There are four cases:

a) (x,z) and (y,w) both in Case 1.
b) (x,z) in Case 1, (y,w) in Case 2.
c) (x,z) in Case 2, (y,w) in Case 1.
d) (x,z) in Case 2, (y,w) in Case 2.

Now suppose the hypothesis of the lemma, i.e. $k \leq l, x \leq y, m \leq n$, and $z \leq w$.

a) $\langle k,x\rangle \oplus \langle m,z\rangle = \langle k+m, x+z\rangle$ and
$$\langle l,y\rangle \oplus \langle n,w\rangle = \langle l+n, y+w\rangle,$$
and the required inequality follows from the four inequalities of the hypothesis and Axioms 1–10.

b) $\langle k,x\rangle \oplus \langle m,z\rangle = \langle k+m, x+z\rangle$, and
$$\langle l,y\rangle \oplus \langle n,w\rangle = \langle l+n+1, y \dot{-} (\tau \dot{-} w)\rangle,$$
and the required inequality follows since $k+m \leq l+n+1$.

c) $\langle k,x\rangle \oplus \langle m,z\rangle = \langle k+m+1, x \dot{-} (\tau \dot{-} z)\rangle$, and
$$\langle l,y\rangle \oplus \langle n,w\rangle = \langle l+n, y+w\rangle.$$
But since $y \geq x$, and $w \geq z$, we have $y+w=\tau$, contradiction. This case does not occur.

d) $\langle k,x\rangle \oplus \langle m,z\rangle = \langle k+m+1, x \dot{-} (\tau \dot{-} z)\rangle$, and
$$\langle l,y\rangle \oplus \langle n,w\rangle = \langle l+n+1, y \dot{-} (\tau \dot{-} w)\rangle.$$
But now $x \leq y$, and (by Lemma 2.3)
$$\tau \dot{-} z \geq \tau \dot{-} w,$$
so
$$x \dot{-} (\tau \dot{-} z) \leq y - (\tau \dot{-} z) \leq y - (\tau \dot{-} w) = y \dot{-} (\tau \dot{-} w)$$
(we are in Case 2), giving the result. □

3.5. Associativity

Verifying this is the most tedious task of all. We can profit a bit from having already proved commutativity. We need to prove:
$$\begin{aligned} &(1)\ \langle k,x\rangle \oplus (\langle l,y\rangle \oplus \langle m,z\rangle) = \\ &(2)\ (\langle k,x\rangle \oplus \langle l,y\rangle) \oplus \langle m,z\rangle = \\ &(3)\ \langle m,z\rangle \oplus (\langle l,y\rangle \oplus \langle k,x\rangle). \end{aligned}$$
So we should calculate (1) via, firstly the Case distinction for (y,z), and then the Case distinction for x and right-hand coordinate of
$$\langle l,y\rangle \oplus \langle m,z\rangle.$$
We then do the same for (3), switching x and z.

The simplest situation for (1) is:

Situation 1. (y, z) in Case 1 and then right-hand coordinate of

$$\langle l, y\rangle \oplus \langle m, z\rangle$$

is in Case 1 with x.

So calculation gives

$$y + z + x < \tau$$

and value of (1) is

$$\langle k + l + m, x + y + z\rangle.$$

This is exactly the same as we get from (3) assuming (y, x) in Case 1, and so is z with right-hand coordinate of

$$\langle l, y\rangle \oplus \langle k, x\rangle,$$

and so we verify one instance of associativity, when

$$x + y + z < \tau.$$

Situation 2. (y, z) in Case 1, and x in Case 2 with right-hand coordinate of

$$\langle l, y\rangle \oplus \langle m, z\rangle.$$

So $y + z < \tau$ but $x + (y + z) = \tau$. Then value of (1) is

$$\langle k + l + m + 1, x \dot{-} (\tau \dot{-} (y + z))\rangle.$$

(Bear in mind that $x \dot{-} (\tau \dot{-} (y + z)) = (y + z) \dot{-} (\tau \dot{-} x))$.

With the same assumptions on x, y, z we try to calculate the value of (3), i.e. of

$$\langle m, z\rangle \oplus (\langle l, y\rangle \oplus \langle k, x\rangle).$$

From the preceding we have $y + z < \tau$ and $x + (y + z) = \tau$.

Now we try to calculate $\langle l, y\rangle \oplus \langle k, x\rangle$.

Subcase 1. $y + x < \tau$.

Then

$$\langle l, y\rangle \oplus \langle k, x\rangle = \langle l + k, y + x\rangle$$

and since $x + (y + z) = \tau$ we have $z + (x + y) = \tau$, whence Case 2 for z and $y + x$, whence

$$\begin{aligned}&\langle m, z\rangle \oplus (\langle l, y\rangle \oplus \langle k, x\rangle)\\ &\quad = \langle m + l + k + 1, z \dot{-} (\tau \dot{-} (x + y))\rangle.\end{aligned}$$

Now, Axiom 11 gives that if $x + y + z = \tau$, $y + z < \tau$, and $y + x < \tau$, then

$$x \dot{-} (\tau \dot{-} (y + z)) = z \dot{-} (\tau \dot{-} (x + y)),$$

so we have (1) = (3) in this subcase.

Subcase 2. $y + x = \tau$.

Then

$$\begin{aligned}\langle l, y\rangle \oplus \langle k, x\rangle &= \langle l+k+1, x \dot{-} (\tau \dot{-} y)\rangle \\ &= \langle l+k+1, y \dot{-} (\tau \dot{-} x)\rangle.\end{aligned}$$

Subcase 2.1. $z + (y \dot{-} (\tau \dot{-} x)) < \tau$.

So (3) is

$$\langle m+l+k+1, z + (y \dot{-} (\tau \dot{-} x))\rangle,$$

and by Axiom 12 we have (1) = (3) in this subcase.

Subcase 2.2. $z + (y \dot{-} (\tau \dot{-} x)) = \tau$.

By Axiom 13 this is impossible.

Situation 3. (y, z) in Case 2, and x in Case 1 with right-hand coordinate of $\langle l, y\rangle \oplus \langle m, z\rangle$.

So $y + z = \tau$, whence

$$\begin{aligned}\langle l, y\rangle \oplus \langle m, z\rangle &= \langle l+m+1, y \dot{-} (\tau \dot{-} z)\rangle \\ &= \langle l+m+1, z \dot{-} (\tau \dot{-} y)\rangle.\end{aligned}$$

Then $x + ((y \dot{-} (\tau \dot{-} z)) < \tau$, and so

$$\langle k, x\rangle \oplus (\langle l, y\rangle \oplus \langle m, z\rangle) = \langle k+l+m+1, x + (y \dot{-} (\tau \dot{-} z))\rangle$$

which equals the value of (1).

With the same assumptions on x, y, z we try to compute (3). The assumptions are $y + z = \tau$ and $x + (y \dot{-} (\tau \dot{-} z)) < \tau$.

Subcase 1. $y + x < \tau$.

Then $\langle l, y\rangle \oplus \langle k, x\rangle = \langle l+k, y+x\rangle$.

Subcase 1.1. $z + (y + x) < \tau$.

This is impossible, since $y + z = \tau$ is assumed.

Subcase 1.2. $z + (y + x) = \tau$.

So (3) equals

$$\langle k+l+m+1, z \dot{-} (\tau \dot{-} (y+x))\rangle = \langle k+l+m+1, (y+x) \dot{-} (\tau \dot{-} z)\rangle.$$

which equals (1) by Axiom 11.

Subcase 2. $y + x = \tau$.

Then

$$\begin{aligned}\langle l, y\rangle \oplus \langle k, x\rangle &= \langle l+k+1, y \dot{-} (\tau \dot{-} x)\rangle \\ &= \langle l+k+1, x \dot{-} (\tau \dot{-} y)\rangle.\end{aligned}$$

Subcase 2.1. $z + (y \dot{-} (\tau \dot{-} x)) < \tau$.

So (3) equals

$$\begin{aligned}&\langle k+l+m+1, (x \dot{-} (\tau \dot{-} y)) + z\rangle \\ &\quad = \langle k+l+m+1, (y \dot{-} (\tau \dot{-} x)) + z\rangle\end{aligned}$$

which equals (1) by Axiom 14.

Subcase 2.2. $z + (y \dot{-} (\tau \dot{-} x)) = \tau$.

But this together with the assumptions of the situation contradicts Axiom 15. So it does not occur.

Situation 4. (y, z) in Case 2 and x in Case 2 with right-hand coordinate of $\langle l, y\rangle \oplus \langle m, z\rangle$.

So $y + z = \tau$ whence

$$\begin{aligned}\langle l, y\rangle \oplus \langle m, z\rangle &= \langle l + m + 1, y \dot{-} (\tau \dot{-} z)\rangle \\ &= \langle l + m + 1, z \dot{-} (\tau \dot{-} y)\rangle.\end{aligned}$$

Then

$$x + (y \dot{-} (\tau \dot{-} z)) = \tau,$$

so

$$\begin{aligned}\langle k, x\rangle \oplus (\langle l, y\rangle \oplus \langle m, z\rangle) &= \langle k + l + m + 2, x \dot{-} (\tau \dot{-} (y \dot{-} (\tau \dot{-} z)))\rangle \\ &= \langle k + l + m + 2, (y \dot{-} (\tau \dot{-} z)) \dot{-} (\tau \dot{-} x)\rangle,\end{aligned}$$

giving the value of (1).

With the same assumptions on (x, y, z), we try to compute (3).

The assumptions are $y + z = \tau$ and $x + (y \dot{-} (\tau \dot{-} z)) = \tau$.

Subcase 1. $y + x < \tau$.

But this is inconsistent with Axiom 6 since $x + (y \dot{-} (\tau \dot{-} z)) = \tau$.

Subcase 2. $y + x = \tau$.

So

$$\begin{aligned}\langle l, y\rangle \oplus \langle k, x\rangle &= \langle l + k + 1, y \dot{-} (\tau \dot{-} x)\rangle \\ &= \langle l + k + 1, x \dot{-} (\tau \dot{-} y)\rangle.\end{aligned}$$

Subcase 2.1. $z + (y \dot{-} (\tau \dot{-} x)) < \tau$.

This is inconsistent by Axiom 15, since

$$x + (y \dot{-} (\tau \dot{-} z)) = \tau.$$

Subcase 2.2. $z + (y \dot{-} (\tau \dot{-} x)) = \tau$.

Then (3) equals

$$\langle k + l + m + 2, (y \dot{-} (\tau \dot{-} x)) \dot{-} (\tau \dot{-} z)\rangle$$

and by Axiom 16 this equals

$$\langle k + l + m + 2, (y \dot{-} (\tau \dot{-} z)) \dot{-} (\tau \dot{-} x)\rangle$$

so (1) = (3).

This concludes the proof that $\oplus$ is associative.

3.6. Cancellation

In the preceding we have established that $(P, \leq_P, \oplus)$ is a commutative ordered monoid with 0 as least element. In Lemma 3.4 (in Section 3.4) we show (en passant) that

$$\langle k, x \rangle \leq \langle k, x \rangle \oplus \langle l, y \rangle$$

for any $\langle k, x \rangle$, $\langle l, y \rangle$.

It remains to prove cancellation or relative complementation, namely:

Lemma 3.5. *If $\langle k, x \rangle \leq \langle l, y \rangle$ then there exists $\langle m, z \rangle$ with*

$$\langle k, x \rangle \oplus \langle m, z \rangle = \langle l, y \rangle.$$

Proof. Assume $\langle k, x \rangle \leq \langle l, y \rangle$. Then $k \leq l$. If $k = l$, then $x \leq y$. So take $m = 0$ and $z = y \dot{-} x$.

If, however, $k < l$, there are two cases.

Case I. $x \leq y$. In this case, we take $m = l - k > 0$ and $z = y \dot{-} x$, and we are done.

Case II. $y < x$.

We shall be concerned with Case II for the rest of the proof. We need $\langle m, z \rangle$ such that

$$\langle l, y \rangle = \langle k, x \rangle \oplus \langle m, z \rangle,$$

and as usual the Case 1/Case 2 distinction (for Case II) on x, z intervenes. But now z is the unknown, with x, y given.

Suppose z can be found in Case 1. So $x + z < \tau$, and then $y = x + z$, so $y \geq x$, contradicting our assumption.

Thus z can be found, if at all, only in Case 2, and then

$$y = x \dot{-} (\tau \dot{-} z) = z \dot{-} (\tau \dot{-} x),$$

so $\tau \dot{-} z = x \dot{-} y$.

So $z = \tau \dot{-} (x \dot{-} y)$. What about m? We want

$$\langle l, y \rangle = \langle k + m + 1, x \dot{-} (\tau \dot{-} z) \rangle,$$

so we need only

$$l = k + m + 1,$$

so $m = (l - k) - 1 \geq 0$. □

This completes the proof of Theorem 3.2. □

Remark 3.6. We are going to use Theorem 3.2 only for discretely ordered TOAG's, in fact only the $[0, \tau]$ coming from models of Presburger arithmetic.

3.7. Presburger truncated ordered abelian groups

In our applications we take Presburger arithmetic (cf. [9]) to be formulated in the language of ordered groups with a distinguished constant 1 (to denote the least positive element). We generally drop the distinction between the group Γ and its non-negative part.

Note that $0 \neq 1$ in models of Presburger.

We now consider Presburger truncated ordered abelian groups, i.e. truncated ordered abelian groups of the form $[0, \tau]$ with distinguished element 1, the least positive element (we do not insist that $1 < \tau$), which are truncations of models of Presburger.

Theorem 3.7. *A truncated ordered abelian group* $[0, \tau]$ *with least positive element* 1 *is a Presburger truncated ordered abelian group if and only if it satisfies the following conditions:*

(1) $[0, \tau]$ *is discretely ordered and every positive element is a successor,*
(2) *For each positive integer* n *and each* x *in* $[0, \tau]$ *there is a* y *in* $[0, \tau]$ *and an integer* $m < n$ *such that*

$$x = ny + m = \underbrace{(y + \cdots + y)}_{n\ times} + \underbrace{(1 + \cdots + 1)}_{m\ times}.$$

Note. If $m = 0$ in $\mathbb{Z}$, then by definition we have that $m = 0$ in $[0, \tau]$.

Proof. Necessity is clear from the axioms of Presburger.

For sufficiency, we argue as follows. Suppose $[0, \tau]$ satisfies Conditions (1) and (2) (and has a distinguished least positive element 1). Build P as in the proof of Theorem 3.2. Clearly 1 is the least positive element of P and P is discretely ordered. Let $\langle k, x\rangle$ be a nonzero element of P, so $k \in \{0, 1, 2, \dots\}$ and $x \in [0, \tau)$. If $x \neq 0$,

$$\langle k, x\rangle = \langle k, x \dot{-} 1\rangle + \langle 0, 1\rangle,$$

so $\langle k, x\rangle$ is a successor. If $x = 0$, and $k \neq 0$, $\langle k, x\rangle$ is the successor of $\langle k-1, \tau-1\rangle$.

So P is discretely ordered, and every positive element is a successor.

To get the Euclidean division results, fix a positive integer n and some $\langle k, x\rangle$ in P. Let $k = na + b$, for non-negative integers a, b with $b < n$.

Now

$$\langle k, x\rangle = \langle k, 0\rangle \oplus \langle 0, x\rangle = \langle na, 0\rangle \oplus \langle b, 0\rangle \oplus \langle 0, x\rangle,$$

and

$$\langle na, 0\rangle = \underbrace{\langle a, 0\rangle \oplus \cdots \oplus \langle a, 0\rangle}_{n \text{ times}}.$$

Also, if $b > 0$,

$$\langle b, 0\rangle = \underbrace{\langle 1, 0\rangle \oplus \cdots \oplus \langle 1, 0\rangle}_{b \text{ times}}$$

and

$$\langle 1, 0\rangle = \langle 0, \tau - 1\rangle + \langle 0, 1\rangle,$$

and if $1 < \tau$ (other case trivial)

$$\tau - 1 = nc + d,$$

for some $c \in [0, \tau - 1]$, and $0 \leq d < n$ (with usual conventions about multiplication by m). Note that $\langle 0, 1\rangle$ is the least positive element in P and when we add it to $\langle 0, \tau - 1\rangle$ we are in Case 2.

Finally, $\langle 0, x\rangle$ is also of the form $n\gamma + \delta$, with $0 \leq \delta < n$ via Condition 2 for $[0, \tau]$.

Thus $\langle k, x\rangle$ is congruent "modulo" n to an integer less than n, and we are done. □

4. Elementary equivalence of Presburger truncated ordered abelian groups

In this section we address the question of when $[0, \tau] \cong [0, \mu]$ holds if both are Presburger truncated ordered abelian groups. We shall give a criterion for this.

Clearly the answer comes, by Theorem 3.7, from an answer to the question: what are the pure 1-types for Presburger arithmetic? The answer to this is well-known [9]. Namely, the pure 1-type of an element x in P, the non-negative part of a model of Presburger, is determined by

(A) Whether or not $x \in \{0, 1, 2, \dots\}$,

(B) The remainder of x modulo n for each positive integer n.

Theorem 4.1. *The elementary theory of a Presburger truncated ordered abelian group* $[0, \tau]$ *is determined by the Presburger* 1*-type of* $\tau - 1$ (*the penultimate element of* $[0, \tau]$), *i.e. by*

(A) *Whether or not* $\tau - 1 \in \{0, 1, 2, \dots\}$,

(B) *The congruence class of* $\tau - 1$ *modulo* n *for each positive integer* n.

Moreover, any Presburger 1*-type can occur for some truncated ordered abelian group* $[0, \tau]$.

Proof. Immediate from the preceding. □

5. An proof suggested by the Referee

We are grateful to the anonymous referee for valuable comments and remarks, for making a connection of our work to work of Rieger [10,11,12] on cyclically ordered abelian groups, and for outlining an alternative proof of the existence of the group in Theorem 3.2 using Rieger's results.

However, in our Theorem 3.2 the TOAG is an initial segment of the group we construct. This does not seem obvious via Rieger. The referee also stated the relevance of the paper by Richard Ball [3] to our work.

Our objective had been to find explicit and computable axioms that would capture exactly the truncations of ordered abelian groups that arise from quotients of valuation rings.

Although this alternative construction is not explicit as ours and does not involve axioms, bringing the ideas and results of Rieger on circular orders into the repertoire of model theory of local rings may well be productive. Rieger's work has been used in [6].

We give the referee's argument below. In this proof, instead of doing the construction in one step as we do, one first defines a cyclically ordered (c.o.) group on a TOAG and then constructs the unwound of this c.o. group. Applying Rieger, one can get a totally ordered group. If one takes the positive part of the unwound, one recovers our set P.

Let $([0,\tau),+,0,\leq)$ be a model of Axioms 1–16. Let $b,c\in[0,\tau)$. Then define a new operation $\tilde{+}$ as follows.

$$b\tilde{+}c=\begin{cases}b+c & \text{if } b+c<\tau,\\ b\dot{-}(\tau\dot{-}c) & b+c=\tau.\end{cases}$$

(This represents cases 1 and 2 that we distinguished in Subsection 3.2).

Let R be a ternary relation on $([0,\tau),<)$ defined by $R(x,y,z)$ if $x<y<z$ or $y<z<x$ or $z<x<y$. Then $([0,\tau),R,\tilde{+},0)$ is a cyclically ordered (c.o.) group. Indeed since $+$ is commutative (Axiom 1), $\tilde{+}$ is also commutative. It can be easily checked that R is compatible with $\tilde{+}$. For $z\in[0,\tau)$, define $-z:=\tau\dot{-}z$. By Axiom 9, $\tau\dot{-}(\tau\dot{-}z)=z$ and so $z\tilde{+}(\tau\dot{-}z)=0$. By the proof of associativity in Subsection 3.5, $\tilde{+}$ is associative (this can also be checked directly—however in our proofs in Subsection 3.5 we prepare for the greater complexities to be met in the proof of the main embedding theorem).

For Rieger's notion of unwound and his result see [10,11,12] and [5, pp. 61–65]. Let us establish the following parallel between the construction of the unwound of a c.o. group and our construction in Section 3.1. It is convenient to follow the notation in Giraudet, Leloup and Lucas [6]. Set $H=([0,\tau),R,\tilde{+},0)$. In Section 3.1, we had $P=\omega\times[0,\tau)$.

In [6, Definition 2.2], the unwound is defined as $uw(H)=\mathbb{Z}\times H$, where the order $\leq_R$ is defined by

$$(m,c)\leq_R(m',c')\Leftrightarrow m<m' \text{ or } (m=m' \text{ and } (c=0 \text{ or } R(0,c,c'))),$$

and the group operation is defined by

$$\begin{aligned}&(m,c)+(m',c')\\ &\quad=\begin{cases}(m+m',c\tilde{+}c') & \text{if } c=0 \text{ or } c'=0 \text{ or } R(0,c,c\tilde{+}c'),\\ (m+m'+1,c\tilde{+}c') & \text{if } c\neq 0 \text{ and } c\tilde{+}c'=0 \text{ or } R(0,c\tilde{+}c',c).\end{cases}\end{aligned}$$

The set of (m,m'), where $m,m'>0$ corresponds to our P and (in the notation of [6]), $z_H:=(1,0)$ corresponds to our τ_P.

By Rieger's theorem [10,11,12] (stated also in [5, pp. 61–65] and [6, Theorem 2.3]), one has $uw(H)/\langle z_H\rangle\cong H$. This gives the construction of the group.

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Paola D'Aquino
Dipartimento di Matematica e Fisica
Università della Campania "L. Vanvitelli"
viale Lincoln, 5
81100 Caserta
Italy
e-mail: paola.daquino@unicampania.it

Jamshid Derakhshan
St Hilda's College
University of Oxford
Cowley Place
Oxford OX4 1DY
UK
e-mail: derakhsh@maths.ox.ac.uk

Angus Macintyre
School of Mathematical Sciences, Queen Mary
University of London
Mile End Road
London E1 4NS
UK
e-mail: a.macintyre@qmul.ac.uk

and

School of Mathematics
University of Edinburgh
James Clerk Maxwell Building, Peter Guthrie Tait Road
Edinburgh EH9 3FD
UK

Received: 20 December 2019.
Accepted: 4 November 2020.