Working Paper 2012:31

Department of Economics
School of Economics and Management

Truth-Seeking Judgment Aggregation over Interconnected Issues

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November 2012

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Abstract

This paper analyses the problem of aggregating judgments over multiple interconnected issues. We enrich the model by introducing the private information underlying individuals' judgments. Individuals share a common preference for reaching true collective judgments, but hold private information about what the truth might be. Information conflicts may occur both between and within individuals. Assuming strategic voting in a Bayesian voting game setting, we determine the voting rules which lead to collective judgments that efficiently incorporate all private information. We characterize the (rare) situations in which such rules exist, as well as the nature of these rules.

Keywords: judgment aggregation, private information, efficient information aggregation, strategic voting *JEL Classification Numbers*: D70, D71

1 Introduction

The main research question of the theory of judgment aggregation is how a group of individuals can make a collective 'yes' or 'no' judgment on several issues on the basis of group members' judgments on these issues. Such problems occur in many collective decision-making bodies ranging from expert panels to referenda, from juries to legislative committees, multi-member courts, boards of companies, and social groups. The jury in a court trial might need to form judgments on two issues; whether the defendant has broken the contract, and whether the contract is legally valid. Each juror has to accept or reject each issue and a collective decision is taken on each issue using a voting rule. This is a simple judgment aggregation problem. Other examples include an expert panel that needs to form judgments on whether the CO_2 emissions are above a given threshold and whether there will be a critical temperature change,

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or The United Nations security council that needs to form judgments on whether country X is threatened by a military coup and whether country X risks an economic crisis. A group of governmental advisors might seek judgments on whether a given bank is solvent and whether the bank has liquidity problems.

Judgment aggregation models allow for the study of complex aggregation problems. Typically, the issues in the group's agenda might be *mutually interconnected*, which means the judgment made on one issue might constrain the judgment on other issues. In the case of governmental advisors, a 'no' judgment on the first issue might restrict the judgment on the second issue to 'yes'; hence, if the bank is judged to be 'not solvent', then it follows that the bank has liquidity problems. Judgment aggregation models also allow for the study of problems where group judgments lead to a group action. They may determine whether a rescue plan for the bank will be implemented by the government. In the court trial example, group judgments lead to the conviction or acquittal of the defendant (according to a commonly accepted legal doctrine that the defendant is guilty if and only if both issues are accepted by the jury). In the UN example, group judgments may determine whether an intervention in country X will take place.

The problem of judgment aggregation has been studied from the perspective of two different approaches so far; the *social-choice theoretic* approach and the *epistemic* approach. The classical social-choice theoretic approach asks questions about whether and how the individuals' judgments can be aggregated *fairly* to obtain a *consistent* collective judgment. Fairness is attained by some axiomatic requirements on the voting rule, such as anonymity or unanimity preservation. Consistency, on the other hand, is a property that requires the collective decision to be free from any logical contradictions.[1] Nearly the entire judgment aggregation theory takes the perspective of the social-choice theoretic approach, which is illustrated recently in the *Symposium on Judgment Aggregation* in Journal of Economic Theory (C. List and B. Polak eds., 2010, vol. 145(2)) (see review below). The *epistemic* approach of *tracking the truth*, on the other hand, is followed by very few works. This approach aims to reach *true* group judgments. In a context where the goal is to aggregate individuals' judgments or opinions over some issues apart from mere preferences, the epistemic approach seems very natural. Consider the court trial example, where the jury has to decide over two issues which will lead to conviction or acquittal of the defendant. It seems that the jury's goal is to find out two independent facts (whether the defendant has broken the contract and whether the contract is legally valid). The epistemic approach aims to optimise the voting rule in a way that the group's collective judgments are true, not that they are fair to voters. To explain the difference between the two approaches, consider the extreme situation where all jurors judge that the defendant has broken the contract. From a social-choice theoretic perspective, the jury should collectively accept the issue in order to respect the unanimity. From an epistemic perspective, a collective 'yes' judgment will be good when it is motivated by truth-tracking type of

[1] For instance; in the governmental advisors example, a 'no' judgment on both issues (the bank is not solvent and the bank does not have liquidity problems) will be inconsistent. In the expert panel example a 'yes' judgment on the first issue (CO_2 emissions are above a given threshold) and a 'no' judgment on the second (there will not be a critical temperature change) will be inconsistent.

questions, such as whether the jurors' judgments are sufficient evidence for breach of contract, and whether they expressed their judgments truthfully.

This paper takes the epistemic approach to analyse judgment aggregation problems. We assume that there is an objective truth to be found out through the collective judgment, called the *state* (of the world), individuals share a common interest of finding out the correct state, but hold possibly conflicting private information about what might be correct. We analyse strategic voting in a Bayesian voting game setting and we want to answer the following question: Does there always exist voting rules which lead to efficient and truthful Bayesian Nash equilibrium of the corresponding game, and if yes, what are these rules? In other words, we want to design voting rules (whenever possible) which first lead to truthful revelation of private information, and second lead to the efficiency in equilibrium. Note that individual reporting of private information need not always be truthful, even when voters have no conflicting aims. This observation is due to Austen-Smith and Banks (1996), who study binary collective choice problems with common interests.[2] Like us, they take a truth-tracking and strategic voting perspective, which is a well-established approach in binary collective choice problems (see review below). As Austen-Smith and Banks (1996) show, if an individual conditions her beliefs on being pivotal – on being able to change the outcome – she may not always find it best to report truthfully. This observation gives rise to the analysis and design of voting rules which lead to simple-minded and truthful behaviour of individuals as well as correct decisions. Bozbay, Dietrich and Peters (2011) introduce this problem into the theory of judgment aggregation and study an agenda with two issues, where issues are independent. We extend their work to the setting where issues are interconnected.

This paper assumes that the group faces *two* issues, and a 'no' answer to both issues is inconsistent. Let the first issue of the governmental advisors example be denoted by the proposition p, which states 'the bank is solvent'. Similarly, let q: 'the bank has liquidity problems'. If the bank is not solvent, then it has liquidity problems; hence, the interconnection is encoded by $(\bar{p} \rightarrow q)$ (where $\bar{p}$ stands for the negation of p). Note that by exchanging the roles of issues with their negations, we can obtain all kinds of interconnection between two issues (except bi-implication).[3] So, by studying this kind of interconnection, we exhaustively cover all possible mutual interconnections between two issues.

We consider two different types of preferences; namely *simple* and *consequentialist preferences* in this paper. In the governmental advisors example, we say that there will be a rescue plan for the bank if and only if p and q are collectively accepted. If only one of these issues are judged to be true, a rescue is either not deserved or not necessary. Suppose the true state of the world is $p\bar{q}$. A voter with simple preferences would only be happy if the collective judgment exactly matches the true state, so, if p and $\bar{q}$ are both collectively accepted. A consequentialist voter, on the other hand, would only mind the correct consequence, which in this case is 'no rescue

[2] Such problems are equivalent to judgment aggregation problems with one issue. The examples above have two issues. If instead only the first issue were on the agenda, it would be a binary collective choice problem.

[3] If instead p: 'the bank is not solvent', then the interconnection would be encoded by $(p \rightarrow q)$.

plan'. Any collective judgment that leads to the same consequence (which in this case is $\bar{p}q$) would be equally good for a consequentialist voter. In this paper, we consider problems where the decision pq leads to one consequence while each of $p\bar{q}$ and $\bar{p}q$ leads to the other consequence as in the example of governmental advisors. When interconnections are introduced into the agenda, this is the only interesting way of defining consequences. For instance, if pq and $p\bar{q}$ led to the same consequence while $\bar{p}q$ led to the other, then the utility would only depend on the decision on p, which would reduce the problem to a binary collective choice problem. All decisions leading to the same consequence is another uninteresting case. Thus, this paper covers non-degenerate consequentialist preferences exhaustively.

Our results depend on which kind of preferences we consider. Under simple preferences, it turns out that no voting rule supports truthful voting. On the other hand, under consequentialist preferences there are voting rules which make truthful voting efficient under a demanding condition over the model parameters. These voting rules accept both issues only when they are unanimously accepted. This result relies heavily on the indifference relation between $p\bar{q}$ and $\bar{p}q$ under consequentialist preferences. The voting rule becomes binary in that it suffices to choose between pq and one of $p\bar{q}$ and $\bar{p}q$. Hence, this possibility is only meaningful in the cases where issues alone are not important. These results come as a surprise when one considers the results in binary collective choice problems with common interests and judgment aggregation problems with two independent issues, where it is always possible to aggregate information efficiently. In binary collective choice problems, the only way for efficient information aggregation is using a quota rule which decides on each issue according to whether the number of 'yes' judgments on the issue exceeds a particular quota. In case of two independent issues the rule depends on the exact preferences in use, though in most cases it should be a quota rule. Introducing interconnections between the issues leads to the impossibility for efficiency of truthful voting in almost all cases.

We now give a review of different bodies of literature that this paper connects to. A judgment aggregation problem is formulated in its present form by List and Pettit (2002, 2004), while the origins of the problem goes back to works by Kornhauser and Sager (1986, 1993) in the area jurisprudence, to Guilbaud (1966), Wilson (1975) and Rubinstein and Fishburn (1986). The classical social-choice theoretic approach in judgment aggregation theory is followed by a variety of works and authors and a series of possibility and impossibility results are successfully obtained (e.g., Dietrich 2006, 2007, 2010, Nehring and Puppe 2008, 2010, Dietrich and List 2007a, 2007c, 2008, Dokow and Holzman 2010a, 2010b, Dietrich and Mongin 2010; for an introductory overview see List and Polak 2010). Dietrich and List (2007b) analyse strategic voting behaviour like us, but in a setting where voters have private values instead of private information. See also related work by Nehring and Puppe (2002, 2007). Ahn and Oliveros (2012) study multi-issue elections where voters have private values over alternatives, and hold common information about them. There are few works in judgment aggregation following the epistemic approach (Bovens and Rabinowicz 2006, List 2005 and Pivato 2011), but these works do not consider private information and strategic incentives. See List and Pettit (2011) for a philosophical analysis of the truth-tracking approach. The work by Bozbay, Dietrich and Peters

(2011) combines judgment aggregation and binary collective choice with common interests. They model voters' common interests and private information with an agenda with independent issues and they study the problem of designing a voting rule (which this paper extends to agendas with mutual interconnections between issues). Ahn and Oliveros (2011) and De Clippel and Eliaz (2011) study elections on multiple issues with common preferences and asymmetric information to compare different voting procedures in terms of efficient aggregation of information, asymptotically as the group size tends to infinity. The main difference between Bozbay, Dietrich and Peters (2011) and these two works is that the former takes a mechanism design approach while the latter two consider fixed mechanisms. For the binary collective choice problem with common interests, Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997) show that it need not be rational for all voters to vote sincerely. This approach is extended to continuous rather than binary private information by Duggan and Martinelli (2001) and Meirowitz (2002). Feddersen and Pesendorfer (1998), Coughlan (2000) and Gerardi (2000) examine the unanimity rule in protecting the innocent in jury trials. Austen-Smith and Feddersen (2005, 2006) analyse pre-voting deliberation and preference heterogeneity in such problems.

The paper proceeds as follows. In Section 2, we introduce our model. Section 3 contains a general existence result about efficient aggregation of information for any kind of common preferences. In Section 4 and 5, we focus on simple and consequentialist preferences respectively. Section 5 also studies the case of quota rules and whether they can be used for efficient information aggregation. Section 6 concludes. All proofs are in Appendix.

2 The Model

2.1 A simple judgment aggregation problem

We consider a group of voters, labeled $i = 1, ..., n$, where $n \geq 2$. This group needs a collective judgment on whether some proposition p or its negation $\bar{p}$ is true, and whether some other proposition q or its negation $\bar{q}$ is true. While doing so, voters know that the combination $\{\bar{p}, \bar{q}\}$ is not possible. The three possible judgment sets are $\{p, q\}, \{p, \bar{q}\}, \{\bar{p}, q\}$, abbreviated by pq, $p\bar{q}$ and $\bar{p}q$, respectively. Similarly, $\{\bar{p}, \bar{q}\}$ is abbreviated by $\bar{p}\bar{q}$. Each voter votes for a judgment set in $\mathcal{J} = \{pq, p\bar{q}, \bar{p}q\}$. A collective decision in $\mathcal{J}$ is taken using a voting rule. A voting rule is defined as a function $f : \mathcal{J}^n \to \mathcal{J}$, which maps each voting profile $\mathbf{v} = (v_1, ..., v_n)$ to a decision $d \equiv f(\mathbf{v})$.

2.2 Common preferences for true collective judgments

There is one 'correct' judgment set in $\mathcal{J}$, which we call the *state (of the world)* and denote by s. The state is unobservable by voters. Voters have identical preferences, represented by a common utility function $u : \mathcal{J} \times \mathcal{J} \to \mathbb{R}$ which maps any decision-state pair (d, s) to its utility $u(d, s)$. The notion of truth-tracking requires the utility to be high if the decision is correct, but one can define different kinds of truth-tracking preferences.[4] We focus on two natural kinds of preferences.

Simple preferences. Under simple preferences, the utility function is given by

$$u(d,s) = \begin{cases} 1 & \text{if } d = s \text{ (correct decision)} \\ 0 & \text{if } d \neq s \text{ (incorrect decision).} \end{cases} \tag{1}$$

Hence, voters with simple preferences want to find out the correct decision. If there is a collective action after voting (such as convicting or acquitting the defendant as in the court trial example), they want to reach the right action through correct reasons.

Consequentialist preferences. To define consequentialist preferences, we first introduce 'consequences'. We assume that the group decision leads to one of two possible consequences which represents group actions. A *consequence function* Co maps the set $\mathcal{J}$ to a two-element set of possible consequences. Consider the example of governmental advisors where p: 'the bank is solvent', and q: 'the bank has liquidity problems'. If both issues are judged to be true, the consequence is to implement a rescue plan for the bank, so $\mathrm{Co}(pq)$ ='rescue plan'. If only one of the issues are judged to be true, then the consequence is 'no rescue plan' since then the government does not see the bank as a good candidate for rescuing because of insolvency, or the rescue is not necessary; so $\mathrm{Co}(p\bar{q}) = \mathrm{Co}(\bar{p}q)$ ='no rescue plan'. It turns out that this consequence function with the property $\mathrm{Co}(pq) \neq \mathrm{Co}(p\bar{q}) = \mathrm{Co}(\bar{p}q)$ is the only interesting consequence function up to isomorphism. (See Section 5 for further discussion.) The consequentialist utility function is given by

$$u(d,s) = \begin{cases} 1 & \text{if } \mathrm{Co}(d) = \mathrm{Co}(s) \text{ (correct consequence)} \\ 0 & \text{if } \mathrm{Co}(d) \neq \mathrm{Co}(s) \text{ (incorrect consequence).} \end{cases} \tag{2}$$

2.3 Private information and strategies

Each voter has a *type*, which is an element of $\mathcal{T} = \{pq, p\bar{q}, \bar{p}q, \bar{p}\bar{q}\}$ and is denoted by t generically. A voter's type represents evidence about whether p is true and whether q is true. For instance, the type $t = p\bar{q}$ represents evidence for p and for $\bar{q}$, and the type $t = \bar{p}\bar{q}$ represents evidence for $\bar{p}$ and for $\bar{q}$, which overall is conflicting information since $\bar{p}\bar{q} \notin \mathcal{J}$. We write $\mathbf{t} = (t_1, ..., t_n) \in \mathcal{T}^n$ for a profile of voters' types.

Nature draws a state-types combination $(s, \mathbf{t})$ in $\mathcal{J} \times \mathcal{T}^n$ according to a probability measure denoted Pr. By convention, the prior probability of state $s \in \mathcal{J}$ is denoted

$$\pi_s = \Pr(s)$$

and is assumed to be in the interval $(0,1)$. If a proposition $r \in \{p, \bar{p}, q, \bar{q}\}$ represents (part of) voter i's type rather than (part of) the true state, we often write r_i for r. We write $\Pr(p_i|p)$ for the probability that voter i has evidence for p given that p is true. The probability of getting evidence for r given that r is true is denoted

$$a_r = \Pr(r_i|r)$$

and by assumption belongs to $(1/2, 1)$ and does not depend on the voter i.

[4] A voter tracks the truth on a proposition p if the following is true: if p were true, the voter would accept p and if p were false, the voter would accept $\bar{p}$ (Nozick, 1981).

We assume voters' types are independent given the state. Moreover, given the truth about p (i.e., either p or $\bar{p}$), a voter's evidence about p (i.e., either p_i or $\bar{p}_i$) is independent of the truth and the evidence about q; and similarly, given the truth about q, a voter's evidence about q is independent of the truth and the evidence about p. The joint distribution of the state and the types is then given by

$$\Pr(s, \mathbf{t}) = \Pr(s) \times \prod_{i=1}^{n} \Pr(t_i|s).$$

Each voter votes for a judgment in $\mathcal{J}$ based on his type. A *(voting) strategy* is a function $\sigma : \mathcal{T} \to \mathcal{J}$, mapping each type $t \in \mathcal{T}$ to the type's vote $v = \sigma(t)$. We write $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$ for a profile of voters' strategies. We now have a well-defined Bayesian game, with a voting rule f and a common utility function u.

For a given type profile $\mathbf{t} \in \mathcal{T}^n$, we call a decision $d \in \mathcal{J}$ *efficient* if it has maximal expected utility conditional on the full information $\mathbf{t}$. We adapt some common notions of voting behaviour to our framework.

- A strategy σ of a voter is *informative* if $\sigma(t) = t$ for all $t \in \mathcal{T} \setminus \{\bar{p}\bar{q}\}$ and $\sigma(\bar{p}\bar{q}) \in \{p\bar{q}, \bar{p}q\}$.

- A strategy profile $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$ is *rational* if each strategy is a best response to the other strategies, i.e., if the profile is a Nash equilibrium of the corresponding Bayesian game. Hence, each voter maximises the expected utility of the collective decision given the strategies of the other voters.

- A strategy profile $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$ is *efficient* if for every type profile $\mathbf{t} = (t_1, ..., t_n)$ the resulting decision $d = f(\sigma_1(t_1), ..., \sigma_n(t_n))$ is efficient (i.e., has maximal expected utility conditional on full information $\mathbf{t}$). Hence, all the information spread across the group is used efficiently: the collective decision is no worse than a decision of a (virtual) social planner who has full information.

A voter with informative strategy votes for her type if her type is non-conflicting, while she *partly* follows it when she has the conflicting evidence $t = \bar{p}\bar{q}$. Here, informativeness is open to behaviour. One can choose between $p\bar{q}$ and $\bar{p}q$ under conflicting evidence. In a setting where information is never conflicting, following the evidence would be simply voting for the type. An informative voter in our setting follows the evidence as much as possible.

We make two assumptions to avoid trivialities. First, we exclude the degenerate case where some decision in $\mathcal{J}$ is not efficient for any type profile. Hence, each decision is efficient for at least one type profile. Second, we exclude efficiency ties, i.e., those special parameter combinations such that some type profile leads to different efficient decisions (with different consequences when we assume consequentialist preferences). Hence, we exclude those instances where a voter is indifferent between two decisions except in the case that these decisions lead to the same consequence.

3 A general (im)possibility

The objective of this paper is to design voting rules which lead to efficient decisions as well as simple-minded, truthful voting behaviour in equilibrium. We generally mean informative voting as simple-minded, truthful behaviour.[5] For a rule to induce simple-minded, truthful behaviour, informative voting should be rational in equilibrium. Our desired voting rule should then lead to efficient decisions and make informative voting rational. Note that informative voting being efficient means that for any given type profile $\mathbf{t}$, *every* profile of corresponding informative strategies is efficient.[6] Following a well-known result by McLennan (1998), an efficient strategy profile is rational and our objective is reduced to finding out when informative voting is efficient.

Remark 1 *(McLennan 1998) For any common utility function $u : \mathcal{J}^2 \to \mathbb{R}$, and for any voting rule $f : \mathcal{J}^n \to \mathcal{J}$, if a strategy profile is efficient, then it is rational.*

Does there always exist voting rules which make informative voting efficient? The answer is 'yes' when we consider the single-issue setting and multiple issue setting with no interconnections. Bozbay, Dietrich and Peters (2011) show that there is *always* an anonymous voting rule for which informative voting is efficient when two issues are independent. We show that this result do not persist when we introduce interconnections between the issues. Informative voting is efficient only under some condition over the model parameters.

Condition 1: For any $\mathbf{t}, \mathbf{t}' \in \mathcal{T}^n$, if $\{i : t_i = pq\} = \{i : t'_i = pq\}$, then some decision $d \in \mathcal{J}$ is efficient for both $\mathbf{t}$ and $\mathbf{t}'$.

Theorem 1 *Consider an arbitrary common utility function $u : \mathcal{J}^2 \to \mathbb{R}$. There exists a voting rule for which informative voting is efficient if and only if Condition 1 holds.*

It is difficult to see when and how often this condition is satisfied unless we narrow down our focus on specific kind of preferences. We study simple and consequentialist preferences in turn in the following section, which allow us to say more about how strong the condition of Theorem 1 is, and about the nature of voting rules making informative voting efficient.

4 Simple preferences

We start by addressing simple preferences, defined by (1), where correct decisions are preferred to incorrect ones. By focusing on simple preferences, what can we say

[5] Alternatively, one can mean sincere voting by simple-minded behaviour. A sincere voter votes only according to her type and as if she determines the outcome alone. Sincere behaviour depends on the utility and requires full awareness of utility. On the other hand, one does not need a definitive utility function to vote informatively.

[6] Given a type profile $\mathbf{t}$, an informative strategy profile is not necessarily uniquely defined since type $\bar{p}\bar{q}$ voters can vote for $p\bar{q}$ or $\bar{p}q$. A voting rule which makes *some* strategy profiles efficient and some not is not interesting since a designer never knows how these voters will vote.

more than the existential claim of Theorem 1? Here is the answer, which states the impossibility of efficient information aggregation under simple preferences.

Theorem 2 *Under simple preferences, there exists no voting rule for which informative voting is efficient.*

It turns out that Condition 1 never holds under simple preferences. This result is not surprising as we now illustrate. Take two type profiles $\mathbf{t}$, $\mathbf{t}'$ for which only the first voter receives evidence for pq. Suppose further, all other voters in $\mathbf{t}$ have evidence for $p\bar{q}$ while all other voters in $\mathbf{t}'$ have evidence for $\bar{p}q$. Condition 1 requires that the efficient decision for both type profiles is the same, while this seems demanding since the two type profiles are almost opposite to each other. Does the impossibility persist under consequentialist preferences? The next section addresses this question.

5 Consequentialist preferences

For the study of consequentialist preferences we consider situations where the decision leads to one of *two* possible consequences. Such problems are very common in practice and widely studied in the judgment aggregation literature, where the two possible consequences are represented by a conclusion proposition, c.[7]. Consequence functions which lead all decisions to the same consequence are degenerate and uninteresting. If the consequence function depends only on the decision between p and $\bar{p}$ (as in, $\mathrm{Co}(pq) = \mathrm{Co}(p\bar{q}) \neq \mathrm{Co}(\bar{p}q)$), or only on the decision between q and $\bar{q}$ (as in, $\mathrm{Co}(pq) = \mathrm{Co}(p\bar{q}) \neq \mathrm{Co}(\bar{p}q)$), then the decision problem reduces to a problem with a single proposition-negation pair which has already been studied in the literature on binary collective choice with common interests. Therefore, there is only one non-degenerate and interesting consequence function up to isomorphism, and this function has the property $\mathrm{Co}(pq) \neq \mathrm{Co}(p\bar{q}) = \mathrm{Co}(\bar{p}q)$. In the governmental advisors example in Section 2.2, the consequence of the decision follows from this kind of consequence function. To state our result, we first define two coefficients:

$$A := \pi_{p\bar{q}} \left(\frac{1 - a_{\bar{q}}}{a_q} \right)^n + \pi_{\bar{p}q} \left(\frac{1 - a_{\bar{p}}}{a_p} \right)^{n-1} \frac{a_{\bar{p}}}{1 - a_p},$$

$$B := \pi_{p\bar{q}} \left(\frac{1 - a_{\bar{q}}}{a_q} \right)^{n-1} \frac{a_{\bar{q}}}{1 - a_q} + \pi_{\bar{p}q} \left(\frac{1 - a_{\bar{p}}}{a_p} \right)^n.$$

Theorem 3 *Under consequentialist preferences, there exists a voting rule for which informative voting is efficient if and only if decision pq is only efficient for the unanimous type profile $\mathbf{t} = (pq, ..., pq)$ (which is the case if and only if $A, B > \pi_{pq}$).*

The theorem states that informative voting can be efficient under consequentialist preferences if pq is the efficient decision *only* when there is perfect evidence for pq.

[7]Consider the lead example of judgment aggregation: a jury is to decide whether the defendant has broken the contract (p) or not ($\bar{p}$) and whether the contract is legally valid (q) or not ($\bar{q}$). The defendant is convicted (c) if and only if both propositions are collectively accepted. The consequence function here is encoded by $c \leftrightarrow (p \wedge q)$.

This is what Condition 1 reduces to under consequentialist preferences. To satisfy this condition, the prior probability of pq should be sufficiently low compared to prior probabilities of $p\bar{q}$ and $\bar{p}q$. For instance, if $\pi_{pq} = \pi_{p\bar{q}} = \pi_{\bar{p}q} = 0.7$, $a_p = a_q = a_{\bar{p}} = a_{\bar{q}} = 0.6$ and $n = 3$, no voting rule makes informative voting efficient, whereas if instead $\pi_{pq} = 0.5$, such a voting rule exists. We present a simple characterization of voting rules which make informative voting efficient when this condition is satisfied.

Proposition 1 *Assume consequentialist preferences and Condition 1. A voting rule $f : \mathcal{J}^n \to \mathcal{J}$ makes informative voting efficient if and only if for every voting profile $\mathbf{v} \in \mathcal{J}^n$, the decision $f(\mathbf{v}) = pq$ if $\mathbf{v} = (pq, ..., pq)$ and $f(\mathbf{v}) \in \{p\bar{q}, \bar{p}q\}$ otherwise.*

Proposition 1 describes voting rules which accept pq only when it is unanimously accepted. Under such rules, only under the extreme situation that every member of the group judges that both issues are correct, they are collectively accepted.

Proposition 1 characterises a large class of voting rules. Some of these rules satisfy some natural properties which we define below:

- *Anonymity:* For all voting profiles $(v_1, ..., v_n) \in \mathcal{J}^n$ and all permutations $(i_1, ..., i_n)$ of the voters, $f(v_{i_1}, ..., v_{i_n}) = f(v_1, ..., v_n)$. Informally, names of voters do not matter.

- *Monotonicity:* For all voting profiles $\mathbf{v}, \mathbf{v}' \in \mathcal{J}^n$, if for each r in $f(\mathbf{v})$ the voters who accept r in $\mathbf{v}$ also accept r in $\mathbf{v}'$, then $f(\mathbf{v}') = f(\mathbf{v})$. Informally, additional votes for the collectively accepted propositions never reverses the collective acceptance of these propositions.

- *Independence:* The decision on each proposition $r \in \{p, q\}$ only depends on the votes on r. So, under independence the group takes two separate votes, one between p and $\bar{p}$ and one between q and $\bar{q}$.

- *Neutrality:* For every voting profile $\mathbf{v}$ and every voting profile $\mathbf{v}'$, if for every voter i the vote v_i contains p if and only if the vote v'_i contains q, then $f(\mathbf{v})$ contains p if and only if $f(\mathbf{v}')$ contains q. Informally, the two issues are treated symmetrically.

While some of the voting rules described in Proposition 1 are anonymous, monotonic and neutral, some of them fail to satisfy any of these properties. The class of anonymous, monotonic and neutral voting rules which make informative voting efficient is characterised in the next theorem by the voting rule described below. For each $\mathbf{v} \in \mathcal{J}^n$,

$$f(\mathbf{v}) = pq \iff n_p^{\mathbf{v}} = n_q^{\mathbf{v}} = n \tag{3}$$

$$f(\mathbf{v}) = p\bar{q} \text{ if } n_p^{\mathbf{v}} > n_q^{\mathbf{v}} \tag{4}$$

$$f(\mathbf{v}) = \bar{p}q \text{ if } n_p^{\mathbf{v}} < n_q^{\mathbf{v}} \tag{5}$$

$$f(\mathbf{v}) \in \{p\bar{q}, \bar{p}q\} \text{ if } n_p^{\mathbf{v}} = n_q^{\mathbf{v}} < n \tag{6}$$

Theorem 4 *Assume consequentialist preferences. A voting rule $f : \mathcal{J}^n \to \mathcal{J}$ which makes informative voting efficient is anonymous, monotonic and neutral if and only if it is the rule defined by (3)-(6).*

Among the efficient aggregation possibilities, anonymity, monotonicity and neutrality can be attained. What about independence? The property of independence characterizes (together with anonymity and monotonicity) quota rules. Under quota rules, separate votes are taken on each proposition using acceptance thresholds. Formally, a quota rule is given by two thresholds $m_p, m_q \in \{0, 1, ..., n+1\}$ with $m_p + m_q \leq n + 1$, and for each voting profile it accepts p $[q]$ if and only if at least m_p $[m_q]$ voters accept it in the profile.[8] Quota rules are monotonic, anonymous and *independent*, but not necessarily neutral.[9] Among all the rules making informative voting efficient, is there a quota rule? The answer is no, following Proposition 1.

Corollary 1 *There exists no quota rule $f : \mathcal{J}^n \to \mathcal{J}$ making informative voting efficient.*

This corollary applies to both kind of preferences as we already have an impossibility in the case of simple preferences. The possibility of efficient information aggregation under consequentialist preferences rely mostly on the indifference of voters between the two judgment sets $p\bar{q}$ and $\bar{p}q$. The judgment sets $p\bar{q}$ and $\bar{p}q$ both lead to the same consequence, so the same utility for each consequentialist voter. Consider a type profile $\mathbf{t}$ with $t_i = p\bar{q}$ for all $i \in \{1, ..., n\}$ and a type profile $\mathbf{t}'$ with $t'_i = p\bar{q}$ for all $i \in \{1, ..., n\}$. Condition 1, the existence condition for efficient information aggregation, would require that the efficient decision for both $\mathbf{t}$ and $\mathbf{t}'$ is the same. This is of course not possible under simple preferences, while it follows under consequentialist preferences. The voting rule which supports truthful voting turns out to be binary in the sense that it simply asks to choose between pq and any of $p\bar{q}$ and $\bar{p}q$. Hence, the issues alone are not important while making a decision, which is not an interesting case. We want to conclude with the following remark which is motivated by our results.

Remark 2 *Informative voting is not efficient almost in all cases with an agenda with mutually interconnected propositions.*

6 Conclusion

We consider a model where a group of voters with common interests wants to form collective judgments over two propositions which are mutually interconnected. Each of these propositions is factually true or false, but the truth value is unknown to voters. Each voter has a type representing evidence about what the true state might

[8]The additional requirement of $m_p + m_q \leq n + 1$ is for leaving out $\bar{p}\bar{q}$ from possible outcomes. This requirement follows from Theorem 2(c) in Dietrich and List (2007c). Alternatively, one could define quota rules as mappings from $\mathcal{J}^n \to \mathcal{J} \cup \{\bar{p}\bar{q}\}$, without the extra requirement. Then, we could say that a quota rule is consistent (never returns $\bar{p}\bar{q}$) if and only if $m_p + m_q \leq n + 1$.

[9]Whenever the acceptance thresholds for propositions are equal, they turn out to be neutral.

be and this is private information. We study the problem of efficient information aggregation when propositions are mutually interconnected. The results depend particularly on how the utility function is specified. It turns out that a voting rule which makes informative voting efficient does not exist under simple preferences while such a rule exists under consequentialist preferences if some condition relating the model parameters and the utility function is satisfied.

We consider two issues, which is the simplest case for multi-issue problems. Such agendas are important in practice as our examples show. We study exhaustively all possible interconnections between two issues (except bi-implication which is a trivial case). Although we do not present formal results concerning more than two issues, we believe that the impossibilities persist since the problem becomes more complex. One has to consider several kinds of interconnections when there are more than two issues in the agenda.

Our results do not come close to the results in single issue setting and multi-issue setting with no interconnections. In the case of a single issue, a quota rule always makes informative voting efficient as shown in Austen-Smith and Banks (1996). In the case of multiple independent issues, there is always a voting rule which makes informative voting efficient and in most cases, this rule is a quota rule as Bozbay, Dietrich and Peters (2011) show. Adding interconnections to the model changes the results drastically. Besides impossibility of a quota rule, in almost all cases there is no voting rule for which informative voting is efficient.

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A Appendix: proofs

We begin by some preliminary derivations and then prove the results

A.1 Preliminary derivations

The joint probability of a state-types vector $(s, \mathbf{t}) = (s_p s_q, t_{1p} t_{1q}, ..., t_{np} t_{nq}) \in \mathcal{J}^{n+1}$ is

$$\Pr(s, \mathbf{t}) = \Pr(s) \Pr(\mathbf{t}|s) = \Pr(s) \prod_i \Pr(t_i|s) = \Pr(s) \prod_i \Pr(t_{ip}|s_p) \Pr(t_{iq}|s_q),$$

where the last two equations follow from independence assumptions. The probability of the three states in $\mathcal{J}$ conditional on the *full* information $\mathbf{t} \in \mathcal{J}^n$ is given as follows, where $k := n_p^{\mathbf{t}}$ and $l := n_q^{\mathbf{t}}$:

$$\Pr(pq|\mathbf{t}) = \frac{\pi_{pq} a_p^k (1 - a_p)^{n-k} a_q^l (1 - a_q)^{n-l}}{\Pr(\mathbf{t})} \tag{7}$$

$$\Pr(p\bar{q}|\mathbf{t}) = \frac{\pi_{p\bar{q}} a_p^k (1 - a_p)^{n-k} (1 - a_{\bar{q}})^l a_{\bar{q}}^{n-l}}{\Pr(\mathbf{t})} \tag{8}$$

$$\Pr(\bar{p}q|\mathbf{t}) = \frac{\pi_{\bar{p}q} (1 - a_{\bar{p}})^k a_{\bar{p}}^{n-k} a_q^l (1 - a_q)^{n-l}}{\Pr(\mathbf{t})}. \tag{9}$$

A.2 Proofs

Proof of Theorem 1. To start with, we introduce some notation. Given a voting profile $\mathbf{v}$, let $\Theta(\mathbf{v})$ denote the set of all type profiles which possibly lead to $\mathbf{v}$ under informative voting. Given a type profile $\mathbf{t}$, let $\Omega(\mathbf{t})$ denote the set of all voting profiles which possibly result from $\mathbf{t}$ under informative voting. Consider a voting rule $f : \mathcal{J}^n \to \mathcal{J}$.

(1) First, let Condition 1 hold. Suppose there is an exogenously given ordering of judgment sets, and let f be the following voting rule: for all $\mathbf{v} \in \mathcal{J}^n$, $f(\mathbf{v}) = d \iff d$ is the highest ordered decision among all decisions which are efficient for some $\mathbf{t} \in \Theta(\mathbf{v})$. Consider any type profile $\hat{\mathbf{t}} \in \mathcal{T}^n$ and suppose informative voting. We want to show that (*) for each $\mathbf{v} \in \Omega(\hat{\mathbf{t}})$, $f(\mathbf{v})$ is efficient for $\hat{\mathbf{t}}$. Let $\mathbf{v} \in \Omega(\hat{\mathbf{t}})$. One can show that all type profiles in $\Theta(\mathbf{v})$ share the same subvector restricted to pq. Since Condition 1 holds, there is some decision d which is efficient for all $\mathbf{t} \in \Theta(\mathbf{v})$, including $\hat{\mathbf{t}}$. It follows from Condition 1 that if any other decision $d' \neq d$ is efficient for *some* $\mathbf{t} \in \Theta(\mathbf{v})$, it is efficient for all $\mathbf{t} \in \Theta(\mathbf{v})$. Then, (*) holds.

(2) Conversely, let f make informative voting efficient. Let $\mathbf{t}, \mathbf{t}'$ be two type profiles in $\mathcal{T}^n$ with $\mathbf{t}_{pq} = \mathbf{t}'_{pq}$. One has to show that (*) there is $d \in \mathcal{J}$ which is efficient for both $\mathbf{t}, \mathbf{t}'$. By construction, for each $\mathbf{v} \in \Omega(\mathbf{t})$, $\mathbf{t}' \in \Theta(\mathbf{v})$; and similarly, for each $\mathbf{v}' \in \Omega(\mathbf{t}')$, $\mathbf{t} \in \Theta(\mathbf{v}')$. Then, $f(\mathbf{v})$ must be efficient for $\mathbf{t}'$ (as well as $\mathbf{t}$) and $f(\mathbf{v}')$ must be efficient for $\mathbf{t}$ (as well as $\mathbf{t}'$) since informative voting is efficient. So, (*) holds. □

Proof of Theorem 2. By Theorem 1, it is sufficient to show that Condition 1 never holds under simple preferences. Suppose for a contradiction, it holds. Consider the two type profiles $\mathbf{t} = (p\bar{q}, ..., p\bar{q})$ and $\mathbf{t}' = (\bar{p}q, ..., \bar{p}q)$. Since $\mathbf{t}_{pq} = \mathbf{t'}_{pq}$ and Condition 1 holds, there is a decision which is efficient for both profiles. Then, $p\bar{q}$ must be efficient for $\mathbf{t}$ since otherwise $p\bar{q}$ wouldn't be efficient for any type profile which contradicts to non-degeneracy assumption. Similarly, $\bar{p}q$ must be efficient for $\mathbf{t}'$. Hence, $p\bar{q}$ and $\bar{p}q$ are both efficient given $\mathbf{t}$ or $\mathbf{t}'$, which contradicts to no-efficiency ties assumption. □

Proof of Theorem 3. Let the condition in Theorem 3 be called Condition 2.

(1) We first prove that (c) implies (a) and (b). Assume Condition 2 holds. This implies that Condition 1 holds. By Theorem 1, there is a voting rule which makes informative voting efficient. Let $\mathbf{t}, \mathbf{t}'$ be type profiles with one $p\bar{q}$ and one $\bar{p}q$ respectively while each of the rest of the types is pq. Without loss of generality, let $\mathbf{t} = (pq, ..., \bar{p}q)$ and $\mathbf{t}' = (pq, ..., p\bar{q})$. By Condition 1, $p\bar{q}, \bar{p}q$ are both efficient for each of the type profiles. Using (7) and (9), we can write the following:

$$E(u(p\bar{q}, S)|\mathbf{t}) > E(u(pq, S)|\mathbf{t}) \tag{10}$$

$$\Leftrightarrow \pi_{p\bar{q}} a_p^{n-1}(1-a_p)(1-a_{\bar{q}})^n + \pi_{\bar{p}q}(1-a_{\bar{p}})^{n-1} a_{\bar{p}} a_q^n > \pi_{pq} a_p^{n-1}(1-a_p) a_q^n \tag{11}$$

$$\Leftrightarrow \pi_{p\bar{q}} \left(\frac{1-a_{\bar{q}}}{a_q}\right)^n + \pi_{\bar{p}q} \left(\frac{1-a_{\bar{p}}}{a_p}\right)^{n-1} \left(\frac{a_{\bar{p}}}{1-a_p}\right) > \pi_{pq}. \tag{12}$$

Similarly,

$$E(u(p\bar{q}, S)|\mathbf{t}') > E(u(pq, S)|\mathbf{t}') \tag{13}$$

$$\Leftrightarrow \pi_{p\bar{q}} a_p^n (1-a_{\bar{q}})^{n-1} a_{\bar{q}} + \pi_{\bar{p}q}(1-a_{\bar{p}})^n a_q^{n-1}(1-a_q) > \pi_{pq} a_p^n a_q^{n-1}(1-a_q) \tag{14}$$

$$\Leftrightarrow \pi_{p\bar{q}} \left(\frac{1-a_{\bar{q}}}{a_q}\right)^{n-1} \left(\frac{a_{\bar{q}}}{1-a_q}\right) + \pi_{\bar{p}q} \left(\frac{1-a_{\bar{p}}}{a_p}\right)^n > \pi_{pq}. \tag{15}$$

So, $A, B > \pi_{pq}$.

(2) We now prove that (a) implies (c). Consider a voting rule $f : \mathcal{J}^n \to \mathcal{J}$ and suppose f makes informative voting efficient. By Theorem 1, Condition 1 holds. Given a type profile $\mathbf{t} \in \mathcal{T}^n$, let $\Gamma(\mathbf{t})$ denote the set of type profiles which have the same subvector on pq as in $\mathbf{t}$. Recall that the number of occurrences for a proposition r in a type profile $\mathbf{t}$ is written $n_r^{\mathbf{t}}$. Now, take a type profile $\hat{\mathbf{t}} \in \mathcal{T}^n$ with k times pq where $1 \le k < n$. The proof proceeds in several steps.

Claim 1: There is a type profile $\mathbf{t} \in \Gamma(\hat{\mathbf{t}})$ with $n_p^{\mathbf{t}} = k$ and $n_q^{\mathbf{t}} = k$.

Any type profile with k times pq and $n-k$ times $\bar{p}\bar{q}$ satisfies this condition and one of these type profiles is obviously in $\Gamma(\hat{\mathbf{t}})$. Now, take $\tilde{\mathbf{t}} \in \mathcal{T}^n$ with $k-1$ times pq.

Claim 2: There is a type profile $\mathbf{t} \in \Gamma(\tilde{\mathbf{t}})$ with $n_p^{\mathbf{t}} = k$ and $n_q^{\mathbf{t}} = k$.

One can easily see there is always a type profile with the exact same pq structure as $\tilde{\mathbf{t}}$ and with only one occurrence of $p\bar{q}$ and only one occurrence of $\bar{p}q$.

Claim 3: Under consequentialist preferences, for all $\mathbf{t}, \mathbf{t}' \in \mathcal{T}^n$ with $n_p^{\mathbf{t}} = n_p^{\mathbf{t}'}$ and $n_q^{\mathbf{t}} = n_q^{\mathbf{t}'}$, $E(u(d, S)|\mathbf{t}) = E(u(d, S)|\mathbf{t}')$ for each $d \in \mathcal{J}$.

The claim follows from the expressions (7)-(9). By Condition 1, there is a decision $d \in \mathcal{J}$ which is efficient for all $\mathbf{t} \in \Gamma(\hat{\mathbf{t}})$. Similarly, there is a decision $d \in \mathcal{J}$ which is efficient for all $\mathbf{t} \in \Gamma(\tilde{\mathbf{t}})$. Combining Claim 1, 2 and 3, one obtains that the same decision $d \in \mathcal{J}$ is efficient for all $\mathbf{t} \in \Gamma(\hat{\mathbf{t}})$ and all $\mathbf{t} \in \Gamma(\tilde{\mathbf{t}})$. Since this is true for all k with $1 \leq k < n$, there is a decision d which is efficient for all $\mathbf{t} \in \mathcal{T}^n \setminus \{(pq, ..., pq)\}$. By non-degeneracy assumption, pq is efficient for $\mathbf{t} = (pq, ..., pq)$. Hence, this decision must be in $\{p\bar{q}, \bar{p}q\}$ since otherwise pq would be efficient for all type profiles which contradicts to non-degeneracy assumption. Hence, Condition 2 holds.

(3) We finally prove that (b) implies (c). Let $A, B > \pi_{pq}$. To show that Condition 2 holds, we first show the following claim.

Claim 4: The expected utility of pq given a type profile $\mathbf{t}$ is an increasing function of $n_p^{\mathbf{t}}$ and $n_q^{\mathbf{t}}$.

The claim follows from the definition of the utility function and from $\Pr(S = pq|\mathbf{t})$ being an increasing function of $n_p^{\mathbf{t}}$ and $n_q^{\mathbf{t}}$. Let $\mathbf{t}, \mathbf{t}' \in \mathcal{T}^n$ be type profiles with one $p\bar{q}$ and one $\bar{p}q$ respectively while each of the rest of the types is pq. Without loss of generality, let $\mathbf{t} = (pq, ..., \bar{p}q)$ and $\mathbf{t}' = (pq, ..., p\bar{q})$. By (7) and (9), one has $E(u(p\bar{q}, S)|\mathbf{t}) > E(u(pq, S)|\mathbf{t})$ and $E(u(p\bar{q}, S)|\mathbf{t}') > E(u(pq, S)|\mathbf{t}')$. By the claim, it follows that $E(u(p\bar{q}, S)|\mathbf{t}) = E(u(\bar{p}q, S)|\mathbf{t}) > E(u(pq, S)|\mathbf{t})$ for all $\mathbf{t} \in \mathcal{T}^n \setminus \{(pq, ..., pq)\}$ which means $p\bar{q}, \bar{p}q$ are efficient for each $\mathbf{t} \in \mathcal{T}^n \setminus \{(pq, ..., pq)\}$. Thus, Condition 2 holds. □

Proof of Proposition 1. Consider a voting rule $f : \mathcal{J}^n \to \mathcal{J}$. Proof if the 'if' part is obvious and left to the reader. To show converse, let f make informative voting efficient. Note that Condition 1 reduces to decision pq being only efficient for the unanimous type profile $\mathbf{t} = (pq, ..., pq)$ under consequentialist preferences by Theorem 3. Then, for all voting profiles obtained by informative voting from any $\mathbf{t} \in \mathcal{T}^n \setminus \{(pq, ..., pq)\}$, $f(\mathbf{v}) \in \{p\bar{q}, \bar{p}q\}$. By non-degeneracy assumption, pq is efficient for $\mathbf{t} = (pq, ..., pq)$. By f making informative voting efficient, $f(\mathbf{v}) = pq$ if $\mathbf{v} = (pq, ..., pq)$. □

Proof of Theorem 4. Consider a voting rule $f : \mathcal{J}^n \to \mathcal{J}$.

(1) First, let f be defined by (3)-(6). Clearly, f is anonymous. It follows from Proposition 1 that informative voting is efficient with f since for all $\mathbf{v} \in \mathcal{J}^n$, $f(\mathbf{v}) = pq$ if and only if $n_p^{\mathbf{v}} = n_q^{\mathbf{v}} = n$; so, if and only if $\mathbf{v} = (pq, ..., pq)$. To show monotonicity of f, take two voting profiles $\mathbf{v}, \mathbf{v}' \in \mathcal{J}^n$ such that for all $r \in f(\mathbf{v})$, the voters who vote for r in $\mathbf{v}$ also vote for r in $\mathbf{v}'$.

Case 1: $f(\mathbf{v}) = pq$. Then $\mathbf{v} = (pq, ..., pq)$. By definition, $\mathbf{v}' = \mathbf{v}$ and $f(\mathbf{v}') = pq$.

Case 2: $f(\mathbf{v}) = p\bar{q}$. The definition of f implies either $n_p^{\mathbf{v}} > n_q^{\mathbf{v}}$ or $n_p^{\mathbf{v}} = n_q^{\mathbf{v}} < n$; and the definition of $\mathbf{v}'$ implies $n_p^{\mathbf{v}'} \geq n_p^{\mathbf{v}}$ and $n_q^{\mathbf{v}'} \leq n_q^{\mathbf{v}}$. Suppose $n_p^{\mathbf{v}} > n_q^{\mathbf{v}}$. Then, $n_p^{\mathbf{v}'} > n_q^{\mathbf{v}'}$ and $f(\mathbf{v}') = p\bar{q}$. Next, suppose $n_p^{\mathbf{v}} = n_q^{\mathbf{v}} < n$. If $\mathbf{v}' \neq \mathbf{v}$, one has $n_p^{\mathbf{v}'} > n_p^{\mathbf{v}}$ or $n_q^{\mathbf{v}'} < n_q^{\mathbf{v}}$ which means $n_p^{\mathbf{v}'} > n_q^{\mathbf{v}'}$ and $f(\mathbf{v}') = p\bar{q}$.

Case 3: $f(\mathbf{v}) = \bar{p}q$. One can show that $f(\mathbf{v}') = \bar{p}q$ analogously to Case 2.

It remains to show neutrality of f. Take two voting profiles $\mathbf{v}, \mathbf{v}' \in \mathcal{J}^n$ such that $\mathbf{v}_r = \mathbf{v}'_{r'}$ for every distinct $r, r' \in \{p, q\}$ and there is no permutation of voters $(i_1, ..., i_n)$ with $(v_{i_1}, ..., v_{i_n}) = (v'_1, ..., v'_n)$. We have to show that (*) f accepts r in $\mathbf{v}$ if and only if f accepts r' in $\mathbf{v}'$. We distinguish 3 cases:

Case 1: $f(\mathbf{v}) = pq$. It is clear that $\mathbf{v}' = \mathbf{v}$, and $f(\mathbf{v}') = pq$.

Case 2: $f(\mathbf{v}) = p\bar{q}$. By definition of f, either $n_p^{\mathbf{v}} > n_q^{\mathbf{v}}$ or $n_p^{\mathbf{v}} = n_q^{\mathbf{v}} < n$. One can see that the latter is not possible since then one could find a permutation of voters $(i_1, ..., i_n)$ with $(v_{i_1}, ..., v_{i_n}) = (v'_1, ..., v'_n)$. Suppose $n_p^{\mathbf{v}} > n_q^{\mathbf{v}}$. By definition of $\mathbf{v}'$, whenever p (q) is accepted in $\mathbf{v}$, q (p) is accepted in $\mathbf{v}'$. This means $n_p^{\mathbf{v}'} < n_q^{\mathbf{v}'}$ and $f(\mathbf{v}') = \bar{p}q$. So, f accepts p in $\mathbf{v}$ and q in $\mathbf{v}'$, and it accepts $\bar{q}$ in $\mathbf{v}$ and $\bar{p}$ in $\mathbf{v}'$. Hence, (*) holds.

Case 3: $f(\mathbf{v}) = \bar{p}q$. One can show that $f(\mathbf{v}') = \bar{p}q$ analogously to Case 2.

(2) Conversely, let f be anonymous, monotonic and neutral, and make informative voting efficient. We have to show that $(*)$ f is defined by (3)-(6). By Proposition 1 and informative voting being efficient, $f(\mathbf{v}) = pq$ if and only if $\mathbf{v} = (pq, ..., pq)$, equivalently $n_p^{\mathbf{v}} = n_q^{\mathbf{v}} = n$. Now, take a voting profile $\mathbf{v} \in \mathcal{J}^n \setminus \{(pq, ..., pq)\}$.

Case 1: $n_p^{\mathbf{v}} > n_q^{\mathbf{v}}$. Suppose for a contradiction, $f(\mathbf{v}) = \bar{p}q$. Let $\mathbf{v}'$ be a voting profile with $n_p^{\mathbf{v}'} = n_q^{\mathbf{v}}$ and $n_q^{\mathbf{v}'} = n_p^{\mathbf{v}}$. We start by proving the following claim.

Claim: For each combination of $k, l \in \{0, ..., n\}$, there is only one voting profile $\mathbf{v} \in \mathcal{J}^n$ with $n_p^{\mathbf{v}} = k$ and $n_p^{\mathbf{v}} = l$ up to the permutations of votes.

The claim follows from the fact that all votes containing $\bar{p}$ are $\bar{p}q$, and similarly, all votes containing $\bar{q}$ are $p\bar{q}$. Hence, subtracting number of p (q) occurrences in a profile from n gives the exact number of $\bar{p}q$ $(p\bar{q})$ votes. Then, there is only one voting profile with $n_p^{\mathbf{v}}$ times q and $n_q^{\mathbf{v}}$ times p up to permutations of votes. Hence, by neutrality and anonymity, $f(\mathbf{v}') = p\bar{q}$. However, by monotonicity of f, $f(\mathbf{v}') = \bar{p}q$ since $n_p^{\mathbf{v}'} \leq n_p^{\mathbf{v}}$ and $n_q^{\mathbf{v}'} \geq n_q^{\mathbf{v}}$, a contradiction. Then, $f(\mathbf{v}) = p\bar{q}$.

Case 2: $n_p^{\mathbf{v}} < n_q^{\mathbf{v}}$. One can show that $f(\mathbf{v}) = \bar{p}q$ analogously to Case 1.

Case 3: $n_p^{\mathbf{v}} = n_q^{\mathbf{v}} < n$. By Proposition 1 and informative voting being efficient, $f(\mathbf{v}) \in \{p\bar{q}, \bar{p}q\}$.

So, $(*)$ is true. $\square$