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Tubes with Darboux Frame

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Abstract

A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ (center curve) of its centers and a radius function $r(t)$ and canal surface is parametrized via Frenet frame of the center curve $C(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or tubular surface. In this study, for a center curve $C(t)$ on arbitrary surface M we define tube with Darboux frame instead of Frenet frame. Subsequently, we compute the curvatures of tube with Darboux frame and obtain some characterizations for special curves on this tube.

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1 Introduction

Canal surface is one kind of the swept surfaces. The class of surfaces formed by sweeping a sphere was first investigated by Monge in 1850. Alternatively, a canal surface is the envelope of a family of one parameter spheres and is useful to represent various objects e.g. pipe, hose, rope or intestine of a body. Again, canal surface is an important instrument in surface modelling for CAD/CAM such as tubular surfaces, torus and Dupin cyclides.

Canal surface around the center curve $C(s)$ is parametrized as

$$K(s,\theta)=C(s)-r(s)r'(s)t(s)\mp r(s)\sqrt{1-r'(s)^2}(\cos\theta n(s)+\sin\theta b(s));0\leq\theta<2\pi,$$

where s is arclenght parameter and t,n,b are Frenet vectors of $C(s)$. If the radius function $r(s)=r$ is a constant, then the canal surface is called tube (pipe) surface and it is parametrized as

$$Tube(s,\theta)=C(s)+r(\cos\theta n(s)+\sin\theta b(s)).$$

Maekawa [3] *et.al.* researched necessary and sufficient conditions for the regularity of tube (tubular) surfaces. Recently, Xu [4] *et.al.* studied these conditions for canal surfaces and also examined principle geometric properties of these surfaces like computing the area and Gaussian curvature.

This work is organized as follows. In section 2 we give some concepts regarding curves and surfaces. Afterwards, we define tube with respect to Darboux frame and compute Gaussian and mean curvatures. In section 3 we obtain some characterizations for special curves lying on tube with Darboux frame.

2 Preliminaries

In this section, we define tube with respect to Darboux frame. After that, we compute the coefficients of first and second fundamental form, Gaussian and mean curvatures for this tube, respectively.

Let M be a regular surface and $\alpha: I\subset\mathbb{R}\longrightarrow M$ be a unit speed curve on the surface. Then, Darboux frame $\{T,\ Y=N\times T,\ N\}$ is well-defined along the curve α where T is the tangent of α and N is the unit normal of M. Darboux equations for this frame are given by

$$\begin{aligned}T' &= k_gY+k_nN\\ Y' &= -k_gT+\tau_gN\\ N' &= -k_nT-\tau_gY,\end{aligned}$$

where k_n is the normal curvature, k_g is the geodesic curvature and τ_g is the geodesic torsion of α.

Let the center curve $C(s)$ be on the surface M. Since the characteristic circles of canal surface lie in the plane which is perpendicular to the tangent of center curve $C(s)$, we can write tube with Darboux frame as

$$D(s,\beta)=C(s)+r\left(\cos\beta Y(s)+\sin\beta U(s)\right),$$

where U is the unit normal of the surface M along the curve $C(s)$. Then from the derivative formulas of Darboux frame, we have

$$\begin{aligned}
D_s &= T + r\left[\cos\beta Y'(s) + \sin\beta U'(s)\right] \\
&= [1 - r\cos\beta k_g - r\sin\beta k_n]\,T - r\sin\beta\tau_g Y + r\cos\beta\tau_g U \\
D_\beta &= -r\sin\beta Y + r\cos\beta U \\
D_{ss} &= \left[-r\cos\beta k_g' - r\sin\beta k_n' + r\sin\beta k_g\tau_g - r\cos\beta k_n\tau_g\right] T \\
&+ \left[k_g - r\sin\beta k_n k_g - r\cos\beta k_g^2 - r\cos\beta\tau_g^2 - r\sin\beta\tau_g'\right] Y \\
&+ \left[k_n - r\cos\beta k_n k_g - r\sin\beta k_n^2 - r\sin\beta\tau_g^2 + r\cos\beta\tau_g'\right] U \\
D_{\beta s} &= [r\sin\beta k_g - r\cos\beta k_n]\,T - r\cos\beta\tau_g Y - r\sin\beta\tau_g U \\
D_{\beta\beta} &= -r\cos\beta Y - r\sin\beta U
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
D_s \times D_\beta &= \left[-r\cos\beta + r^2\sin\beta\cos\beta k_n + r^2\cos^2\beta k_g\right] Y \\
&\quad + \left[-r\sin\beta + r^2\sin\beta\cos\beta k_g + r^2\sin^2\beta k_n\right] U \\
\|D_s \times D_\beta\| &= r\left(1 - r\cos\beta k_g - r\sin\beta k_n\right).
\end{aligned}$$

Thus, the unit normal N, the coefficients of first and second fundamental form, the Gaussian and mean curvature of $D(s,\beta)$ are obtained as follows.

$$\begin{aligned}
N &= \frac{D_s \times D_\beta}{\|D_s \times D_\beta\|} = -\cos\beta Y - \sin\beta U \\
E &= D_s \cdot D_s = (1 - r\cos\beta k_g - r\sin\beta k_n)^2 + r^2\tau_g^2 \\
F &= D_s \cdot D_\beta = r^2\tau_g \\
G &= D_\beta \cdot D_\beta = r^2 \\
e &= N \cdot D_{ss} = (k_g\cos\beta + k_n\sin\beta)\left[r(k_g\cos\beta + k_n\sin\beta) - 1\right] + r\tau_g^2 \\
f &= N \cdot D_{\beta s} = r\cos^2\beta\tau_g + r\sin^2\beta\tau_g = r\tau_g \\
g &= N \cdot D_{\beta\beta} = r\cos^2\beta + r\sin^2\beta = r
\end{aligned} \tag{2.2}$$

$$\mathbb{K} = \frac{eg - f^2}{EG - F^2} = \frac{k_g\cos\beta + k_n\sin\beta}{r\left(1 - r\cos\beta k_g - r\sin\beta k_n\right)}$$

$$\mathbb{H} = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{2r\left(k_g\cos\beta + k_n\sin\beta\right) - 1}{2r\left(rk_g\cos\beta + rk_n\sin\beta - 1\right)}, \tag{2.3}$$

where k_g, k_n and τ_g are the geodesic curvature, normal curvature and geodesic torsion of $C(s)$, respectively.

3 Some Characterizations of Special Curves on $D(s,\beta)$

In this section, we investigate the relation between parameter curves and special curves such as geodesics, asymptotic curves and lines of curvature on the tube $D(s,\beta)$.

Theorem 3.1. *For the regular tube $D(s,\beta)$,*
(1) $\beta-$ parameter curves are also geodesics.
(2) $s-$ parameter curves are also geodesics $\Longleftrightarrow$ k_g, k_n and τ_g of $C(s)$ satisfy the equation system

$$\begin{aligned}\sin\beta k_g-\cos\beta k_n+r\cos 2\beta k_nk_g+\frac{1}{2}r\sin 2\beta(k_n^2-k_g^2)-r\tau_g^{'}= &\quad 0\\ \cos\beta(k_g^{'}+k_n\tau_g)+\sin\beta(k_n^{'}-k_g\tau_g)= &\quad 0.\end{aligned}\tag{3.1}$$

Proof. For $s-$ and $\beta-$ parameter curves, we get

$$\begin{aligned}N\times D_{\beta\beta} &= r\sin\beta\cos\beta T-r\sin\beta\cos\beta T=0\\ N\times D_{ss} &= (\sin\beta k_g-\cos\beta k_n+r\cos 2\beta k_nk_g+\frac{1}{2}r\sin 2\beta(k_n^2-k_g^2)-r\tau_g^{'})T\\ &\quad +r(\sin\beta\cos\beta k_g^{'}+\sin^2\beta k_n^{'}-\sin^2\beta k_g\tau_g+\sin\beta\cos\beta k_n\tau_g)Y\\ &\quad -r(\cos^2\beta k_g^{'}+\sin\beta\cos\beta k_n^{'}-\sin\beta\cos\beta k_g\tau_g+\cos^2\beta k_n\tau_g)U.\end{aligned}$$

(1) Since $N\times D_{\beta\beta}=0$, $\beta-$ parameter curves are also geodesics.
(2) Because T,Y and U are linearly independent, $N\times D_{ss}=0\Longleftrightarrow$

$$\begin{aligned}\sin\beta k_g-\cos\beta k_n+r\cos 2\beta k_nk_g+\frac{1}{2}r\sin 2\beta(k_n^2-k_g^2)-r\tau_g^{'} &= 0\\ r(\sin\beta\cos\beta k_g^{'}+\sin^2\beta k_n^{'}-\sin^2\beta k_g\tau_g+\sin\beta\cos\beta k_n\tau_g) &= 0\\ r(\cos^2\beta k_g^{'}+\sin\beta\cos\beta k_n^{'}-\sin\beta\cos\beta k_g\tau_g+\cos^2\beta k_n\tau_g) &= 0.\end{aligned}$$

By the last two equations, we have

$$\cos\beta(k_g^{'}+k_n\tau_g)+\sin\beta(k_n^{'}-k_g\tau_g)=0.$$

Then k_g, k_n and τ_g hold the equation system

$$\begin{aligned}\sin\beta k_g-\cos\beta k_n+r\cos 2\beta k_nk_g+\frac{1}{2}r\sin 2\beta(k_n^2-k_g^2)-r\tau_g^{'} &= 0\\ \cos\beta(k_g^{'}+k_n\tau_g)+\sin\beta(k_n^{'}-k_g\tau_g) &= 0.\end{aligned}$$

□

Corollary 3.2. *Let $C(s)$ be a geodesic on M. If $s-$ parameter curves are also geodesics on $D(s,\beta)$, then the curvatures κ and τ of $C(s)$ satisfy the equation*

$$r\sin^2\beta\kappa^2 - 2\sin\beta\kappa + r\tau^2 = c,$$

where c is a constant.

Proof. Since the center curve $C(s)$ is a geodesic, $k_g = 0$. If we replace $k_g = 0$ in Eq (3.1), we obtain

$$\begin{aligned} \kappa\cos\beta(1 - r\kappa\sin\beta) + r\tau^{'} &= 0 \\ \kappa^{'}\sin\beta + \kappa\tau\cos\beta &= 0. \end{aligned}$$

In the first equation above, if we leave alone $\kappa\cos\beta$ and substitute this in the second equation we get

$$\kappa^{'}\sin\beta - r\kappa\kappa^{'}\sin^2\beta - r\tau\tau^{'} = 0.$$

If we integrate the last equation, it follows that

$$r\sin^2\beta\kappa^2 - 2\sin\beta\kappa + r\tau^2 = c.$$

□

Corollary 3.3. *Let the center curve $C(s)$ be an asymptotic curve on M. If $s-$ parameter curves are also asymptotic curves on $D(s,\beta)$, then the curvatures κ and τ of $C(s)$ satisfy the equation*

$$r\cos^2\beta\kappa^2 - 2\cos\beta\kappa + r\tau^2 = c,$$

where c is a constant.

Proof. Since the center curve $C(s)$ is an asymptotic curve, $k_n = 0$. If we replace $k_n = 0$ in Eq (3.1), we obtain

$$\begin{aligned} \kappa\sin\beta(1 - r\kappa\cos\beta) - r\tau^{'} &= 0 \\ \kappa^{'}\cos\beta - \kappa\tau\sin\beta &= 0. \end{aligned}$$

If we leave alone $\kappa\sin\beta$ and substitute this in the second equation we get

$$\kappa^{'}\cos\beta - r\kappa\kappa^{'}\cos^2\beta - r\tau\tau^{'} = 0.$$

and then if we integrate the last equation, it concludes that

$$r\cos^2\beta\kappa^2 - 2\cos\beta\kappa + r\tau^2 = c.$$

□

Theorem 3.4. *For the regular tube $D(s,\beta)$,*
(1) $\beta-$ parameter curves cannot also be asymptotic curves.
(2) $s-$ parameter curves are also asymptotic curves $\Longleftrightarrow$ $D(s,\beta)$ is generated by a moving sphere with the radius function

$$r=\frac{k_g\cos\beta+k_n\sin\beta}{(k_g\cos\beta+k_n\sin\beta)^2+\tau_g^2}=c \tag{3.2}$$

such that r is a constant.

Proof. (1) As $N\centerdot D_{\beta\beta}=r\cos^2\beta+r\sin^2\beta=r\neq 0$, $\beta-$ parameter curves cannot also be asymptotic curves.
(2) $s-$ parameter curves are also asymptotic curves $\Longleftrightarrow$

$$N\centerdot D_{ss}=(k_g\cos\beta+k_n\sin\beta)\left[r(k_g\cos\beta+k_n\sin\beta)-1\right]+r\tau_g^2=0.$$

From this, we get the radius function

$$r=\frac{k_g\cos\beta+k_n\sin\beta}{(k_g\cos\beta+k_n\sin\beta)^2+\tau_g^2}=c$$

such that r is a constant. □

Corollary 3.5. *Let $s-$ parameter curves be also asymptotic curves on $D(s,\beta)$.*
(1) If the center curve $C(s)$ is a geodesic on M, then

$$r=\frac{\kappa\sin\beta}{\kappa^2\sin^2\beta+\tau^2}=c.$$

(2) If the center curve $C(s)$ is an asymptotic curve on M, then

$$r=\frac{\kappa\cos\beta}{\kappa^2\cos^2\beta+\tau^2}=c.$$

(3) If the center curve $C(s)$ is a line of curvature on M, then

$$r=\frac{1}{k_g\cos\beta+k_n\sin\beta}=c.$$

Proof. Because $s-$ parameter curves are also asymptotic curves, from Eq (3.2)

$$r=\frac{k_g\cos\beta+k_n\sin\beta}{(k_g\cos\beta+k_n\sin\beta)^2+\tau_g^2}=c.$$

(1) Since $C(s)$ is a geodesic, $k_g=0$. So, $k_n=\kappa$ and $\tau_g=\tau$. If we replace these in Eq (3.2) we get

$$r=\frac{\kappa\sin\beta}{\kappa^2\sin^2\beta+\tau^2}=c.$$

Let us give an example. For $\beta = \dfrac{\pi}{2}$, it follows that $r = \dfrac{\kappa}{\kappa^2 + \tau^2}$ is a constant and so the center curve $C(s)$ becomes a Mannheim curve. In that case, when $C(s)$ is a Mannheim curve, the $s-$ parameter curve $\beta = \dfrac{\pi}{2}$; $D(s, \dfrac{\pi}{2}) = C(s) + rU(s)$ is an asymptotic curve on $D(s, \beta)$.
(2) Since $C(s)$ is a asymptotic curve, $k_n = 0$. Hence, $k_g = \kappa$ and $\tau_g = \tau$. If we replace these in Eq (3.2) we get

$$r = \frac{\kappa \cos \beta}{\kappa^2 \cos^2 \beta + \tau^2} = c.$$

Again, for $\beta = 0$, it follows that $r = \dfrac{\kappa}{\kappa^2 + \tau^2}$ is a constant and therefore $C(s)$ becomes a Mannheim curve. In this situation, while $C(s)$ is a Mannheim curve the $s-$ parameter curve $\beta = 0$; $D(s, 0) = C(s) + rY(s)$ is an asymptotic curve on $D(s, \beta)$.
(3) Since $C(s)$ is a line of curvature $\tau_g = 0$. If we put this in Eq (3.2) we get

$$r = \frac{1}{k_g \cos \beta + k_n \sin \beta} = c.$$

$\square$

Theorem 3.6. *The parameter curves of $D(s, \beta)$ are also lines of curvature $\Longleftrightarrow$ The center curve $C(s)$ is a line of curvature on M.*

Proof. From Eq (2.2) we have

$$\begin{aligned} F &= r^2 \tau_g \\ f &= r \tau_g. \end{aligned}$$

According to theorem of line of curvature, the parameter curves on a surface are also lines of curvature if and only if $F = f = 0$. From $F = f = 0$, it concludes that $\tau_g = 0$, i.e , $C(s)$ is a line of curvature on M. $\square$

Theorem 3.7. *For the regular tube $D(s, \beta)$,*
(1) If the center curve $C(s)$ is a geodesic on M, then the Gaussian and mean curvature of $D(s, \beta)$ are as follows.

$$\begin{aligned} \mathbb{K} &= \frac{\kappa \sin \beta}{r\left(1 - r\kappa \sin \beta\right)} \\ \mathbb{H} &= \frac{2r\kappa \sin \beta - 1}{2r\left(r\kappa \sin \beta - 1\right)}. \end{aligned}$$

(2) If the center curve $C(s)$ is an asymptotic curve on M, then the Gaussian and mean curvature of $D(s, \beta)$ are as follows.

$$\begin{aligned} \mathbb{K} &= \frac{\kappa \cos \beta}{r\left(1 - r\kappa \cos \beta\right)} \\ \mathbb{H} &= \frac{2r\kappa \cos \beta - 1}{2r\left(r\kappa \cos \beta - 1\right)}. \end{aligned}$$

Proof. By Eq (2.2) and Eq (2.3), the Gaussian and mean curvature for $D(s,\beta)$ are

$$\mathbb{K} = \frac{k_g \cos\beta + k_n \sin\beta}{r\left(1 - r\cos\beta k_g - r\sin\beta k_n\right)}$$
$$\mathbb{H} = \frac{2r\left(k_g \cos\beta + k_n \sin\beta\right) - 1}{2r\left(rk_g \cos\beta + rk_n \sin\beta - 1\right)}$$

respectively.
(1) Because $C(s)$ is a geodesic, $k_g = 0$, $k_n = \kappa$ and $\tau_g = \tau$. If we substitute these above, it gathers that

$$\mathbb{K} = \frac{\kappa\sin\beta}{r\left(1 - r\kappa\sin\beta\right)}$$
$$\mathbb{H} = \frac{2r\kappa\sin\beta - 1}{2r\left(r\kappa\sin\beta - 1\right)}.$$

(2) Since $C(s)$ is an asymptotic curve, $k_n = 0$, $k_g = \kappa$ and $\tau_g = \tau$. Then, it gathers that

$$\mathbb{K} = \frac{\kappa\cos\beta}{r\left(1 - r\kappa\cos\beta\right)}$$
$$\mathbb{H} = \frac{2r\kappa\cos\beta - 1}{2r\left(r\kappa\cos\beta - 1\right)}.$$

Here, when $C(s)$ is an asymptotic curve, the Gaussian and mean curvatures of $D(s,\beta)$ are equal with

$$Tube(s,\theta) = C(s) + r(\cos\theta n(s) + \sin\theta b(s)).$$

(see [1]). □

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