

Turbulence and Multiscaling in the Randomly Forced Navier-Stokes Equation

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We present a pseudospectral study of the randomly forced Navier-Stokes equation (RFNSE) stirred by a stochastic force with zero mean and a variance $\sim k^{4-d-y}$, with k the wave vector and the dimension $d = 3$. We provide the first evidence for multiscaling of velocity structure functions for $y \geq 4$. We extract the multiscaling exponent ratios ζ_p/ζ_2 by using extended self-similarity, examine their dependence on y , and show that, if $y = 4$, they are in agreement with those obtained for the Navier-Stokes equation forced at large spatial scales (3DNSE). Also well-defined vortex filaments, which appear clearly in studies of the 3DNSE, are absent in the RFNSE. [S0031-9007(98)07657-1]

Kolmogorov's classic work (K41) on homogeneous, isotropic fluid turbulence focused on the scaling behavior of velocity \mathbf{v} structure functions $S_p(r) = \langle |\mathbf{v}_i(\mathbf{x} + \mathbf{r}) - \mathbf{v}_i(\mathbf{x})|^p \rangle$, where the angular brackets denote an average over the statistical steady state [1]. He suggested that, for separations $r \equiv |\mathbf{r}|$ in the *inertial range*, which is substantial at large Reynolds numbers Re and lies between the forcing scale L and the dissipation scale η_d , these structure functions scale as $S_p \sim r^{\zeta_p}$, with $\zeta_p = p/3$. Subsequent experiments [2] have suggested instead that multiscaling obtains with $p/3 > \zeta_p$, which turns out to be a nonlinear, monotonically increasing function of p ; this has also been borne out by numerical studies of the three-dimensional Navier-Stokes equation forced at large spatial scales (3DNSE) [2,3]. The determination of the exponents ζ_p has been one of the central, but elusive, goals of the theory of turbulence. One of the promising starting points for such a theory is the randomly forced Navier-Stokes equation (RFNSE) [4–6], driven by a Gaussian random force whose spatial Fourier transform $\mathbf{f}(\mathbf{k}, t)$ has zero mean and a covariance $\langle \mathbf{f}_i(\mathbf{k}, t) \mathbf{f}_j(\mathbf{k}', t') \rangle = Ak^{4-d-y} P_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta(t - t')$; here \mathbf{k}, \mathbf{k}' are wave numbers, t, t' times, i, j Cartesian components in d dimensions, and $P_{ij}(\mathbf{k})$ the transverse projector which enforces the incompressibility condition. One-loop renormalization-group (RG) studies of this RFNSE yield [4,5] a K41 energy spectrum, namely, $E(k) \sim k^2 S_2(k) \equiv k^2 \langle |\mathbf{v}(\mathbf{k})|^2 \rangle \sim k^{-5/3}$, if we set $d = 3$ and $y = 4$; this has also been verified numerically [6]. Nevertheless, these RG studies have been criticized for a variety of reasons [7,8] such as using a large value for y in a small- y expansion and neglecting an infinity of marginal operators (if $y = 4$). These criticisms of the *approximations* used in these studies might well be justified, but they clearly cannot be used to argue that the RFNSE is *in itself* inappropriate for a theory of turbulence. It is our purpose here to test whether structure functions in the RFNSE display the same multiscaling as in the 3DNSE for some value of y ; if so, then the RFNSE can, defensibly,

be used to develop a statistical theory of inertial-range multiscaling in homogeneous, isotropic fluid turbulence.

We have carried out an extensive pseudospectral study of the RFNSE and compared our results with earlier numerical studies [3,9] of the 3DNSE and experiments [2]. We find several interesting and new results: We show that structure functions in the RFNSE display multiscaling for $y \geq 4$. We obtain ζ_2 from $S_2(k)$ (Fig. 1) and the exponent ratios ζ_p/ζ_2 by using the extended-self-similarity (ESS) procedure (Fig. 2a) [9,10]. We find that ζ_p/ζ_2 is close to the 3DNSE result (Fig. 2b) for $y = 4$ at least for $p \leq 7$. Furthermore we show that the qualitative behaviors of the probability distributions $P(\delta v_\alpha(r))$, where $\delta v_\alpha(r) \equiv v_\alpha(\mathbf{x}) - v_\alpha(\mathbf{x} + \mathbf{r})$, are similar in the two models (Fig. 2c), but the shapes of constant- $|\omega|$

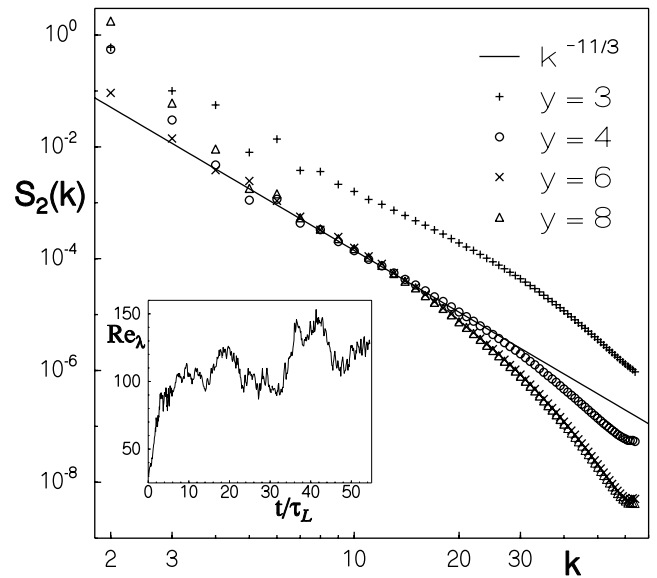


FIG. 1. Log-log plots (base 10) of $S_2(k)$ versus k for different values of y . The line indicates the K41 result $S_2(k) \sim k^{-11/3}$. \mathbf{k} indicates the shell number, which is twice the wave number ($= \frac{2\pi}{L} n$). The inset shows a representative plot of Re_λ versus time (t) for $y = 4$.

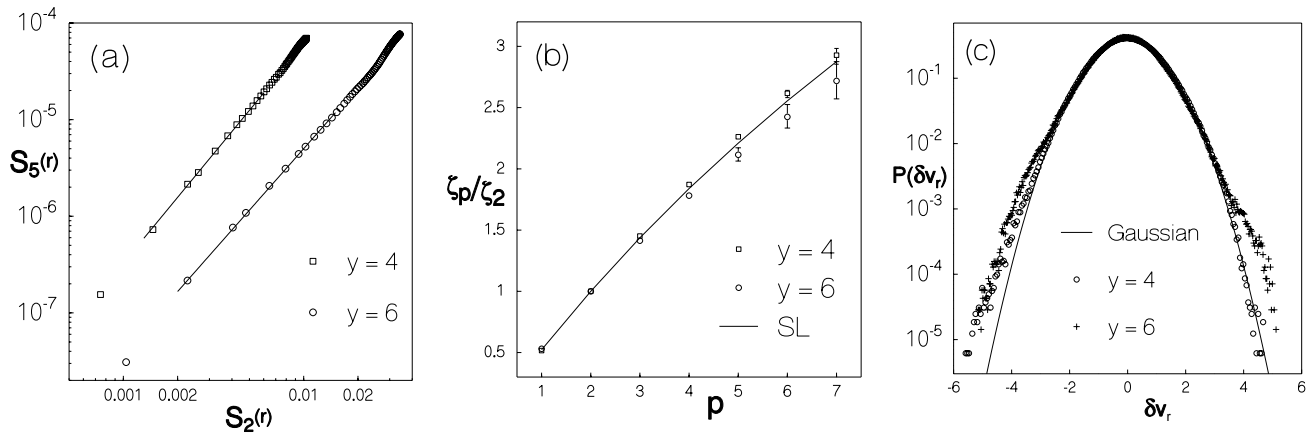


FIG. 2. (a) Log-log plots (base 10) of $S_5(r)$ versus $S_2(r)$ illustrating the ESS procedure; full lines indicate fits to points in the extended inertial range; (b) inertial-range exponent ratios ζ_p/ζ_2 versus p for the RFNSE with $y = 4$ and 6 [extracted from plots such as (a)]; the line indicates the SL formula; (c) semilog plots of the distribution $P(\delta v_r)$ [i.e., $P(\delta v_\alpha(r))$] averaged over α for r in the dissipation range and $y = 4$ and 6 ; a Gaussian distribution is shown for comparison.

surfaces, where ω is the vorticity, are markedly different (Fig. 3); the stochastic force destroys well-defined filamentary structures that obtain in 3DNSE studies. This has implications for the She-Leveque (SL) [11] formula for ζ_p as we discuss below.

We use a pseudospectral method [12] to solve the RFNSE numerically on a 64^3 grid with a cubic box of linear size $L = 2\pi$ and periodic boundary conditions; we have checked in representative cases that our results are unchanged if we use an 80^3 grid or aliasing. Aside from the stochastic forcing [13], our numerical scheme is the same as in Ref. [9]. Our dissipation term, $(\nu + \nu_H k^2)k^2 \mathbf{v}(\mathbf{k})$ in wave-vector (k) space, includes both the viscosity ν and the hyperviscosity ν_H ; the exponents ζ_p are unaffected by ν_H if $\nu > 0$ [9,14]. For a fixed grid size we can attain higher Taylor-microscale Reynolds numbers Re_λ in the RFNSE, and hence a larger inertial range, than in the 3DNSE ($\text{Re}_\lambda \approx 120$ compared to $\text{Re}_\lambda \approx 22$ in our study), as noted earlier [6] for $y = 4$. This advantage is reduced somewhat by the need to aver-

age statistical observables longer in the RFNSE than in the 3DNSE. In the latter case it normally suffices to average over a few box-size eddy turnover times τ_L ; this is not enough for the RFNSE since (a) Re_λ fluctuates strongly over time scales considerably larger than τ_L (inset in Fig. 1) and (b) the length of the $\mathbf{f}(\mathbf{k}, t)$ time series required to obtain a specified variance for the stochastic force is quite large ($\approx 6\tau_L$ to achieve the given variance within 1%–2%). We have collected data for averages over $(25\text{--}33)\tau_L$ (for different values of y), after initial transients have been allowed to decay [over times $\approx (10\text{--}20)\tau_L$]. Our $\tau_L \approx 10\tau_I$, the integral-scale time used in some studies [12]; $\tau_I \equiv L_I/\nu_{\text{rms}}$, where the integral scale $L_I \equiv [\int dk k E(k)/\int dk E(k)]^{-1}$ and ν_{rms} is the root-mean-square velocity. We have checked explicitly that the RFNSE captures the hierarchy of time scales present in the 3DNSE. In spite of the delta-correlated stochastic force in the RFNSE, the variation of $\mathbf{v}(\mathbf{k})$ as a function of time is similar in both the RFNSE and the 3DNSE: There is a hierarchy of time scales which increase with decreasing $k \equiv |\mathbf{k}|$. In the RFNSE, the stochastic force puts a high-frequency ripple on $\mathbf{v}(\mathbf{k})$ even for small k , but this does not affect its overall variation significantly, nor does it affect the multiscaling exponent ratios if $y = 4$, as we show below.

We begin by investigating the inertial-range scaling of the k -space structure function $S_2(k) \sim k^{-\zeta'_2}$. Given this power-law form, the exponent ζ'_2 is easily related to the r -space exponent ζ_2 by $\zeta_2 = \zeta'_2 - 3$. Our data in Fig. 1 for $4 \leq y$ are consistent with $\zeta'_2 = 11/3$ [i.e., the K41 value since $E(k) \sim k^2 S_2(k) \sim k^{-5/3}$]. For $y = 4$ this result has been reported earlier [6]. The y independence of ζ'_2 above some critical y_c [our data for $S_2(k)$ suggest $y_c \approx 4$] is theoretically satisfying since the variance of the stochastic force in the RFNSE rises rapidly at small k , so we might expect that, for sufficiently large y , it approximates the conventional forcing of the 3DNSE at large spatial scales. This has been explored in the $N \rightarrow \infty$

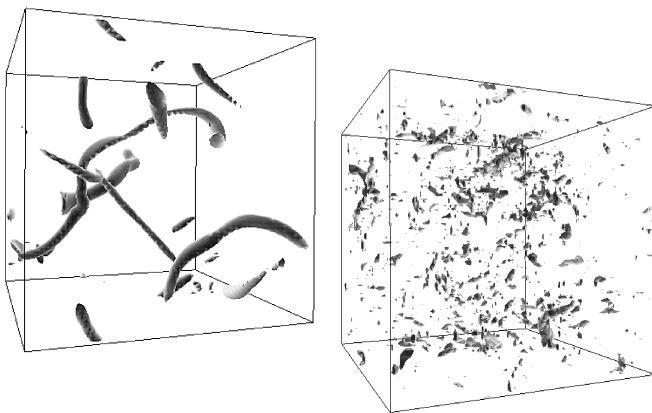


FIG. 3. Iso- $|\omega|$ surfaces obtained from instantaneous snapshots of the vorticity fields showing filaments for the 3DNSE (left) and no filaments for the RFNSE with $y = 4$ (right).

limit of an N -component RFNSE [7]. This study suggests $\zeta_2' = 7/2$ for $y \geq y_c = 4$; given our error bars (Table I) it is difficult to distinguish this from the $O(y)$ RG prediction $\zeta_2' = 11/3$ though our data are closer to the latter. For $0 < y \leq 3$ both the one-loop RG [5] and the $N \rightarrow \infty$ theory [7] predict $\zeta_2' \sim 1 + 2y/3 + O(y^2)$, in fair agreement with our numerical results, especially for small y (Table I). Note that, for $0 < y < 4$, there is no *invariant* energy cascade as in conventional K41: The dominance of dissipation at large k does lead to an energy cascade, but the energy flux depends on the length scale r ; specifically $\Pi(r) \approx Ar^{y-4}$, with A the scale-independent part of the variance of the stochastic force. A K41-type argument [15] now yields an energy-transfer rate $\sim \langle \delta v_r^3 \rangle / r \sim r^{(y-4)}$, whence $S_3(r) \sim r^{(y-3)}$ and, if we assume simple scaling as in K41, $S_2(r) \sim r^{(y-3)2/3}$, i.e., $\zeta_2' = 1 + 2y/3$, as in the $O(y)$ RG prediction. This formula breaks down for $y < 0$; however, the RG predicts correctly that the linear-hydrodynamics result obtains in this regime.

We ensure that systematic errors do not affect ζ_2' as follows. If k_{\max} is the largest wave-vector magnitude in our numerical scheme, we find that $L_I k_{\max}$ decreases with decreasing y ; this shortens the inertial range which can be used to obtain ζ_2' . The lower the value of y the more difficult it is to obtain a dissipation range free of finite-resolution errors. For $y < 4$, we define $k_d \equiv \eta_d^{-1}$ to be the inverse length scale at which the energy-transfer time $t_r \sim (r/\nu_r) \sim [Ar^{(y-6)}]^{1/3}$ equals the diffusion time $t_D \sim [\nu k^2 + \nu_H k^4]^{-1}$; this yields $\nu_0 k_d^2 + \nu_H k_d^4 = [Ak_d^{6-y}]^{1/3}$, which when solved numerically shows that, for fixed A , k_d increases as y decreases (in Table I A is not fixed). Statistical steady states, with ill-resolved dissipation ranges that do not have a decaying tail [9], can be obtained by adjusting A . In such cases $k_d \gg k_{\max}$ and we get spurious results for ζ_2' . We find that, if we increase the hyperviscosity ν_H , k_d is sufficiently close to k_{\max} so that we can resolve both inertial and dissipation ranges and obtain reliable values for ζ_2' . Table I shows the range over which we fit our data for $S_2(k)$. Since our data for ζ_2' indicate that $y_c \approx 4$, we investigate multiscaling only for $y \geq 4$.

Our data for ζ_2' in Table I suggest that naive estimates for the ζ_p require longer inertial ranges than are available in our studies. However, we find that, as in the 3DNSE, the extended-self-similarity procedure [3,9,10] can be used fruitfully here to extract the exponent ratios

ζ_p/ζ_q from the slopes of log-log plots of $S_p(r)$ versus $S_q(r)$ (see Fig. 2) since this extends the apparent inertial range. We compare the resulting ζ_p/ζ_2 in Fig. 2b with the She-Leveque formula [11], which provides a convenient parametrization for the experimental values for ζ_p . Figure 2b shows that, with $y = 4$, our RFNSE exponent ratios lie very close to those for the 3DNSE and, to this extent, these two models are in the same *universality class*. We obtained ζ_p/ζ_2 by a regression fit. We have also checked that a local-slope analysis of ESS plots like Fig. 2a yields exponent ratios nearly indistinguishable from those shown in Fig. 2b. The error bars in Fig. 2b give a rough estimate of the systematic error associated with the choice of the precise range of points which fall in the extended inertial range; they were obtained by varying the number of points used in our regression fits. The exponent ratios for $y < 4$ lie away from the 3DNSE values. One might expect naively that, at very large values of y , the inertial-range behaviors of structure functions of all orders should be the same as in the 3DNSE. However, strictly speaking, this is not obvious *a priori*, neither from renormalization-group calculations [4,5,8] nor from $N \rightarrow \infty$ calculations [7]. The former are not very helpful for large y since an infinity of marginal operators appears at $y = 4$; all these become relevant for $y > 4$. The $N \rightarrow \infty$ studies have been restricted to $p = 2$. For $p > 3$, our data for $\zeta_p/\zeta_2(y = 6)$ fall systematically below those for $\zeta_p/\zeta_2(y = 4)$ or the SL line. Also the probability distributions of $P(\delta v_r)$ (Fig. 2c) have non-Gaussian tails for r in the dissipation range, and for $y > 4$ the deviations from a Gaussian distribution increase systematically with y . Thus, at the resolution of our calculation, the RFNSEs with $y = 4$ and $y = 6$ are in different universality classes. However, we point out that our data for $y = 6$ are more noisy and yield a smaller inertial range ($k_d \approx 20$) than those for $y = 4$ (Table I). So longer runs with finer grids might well be required to settle this issue conclusively.

Strictly speaking the RFNSE with $y = 4$ falls in the same universality class as the 3DNSE only in the ESS sense. For arbitrary y the energy flux through the k th shell is $\Pi_k \equiv \Pi(r = k^{-1}) \sim \int_{1/L}^k \langle |\mathbf{f}(\mathbf{k})|^2 \rangle d^3k$, where r is in the inertial range and we have used Novikov's theorem [15], i.e., $\langle \mathbf{f}(\mathbf{k}) \cdot \mathbf{v}(-\mathbf{k}) \rangle \sim \langle |\mathbf{f}(\mathbf{k})|^2 \rangle$. For $y > 4$, Π_k saturates to a constant for $kL \gg 1$, but for $y = 4$, $\Pi_k \sim \log(kL)$ in the RFNSE [16]. This is to be

TABLE I. The dissipation-scale wave number k_d (see text), the integral-scale wave number $k_I \equiv L_I^{-1}$, the apparent inertial range over which we fit our data for $S_2(k)$, the hyperviscosities ν_H , the exponent ζ_2' that we compute, and its $O(y)$ RG value, for $1 \leq y \leq 4$. The viscosity ν is 5×10^{-4} in all these runs which use a 64^3 grid.

y	k_d	k_I	Fitting range	ν_H	ζ_2' (This study)	ζ_2' from $O(y)$ RG
4	49.0	1.16	$(0.1-0.5)k_d$	10^{-6}	3.6 ± 0.1	≈ 3.67
3	38.7	1.90	$(0.16-0.52)k_d$	3×10^{-6}	3.0 ± 0.1	≈ 3
2	35.0	5.90	$(0.17-0.63)k_d$	8×10^{-6}	2.3 ± 0.1	≈ 2.33
1	35.4	10.3	$(0.2-0.7)k_d$	8×10^{-6}	1.6 ± 0.15	≈ 1.67

contrasted with the 3DNSE where $\Pi_k = \text{const}$. Thus the inertial-range behaviors of all correlation functions in the two models are not the same. A K41-type dimensional analysis suggests that for $y = 4$ the energy flux $\Pi_k \sim \langle \delta v_r^3 \rangle / r \sim \log(r/L)$; if we assume that there is no multiscaling, then $S_p(r) \sim [r \log(r/L)]^{p/3}$. Multiscaling will clearly modify this simple prediction, but some weak deviation from the von Karman–Howarth form $S_3(r) \sim r$ must remain, since the standard derivation of this relation [15] does not go through [17] with the RFNSE result for Π_k . Since our data show that the ESS procedure works for the RFNSE, these weak deviations must cancel in the ratios of structure functions, and, as noted above, for $y = 4$ the ζ_p/ζ_2 agree with the SL result for the 3DNSE.

Filamentary structures (Fig. 3) [18] in iso- $|\omega|$ plots are important in phenomenological models for multiscaling in fluid turbulence. For example, the SL formula [11] is obtained by postulating a hierarchical relation among the moments of the scale-dependent energy dissipation; this yields a difference equation for the exponents τ_p , which are simply related to the exponents ζ_p ; one of the crucial boundary conditions used to solve this equation requires the codimension of the most intense structures. If these are taken to be vorticity filaments, their codimension is 2 and one gets the SL formula. Filaments have been observed in experiments also [19]. We have shown above that the exponent ratios ζ_p/ζ_2 that we obtain from the RFNSE with $y = 4$ agree with the SL formula. One might expect, therefore, that filamentary structures should appear in iso- $|\omega|$ plots for the RFNSE. However, this is not the case as can be seen from the representative plot shown in Fig. 3. The stochastic forcing seems to destroy the well-defined filaments observed in the 3DNSE *without changing the multiscaling exponent ratios*. Therefore, the existence of vorticity filaments is not crucial for obtaining these exponents, which is perhaps why simple shell models [9,20] also yield good estimates for ζ_p .

In summary, then, we have shown that the RFNSE with $y = 4$ exhibits the same multiscaling behavior as the 3DNSE, at least in the ESS sense. Probability distributions like $P(\delta v_r)$ (Fig. 2c) are also qualitatively similar in the two models, in so far as they show deviations from Gaussian distributions for r in the dissipation range. It would be interesting to see if the RFNSE model can be obtained as an effective, inertial-range equation for fluid turbulence. We have tried to do this by a coarse-graining procedure that has been used [21] to map the Kuramoto–Sivashinsky (KS) equation onto the Kardar–Parisi–Zhang (KPZ) equation; however, it turns out that the 3DNSE \rightarrow RFNSE

mapping, if it exists, is far more subtle than the KS \rightarrow KPZ mapping as we discuss elsewhere [17].

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