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TURBULENCE AND THE DYNAMICS OF COHERENT STRUCTURES PART II: SYMMETRIES AND TRANSFORMATIONS*

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1. Introduction. The accurate determination of coherent structures depends upon having a sufficiently large database. As we will see in this part, symmetry considerations can considerably extend the amount of available data. In addition, a priori consideration of symmetries can be significant in designing numerical or physical experiments. In the following we outline, on a case by case basis, and sometimes only in brief, the effect of such deliberations for a number of standard geometries. We then show how such data can be transformed for use in related geometries.

2. Plane Poiseuille (channel) flow. We consider incompressible flow governed by (I:2.1)[1]. The direction of mean flow is denoted by x, the spanwise direction by y and the vertical by z. If $\mathbf{v}^{(n)}(\mathbf{x})$ is a flow realization or *snapshot* at some instant, then the transformations

$$T_x : \mathbf{v}^{(n)} \to \mathbf{v}^{(n)}(x + l_1, y, z), \qquad T_y : \mathbf{v}^{(n)} \to \mathbf{v}^{(n)}(x, y + l_2, z) \tag{2.1}$$

also give admissible flows for any values of l_1 and l_2. Thus, as part of the ensemble average determining $\mathbf{K}$, we can average over all translations of the form (2.1). To accomplish this consider, for example, the effect of averaging over T_x,

$$\hat{K}_{ij}(\mathbf{x}, \mathbf{x}') = \int u_i(x + l, y, z)\bar{u}_j(x' + l, y', z')\, dl, \tag{2.2}$$

which under the variable change $l + x' = s$ yields

$$\hat{K}_{ij}(\mathbf{x}, \mathbf{x}') = \int u_i(x - x' + s, y, z)\bar{u}_j(s, y', z')\, ds = \hat{K}_{ij}(x - x', y, z, y', z').$$

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[1]We will use I to refer to Pt. I, [1], both in regard to equations and to references.

Limits of integration have not been specified since, in the two cases of interest, periodic and infinite boundary conditions, they play no role in the analysis. A similar averaging over T_y results in a translation kernel in y. It therefore follows that averaging over the groups (2.1) implies

$$\mathbf{K}(\mathbf{x},\mathbf{x}') = \mathbf{K}(x-x', y-y', z, z'), \tag{2.3}$$

i.e., the two point correlation is translationally invariant or homogeneous in the horizontal directions.[2]

In typical numerical experiments [I: 35–37] the flow is taken to be periodic in the x and y directions. We denote by $2L_1$ and $2L_2$ the periodicities in the x and y directions. Then it follows that if the flow is expressed as

$$u_j(\mathbf{x},t) = \sum_{k,m} \mu_j^{(0)}(k,m;\, z,t)\exp\left[\frac{ik\pi}{L_1}x + \frac{im\pi}{L_2}y\right] \tag{2.4}$$

(the reason for the zero superscript will become apparent), then the correlation may be put in the form

$$K_{ij}(\mathbf{x},\mathbf{x}') = \langle u_i(\mathbf{x})\bar{u}_j(\mathbf{x}')\rangle = \sum_{k,m} \kappa_{ij}(k,m;\, z,z')\exp\left[\frac{ik\pi}{L_1}(x-x') + \frac{im\pi}{L_2}(y-y')\right] \tag{2.5}$$

where

$$\kappa_{ij} = \langle \mu_i^{(0)}(k,m;\, z)\bar{\mu}_j^{(0)}(k,m;\, z')\rangle. \tag{2.6}$$

Before going further we point out a subtle but important distinction that arises in the derivation of (2.3). As obtained in (2.5), the form of the correlation matrix, $\mathbf{K}$, depends on having the representation in the form (2.4). On the other hand, if the data set being treated arises from experiment or the method of snapshots the group average (see (2.2)), over l_1 and l_2, (2.1), used to obtain (2.3), has to be performed. In either case this also results in an effective increase in the data, an estimate of which is discussed in Sec. 6.

This geometry, viz, flow through a channel, is also invariant under the dihedral group of transformations, D_2[2]. In particular, if the origin is placed at the center of the channel the flow is invariant under vertical reflection (i.e., in the (x, y)-plane)

$$R_z : (x,y,z,u,v,w) \to (x,y,-z,u,v,-w), \tag{2.7}$$

spanwise reflection

$$R_y : (x,y,z,u,v,w) \to (x,-y,z,u,-v,w), \tag{2.8}$$

and rotation about the x-axis

$$R_z R_y : (x,y,z,u,v,w) \to (x,-y,-z,u,-v,-w). \tag{2.9}$$

[2] If a flow is entirely homogeneous the eigenfunctions are sinusoids and the eigenvalues are continuous. The latter are given by the Fourier transform of the correlation matrix.

If we write

$$\begin{aligned} R_z\boldsymbol{\mu}^0 &= \boldsymbol{\mu}^{(1)} = \left(\mu_1^{(0)}(k,m;z),\ \mu_2^{(0)}(k,m,-z),\ -\mu_3^{(0)}(k,m,-z)\right), \\ R_y\boldsymbol{\mu}^0 &= \boldsymbol{\mu}^{(2)} = \left(\mu_1^{(0)}(k,-m,z),\ -\mu_2^{(0)}(k,-m,z),\ \mu_3^{(0)}(k,-m,z)\right), \\ R_zR_y\boldsymbol{\mu}^0 &= \boldsymbol{\mu}^{(3)} = \left(\mu_1^{(0)}(k,-m,-z),\ -\mu_2^{(0)}(k,-m,-z),\ -\mu_3^{(0)}(k,-m,-z)\right), \end{aligned} \tag{2.10}$$

then alternately,

$$\begin{aligned} R_z\mathbf{u} &= \sum_{k,m} \boldsymbol{\mu}^{(1)}\exp\left(\frac{ik\pi}{L_1}x + \frac{im\pi}{L_2}y\right), \\ R_y\mathbf{u} &= \sum_{k,m} \boldsymbol{\mu}^{(2)}\exp\left(\frac{ik\pi}{L_1}x + \frac{im\pi}{L_2}y\right), \\ R_zR_y\mathbf{u} &= \sum_{k,m} \boldsymbol{\mu}^{(3)}\exp\left(\frac{ik\pi}{L_1}x + \frac{im\pi}{L_2}y\right). \end{aligned} \tag{2.11}$$

Since all these now represent admissible flow realizations we can write instead of (2.5) that

$$\kappa_{ij} = \left\langle \frac{1}{4} \sum_{p=0}^{3} \mu_i^{(p)}(k,m;\ z)\bar{\mu}_j^{(p)}(k,m;\ z')\right\rangle, \tag{2.12}$$

i.e., we include in the ensemble the flows produced by the group actions.

It is clear that for a kernel of the form (2.3) the eigenfunctions have the form

$$\boldsymbol{\phi} = \boldsymbol{\psi}(k,m;\ z)\exp\left[\frac{ik\pi}{L_1}x + \frac{im\pi}{L_2}y\right]. \tag{2.13}$$

The determination of $\boldsymbol{\psi}$ then follows from

$$\int_{-L_3}^{L_3} dz'\kappa_{ij}(k,m;\ z,z')\psi_j(k,m;\ z') = \lambda(k,m)\psi_i(k,m;\ z) \tag{2.14}$$

where L_3 is the channel half width. In an actual calculation, (2.14) is directly reducible to a finite matrix problem. (Typically, the vertical dependence is expanded in a finite set of Chebyshev polynomials [I : 31].) Thus $\boldsymbol{\kappa}$ represents a degenerate kernel, the eigenfunctions are then expressible in the same finite set, and the problem reduces to a finite matrix problem. The *direct method* of Pt. I is of practical use in this problem provided the number of approximating functions in the vertical direction does not turn out to be excessive. It should be noted that in this approach

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}(k,m;\ z,z') \tag{2.15}$$

so that an eigenfunction calculation is required for each quantum pair (k,m).

Symmetry considerations continue to play a simplifying role in the actual calculation of the eigenfunctions. To show this let us first suppose that

$$\mathbf{V} = \left(V_1(\mathbf{x}), V_2(\mathbf{x}), V_3(\mathbf{x})\right) \tag{2.16}$$

is an eigenfunction of $\mathbf{K}(\mathbf{x},\mathbf{x}')$ corresponding to the eigenvalue λ, viz

$$\int \mathbf{K}(\mathbf{x},\mathbf{x}')\mathbf{V}(\mathbf{x}')\,d\mathbf{x}' = \lambda\mathbf{V}(\mathbf{x}). \tag{2.17}$$

It then follows from (2.7) that

$$\mathbf{V}' = R_z\mathbf{V} = \left(V_1(x, y, -z), V_2(x, y, -z), -V_3(x, y, -z)\right) \tag{2.18}$$

also is an eigenfunction, i.e., satisfies (2.17) with the same eigenvalue λ. If (2.16) and (2.18) are added and subtracted

$$\mathbf{V}^{\pm} = \mathbf{V} \pm \mathbf{V}',$$

then $\mathbf{V}^{\pm}$ are eigenfunctions in which the first two components are $\left\{{\text{even} \atop \text{odd}}\right\}$ and the last $\left\{{\text{odd} \atop \text{even}}\right\}$ in z. Equivalently this could have been assumed in the statement of (2.16). Similarly (2.8) shows that we can also assume that V_1, V_3 are $\left\{{\text{even} \atop \text{odd}}\right\}$ and V_2 $\left\{{\text{odd} \atop \text{even}}\right\}$ in y.

The problem has in part been formatted in terms of trigonometric functions and we also comment on this. Since a flow is real it follows that in say (2.4)

$$\bar{\mu}_j(-k, -m) = \mu_j(k, m).$$

From this it follows that we can restrict attention to *quantum* numbers (k, m) such that $k \geqslant 0$. Further it follows from (2.10) that the eigenfunction $\psi(k, -m; z)$ can be calculated from $\psi(k, m; z)$. Hence we can restrict attention to

$$(k, m) \geqslant 0. \tag{2.19}$$

Also from the aforementioned parity considerations in the z-direction the integral in (2.14) can be reduced to the interval $(0, L_3)$ or an equivalent of this.

3. Poiseuille flow in a rectangular channel. Next we relax the condition of periodicity in the y-direction and consider flow through the cross section $|y| < L_2$, $|z| < L_3$ with homogeneous boundary conditions on the boundaries. Translational invariance in the streamwise or x-direction is still preserved and we now have

$$\mathbf{K}(\mathbf{x}, \mathbf{x}') = \mathbf{K}(x - x', y, y', z, z') \tag{3.1}$$

and, in particular, if periodicity in the x-direction is assumed

$$\mathbf{K} = \sum_k \boldsymbol{\kappa}(k; y, y', z, z')\exp\left[\frac{ik\pi}{L_1}(x - x')\right]. \tag{3.2}$$

Although spanwise translational invariance has been lost, the geometry is still invariant under the dihedral group D_2. An average analogous to that of (2.12) still applies. In this instance only the streamwise variable separates out and it is necessary to deal with the eigentheory of $\boldsymbol{\kappa}(k; y, y', z, z')$ for each k. It is unlikely that any fully turbulent case can now be accurately treated by the *direct method*. The method of *snapshots* is therefore appropriate in this case.

The integral equation for the eigenfunctions has the form

$$\int_{-L_3}^{L_3} dz \int_{-L_2}^{L_2} dy\, \boldsymbol{\kappa}(k; y, y', z, z')\psi(k; y', z') = \lambda\psi(k; y, z). \tag{3.3}$$

From the reality condition we can restrict attention to $k \geqslant 0$. Further, under the parity conditions discussed in the previous section and which still apply, we can, for each eigenfunction, reduce the domain of integration in (3.3) to the quadrant

$$0 \leqslant y \leqslant L_2, \quad 0 \leqslant z \leqslant L_3.$$

An interesting extension of the case under study is obtained if $L_2 = L_3 = L$. In this case the problem is invariant under the dihedral group D_4[2]. If we denote by R_0 the rotation of 90° about the x-axis

$$R_0 : (x, y, z, u, v, w) \to (x, -z, y, -u, -w, v) \tag{3.4}$$

then the dihedral group D_4 contains the eight elements

$$D_4 : R_0^0, R_0^1, R_0^2, R_0^3, R_y, R_0 R_y, R_0^2 R_y, R_0^3 R_y. \tag{3.5}$$

The specific forms of this group follow directly from the definitions of R_0, (3.3), and R_y, (2.8). By choosing a square cross section the data is stretched by a factor of eight in addition to the extension afforded by translation in the x- direction. Thus instead of (2.12) the average is taken over the eight group elements (3.5). We mention in passing that although the group contains eight elements the actual numerical coding of the group actions is actually simpler if one notes that the group only contains two generators. Thus R_0 and R_y could be coded as subroutines and by repeated application of these subroutines all members of the group can be generated. Finally we mention that in this case the domain of integration in (3.3) can be reduced to an octant.

4. Bénard problem (convection). As a further illustration we consider the Bénard or convection problem. This is governed by the Boussinesq equations [3],

$$\nabla \cdot \mathbf{u} = 0,$$
$$\frac{d\mathbf{u}}{dt} - \nabla P = P_r R_a T \mathbf{e}_z + P_r \nabla^2 \mathbf{u}, \tag{4.1}$$
$$\frac{dT}{dt} + w = \nabla^2 T.$$

P_r denotes the Prandtl number, and R_a the Rayleigh number. As before z denotes the vertical direction with origin in the mid-plane and w the corresponding velocity. As usual T denotes the departure from the (nonconvective) equilibrium temperature profile. In the turbulent regime

$$\langle \mathbf{u} \rangle = 0, \quad \langle T \rangle = \overline{T}(z), \tag{4.2}$$

and we write

$$T = \overline{T}(z) + \theta(\mathbf{x}, t). \tag{4.3}$$

As mentioned in Pt. I the appropriate flow variable now is $\mathbf{v} = (\mathbf{u}, \theta)$.

For the horizontally unbounded case the kernel is translationally invariant so that

$$\mathbf{K} = \langle \mathbf{v}(\mathbf{x}) \bar{\mathbf{v}}(\mathbf{x}') \rangle = \mathbf{K}(x - x', y - y', z, z') \tag{4.4}$$

and, for example, if periodicity of $2L_1$ and $2L_2$ in the horizontal is assumed, then

$$\mathbf{K} = \sum_{k,m} \boldsymbol{\kappa}(k, m;\, z, z') \exp\left[ik\pi(x - x')/L_1 + im\pi(y - y')/L_2\right]. \tag{4.5}$$

Also if we write

$$\mathbf{v} = \sum_{k,m} \boldsymbol{\nu}(k, m; z) \exp[ik\pi x/L_1 + im\pi y/L_2] \tag{4.6}$$

then

$$\boldsymbol{\kappa} = \langle \mathbf{v}(k, m;\ z)\bar{\mathbf{v}}(k, m;\ z')\rangle, \tag{4.7}$$

a consequence of averages of the form (2.2).

The flow geometry is invariant under the dihedral group D_2 in the (x, y)-plane and if $L_1 = L_2$ then the invariance group is D_4, the dihedral group with eight elements. In addition, this flow has a symmetry in the vertical direction,

$$\hat{R}_z : \mathbf{v} \to (u(x, y, -z), v(x, y, -z), -w(x, y, -z), -\theta(x, y, -z)). \tag{4.8}$$

(The circumflex is to contrast this with (2.7) in which temperature, θ, doesn't appear.) Thus for the case in which the (x, y)-plane projection is a square, the group of symmetries has *sixteen* elements with the three generators R_0, R_z, $\hat{R}_y$. [This in effect is the outer product of R_z with D_4.] If the rectangular projection in the (x, y)-plane is not square the group of symmetries has eight elements with the three generators R_0^2, R_z, R_y, and is the outer product of R_z with D_2.

In any case, if we denote the elements of the group by $T^{(p)}$, it then follows that the correlation matrix is

$$\boldsymbol{\kappa} = \langle \frac{1}{P} \sum_{p=1}^{P} T^{(p)}\mathbf{v}(k, m;\ z)T^{(p)}\bar{\mathbf{v}}(k, m;\ z')\rangle \tag{4.9}$$

where P is the number of elements of the group. Since a square (x, y)- projection yields a 16-element symmetry group this would seem to be a desirable case to pursue. On the one hand it affords a 16-fold extension of the available data and on the other it considerably reduces the labor in the eigenfunction calculation. Just as we arrived at (2.19) for the channel problem we can show in this instance that all eigenfunctions of the kernel (4.9) can be determined from $0 \leqslant m \leqslant n$ (4.10), i.e., from an octant. Moreover, it follows from (4.8) that the eigenfunctions of (4.9) can be formatted in terms of odd and even functions in the z- direction, which further reduces the eigenfunction calculation.

5. Other geometries. We now briefly explore the symmetry properties of some of the more commonly studied flows.

Flow past bodies of revolution. In this case, we take the uniform upstream flow in the x-direction and consider cylindrical coordinates (x, r, θ) with velocity components (u, v_r, v_θ). Then for bodies of revolution such as a sphere the transformation

$$T_\theta : (x, r, \theta, u, v_r, v_\theta) \to (x, r, \theta + \alpha, u, v_r, v_\theta) \tag{5.1}$$

for any $|\alpha| \leqslant \pi$ produces an admissible flow. In addition, we have the discrete symmetry

$$R_\theta : (x, r, \theta, u, v_r, v_\theta) \to (x, r, -\theta, u, v_r, -v_\theta). \tag{5.2}$$

Flow in a circular pipe. If the generator of the pipe lies in the x-direction then translation in the x-direction, T_x, (2.1), produces admissible flows as does the rotation, T_θ, (5.1), and the reflection, R_θ, (5.2).

Flow past a circular cylinder. We denote the upstream uniform flow direction by x and the perpendicular direction containing the axis of the cylinder by y. Then other admissible flows are generated by: translation in the y-direction, T_y; reflection in the (x, z)-plane), R_y, (2.8); reflection in the (x, y)-plane, R_z, (2.7); and rotation of 180° about the x-axis, $R_z R_y$, (2.9). In other words, T_x along with this dihedral group D_2 on the (y, z)-plane.

Plane Couette flow. We take (x, y) to be the plane of flow with x the direction of flow and z the spanwise direction. The invariance group is now composed of translation in the z-direction, T_z; reflection in the (x, y)-plane, R_z, (2.7); rotation through an angle π about the z-axis, T_π; and the product $R_z T_\pi$.

Taylor-Couette flow. Take the axis of rotation in the x-direction and cylindrical coordinates (x, r, θ). The invariance group is now composed of translation in the x-direction, T_x, (2.1); reflection in the plane normal to x, R_x: rotation around the x-axis, T_θ, (5.1).

6. Transformations. In anticipation of the results of Pt. III we mention the application of the eigenfunctions or coherent structures as a basis set in the Galerkin procedure for general flows. Two considerations are of immediate concern: (1) situations in which the flow parameters are changed and, (2) situations in which the geometry is altered. The former is considered in Pt. III, while in this section we make some brief remarks on changes in geometry. In what follows it is probably useful to imagine the eigenfunctions as residing in computer files, i.e., they are known numerically at a finite set of lattice points. In speaking of the transformation of eigenfunctions we mean the mapping of these discrete locations and the values of the flow at them.

Under a sufficiently regular mapping of three-space

$$\mathbf{x} = \mathbf{F}(\mathbf{X}) \tag{6.1}$$

the complete set $\{\mathbf{V}_n\}$ maps to

$$\{\mathbf{V}_n(\mathbf{F}(\mathbf{X}))\} = \{\mathbf{W}_n(\mathbf{X})\}, \tag{6.2}$$

some new (complete) basis set. Boundary conditions are preserved since they were taken to be homogeneous or periodic and are being sent to new appropriate locations. Orthogonality is in general lost (unless the Jacobian is taken to be the weight function), but may be recovered through the use of the Gram-Schmidt procedure under the inner product, (I:2.8), and taken with respect to the new independent variables. In following such a program, it is desirable that problems with like boundary conditions and similar nature be treated. The more removed the transformed problem, the greater the expected number of basis functions necessary to fit the situation. To illustrate some of these points we consider some simple situations.

Rectilinear Boundaries. As a first, and very simple, example we consider a problem with rectilinear boundaries as, for example, the convection or channel problem. In the simplest case we wish to transform a *box* to another having a different aspect. Denote the transformation taking the original *box* to the new *box* by

$$\mathbf{X} = \mathbf{X}(\mathbf{x}) = (X(\mathbf{x}), Y(\mathbf{x}), Z(\mathbf{x})).$$

The coherent structures then transform to $\{V^{(n)}(\mathbf{x}(\mathbf{X}))\}$, which in general are not incompressible. In fact, if we denote the velocity components of a typical coherent structure by V_j, $j = 1, 2, 3$, then

$$0 = \frac{\partial V_j}{\partial x_j} = \frac{\partial V_j}{\partial X_k}\frac{\partial X_k}{\partial x_j}. \tag{6.3}$$

For the simple case being discussed here $X_{k,j}$ is a constant matrix. Therefore if we take

$$W_i = W_i(\mathbf{X}) = X_{k,j}V_j(\mathbf{x}(\mathbf{X})) \tag{6.4}$$

it then follows that

$$\frac{\partial W_j}{\partial X_j} = 0. \tag{6.5}$$

Since $\mathbf{V}$ satisfies the boundary conditions the linear combination of these $\mathbf{W}$ given by (5.4) does also.

It is therefore clear that we can in this way construct a complete set of basis functions $\{\mathbf{W}^{(n)}\}$ for the transformed problem. These will satisfy the boundary conditions and, as well, each will be incompressible. In general only orthogonality is lost, although for the simpler versions of the convection and channel problems even orthogonality is preserved. However, even when all this is true the set $\{\mathbf{W}^{(n)}\}$ will not in general be the coherent structures of the problem. This last point will be further taken up in Pt. III.

Three-dimensional flows in two-dimensional geometries. As a second, and more serious, example of a mapping method we consider problems for which the boundaries are two-dimensional, but the flow is not. Examples are flow in a pipe or flow past a cylinder. The problem is then to generate a basis set for other two-dimensional geometries of the same connectivity, for example, to treat flow in a pipe of noncircular cross section from results on the circular cross section, or to treat flow past an airfoil based on flow past a circular cylinder.

As in the previous example, we consider incompressible flow and assume a basis set $\{\mathbf{V}^{(n)}(\mathbf{x})\}$ of functions. In addition to being incompressible, these satisfy homogeneous boundary conditions on boundaries ∂b lying say in the (x, y)-plane,

$$\mathbf{V}^{(n)}(x, y, z) = 0; \quad (x, y) \in \partial b. \tag{6.6}$$

We consider a flow of like nature in a second domain of the same connectivity and boundaries ∂B. Then, under mild conditions, the Riemann mapping theorem assures us of a conformal mapping

$$x = \phi(X, Y), \qquad y = \psi(X, Y) \tag{6.7}$$

such that ∂B maps to ∂b, and ϕ and ψ satisfy the Cauchy-Riemann equations. For example, a circular pipe can be conformally mapped to an elliptic pipe and a circular cylinder to an airfoil under the Jukowski transformation [4]. Extremely efficient numerical codes are available for carrying out more general transformations [5].

If we write a typical eigenfunction as

$$\mathbf{V} = (V_1, V_2, V_3) \tag{6.8}$$

and set the Jacobian of the transformation to

$$J = \left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2 \tag{6.9}$$

then

$$\mathbf{W} = (W_1, W_2, W_3) = \mathbf{T}V = \left(V_1\frac{\partial\phi}{\partial X} + V_2\frac{\partial\psi}{\partial X}, V_1\frac{\partial\phi}{\partial Y} + V_2\frac{\partial\psi}{\partial Y}, JV_3\right) \tag{6.10}$$

is incompressible, i.e.,

$$\nabla_{\mathbf{x}}\cdot\mathbf{W} = \frac{\partial W_1}{\partial X} + \frac{\partial W_2}{\partial Y} + \frac{\partial W_3}{\partial Z} = 0 \tag{6.11}$$

in the new coordinate system as may be verified directly. Since $\mathbf{V}$ satisfies homogeneous boundary conditions so also does $\mathbf{W}$. It therefore follows that

$$\{\mathbf{W}^n\} = \{T\mathbf{V}^{(n)}\} \tag{6.12}$$

constitute an admissible set of basis functions. (In general they are no longer orthogonal.)

It is of interest to point out that the case of the circular pipe, for which extensive experimental data exist [6] and for which numerical data also exist [7] is a candidate for the *direct method*. On the other hand, the elliptical cross section (or any other cross section for that matter) is out of reach by this method. The transformation method just discussed should nevertheless produce a reasonable set of functions based on the circular case.

7. Further comments.

1. Throughout our discussion there has been an emphasis on the discrete groups and their extension of the available data. However, we have also applied the continuous groups of transformations, (2.1). It is of importance, for example, in error estimates to assess how these extend the available data. Toward this end denote a typical spatial correlation length by λ. Then a translation in a horizontal direction of λ produces a new member of the ensemble. For example if $\mathbf{v}(\mathbf{x})$ represents an admissible flow, then $\mathbf{v}(x+\lambda, y, z)$ represents an admissible flow which is statistically independent. Hence if $2L_1$ and $2L_2$ denote the spatial periods in the horizontal directions the available data is extended by the factor $(2L_1 \cdot 2L_2)/\lambda^2$. That is, averaging over the groups (2.1), as typified by (2.2), results in the members of the ensemble being increased by this factor.

2. Another result of applying the symmetry groups of a flow geometry is in zeroing out appropriate quantities. A situation often encountered in data analysis is that of finding a quantity, which should have a zero mean, but tending to zero slowly and *not crossing the zero axes*. An excellent example of this is given in chapter III of Feller's book [8], where coin tossing is considered. It is shown there that in considering the gain in tossing a fair coin, *infinity* is the expected number of trials between zero crossings. In the context of say the convection problem we should find $\langle\mathbf{u}\rangle = 0$, or $\langle u_1u_2\rangle = 0 = \langle u_2u_3\rangle$, but in practice we find an annoying persistence of signature. However, after the application of group averaging, such quantities zero out. Moreover a general *cleanup* in the form of $\mathbf{K}$ or $\boldsymbol{\kappa}$ occurs after imposing the symmetries of the problem.

3. The continuity equation for incompressible flows,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

imposes a linear relation on the velocity components. This in turn implies a degeneracy in the operator **K**. Thus for the channel problems one third of the eigenvalues should be zero and for the convection problem one fourth should vanish. This therefore suggests that a further reduction in the formulation is possible. For example, consider an incompressible flow for which plane waves

$$\exp[i\mathbf{p}\cdot\mathbf{x}] = \exp[i(p_1x_1 + p_2x_2 + p_3x_3)]$$

are a suitable basis. Then

$$\mathbf{V} = (a\boldsymbol{\alpha} + b\mathbf{p}\wedge\boldsymbol{\alpha})\exp[i\mathbf{p}\cdot\mathbf{x}] \tag{7.1}$$

with

$$\boldsymbol{\alpha} = (p_2 - p_3, p_3 - p_1, p_1 - p_2) \tag{7.2}$$

for arbitrary a and b represents an incompressible flow. Thus, for a suitable chosen set $\{\mathbf{p}_n\}$,

$$\mathbf{V} = \Sigma[a_n(t)\boldsymbol{\alpha}_n + b_n(t)(\mathbf{p}_n\wedge\boldsymbol{\alpha}_n)]\exp(i\mathbf{p}_n\cdot\mathbf{x}) \tag{7.3}$$

can be introduced into the dynamical equations and solved. The reduction from three to two components is dealt directly in this way. A reduction to the same extent would then be inherited by **K** or $\boldsymbol{\kappa}$.

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