QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLV, NUMBER 3
OCTOBER 1987, PAGES 583-590

TURBULENCE AND THE DYNAMICS OF COHERENT STRUCTURES PART III: DYNAMICS AND SCALING*

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1. Dynamics of coherent structures. Following the methods presented in [1] and [2], it is now assumed that we have a complete set of vector eigenfunctions $\{\mathbf{V}_n\}$ of the correlation matrix operator **K**. These functions fit the boundary conditions of the problem and if the flow is incompressible they also each satisfy the continuity equation. We can therefore expand the flow $\mathbf{v}$ in this set,

$$\mathbf{v} = \sum_{n=1}^{\infty} a_n(t)\mathbf{V}_n(\mathbf{x}), \tag{1.1}$$

and substitute into the Navier-Stokes equations, written symbolically as

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{D}(\mathbf{v}). \tag{1.2}$$

When this is projected along the eigenfunctions, $\mathbf{V}_n$,

$$\left(\mathbf{V}_k, \frac{\partial}{\partial t}\mathbf{v} - \mathbf{D}(\mathbf{v})\right) = 0, \quad k = 1, 2, \ldots, \tag{1.3}$$

we obtain the ordinary differential equations,

$$\frac{da_k}{dt} = F_k(\mathbf{a}), \quad k = 1, 2, \ldots, \tag{1.4}$$

where F_k is at most quadratic in the coefficients $\mathbf{a}$. If the infinite system (1.4) is truncated after say $k = N$ and all coefficients a_k are set to zero for $k > N$, the resulting finite system yields the Galerkin approximation to (1.2), to order N. Equivalently we can write

$$\mathbf{v}_N = \sum_{k=1}^{N} a_k \mathbf{V}_k \tag{1.5}$$

* Received October 1, 1986.

and instead of (1.3) consider

$$\left(\mathbf{V}_k, \frac{\partial \mathbf{v}_N}{\partial t} - \mathbf{D}(\mathbf{v}_N)\right) = 0, \quad k = 1, 2, \ldots, N. \tag{1.6}$$

This then furnishes the outline of a practical procedure for following the turbulent evolution of a flow. However, a good deal of important detail has been sacrificed in giving the above bare bones treatment. First we observe that the ordering of the eigenfunctions plays a role and in the absence of other considerations (some of which will be mentioned later) we take them to be ordered according to the eigenvalues,

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots .$$

In this way we maximize the fraction of retained *energy*

$$\langle(\mathbf{v}_N, \mathbf{v}_N)\rangle / \langle(\mathbf{v}, \mathbf{v})\rangle = \left(\sum_{k=1}^{N} \lambda_k\right) \Big/ \left(\sum_{k=1}^{\infty} \lambda_k\right). \tag{1.7}$$

Next we point out the ambiguity in whether the expansion (1.1) is in the total flow quantities or the fluctuation from the mean quantities or some other representation of the flow. As the set $\{\mathbf{V}_n\}$ is complete, we have the liberty to make a variety of choices. (For open flows with nonzero mean at infinity some obvious modifications are required.) Since only a finite set of basis functions, $\{\mathbf{V}_n\}$, is to be retained, a question of importance in regard to the first suggestion is how well fit the mean flow will be by the truncated set of eigenfunctions. At the stage of the development at which the coherent structures are determined, the mean flow is known to us from either numerical or physical experiments. Since it is known that the fluctuations away from the mean are well fit by the eigenfunctions, this strongly suggests that we expand only the fluctuations.

This then leads naturally to the next issue, which is the effect on the eigenfunctions of changes in physical parameters such as Reynolds number, Rayleigh number, Taylor number, etc. (These we denote generically by the vector $\mathbf{R}$.) As a basic point of reference it should be noted that

$$\mathbf{V}_n = \mathbf{V}_n(\mathbf{x}; \mathbf{R}_0) = \mathbf{V}_n^0 \tag{1.8}$$

where $\mathbf{R}_0$ represents the specific values at which the calculation of coherent structures was performed. If $\mathbf{R} \neq \mathbf{R}_0$ *the set* $\{\mathbf{V}_n^0\}$ *are no longer, in general, the coherent structures of the flow*. In particular $\langle a_k \bar{a}_l \rangle$, for $k \neq l$, is in general zero only if $\mathbf{R} = \mathbf{R}_0$. However, $\{\mathbf{V}_n^0\}$ still form a *complete, orthonormal set satisfying the boundary conditions*, and if the flow is incompressible they also *satisfy the incompressibility condition*. The calculated set $\{\mathbf{V}_n^0\}$ might therefore be expected to be useful over a range of parameter space, $\mathbf{R}$. This was the case for the Ginzburg-Landau equation where the method under discussion was applied [3]. In that case a relatively large region of the parameter space (which is three-dimensional) was very well described by a small set of eigenfunctions calculated for one particular set of parameters.

2. Channel flow. In order to make these notions more concrete we consider in some detail the specific example of channel flow. We explore, in particular, the solution strategies available to us. The geometry of this flow and the various symmetries have been discussed in [II: Sec. 2]. In brief, the flow is governed by

$$\nabla \cdot \mathbf{U} = 0, \qquad \frac{\partial \mathbf{U}}{\partial t} = \mathbf{E}[\mathbf{U}] = -\mathbf{U} \cdot \nabla \mathbf{U} - \nabla P + \frac{1}{R_e} \nabla^2 \mathbf{U} \tag{2.1}$$

with homogeneous boundary conditions at $x_3 = \pm H$. The spanwise boundary conditions, which in practice are homogeneous or periodic, do not enter into our deliberations.

For channel flow, in our notation, $\mathbf{v} = \mathbf{u}$, the fluctuation velocity field itself. The total velocity is expressed as

$$\mathbf{U} = \bar{U}\mathbf{e}_1 + \mathbf{u} \tag{2.2}$$

where $\bar{U} = \langle \mathbf{U} \rangle = \bar{U}(x_3)$ is the mean flow field which is in the x_1-direction. This, when inserted into (2.1), yields

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{\partial}{\partial t} u_i + \bar{U} \frac{\partial}{\partial x_1} u_i + u_3 \frac{\partial}{\partial x_3} \bar{U} \delta_{i1} + \frac{\partial}{\partial x_j} u_i u_j + \frac{\partial}{\partial x_i} (\bar{P} + p)$$

$$= \frac{1}{R_e} \nabla^2 (\bar{U} \delta_{i1} + u_i) \tag{2.3}$$

with [I:26]

$$\frac{\partial \bar{P}}{\partial x_1} + \frac{\partial}{\partial x_3} \overline{u_1 u_3} = \frac{1}{R_e} \frac{\partial^2}{\partial x_3^2} \bar{U} \tag{2.4}$$

and

$$\bar{P} + \overline{u_3^2} = P_0(x_1). \tag{2.5}$$

The pressure field at the wall $P_0(x_1)$ is linear and we take

$$\frac{\partial \bar{P}}{\partial x_1} = \frac{\partial P_0}{\partial x_1} = -1. \tag{2.6}$$

This corresponds to a spatial scaling based on the half width of the channel and a velocity scaling based on the friction velocity

$$u^* = \sqrt{\frac{Hk}{\rho}} \tag{2.7}$$

where k is the dimensional pressure gradient and H is the half channel width. Equation (2.4) may now be integrated twice to give

$$\bar{U}(x_3) = R_e \left\{ \frac{(1 - x_3^2)}{2} + \int_{-1}^{x_3} \overline{u_1 u_3}\, dx_3 \right\}. \tag{2.8}$$

The coherent structures $\{\mathbf{V}_n\}$ have been obtained at a particular reference value of the Reynolds number $R_e = R_e^0$. To underline this point we write

$$\{\mathbf{V}_n^0\} = \{\mathbf{V}_n(\mathbf{x};\ R_e^0)\}. \tag{2.9}$$

More emphatically, if we expand $\mathbf{u}$,

$$\mathbf{u} = \sum_{n=0} a_n^0 \mathbf{V}_n^0, \tag{2.10}$$

substitute in (2.3), with $\bar{U} = \bar{U}^0 = \bar{U}(R_e^0)$, then the resulting system corresponding to (1.4) should produce statistically the same system used to generate (2.9). In particular, we should find the coefficients $\{a_n^0\}$ uncorrelated,

$$\langle a_k^0 \bar{a}_l^0 \rangle = \lambda_k \delta_{kl}, \tag{2.11}$$

and also that (2.8) is self-consistently satisfied. When $R_e \neq R_e^0$ this picture is altered and the possible approaches in this instance deserve some discussion.

3. Solution strategies. Perhaps the most straightforward approach to dealing with off reference values of R_e is to directly expand $\mathbf{U}$ in terms of the set $\{\mathbf{V}_n^0\}$. Thus we write

$$\mathbf{U} \approx \mathbf{U}_N = \sum_{n=1}^{N} A_n(t)\mathbf{V}_n^0 \tag{3.1}$$

and apply the Galerkin procedure to (2.1),

$$\left(V_j^0, \frac{\partial \mathbf{U}_N}{\partial t} - E[\mathbf{U}_N] \right) = 0, \quad j = 1, \ldots, N. \tag{3.2}$$

Since the set $\{V_n^0\}$ is complete, the solution to the problem can in general be obtained in this way. However, a problem with this approach is that the mean flow $\bar{U}$ will not be well fit unless care is taken in selecting the V_n^0 which will accomplish this. Specifically, the choice of a set from $\{V_n^0\}$ to be used in the Galerkin procedure can not be based solely on the ordering of eigenvalues indicated in Sec. 1, since the eigenfunctions needed to fit $\bar{U}(x_3)$ will almost certainly correspond to relatively small eigenvalues.

As a modification to this approach we can instead expand as follows:

$$\mathbf{U} = \bar{U}^0\mathbf{e}_1 + \sum b_n \mathbf{V}_n^0, \tag{3.3}$$

i.e., use the known mean flow determined at reference conditions to expand about. After applying the Galerkin procedure (3.2) this also leads to just evolutionary equations. However, in this instance,

$$\Delta\bar{U} = \bar{U} - \bar{U}^0 \tag{3.4}$$

will require fewer functions to fit it and, at least for relatively small departures from the reference case, R_e^0, this should work well.

For the hoped for success in wider departures from R_e^0 we consider a third approach which is basically an extension of the previous strategy. For this purpose we re-express (2.3) as

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}(P + p) - \frac{\partial}{\partial x_j}(u_i u_j) + \frac{1}{R_e}\nabla^2 u_i + L_i(\mathbf{u})U = G_i[\mathbf{u}, U] \tag{3.5}$$

where L_i is an operator, at most linear in $\mathbf{u}$, which acts on $\bar{U}$. Following the outline in Sec. 1, the Galerkin approximation is obtained by projecting $\mathbf{u}$ onto the first N eigenfunctions

$$\mathbf{u}_N = \sum_{n=1}^{N} a_n \mathbf{V}_n^0 \tag{3.6}$$

and performing a corresponding projection on (3.6),

$$\left(\mathbf{V}_k^0, \frac{\partial \mathbf{u}_N}{\partial t}\right) = \frac{\partial a_k}{\partial t} = \left(V_k^0, \mathbf{G}[\mathbf{u}_N, U]\right) = F_k(\mathbf{a};\ Re). \tag{3.7}$$

Since $\overline{U}$ is unknown we take $\overline{U}^0$ to be the initial guess, i.e., we take

$$F_k = \left(\mathbf{V}_k^0, G\left[\mathbf{u}_N, \overline{U}^0\right]\right). \tag{3.8}$$

At this state the development is exactly the same as that which follows from (3.3). This solution is followed until say the turbulent shear stress

$$\overline{u_1 u_3} = \lim_{T\uparrow\infty} \int_0^T u_1(t) u_3(t)\, dt \tag{3.9}$$

settles down. (The symmetry conditions in [II: Sec. 2] should substantially speed up the convergence of (3.9).) At this point the mean flow $\overline{U}$ can be computed from (2.8). When substituted into (3.8) instead of $\overline{U}^0$ a new dynamical system results. This then forms the basis of an iterative procedure. It is worth noting that this procedure does not require any special choice of V_n^0 to fit the mean flow. In a real sense these are generated in the product of $u_1 u_3$ which occurs in (3.9). Another point worth commenting on is that the system (3.7) is actually cubic in the a_k's. To see this observe that the *driving term* of F_k is

$$\left(V_k^0, L(\mathbf{u}_N)\overline{U}\right)$$

which from (3.9) and (2.8) is seen to be cubic.

Finally, it is important to note that for incompressible flows the calculation of pressure is avoidable. The advantage in having a complete vector set of orthogonal incompressible expansion functions has been pointed out by several authors [4], [5]. In practice such a representation is difficult to find, but, as we see, it results in a natural way from the procedure presented here. (Note that pressure can also be avoided in the convection problem since the mechanical part of the flow is incompressible.) For channel flow, since P_0 is linear in x_1, a contribution actually does appear. To see this we substitute (2.5) and (2.6) into the pressure portion of F_k, (3.7), to obtain

$$\left(\mathbf{V}^0, \nabla(\overline{P} + p)\right) = \left[\mathbf{V}^0, \nabla\left(P_0 + u_3^2 + p\right)\right] = [\mathbf{V}, \nabla P_0] = -\int V_1^0\, d\mathbf{x}, \tag{3.10}$$

where V_1^0 is the x-component of $\mathbf{V}^0$. This follows from (I:2.22) where it is shown that divergence free vector fields satisfying admissible boundary conditions are orthogonal to the space of vectors which can be expressed as gradients. [If we write $u \cdot \nabla u = \omega \wedge u + \nabla(u^2/2)$, the latter term will also drop from (3.10).] If a spectral expansion in x_1 and x_3 is used, the integral in (3.10) becomes one-dimensional, i.e., in the x_3-direction.

4. Off reference coherent structures. As has been mentioned several times, when the parameters of the problem are not at their reference values, $\mathbf{R} \neq \mathbf{R}^0$, the basis functions no longer have the interpretation of being the coherent structures of the flow field. In terms of the expansion (1.1) the ensemble average of the correlation is

$$\langle \mathbf{v}\bar{\mathbf{v}} \rangle = \sum C_{mn} \mathbf{V}_m^0 \overline{\mathbf{V}}_n^0 \tag{4.1}$$

with

$$C_{mn} = \langle a_m \bar{a}_n \rangle. \tag{4.2}$$

Only when $\mathbf{R} = \mathbf{R}^0$ can we expect that C_{mn} is diagonal.

Within this framework, however, it is straightforward to generate the new coherent structures. The matrix **C** is Hermitian and may be put in diagonal form under similarity

$$\mathbf{C} = \mathbf{L}^{-1}\Lambda\mathbf{L} \tag{4.3}$$

where the rows of **L** are the orthonormal eigenvectors of **C** and Λ is the diagonal matrix of eigenvalues. Specifically we denote the eigenvectors of **C** by **l** and the corresponding eigenvalues by λ so that

$$\mathbf{C}\mathbf{l}^{(k)} = \lambda_k \mathbf{l}^{(k)}. \tag{4.4}$$

From this it follows that if we define

$$\mathbf{W}_k = \sum_n l_n^{(k)} \mathbf{V}_n^0 \tag{4.5}$$

then

$$\langle \mathbf{v}\bar{\mathbf{v}} \rangle = \sum_k \mathbf{W}_k \lambda_k \mathbf{W}_k \tag{4.6}$$

so that $\{\mathbf{W}_k\}$ are the coherent structures appropriate to the new values of the parameters **R**. Naturally in any practical case we are dealing with approximate forms of the coherent structures, just as $\{\mathbf{V}_n\}$ are themselves approximate forms when $\mathbf{R} = \mathbf{R}^0$.

5. Discussion. A point of some interest which was discussed briefly in the Introduction in Pt. I concerns data compression or reduction. In a typical numerical simulation of the Navier-Stokes equations, the equations are integrated forward in time and the results saved at appropriate time steps. The eigenfunction approach serves as an alternate method for archiving data of this sort. What is being suggested is that, after collection, the data may be processed to determine the eigenfunctions of the correlation matrix. The data may then be represented in terms of these *standard vectors*. As an indication of the types of savings achieved in this way we mention an application of this method to the convection problem. A grid of $(32)^3$ points was considered, and since four quantities $(\mathbf{u}, \theta)$ are calculated at each point, $O(10^5)$ numbers are stored for each time frame. Our analysis however indicates that 95% of the flow based on energy can be accurately captured by retaining $O(10^2)$ modes. Thus by this reasonable criterion the data storage is reduced by a factor of $O(10^3)$.

The recalculation of the coherent structures discussed in Sec. 4 also bears on this point. For it was shown in Pt. II that at any value of the parameters, **R**, the optimal fit to the data is given by the coherent structures. Thus the recalculation given in Sec. 4 also serves as a basis for an efficient method of data compression.

6. Final remarks. To close this series of papers we make some general remarks on coherent structures and their use as a basis system.

We have, throughout this series of papers, associated a coherent structure with the eigenfunctions of the correlation operator. This is both a rational and objective definition but may not be what is *observed* in the laboratory. What is reported to be a coherent structure can have a very subjective slant, and as a result differences of opinion can be expected. In certain cases, however, there is a general agreement about what is a coherent structure. A particular example is the hairpin vortices found in transitional and turbulent boundary layers. These have been observed both in experiment [6, 7] and in numerical calculations [8, 9].

It is reasonably obvious that hairpin vortices will not emerge as coherent structures as defined here. The basis for this assertion is that the eigenfunction dependence in the spanwise direction (for example as in [8, 9]) are single sinusoids and it is clear that the hairpin vortices have a richer structure in the spanwise direction than one harmonic. A similar set of remarks holds for the steamwise dependence which is also sinusoidal. As we saw in [2] this sinusoidal dependence is a consequence of translational invariance. Lumley [I : 4, 5, 13] introduces a shot noise hypothesis to deal with homogeneous directions. From this starting point he develops an elegant construction of a *coherent structure* as an appropriate superposition of eigenfunctions. Higher-order statistics enter in the calculation of the coefficients. Since hairpin vortices evolve in time it is not clear that such a treatment is appropriate. Rather it is more likely that the eigenfunctions are basic *modes* having roughly the same time course, that interact in time (they are uncorrelated for longer times) and thus make up the genuine coherent structure. If this speculation proves valid, it in no way detracts from the basic nature of the eigenfunctions. It has been shown, that in a statistical sense, they are persistent structures. For long times they only weakly interact with other such structures (at least to within second-order statistics). On short time scales intense interaction is entirely possible.

Another basic property of the derived eigenfunctions is their efficiency as a basis set in describing a flow. This was mentioned in the previous section and we now continue this discussion. We can ask why the eigenfunctions are so good at representing the flow or, what is equivalent, why is the straightforward approach of choosing, say sinusoids, so poor at *fitting* a flow? To attempt an answer to this we recall Landau's estimate [10] on the degrees of freedom present in a given flow. Using the Kolmogorov scale as an estimate of the smallest relevant scale he arrived at $(R/R_c)^{9/4}$ degrees of freedom, where R_c is the critical Reynolds number. The use of sinusoids is essentially an attempt to meet the requirement of matching the number of degrees of freedom. While on this topic, we also mention the estimate given by Constantin et al [I; 19] for the dimension of the attractor, viz $(R/R_c)^3$. This estimate, which is less optimistic than Landau's, essentially arrived at without intuitive physical arguments, makes use of the Stoke's eigenfunctions. These are the eigenfunctions of the spatial operator of the fluid equations in the limit $R \downarrow 0$. Since the interest is in the opposite limit, $R \uparrow \infty$, it is easy to imagine that these could be a limitingly poor way in which to measure out the dimension of the attractor set, which is generated for $R \uparrow \infty$.

The eigenfunctions which have been considered here are in a very specific way generated to fit the statistical features of the flow at relatively high value of (R/R_c). Unlike Landau's calculation, (which does not attempt to compute dimension), and that of Constantin et al [I; 19] the effect of correlations is now included. This is apparent from the form of the coherent structures which were derived in [I; 20, 25]. In general these included a wide range of scales. Thus the effect of correlations appears to be significant in reducing the number of dimensions needed to describe the system, and we conjecture that the attractor dimension will be considerably less than $O([R/R_c]^{9/4})$ in real problems.

The approach taken in these papers made no direct attempt to parametrize the underlying attractor in a problem. In a sense the only role played by the attractor was in providing an inspiration for seeking a low-dimensional description. A simple characterization of our method is that it finds a *principal axes* representation of the correlation operator and then ignores directions that are *flat* enough (i.e., small eigenvalues). Thus the attractor can be viewed as being encapsulated by a finite ellipsoid in function space (with axes proportional to the roots of the eigenvalues). If not much of the attractor *leaks out* of the ellipsoid, the approximation can be expected to succeed.

Note added in proof. In a recent very interesting report, Aubry et al [11] consider a finite-dimensional description of the wall region of a turbulent flow. The approach taken by these authors, although different in many ways, parallels that given here for the dynamical description of coherent structures.

Acknowledgment. I gratefully acknowledge my debt to Bruce W. Knight whose influence, direct and indirect, on these three papers was felt in many places. Special thanks are also owed to Martin Maxey who first interested me in this problem and to whom I turned for expert knowledge on turbulent flows.

The research contained here was supported by DARPA URI under contract number N00014-86-K0754.

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