TURBULENCE IN AN ATMOSPHERE WITH A NON-UNIFORM TEMPERATURE

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(Received 1 April, 1971)

Abstract. In this work the effect of static stability on the development of atmospheric turbulence is investigated. This influence is considered quantitatively by generalizing Prandtl's semi-empirical theory, i.e., by using a correcting factor in the form of a universal function of the Richardson number. An evaluation of the thickness of the layer of dynamic turbulence under various external conditions is successfully achieved quantitatively.

List of symbols

A Austausch coefficient $= \varrho K$
c_p specific heat at constant pressure
D dissipation of turbulent energy into heat
E' kinetic energy of turbulent fluctuations
F_s flux of quantity s
g acceleration due to gravity
k von Karman's constant
K coefficient of turbulence (eddy transfer coefficient)
K_s eddy diffusivity
K_T eddy thermal diffusivity
l mixing length
$L_1 = v_*^3/kgu$
$Pr = \mu c_p/\lambda$
q heat flux
$Ri = g/T\,(\partial\theta/\partial z)/(\partial v/\partial z)^2$
Ri_{cr} critical Richardson number
s gravimetric concentration of a substance
T absolute temperature; also: transformation of energy from mean flow into turbulent energy
$u = -q/c_p\varrho T$
U potential energy of the flow
v mean velocity
v' velocity fluctuation

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Boundary-Layer Meteorology **2** (1971) 7–29.

$v_* = (\tau/\varrho)^{1/2}$ friction velocity
z height coordinate
$\alpha = \alpha_T = K_T/K$
$\alpha_s = K_s/K$
$\beta = (\alpha Ri_{cr})^{-1}$
γ_a adiabatic lapse rate
$\zeta = z/L_1$
$\eta = Ri/Ri_{cr}$
θ potential temperature
λ molecular thermal conductivity
μ molecular viscosity
$\xi = \beta\zeta$
ϱ density
τ surface stress

1. Introduction

The problem of the dependence of the eddy diffusivity in the atmosphere on the gradient of the wind velocity, the distribution of temperature, and the height above ground-level is investigated in the present work.

The empirical data cited in meteorological literature concerning the distribution of the average wind velocity (see Brunt, 1941) show that the temperature distribution term defining the degree of atmospheric stability is a highly important factor.

Theoretical research on turbulence in an incompressible environment with inhomogeneous density (Prandtl, Tollmien, Taylor and others; see Goldstein, 1938) always leads to a definite dimensionless criterion characterizing the influence of environmental heterogeneity on turbulence:

$$g\,\frac{\partial \ln \varrho/\partial z}{(\partial v/\partial z)^2} \tag{1}$$

where g is the acceleration of gravity, ϱ is the density of the fluid, $\partial v/\partial z$ is the mean velocity gradient.

Richardson (1920) showed that an analogous criterion for the terrestrial atmosphere – the compressible environment – has the following appearance:

$$Ri = \frac{g}{T}\,\frac{\partial\theta/\partial z}{(\partial v/\partial z)^2} \tag{1a}$$

where θ is the potential temperature and T is the mean absolute temperature. The Richardson number Ri determines the qualitative character of turbulence in the atmospheric boundary layer. Three cases ought to be discussed here:

(1) Richardson number, positive: $Ri > 0$. The stratification is stable, hindering the development of turbulence. Turbulence is completely suppressed when Ri is larger than some critical value $(Ri)_{cr}$. The value of this critical number is estimated by various

authors on the basis of theoretical considerations. However, reliable values for $(Ri)_{cr}$ can be anticipated only from appropriate treatment of the experimental data.

(2) Richardson number, negative: $Ri<0$. The potential temperature decreases with height, i.e., the gradients are superadiabatic and the stratification is unstable. The energy of the turbulence is increased at the expense of the energy of instability.

(3) The limiting neutral case: $Ri=0$. The distribution of temperature with height is adiabatic. The stratification is neutral and does not influence the development of vertical movement (turbulence). The turbulent processes in such an atmosphere basically occur exactly as in an incompressible environment with a constant density. This case is considered in greater detail in the so-called 'semi-empirical' theory of turbulence.

In the present work an attempt is made to generalize the basic equations of the semi-empirical theory of turbulence to the case of a diabatic atmosphere, by introducing a correction factor which depends on the Richardson number.

With the aid of the resulting equations, the problem concerning the distribution of wind and temperature in the surface layer is examined. It turns out that for the stable case the eddy diffusivity increases linearly with height near the Earth's surface and approaches asymptotically a constant at a height of several dozen meters. The order of magnitude for the maximum value of the eddy diffusivity also agrees with reality.

2. The Semi-Empirical Theory of the Neutral Atmospheric Surface Layer

Turbulent processes in a uniform liquid may be described by Prandtl's equation with an accuracy completely satisfactory for practical applications:

$$\tau = \varrho l^2 \left(\frac{\partial v}{\partial z}\right)^2 \tag{2}$$

composing the basis of the semi-empirical theory of turbulence (cf. Velikanov, 1936).*

Here τ is the turbulent shear stress, l is the 'mixing length' according to Prandtl** and $\partial v/\partial z$ is the velocity gradient.

Later on we will call l 'the turbulent length scale' at a given point of the flow. For the flow over an infinite flat surface, the length scale is proportional to the distance to the wall:

$$l = kz \tag{3}$$

where k is von Karman's constant which can be taken as approximately equal to 0.4.

Prandtl's equations can be obtained from similarity theory. If it is to be assumed that the Reynolds stress depends only on the gradient of velocity, the density of the liquid, and the 'scale of turbulence', i.e., a characteristic length, then dimensional

* This reference is to a collection of Russian translations of papers on turbulence which includes, in particular, classical papers by Prandtl, von Karman, and Taylor.

** See also: Prandtl, L.: 1929, 'Einfluss Stabilisierender Kräfte auf die Turbulenz, in *Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete*.

analysis shows that only Prandtl's formula can be (with accuracy up to a numerical constant) the unknown relation.

It is possible to include the numerical coefficient in the value l. Also Formula (3) is obtained from a similar consideration of dimensionality, since direct proportionality is the sole relation between two lengths (l, z) which is not changed by the similarity transformations (the flow over a surface assumes such transformations).

Following Boussinesq's idea, developed in detail in application to meteorology by W. Schmidt*, it is possible to characterize turbulence with the aid of some conditional coefficients of 'turbulent exchange', namely 'eddy viscosity', and 'eddy thermal conductivity', introducing them formally by analogy with molecular coefficients of diffusion, viscosity, and thermal conductivity.

The coefficient of eddy viscosity A (Austausch coefficient) is defined by the equation

$$\tau = A\frac{\mathrm{d}v}{\mathrm{d}z}. \tag{4}$$

Formula (4) does not express any physical fact connected with the nature of turbulence but in essence represents a definition of some new variable – 'the coefficient of eddy viscosity' A.

Comparing formula (4) with Prandtl's Equation (2), we obtain the following expression for the coefficient A:

$$A = \varrho l^2 \left(\frac{\mathrm{d}v}{\mathrm{d}z}\right). \tag{5}$$

The analog of the kinematic viscosity, introduced by Taylor**

$$K = \frac{A}{\varrho} \tag{6}$$

carries the name 'coefficient of turbulence' and in the classical theory of atmospheric turbulence (of Schmidt, 1925; Richardson, 1920; and others) is usually identified with the coefficient of eddy diffusivity (see Brunt, 1941).

Instead of shear stress τ, it is convenient to introduce the 'friction velocity' v_*, defined by

$$v_* = \sqrt{\tau/\varrho}. \tag{7}$$

Thus, with these symbols Prandtl's equation is written

$$v_*^2 = l^2 \left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2$$

or

$$l\frac{\mathrm{d}v}{\mathrm{d}z} = v_* \tag{8}$$

* Schmidt, W.: 1925, *Der Massenaustausch in freier Luft und verwandte Erscheinungen*, Henri Grand Verlag, Hamburg.

** Taylor, G. I.: 1915, 'Eddy Motion in the Atmosphere', *Phil. Trans. Roy. Soc.* **A215**, 1–26.

and the coefficient of turbulence K is determined by the formula

$$K = v_* l. \tag{9}$$

The aforementioned 'coefficient of turbulence' K is introduced in a purely formal fashion and in contrast to the molecular viscosity it represents a variable and depends both on the dynamics of the flow – i.e., friction velocity – and also on the geometrical coordinates. Therefore, it makes no sense to use the 'eddy viscosity coefficient' for purely dynamic calculations, when it is considerably more convenient to employ Prandtl's Equation (2) directly.

However, in calculating the turbulent transfer of different substances (moisture, heat, smoke), the methods of 'classical' theory, created by Schmidt, 1925; Taylor, 1915; Richardson, 1920; and others, based on the concept of the coefficient of eddy diffusivity (coefficient of exchange) can prove helpful if some new supplementary hypotheses are to be accepted.

A comprehensive description of the basic ideas of the 'classical' theory of atmospheric turbulence may be found in the work of Keller 1930, in which three postulates are formulated in regard to the properties of transferable substances:

(1) While moving unmixed with the surrounding air, a substance possesses the property of indestructibility; its quantity in an elementary volume remains unchanged.

(2) The total quantity of a substance is preserved during the mixing of two masses of air (continuity property).

(3) A substance is 'passive'; its admixture to the environment (to the air) does not have an essential influence on the development of turbulence.

With these basic assumptions, using a statistical method, the classical theory of turbulent mixing leads to the usual diffusion equation, describing the transfer of a substance in a vertical direction during the process of turbulent mixing

$$F_s = - K_s \varrho \frac{\partial s}{\partial z} \tag{10}$$

where F_s is the mean flux of the substance, s is the gravimetric concentration of the substance, and K_s is the 'coefficient of eddy diffusivity'. K_s does not depend on the nature of the transported substance, if only it satisfies the aforementioned requirements. K_s is some statistical characteristic of turbulence and may depend on the height above ground-level of the point in the flow under consideration, the wind speed, and the vertical temperature gradient.

The theory of turbulent mixing, evidently, may be applied without serious objections to material contaminants suspended in the air (fine dust, smoke, bacteria, moisture, trace elements mixed with the air, etc.), since the three aforementioned conditions are fulfilled for them with reasonable accuracy.

However, many authors, including even the founder of the theory, W. Schmidt, go considerably farther and postulate an application of the theory of turbulent transfer to such 'substances' as momentum and heat. With such a point of view the shear stress τ is interpreted as the flux of momentum and the coefficient of turbulence K, formally determined from Equations (4) and (6), should coincide with the eddy diffusivity K_s

$$K_s = K = v_* l .$$

Similarly to the determination of K, it is possible to determine a coefficient K_T for processes of turbulent heat exchange on the basis of the equation for turbulent heat transfer:

$$q = - c_p \varrho K_T \frac{\partial \theta}{\partial z} \tag{11}$$

where q is the mean heat flux, c_p is the specific heat at constant pressure of the air, ϱ is the density of the air, and $\partial\theta/\partial z$ is the vertical gradient of the potential temperature.

If Schmidt's hypothesis is to be accepted, in accordance with which the general equation for turbulent transfer (10) is applicable also to heat, then,

$$K_T = K_s .$$

The assumption concerning numerical equality of the exchange coefficients for momentum, material contaminants, and heat cannot be considered as strictly established. While the applicability of the second postulate to the momentum and heat is indubitable, the fulfillment of the first postulate for momentum is far from evident. The exchange of momentum between two air masses can occur with a purely dynamic influence without considerable intermixing of these masses. Also, the third postulate cannot be strictly applied to heat, inasmuch as the distribution of temperature affects the development of turbulence.

Therefore, it is possible to speak only about the approximate coincidence of the order of magnitude of the coefficients K, K_s, and K_T, but their numerical values can be different from each other. This agrees with the observations and is quite natural from the point of view of the similarity theory.

The dynamic coefficient of turbulence K, formally defined with the aid of Formulas (4) and (6), and Schmidt's coefficients of turbulent diffusion K_s and K_T have the same dimension $L^2 T^{-1}$ and simultaneously become zero, when turbulence is absent. The ratio

$$\frac{K_s}{K} = \alpha_s$$

is a dimensionless parameter which should be some function of the dimensionless characteristics of a flow.

The only external dimensionless characteristic of a homogeneous turbulent flow having low velocities, in a weakly compressed liquid not possessing a free interface, is the Reynolds number.

The semi-empirical theory of turbulence is applied in the range of very high Reynolds numbers when all the dimensionless values – functions of Reynolds numbers – are roughly equal to their limits as $Re \to \infty$ (the 'self-similarity' of the flow). The Reynolds number under normal atmospheric conditions is extremely large.

Under the given assumptions

$$\alpha_s \approx \text{const} \qquad K_s = \alpha_s K ,$$

the equation of transfer (10) acquires the following appearance after the substitution of the expression for K_s in it:

$$F_s = -\alpha_s \varrho l^2 \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \frac{\mathrm{d}s}{\mathrm{d}z} \tag{12}$$

where α_s is the dimensionless coefficient of the order of unity, depending upon the nature of the substance. It shall be different for momentum, material contaminants and heat.

In the case of a homogeneous environment, the coefficient α_s is equal to unity for momentum. Its corresponding value for the process of heat transfer is denoted by α_T. Then the equation for the turbulent heat transfer is written

$$q = -\alpha_T c_p \varrho l^2 \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \frac{\partial\theta}{\partial z}. \tag{13}$$

Generally speaking, the coefficient α_T depends upon Prandtl's number for a given environment

$$Pr = \frac{\mu c_p}{\lambda}$$

where μ is the molecular viscosity, and λ is the molecular thermal conductivity.

3. Generalization of Equations of the Turbulence Theory

Among the assumptions formulated above, the hypothesis of 'neutrality', i.e., the passive role of a transported substance, is highly important. This condition is not fulfilled in many important practical cases, and in the above equations it is necessary to introduce a correction which accounts for the effect of the transfer of heat or material substance on the development of turbulence.

This effect is due to the fact that because of concentration differences in the transfer of substance, or temperature differences (in the transfer of heat in various points) of a liquid, supplementary buoyancy forces promoting or hampering the development of turbulence arise.

For the calculation of this factor a dimensionless criterion is suggested by Prandtl

$$-g\frac{\partial\, ln\, \varrho}{\partial z}\left(\frac{\partial v}{\partial z}\right)^{-2}$$

which is the sole dimensionless combination of the following local characteristics of a flow with a variable density:

$$\varrho, \frac{\partial \varrho}{\partial z}, g, \frac{\partial v}{\partial z}.$$

An analogous criterion for the atmosphere was suggested by Richardson*

$$Ri = \frac{g}{T} \frac{\partial\theta/\partial z}{(\partial v/\partial z)^2}$$

where $\partial\theta/\partial z$ is the gradient of potential temperature. It is possible to write the Richardson number in the following form:

$$Ri = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \gamma_a\right)\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^{-2} \tag{14}$$

where $\gamma_a = 0.01\,°\mathrm{C/m}$ is the adiabatic lapse rate.

In light of the considerations of similarity theory, it is natural to assume that all dimensionless characteristics of a turbulent flow with a variable density (or variable potential temperature in the case of the atmosphere) are definite functions of a basic dimensionless parameter, i.e., the Richardson number.

Furthermore, if it is to be postulated that coefficients of exchange for different 'substances' (momentum, heat, moisture, smoke, and so forth), formally defined on the basis of the turbulent transfer Equation (12), are proportional to each other in the case of a stratified atmosphere as well, it is sufficient to know the 'universal' function of the Richardson number $\phi(Ri)$ alone, which is the correction coefficient in the turbulent transfer equation. The coefficient of turbulence under conditions in the atmosphere with an adiabatic distribution of temperature will be designated as K_0 (Richardson number equal to zero). Then, the coefficient of turbulence K may be expressed by the formula

$$K = \phi(Ri)\,K_0 \tag{15}$$

for an inhomogeneous atmosphere under the same dynamic conditions (the same mean velocity distribution).

The determination of the dimensionless function ϕ is a highly complex task and requires a deeper study of the structure of turbulence. An attempt to determine theoretically the function $\phi(Ri)$ is made in Section 5.

However, some general information in regard to the function $\phi(Ri)$ may be obtained from completely elementary considerations. By the very definition of the function $\phi(Ri)$ with $Ri=0$, $\phi(0)=1$. When the Richardson number is increased, the stratification of the atmosphere becomes more stable and due to this, the intensity of turbulence is reduced. Therefore, $\phi(Ri)$ is a monotonically decreasing function. In general, turbulence is not observed for Richardson numbers larger than some critical value, Ri_{cr}, and consequently,

$$\phi(Ri) = 0 \quad \text{for} \quad Ri > Ri_{cr}. \tag{16}$$

A number of theoretical works, regarding the stability of the motion of a non-uniform liquid, are devoted to the simulation of the value of the critical Richardson

* Richardson, L. F.: 1920, 'The Supply of Energy from and to Atmospheric Eddies', *Proc. Roy. Soc.* **A97**, 354–73.

number. According to Richardson (1920),

$$Ri_{cr} \approx 1\,;$$

According to Prandtl, it is equal to $\frac{1}{2}$. According to the works of Taylor* and Goldstein**,

$$Ri_{cr} \approx \tfrac{1}{4}\,;$$

According to Tollmien (see Goldstein, 1931)†, $Ri_{cr} = \frac{1}{24}$. Corresponding processing of Sverdrup's data leads to

$$Ri_{cr} = \tfrac{1}{11}\,,$$

which is used later in numerical calculations. The determination of the critical *Ri* number is an important problem for atmospheric physics and may be solved only experimentally on the basis of the processing of reliable data for simultaneous measurements of wind and temperature distribution in the lower layer of the atmosphere (through observations from towers and captive balloons).

Now using Formula (15) and designating the ratio of the exchange coefficients for heat and momentum as α, the equations of turbulent friction (i.e., momentum transfer) and of heat transfer shall be written in their final form:

$$\tau = \phi(Ri)\,\varrho l^2 \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \frac{\mathrm{d}v}{\mathrm{d}z} \tag{17}$$

$$q = -\,\alpha\phi(Ri)\,c_p\varrho l^2 \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{18}$$

where

$$Ri = \frac{g}{T}\frac{\partial\theta/\partial z}{(\partial v/\partial z)^2}\,; \qquad l = kz\,.$$

The formulae obtained may serve for the calculation of the wind and temperature distribution in the layer near the Earth's surface (the surface layer).

4. The Distribution of the Exchange Coefficient in the Surface Layer

The surface layer plays a dual role in the atmosphere; firstly, friction stress is transmitted through it to the free atmosphere – as a result of the dynamic influence of the underlying surface on the air mass; secondly, heat exchange between the soil and the free atmosphere occurs through it.

* Taylor, G. I.: 1931, 'Effect of Variation in Density on the Stability of Supercooled Streams of Fluids', *Proc. Roy. Soc.* **A132**, 499–523.

** Goldstein, S.: 1931, 'On the Stability of Superposed Streams of Fluids of Different Densities', *Proc. Roy. Soc.* **A132**, 524–48.

† It is not clear which work of Tollmien is referred to here. H. Schlichting reported the same result in 1935 in 'Turbulenz by Warmeschichtung', *Proc. Fourth Int. Congress, Appl. Mech.*, Cambridge. p. 245, or in *Z. Angew. Math. Mech.* **15**, pp. 313–338.

The friction stress at the ground and the heat flux across an areal unit of the horizontal surface in a unit of time shall be designated as τ_0 and q, respectively. It shall be assumed that within the surface layer a certain stationary regime is established whereby external forces are negligible in comparison with the forces of internal (turbulent) friction and, moreover, the internal sources of heat (the heat of water vapor condensation and the absorption of radiation) are absent. With these assumptions defining in essence the concept of the surface layer of the atmosphere, the flow of momentum and heat remains unchanged with height:

$$\tau(z) = \tau_0 = \text{const} \tag{19}$$

$$q(z) = q_0 = \text{const}. \tag{20}$$

The conditions expressed by Formulae (19) and (20) are fulfilled in practice, evidently with sufficient accuracy, if z is not too great (they determine the surface layer). Let us write down the equations for momentum and heat transfer:

$$\tau = \tau_0 = \phi(Ri)\,\varrho\,{}^2 \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \frac{\mathrm{d}v}{\mathrm{d}z} \tag{17'}$$

$$q = q_0 = -\,\alpha\phi(Ri)\,c_p\varrho l^2 \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \frac{\mathrm{d}\theta}{\mathrm{d}z}. \tag{18'}$$

It is convenient to introduce 'the friction velocity' v_* and 'the heat flux velocity' in lieu of τ and q into the investigation by defining

$$v_* = (\tau/\varrho)^{1/2},$$

$$u = -\,q/c_p\varrho T.$$

Since the height of the surface layer is not great (of the order of a few tens of meters), the changes of absolute density and temperature within the layer are small and can be considered negligible. Therefore

$$v_* = v_{*0} = \text{const},$$

$$u = u_0 = \text{const}.$$

Using the new designations, Equation (18′) is divided term by term by (17′), yielding

$$\frac{1}{T}\frac{\partial\theta}{\partial z}\left(\frac{\partial v}{\partial z}\right)^{-1} = \frac{u}{\alpha v_*} = \text{const}$$

whence

$$\theta(z) = \frac{u_0 T}{\alpha v_*^2}\, v(z) + \text{const}. \tag{21}$$

This equation expresses the result that the distribution of temperature is similar, under the introduced assumption, to the distribution of the wind in the surface layer.

Using the values v_* and u introduced above, it is possible to write Equation (17) in the following form

$$\{\phi(Ri)\}^{1/2}\, l\,\frac{\mathrm{d}v}{\mathrm{d}z} = v_*. \tag{22}$$

Having divided Equation (18) term by term by the cube of the right and left parts of (22), the following equation for the Richardson number is obtained:

$$\frac{Ri}{\{\phi(Ri)\}^{1/2}} = \frac{1}{\alpha} g \frac{lu}{v_*^3}; \qquad l = kz. \tag{23}$$

Equation (23) may have no more than one solution for Ri, since the left part is a monotonically increasing function of Ri. Evidently when $l = 0$

$$Ri(0) = 0.$$

Also, since

$$Ri = \frac{1}{\alpha} g \frac{lu}{v_*^3} \{\phi(Ri)\}^{1/2} = \frac{k}{\alpha} z \frac{gu}{v_*^3} \{\phi(Ri)\}^{1/2}$$

$$\frac{\partial Ri}{\partial z} = \frac{k}{\alpha} g \frac{u}{v_*^3} \{\phi(Ri)\}^{1/2} + \frac{k}{\alpha} g \frac{u}{v_*^3} z \frac{\phi'(Ri)}{2\{\phi(Ri)\}^{1/2}} \tag{24}$$

and hence

$$\left(\frac{\partial Ri}{\partial z}\right)_{z=0} = \frac{k}{\alpha} g \frac{u}{v_*^3}; \qquad \phi(0) = 1. \tag{25}$$

The value of Ri near ground-level approaches zero (at ground-level itself, $Ri = 0$). Consequently, in an inhomogeneous atmosphere with a given distribution of the potential temperature within the surface layer, a sub-layer exists in which the influence of the atmospheric stratification (stable or unstable) is small and the turbulence is determined only by dynamic factors (Ri small).

With Expression (25) for the derivative of Ri, it is possible to determine a new conditional characteristic, i.e., 'the height of the sub-layer of dynamic turbulence' (length scale).

$$L_1 = \frac{1}{\alpha \left(\frac{\partial Ri}{\partial z}\right)_{z=0}} = \frac{v_*^3}{kgu}. \tag{26}$$

The scale L_1 determined in such a manner does not depend on the shape of the 'universal' function $\phi(Ri)$. L_1 is directly expressed by τ_0 and q_0

$$L_1 = -\frac{c_p T \tau^{3/2}}{kg\varrho^{1/2} q}. \tag{27}$$

The value L_1 was calculated by Formula (27) for different conditions. Instead of the friction velocity v_*, the 'wind speed' v_0, taken as equal to 20 v_*, is cited in Table I. The coefficient 20 corresponds to Taylor's formula

$$\tau = 0.0025\, \varrho v_0^2.$$

v_0 is related to the wind velocity at a fixed height ($H \approx 30$ m) depending on the roughness of the surface. The heat flux in Table I is given in cal cm^{-2} min^{-1}. The quantity h in Table I is the rate of ice melting produced by heat flux $|q|$ (in mm hr^{-1} of water column thickness).

TABLE I

Height of the sub-layer of dynamic turbulence (in meters)

$\lvert q \rvert$ cal/cm^2 min		0.01	0.02	0.05	0.1	0.15	0.2	0.3
h mm/hr		0.0754	0.1508	0.377	0.754	1.131	1.508	2.262
u cm/s		2.19×10^{-3}	4.38×10^{-3}	10.9×10^{-3}	21.9×10^{-3}	32.8×10^{-3}	43.8×10^{-3}	65.7×10^{-3}
$v_0 = 20v_*$ m/s	v_* cm/s							
0.5	2.5	0.182	0.091	0.036	0.018	0.012	0.008	0.006
1	5	1.45	9.73	0.29	0.14	0.10	0.07	0.05
2	10	11.63	5.81	2.34	1.16	0.78	0.58	0.39
3	15	39.2	19.6	7.9	3.9	2.6	1.9	1.3
4	20	93.0	46.5	18.7	9.2	6.3	4.6	3.1
5	25		90.8	36.5	18.2	12.1	9.1	6.0
6	30			63.1	31.4	21.0	15.7	10.5
10	50					97.0	72.7	48.4

In the calculation the following values for the constants were used:

$$k = 0.4\,; \qquad c_p = 0.24 \text{ cal g}^{-1}\text{ C}^{-1}\,; \qquad g = 981 \text{ cm s}^{-2}\,;$$
$$T = 290\,\text{K}\,; \qquad \varrho = 1.29 \times 10^{-3} \text{ g cm}^{-3}\,.$$

In the range of the most probable values for wind velocity and heat flux, the length L_1 ranged between 8 and 50 m.

Let us now consider the distribution of the Richardson number and the coefficient of turbulence with height. The following notations are introduced for convenience:

$$\frac{Ri}{Ri_{cr}} = \eta\,;$$
$$\frac{z}{L_1} = \zeta\,;$$
$$\phi(Ri) = \phi_1(\eta)\,;$$
$$\beta = \frac{1}{\alpha Ri_{cr}}\,;$$

where Ri_{cr} is the critical Richardson number, ζ is a dimensionless height, and β is a numerical constant of the order of unity. In accordance with the previously formulated properties of the function $\phi(Ri)$, the function $\phi_1(\eta)$ satisfies the following conditions:

1° $\phi_1(0) = 1$

2° $\phi_1'(\eta) \leqslant 0$

3° $\phi_1(\eta) = 0, \quad \text{for} \quad \eta > 1\,.$

First the case of a stable distribution of temperature shall be examined, when $Ri > 0$ and the heat flux is directed from the atmosphere to the ground. It is now possible to write Equation (23) in the following dimensionless form:

$$\eta\,\{\phi_1(\eta)\}^{-1/2} = \beta\zeta\,. \tag{23a}$$

A new function $\eta = \psi(\xi)$ is introduced – the results of the solution of the equation

$$\eta \{\phi_1(\eta)\}^{-1/2} = \xi$$

in regard to η. Then the transformation of the solution of (23a) to initial variables gives the following expression for the Richardson number:

$$Ri(z) = Ri_{cr}\,\psi(\beta z/L_1),$$
$$L_1 = \frac{v_*^3}{kgu},$$
$$\beta = \frac{1}{\alpha Ri_{cr}}.$$

Using the earlier proven properties of the function ϕ_1, the general statement concerning the function ψ is easily obtained

$$\psi(0) = 0; \qquad \psi(\xi) \leqslant 1; \qquad \psi'(\xi) \geqslant 0;$$
$$\psi(\xi) \to 1 \quad \text{for} \quad \xi \to \infty$$
$$\frac{\psi(\xi)}{\xi} \to 1 \quad \text{for} \quad \xi \to 0. \tag{28}$$

Thus, the Richardson number approaches asymptotically its critical value in the surface layer of a stable atmosphere, for z much greater than L_1.

The variation of the coefficient of turbulence with height can be also examined. Due to the definition of the coefficient of turbulence:

$$K = \frac{\tau}{\varrho \left|\frac{dv}{dz}\right|} = \frac{v_*^2}{\left|\frac{dv}{dz}\right|};$$

or in dimensionless form

$$K(\xi) = \frac{v_*}{\left|\frac{dv}{dz}\right| L_1} = k\zeta\sqrt{\phi_1(\eta)}.$$

But according to Equation (23a) $\sqrt{(\phi_1(\eta))}\,\zeta = \eta/\beta$ and hence returning to dimensional form

$$K(z) = \frac{kv_*L_1}{\beta}\,\eta = \frac{kv_*L_1}{\beta}\,\psi\left(\beta\frac{z}{L_1}\right). \tag{29}$$

Let us now investigate the height variation of the exchange coefficient for large and small values of height z (in regard to scale L_1). With small values of z, $z \ll L_1$, on the basis of (28),

$$\psi\left(\beta\frac{z}{L_1}\right) \simeq \beta\frac{z}{L_1}; \qquad K(z) \simeq kv_*z$$

with large values of z, $z \gg L_1$, $\psi \to 1$,

$$K(z) \to \frac{kv_*L_1}{\beta}.$$

TABLE II

Maximum value of the coefficient of turbulence $K_\infty = u_*^4/\beta g u$

$\|q\|$ cal/cm² min		0.01		0.02		0.05	
h mm/hr		0.0754		0.1508		0.377	
u cm/s		2.19×10^{-3}		4.38×10^{-3}		10.9×10^{-3}	
$v_0 = 20v_*$ m/s	v_* cm/s	K_∞	A_∞	K_∞	A_∞	K_∞	A_∞
0.5	2.5	0.00182	0.0224	0.00091	0.0112	0.00036	0.00443
1	5	0.029	0.3567	0.014	0.1722	0.006	0.0738
2	10	0.465	5.7195	0.232	2.8536	0.093	1.1439
3	15	2.35	28.905	1.18	14.514	0.47	5.781
4	20	7.44	91.512	3.72	45.756	1.49	18.327
5	25	–	–	9.08	111.684	3.64	44.895
6	30	–	–	–	–	7.55	93.111
10	50	–	–	–	–	–	–

Thus, the coefficient of turbulence increases linearly with height for small values of z, and approaches a maximum value asymptotically for large values of z:

$$K_\infty = \frac{k v_* L_1}{\beta} \tag{30}$$

or, after using (27) and the definitions for u and v_*

$$K_\infty = \frac{1}{\beta} \frac{v_*^4}{g u}. \tag{31}$$

The results of the calculations of K_∞ are cited in Table II; an approximate value for β, $\beta = 1$, is used; K_∞ is given in m² s⁻¹ and the exchange coefficient A in CGS units.

The theoretical values given in the table for the turbulence characteristics possess the same order of magnitude as those observed in the atmosphere. Thus, for example with a wind velocity of $v_0 = 5\ \mathrm{m\,s^{-1}}$, $v_* = 0.25\ \mathrm{m\,s^{-1}}$ and with a heat flux directed downward,

$$q = -0.1\ \mathrm{cal\,cm^{-2}\,min^{-1}}$$

the coefficient of turbulence is

$$K_\infty = 1.82\ \mathrm{m^2\,s^{-1}}, \qquad A_\infty = 22.4\ \mathrm{g\,cm^{-1}\,s^{-1}}.$$

The values of the scale L_1 (thickness of the sub-layer of dynamic turbulence) and of the maximum value of the temperature gradient $(\partial\theta/\partial z)_\infty$ for the same conditions are

$$L_1 = 18.2\ \mathrm{m}, \quad \text{and} \left(\frac{\partial\theta}{\partial z}\right)_\infty = 0.31\ ^\circ\mathrm{C}\ (100\ \mathrm{m})^{-1}$$

$$\frac{\partial T}{\partial z} = -0.69\ \mathrm{C}\ (100\ \mathrm{m})^{-1}.$$

for a stable atmosphere in $m^2 s^{-1}$, $A = K\varrho$ g $cm^{-1} s^{-1}$

0.1		0.15		0.2		0.3	
0.754		1.131		1.508		2.262	
21.9×10^{-3}		32.8×10^{-3}		43.8×10^{-3}		65.7×10^{-3}	
K_∞	A_∞	K_∞	A_∞	K_∞	A_∞	K_∞	A_∞
0.00018	0.00221	0.00012	0.00158	0.00008	0.001084	0.00006	0.000738
0.003	0.0369	0.002	0.0246	0.001	0.0123	0.001	0.0123
0.046	0.5658	0.031	0.3813	0.023	0.2829	0.0015	0.1845
0.23	2.829	0.16	1.968	0.12	1.476	0.08	0.98
0.74	9.102	0.50	6.150	0.37	4.551	0.25	3.08
1.82	22.386	1.21	14.883	0.91	11.193	0.60	7.38
3.77	46.371	2.51	30.873	1.88	23.124	1.25	15.38
–	–	19.41	238.743	14.53	178.719	9.69	119.19

Thus, the order of magnitude of the temperature gradient calculated according to K_∞ also agrees with observations. In accordance with Sverdrup's observations, the value $Ri_{cr} = \frac{1}{11}$ was used during the calculations of the gradient.

The aforementioned variation of the exchange coefficient with height, i.e., a linear increase at low altitudes with an asymptotic approach to a constant at a certain elevation, is suggested by some authors, but here, evidently, a theoretical explanation of this fact is successfully presented for the first time and also a theoretical value is given for the order of magnitude of K_∞ and L_1. The diurnal variation of temperature with a variable exchange coefficient calculated theoretically by Dorodnitsyn* agrees well with the data. Dorodnitsyn used a law of height variation of the exchange coefficient which is similar to the above indicated linear increase up to a height $L =$ 20 m and a constant $K_\infty = 8.1$ $m^2 s^{-1}$ for greater heights.

In the above developed theory the changes of the shear stress with height were not considered. Due to the earth's Coriolis force, shear stress decreases with altitude, so that the height of the 'surface layer', to which this discourse is related, is limited to 100 m or so.

5. An Equation for the Energy Balance in Turbulent Flow with Variable Density

Equation (17) may also be obtained from the examination of a simplified statistical scheme of turbulence in a non-homogeneous environment on the basis of the energy balance. In this case an analytical solution is obtained for the dimensionless function $\phi(Ri)$.

* Dorodnitsyn, A. A.: 1941, 'The Theory of the Diurnal Variation of Temperature in the Mixing Layer', *Dokl. Akad. Nauk SSSR* **30**, 410–3. In this paper the vertical profile of the coefficient K was assumed to be of the form $K(z) = K_\infty[1 - \exp(-z/L)]$.

To simplify the calculations, an incompressible fluid with a variable density depending upon the temperature will be examined first. The transition to a compressible environment, i.e., the atmosphere, may be made by substitution of the Richardson number Equation (1a) instead of the dimensionless Prandtl Equation (1) criterion of stability.

The equation of the balance of turbulent energy for a non-homogeneous fluid may be written as

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{\mathrm{d}E'}{\mathrm{d}t} = T - D \tag{32}$$

where U is the potential energy of the flow, E' is the kinetic energy of the turbulent fluctuations, T is the transformation of energy of the mean flow into turbulent energy and D is the dissipation of turbulent energy into heat. An equation of the energy balance analogous to Equation (32) was utilized by Richardson when he deduced the basic criteria of turbulence. Such a statement refers to the phenomenon of the growth or suppression of turbulence in a non-homogeneous environment.

The balance equation will be utilized for the study of a steady-state regime of turbulence. The dissipation of turbulent energy in a developed turbulent regime shall be considered (on the basis of similarity considerations at Reynolds number $Re \to \infty$) to be directly proportional to the cube of a velocity fluctuation and inversely proportional to a characteristic length, i.e., the scale of turbulence.

If the expression used by Richardson for the change of potential energy dU/dt and the transformation of the energy of a mean flow is to be accepted, then the equation of the energy balance may be written as follows:

$$\frac{\mathrm{d}E'}{\mathrm{d}t} = K\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - K_T \frac{g}{T}\frac{\partial\theta}{\partial z} - D \tag{33}$$

where dv/dz is the mean velocity gradient, K is the dynamic coefficient of turbulence (i.e., eddy viscosity), and K_T is the coefficient of eddy thermal diffusivity.

If the length scale of turbulence (Prandtl's mixing length) is designated as l and the scale velocity fluctuations as v', then

$$\begin{aligned} K &= v'l \\ K_T &= \alpha v' \\ D &= \beta' \frac{v'^3}{l} \\ E' &= \frac{v'^2}{2}. \end{aligned}$$

In the first approximation the numerical coefficients α and β' shall be considered as constants. Substituting the above-written expressions in (33), one obtains for the stationary regime

$$\frac{\partial E'}{\partial t} = v' l \left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - \alpha v' l \frac{g}{T}\frac{\partial \theta}{\partial z} - \beta' \frac{v'^3}{l} = 0$$

or

$$v' \left[l^2 \left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - \alpha l^2 \frac{g}{T}\frac{\partial \theta}{\partial z} - \beta' v'^2 \right] = 0 \tag{34}$$

Equation (34) is broken into two equations:

$$v' = 0$$

corresponding to a laminar regime, and

$$l^2 \left[\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - \alpha \frac{g}{T}\frac{\partial \theta}{\partial z} \right] = \beta' v'^2 \tag{35}$$

describing the developed turbulent regime.

Equation (35) has a real solution only when

$$\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - \alpha \frac{g}{T}\frac{\partial \theta}{\partial z} \geqslant 0\,.$$

Hence, with a turbulent regime,

$$Ri = \frac{g}{T}\frac{\partial\theta/\partial z}{(\partial v/\partial z)^2} \leqslant Ri_{cr} = \frac{1}{\alpha} \tag{36}$$

and with $Ri > Ri_{cr}$ there is a single real solution

$$v' = 0$$

corresponding to the laminar regime.

For the turbulent regime, using (35) and (36) it is seen that

$$v' = \frac{l}{(\beta')^{1/2}} \left|\frac{\mathrm{d}v}{\mathrm{d}z}\right| \left(1 - \frac{Ri}{Ri_{cr}}\right)^{1/2} \tag{37}$$

so that a coincidence with Prandtl's well-known equation for $Ri = 0$ will be obtained only if $\beta' = 1$.

Thus by examining the energy balance, $\phi_1(\eta)$ is found to be

$$\phi_1(\eta) = (1 - \eta)^{1/2}\,; \qquad \eta = \frac{Ri}{Ri_{cr}}\,. \tag{38}$$

Simultaneously, an approximate value for the critical Richardson number is obtained:

$$Ri_{cr} \approx \frac{1}{\alpha} = \frac{K}{K_T}\,.$$

The works of Sverdrup (1936) and of Montgomery* must be mentioned, in which an attempt to obtain the correct function (Ri) is also made. Sverdrup's result is

$$\phi(Ri) = \frac{1}{(1 + \beta Ri)^{1/2}}$$

which satisfies the above given formula only for small values of the Richardson number; thus $\beta = 1/Ri_{cr}$. When large Richardson numbers are used, the formula of Sverdrup is not valid, because it does not describe the critical phenomena when $Ri > (Ri)_{cr}$ and leads to an entirely unrealistic result when $Ri \to -1/\beta$ in the unstable atmosphere. Sverdrup's observations of the surface layer indicate that

$$\beta = 11,$$

hence

$$Ri_{cr} = 0.09.$$

If Equation (38) is accepted, then it is possible to calculate the distribution of the temperature and wind in the surface layer where the influence of Coriolis force is negligible. These calculations are given in the following section.

6. The Distribution of Temperature and Wind Velocity in the Surface Layer

The equations of turbulent transfer of momentum and heat (17) and (18) are used for calculating the wind and temperature distribution and Equation (38) is used for the correction function ϕ.

The dimensionless function ψ, introduced above (following (23a)) and determining the variation of Richardson's number, may now be calculated by the solution of Equation (23):

$$\frac{\eta}{(1 - \eta)^{1/4}} = \xi, \qquad \eta = \psi(\xi). \tag{39}$$

Two cases are examined separately:

(1) *The stable atmosphere.* ($Ri > 0$, $\eta > 0$). It is convenient to introduce the auxiliary parameter, u',

$$1 - \eta = u'^4; \qquad \xi = \frac{\eta}{u'}.$$

Then the following parametric representation of the function ψ can be given;

$$\eta = \psi(\xi); \qquad \xi = \frac{\eta}{u'} = \frac{1}{u'} - u'^3; \qquad \xi = 0; u' = 1$$

$$\eta = 1 - u'^4; 0 < u' < 1; \qquad \xi = \infty; u' = 0.$$

A diagram of the resulting function $Ri/Ri_{cr} = \psi(z/L_1)$ is presented in Figure 1.

* The work that is referred to here is probably Rossby, C. G. and Montgomery, R. B.: 1935, 'The Layer of Frictional Influence in Wind and Ocean Currents', in *Papers in Physical Oceanography and Meteorology*, MIT and Woods Hole Oceanographic Institution, Volume 3, No. 3.

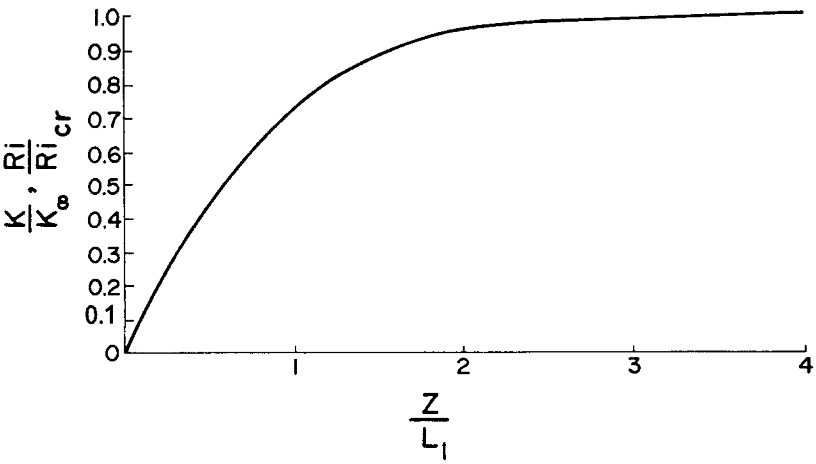

Fig. 1. The dependence of the coefficient of turbulent K on height for a stable atmosphere.

On the basis of Equation (36) and Equations (29) and (30), also

$$\frac{K}{K_\infty} = \psi\left(\frac{z}{L_1}\right).$$

A differential equation exists for the distribution of wind:

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{v_*^2}{K(z)}$$

hence

$$\frac{1}{v_*}\frac{\mathrm{d}v}{\mathrm{d}\xi} = \frac{1}{k}\frac{K_\infty}{K(z)} = \frac{1}{k}\frac{1}{\psi(\xi)}.$$

For the difference of the wind speeds at two levels one obtains

$$v(z_2) - v(z_1) = \frac{v_*}{k}\int_{\xi_1}^{\xi_2}\frac{\mathrm{d}\xi}{\psi(\xi)} = \frac{v_*}{k}\left[u'\left(\frac{z_2}{L_1}\right) - u'\left(\frac{z_1}{L_1}\right)\right].$$

An integral solution is easily obtained by the introduction of the parameter u'. The values of the function $\psi(\xi)$ and $u'(\xi)$ are given in Table III.

The height variation of wind velocity in a stable atmosphere is shown in Figure 2. A simple investigation shows that for low heights (in regard to L_1) the curve has a logarithmic character, while for greater heights it approaches asymptotically a straight line.

The curve of the temperature distribution with the above mentioned assumptions is similar to the curve of the wind velocity. From Equation (21) it follows

$$\theta(z_2) - \theta(z_1) = T\frac{1}{\alpha k}\frac{u}{v_*}\left[u'\left(\frac{z_2}{L_1}\right) - u'\left(\frac{z_1}{L_1}\right)\right].$$

TABLE III

Values of the universal dimensionless function $\psi(\xi)$ and of $u'(\xi)$ for a stable atmosphere

ξ	$\psi(\xi)$	$u'(\xi)$	ξ	$\psi(\xi)$	$u'(\xi)$
0.05	0.055	1.600	1.4	0.86	5.41
0.10	0.102	2.370	1.5	0.88	5.23
0.15	0.144	2.742	1.6	0.90	5.63
0.20	0.189	3.065	1.7	0.92	5.74
0.25	0.231	3.320	1.8	0.93	5.85
0.30	0.278	3.500	1.9	0.94	5.95
0.35	0.320	3.662	2.0	0.95	6.06
0.40	0.359	3.803	2.1	0.96	6.16
0.45	0.398	3.928	2.2	0.96	6.27
0.50	0.435	4.045	2.3	0.97	6.37
0.55	0.470	4.157	2.4	0.97	6.47
0.60	0.502	4.258	2.5	0.98	6.57
0.65	0.533	4.360	2.6	0.98	6.68
0.70	0.565	4.450	2.7	0.98	6.78
0.75	0.597	4.560	2.8	0.98	6.88
0.80	0.626	4.608	2.9	0.99	6.99
0.85	0.650	4.695	3.0	0.99	7.09
0.90	0.677	4.769	3.5	0.99	7.60
0.95	0.700	4.839	4.0	1.00	8.10
1.00	0.723	4.908	4.5	1.00	8.60
1.1	0.76	5.03	5.0	1.00	9.10
1.2	0.80	5.16	5.5	1.00	9.60
1.3	0.84	5.29	6.0	1.00	10.10

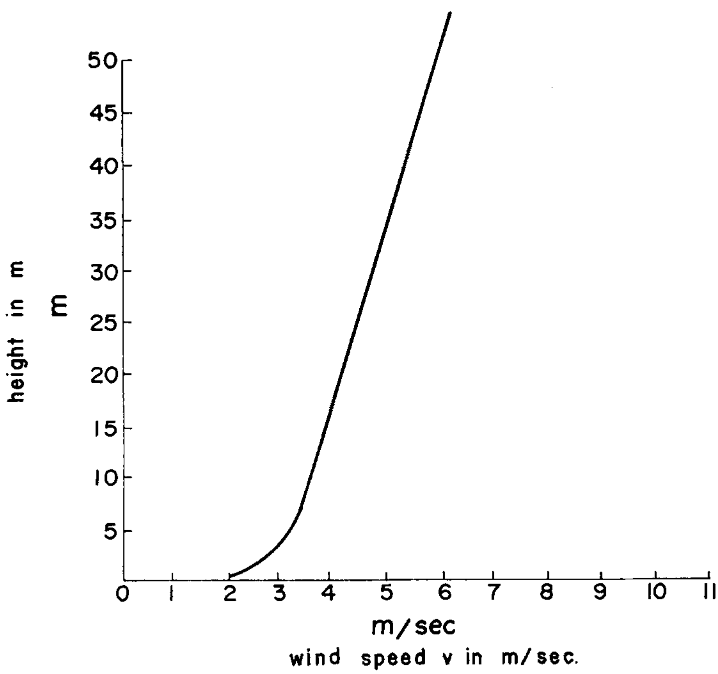

Fig. 2. The distribution of wind velocity in a stable atmosphere.

(2) *The unstable atmosphere.* ($Ri<0$, $\eta<0$, the heat flux is directed upwards). The function ψ is again determined by the Equation (39):

$$\frac{|\eta|}{(1+|\eta|)^{1/4}}=\xi\,.$$

The auxiliary parameter u' is introduced:

$$1+|\eta|=u'^4\,,$$
$$|\eta|=u'^4-1\,,$$
$$\xi=\frac{u'^4-1}{u'}=u'^3-\frac{1}{u'}$$

then

$$\begin{array}{l} u'=1 \quad \text{for} \quad \xi=0 \\ u'=\infty \quad \text{for} \quad \xi=\infty \end{array} \qquad 1<u'<\infty\,.$$

For large values of ξ

$$u'\approx|\xi|^{1/3} \quad \text{and} \quad |\eta|\approx|\xi|^{4/3}\,.$$

Consequently, the exchange coefficient possesses the following asymptotic expression for an unstable atmosphere:

$$K(z)=k^{4/3}(gu)^{1/3}\,z^{4/3}\,. \tag{40}$$

The calculations of wind velocity and potential temperature distribution are made with the above-cited formulae. They are graphically presented in Figure 3. With

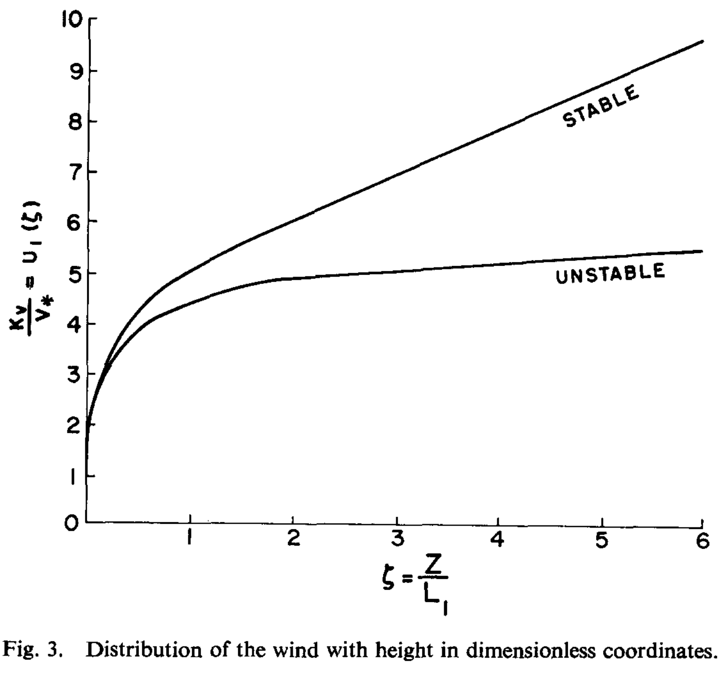

Fig. 3. Distribution of the wind with height in dimensionless coordinates.

large values of z, the wind (and potential temperature distribution) approach asymptotically a constant value in an unstable atmosphere.

Once more, when a sufficiently complete theoretical picture of turbulence in the surface layer is desired, it should be noted that the Coriolis force, which is extremely important in the calculation of the wind distribution at high altitudes, has been neglected.

7. Conclusion

The character and intensity of the turbulent processes in the atmosphere depend to a considerable degree upon the vertical distribution of temperature. Dimensional considerations show that the influence of the atmospheric stability on turbulence may be considered approximately by introducing a correction coefficient into a known equation of turbulent transfer, which depends upon the Richardson number, i.e., a dimensionless characteristic of atmospheric stability.

The equation obtained during this procedure contains a single 'universal', dimensionless function of the Richardson number $\phi(Ri)$, which, generally speaking, should be determined from experiment on the basis of a series of simultaneous observations of the temperature and wind distribution in the surface layer of the atmosphere (up to 100 m).

The general analysis of the equations applied to the problem concerning the distribution of the exchange coefficient in the surface layer leads to a definite expression for a length scale L_1 (the thickness of the layer of dynamic turbulence). The influence of the thermal factor on turbulence is negligible for heights small in comparison with this scale. The exchange coefficient asymptotically approaches a constant, independent of the particular choice of the universal function $\phi(Ri)$ for heights large in comparison with L_1 in a stable atmosphere.

When observational data with regard to the temperature and wind distribution in the surface layer are processed, the dimensionless heights, expressed in fractions of the scale L_1, ought to be utilized.

When the equation of the balance of turbulent energy and rough assumptions of the semi-empirical theory of turbulence are utilized, a definite expression for the 'universal function' is obtained

$$\phi(Ri) = \left(1 - \frac{Ri}{Ri_{cr}}\right)^{1/2} \quad \text{for} \quad Ri < Ri_{cr}$$

$$\phi(Ri) = 0 \quad \text{for} \quad Ri > Ri_{cr}$$

where Ri_{cr} is the critical Richardson number, determined from experiment. To check the results of the theory, reliable material obtained from simultaneous observations of the wind and temperature in the surface layer (on towers or captive balloons) is necessary.

References

Brunt, D.: 1941, *Physical and Dynamical Meteorology*, Cambridge University Press, London.
Goldstein, S.: 1938, *Modern Developments in Fluid Dynamics*, Vol. **2**, Oxford University Press.

Keller, L. V.: 1930, 'Theory of Convection and Turbulence', *Trudy Glavn. Geofis. Observ.* (Works of Main Geophys. Observ.), No. 4.

Sverdrup, H. U.: 1936, 'Austausch und Stabilität in der untersten Luftschicht', *Meteorolog. Z.* **53**.

Velikanov, M. A. (Ed.): 1936, *A Collection of Russian Translations of Classical Papers on Turbulence*, ONTI Publishing House.