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FRAMES OF REFERENCE

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ABSTRACT

The effect of an arbitrary change of frame on the structure of turbulence models is examined from a fundamental theoretical standpoint. It is proven, as a rigorous consequence of the Navier-Stokes equations, that turbulence models must be form invariant under arbitrary translational accelerations of the reference frame and should only be affected by rotations through the intrinsic mean vorticity. A direct application of this invariance property along with the Taylor-Proudman Theorem, material frame-indifference in the limit of two-dimensional turbulence and Rapid Distortion Theory is shown to yield powerful constraints on the allowable form of turbulence models. Most of the commonly used turbulence models are demonstrated to be in serious violation of these constraints and consequently are inconsistent with the Navier-Stokes equations in non-inertial frames. Alternative models with improved non-inertial properties are developed and some simple applications to rotating turbulent flows are considered.

i

1. INTRODUCTION

Turbulence plays a fundamental role in a variety of physical systems which evolve in non-inertial frames of reference. Various types of fluid machinery and geophysical systems (e.g., gas turbines, propellers, ocean currents, and atmospheric weather fronts which can have a profound effect on our daily lives) generate turbulence in non-inertial reference frames that are undergoing time-dependent rotations and translations relative to an inertial framing. Consequently, it is essential that a clear understanding of such non-inertial effects on turbulence be gained if these physical systems are to be modeled properly. Furthermore, due to the analogy between rotations and curvature, a physical model which does not properly account for non-inertial effects is likely to yield erroneous predictions for problems involving curvature in inertial frames of reference.

To date, there have been no comprehensive studies of non-inertial effects on turbulence modeling based on a rigorous analysis of the Navier-Stokes equations. Most of the previous studies consisted of rigorous mathematical analyses of the highly simplified limiting case of two-dimensional turbulence (see Speziale 1981, 1983) or more applied studies of three-dimensional turbulence where the effects of rotations of the reference frame were accounted for by a variety of ad hoc empiricisms (c.f., Majumdar, Pratap, and Spalding 1977, Howard, Patankar, Bordynuik 1980, and Galmes and Lakshminarayana 1983). There have been several studies applying second-order closure models to turbulent flows in rotating frames which are substantially less empirical in nature (c.f., Mellor and Yamada 1974, So 1975, So and Peskin 1980, and Launder, Tselepidakis, and Younis 1987). However, it was recently proven by Speziale (1985) that these particular second-order closure models are fundamentally

inconsistent with the Navier-Stokes equations in a rapidly rotating frame. Consequently, such models cannot be applied to turbulent flows in arbitrary non-inertial frames of reference without the need for making ad hoc adjustments in the constants. Since direct numerical simulations of such turbulent flows, at the high Reynolds numbers and in the complex geometries of scientific and engineering interest, will not be possible for at least the next several decades, it is essential that turbulence models be developed whose properties in non-inertial frames of reference are consistent with the Navier-Stokes equations. This forms the raison d´etre of the present study.

In this paper, it will be proven that turbulence models should be form invariant under arbitrary translational accelerations of the reference frame relative to an inertial framing (i.e., the exact invariance group of turbulence models is the extended Galilean group). Rotations of the reference frame will be shown to affect turbulence models only through the intrinsic mean vorticity. It will be shown that these rotationally-dependent non-inertial effects must vanish for a two-dimensional turbulence (i.e., material frame-indifference in the limit of two-dimensional turbulence; see Speziale 1981, 1983) and should be consistent with Rapid Distortion Theory and the Taylor-Proudman Theorem (c.f., Greenspan 1968). A systematic application of these ideas will be shown to provide powerful constraints on the allowable form of turbulence models. A variety of the popularly used turbulence models (e.g., zero, one, or two equation turbulence models along with second-order closures) will be shown to be in serious violation of these constraints which can give rise to spurious physical results in rotating frames. Improved two-equation models and second-order closure models will be presented along with some brief applications to rotating turbulent flows.

2. CONSTRAINTS ON TURBULENCE MODELS IN NON-INERTIAL REFERENCE FRAMES

We will consider the incompressible turbulent flow of a homogeneous viscous fluid governed by the Navier-Stokes and continuity equations which take the form (c.f., Batchelor 1967)

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} - \dot{\mathbf{\Omega}} \times \mathbf{x} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) - \dot{\mathbf{U}}_0 - 2\mathbf{\Omega} \times \mathbf{v} \qquad (1)$$

$$\nabla \cdot \mathbf{v} = 0 \qquad (2)$$

in an arbitrary non-inertial reference frame (see Figure 1). In Equations (1)-(2), $\mathbf{v}$ is the velocity vector, P is the modified pressure, $\mathbf{\Omega}(t)$ is the rotation rate of the non-inertial frame relative to an inertial framing, $\dot{\mathbf{U}}_0(t)$ is the translational acceleration of the origin of the non-inertial frame relative to an inertial framing, and ν is the kinematic viscosity of the fluid. It should be noted that the Navier-Stokes equations are altered by the presence of four frame-dependent terms on the right-hand-side of (1) which, respectively, are referred to as the Eulerian, centrifugal, translational, and Coriolis accelerations. The continuity equation is frame-indifferent, i.e., it has no non-inertial terms and, hence, is of the same form in all frames of reference independent of whether or not they are inertial.

As in the usual treatments of turbulence, the velocity field $\mathbf{v}$ and pressure P will be decomposed into ensemble mean and fluctuating parts as follows:

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{u}, \qquad P = \overline{P} + p \qquad (3)$$

where

$$\overline{\mathbf{v}} = \lim_{N \to \infty} \frac{1}{N} \sum_{\alpha=1}^{N} \mathbf{v}^{(\alpha)}, \qquad \overline{P} = \lim_{N \to \infty} \frac{1}{N} \sum_{\alpha=1}^{N} P^{(\alpha)} \qquad (4)$$

are ensemble averages taken, in practice, over a large number of N realizations of the turbulence (c.f., Hinze 1975). For a statistically steady or homogeneous turbulence, the ergodic hypothesis can be invoked and time averages or spatial averages, respectively, can be substituted. The mean velocity $\overline{v}$ and mean pressure $\overline{P}$ are solutions of the Reynolds equation and mean continuity equation which, respectively, take the form

$$\frac{\partial \overline{v}}{\partial t} + \overline{v} \cdot \nabla \overline{v} = -\nabla \overline{P} + \nu \nabla^2 \overline{v} + \nabla \cdot \tau - \dot{\Omega} \times x - \Omega \times (\Omega \times x) - \dot{U}_o - 2\Omega \times \overline{v} \tag{5}$$

$$\nabla \cdot \overline{v} = 0 \tag{6}$$

in any arbitrary non-inertial reference frame where

$$\tau = -\overline{uu} \tag{7}$$

is the Reynolds stress tensor. Equations (5)-(6) are obtained by substituting the decomposition (3) into the Navier-Stokes equations and then taking an ensemble average. The fluctuating velocity u and fluctuating pressure p are solutions of the following equations (valid in an arbitrary non-inertial frame):

$$\frac{\partial u}{\partial t} + \overline{v} \cdot \nabla u = -u \cdot \nabla u - u \cdot \nabla \overline{v} - \nabla p + \nu \nabla^2 u - \nabla \cdot \tau - 2\Omega \times u \tag{8}$$

$$\nabla \cdot u = 0 \tag{9}$$

which are referred to as the fluctuating momentum and fluctuating continuity equation, respectively. Equations (8)-(9) are obtained by subtracting

Equations (5)-(6) from Equations (1)-(2), respectively. From Equation (8), it is clear that the evolution of the fluctuating velocity **u** (for a given mean velocity field) is only affected by the reference frame through the Coriolis acceleration $2\mathbf{\Omega} \times \mathbf{u}$. Eulerian, centrifugal, and translational accelerations only have an indirect effect on the fluctuating velocity through the changes that they induce in the mean velocity.

At this point, the concepts of the Oldroyd derivative and intrinsic vorticity will be introduced. The Oldroyd derivative of the fluctuating velocity is defined by

$$\frac{D_c \mathbf{u}}{Dt} \equiv \frac{\partial \mathbf{u}}{\partial t} + \overline{\mathbf{v}} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \overline{\mathbf{v}} \qquad (10)$$

and represents the frame-indifferent convected time rate of **u** following the mean velocity with respect to both position and orientation. Unlike the substantial derivative $D\mathbf{u}/Dt \equiv \partial \mathbf{u}/\partial t + \overline{\mathbf{v}} \cdot \nabla \mathbf{u}$, the Oldyroyd derivative is independent of the observer; relative to any two independent non-inertial reference frames **x** and $\mathbf{x}^*$ (whose motions can differ by an arbitrary time-dependent rotation and translation) the Oldroyd derivative of a given fluctuating velocity field is the same, i.e.,

$$\frac{D_c \mathbf{u}^*}{Dt^*} = \frac{D_c \mathbf{u}}{Dt} . \qquad (11)$$

The intrinsic vorticity **W** (also referred to as the absolute or potential vorticity) is the vorticity relative to an inertial framing and is obtained by adding $2\mathbf{\Omega}$ to the local vorticity $\omega \equiv \nabla \times \mathbf{v}$ in the non-inertial frame. Hence, we have (c.f., Tritton 1977)

$$W = \omega + 2\Omega \; . \tag{12}$$

By taking the ensemble mean and dual of (12), the intrinsic mean vorticity vector and tensor are obtained which, respectively, are given in component form by the equations

$$\overline{W}_k = \overline{\omega}_k + 2\Omega_k \tag{13}$$

$$\overline{W}_{k\ell} = \overline{\omega}_{k\ell} + \varepsilon_{m\ell k}\Omega_m \tag{14}$$

where $\overline{\omega}_{k\ell} \equiv \frac{1}{2}(\partial \overline{v}_k/\partial x_\ell - \partial \overline{v}_\ell/\partial x_k)$ is the local mean vorticity tensor and $\varepsilon_{m\ell k}$ is the permutation tensor. A straightforward substitution of (10) and (14) into (8) gives rise to the alternative form of the fluctuating momentum equation

$$\frac{D_c u}{Dt} = - u \cdot \nabla u - 2(\overline{S} + \overline{W}) \cdot u - \nabla p + \nu \nabla^2 u - \nabla \cdot \tau \tag{15}$$

which is valid in <u>any</u> non-inertial frame of reference. In (15), $\overline{W}$ is the intrinsic mean vorticity tensor and $\overline{S}$ is the mean rate of strain tensor whose components are given by

$$\overline{S}_{k\ell} = \frac{1}{2}(\frac{\partial \overline{v}_k}{\partial x_\ell} + \frac{\partial \overline{v}_\ell}{\partial x_k}) \tag{16}$$

in all frames of reference independent of whether or not they are inertial. As a result of (15), it is clear that the evolution of the fluctuating velocity (and higher-order moments constructed from it) only depend on the reference frame through the intrinsic mean vorticity $\overline{W}$.

The Reynolds and continuity equations (5)-(6) are not closed as a result of the additional unknowns represented by the six components of the Reynolds stress tensor $\tau_{k\ell}$. In virtually all previous studies of turbulence modeling beginning with Boussinesq, it was tacitly assumed that the Reynolds stress tensor is uniquely determined by the global history of the mean velocity field. This assumption is generally consistent with the Navier-Stokes equations in an inertial framing as pointed out by Lumley (1970). Hence, in an inertial frame of reference, we have

$$\tau = \tau[\overline{v}(x',t'); x,t] \qquad x' \in D, \ t' \in (-\infty,t) \qquad (17)$$

where D is the fluid domain and a bracket [·] denotes a functional (i.e., any quantity determined by the global history of a function). It should be noted at this point that both τ and $\overline{v}$ are kinematical quantities whose transformation properties under a change of frame are mathematically determined. To be more specific, given that x is an inertial frame and x^* is an arbitrary non-inertial frame, it is a simple matter to show from basic kinematics that (c.f., Speziale 1979)

$$\tau^* = \tau \qquad (18)$$

$$\overline{v}^* = \overline{v} - \Omega \times x - U_0. \qquad (19)$$

Consequently, once the inertial form of (17) is specified its non-inertial form is automatically determined. It thus follows that if the non-inertial form of (17) is incorrect, its inertial form must also be incorrect since the two are not independent.

Since τ is uniquely determined from the fluctuating velocity (i.e., $\tau = -\overline{uu}$), it is clear that its invariance group must be at least as large as the invariance group of the fluctuating momentum equation (8) (of course, (17) constitutes a special solution of (8) and, hence, it could have a larger invariance group; c.f., Rosen 1980). Consequently, turbulent closure models for τ must be at least form invariant under arbitrary translational accelerations of the reference frame. Hence, Equation (17) must transform in the form invariant manner

$$\tau[\vec{\overline{v}}^{*}(x',t'); x,t] = \tau[\overline{v}(x',t'); x,t], \quad x' \in D, \ t' \in (-\infty,t) \tag{20}$$

under the _extended_ Galilean group of transformations

$$x^{*} = x + c(t) \tag{21}$$

where $\dot{U}_0 \equiv -\ddot{c}$ is the translational acceleration of the non-inertial frame relative to an inertial framing. Constraint (20) would, for example, forbid turbulent closure models from having any explicit dependence on the mean acceleration $D\overline{v}/Dt$. Since any dependence on the rotation rate of the reference frame must arise from the intrinsic mean vorticity, it follows that in an _arbitrary non-inertial frame_, turbulent closure models for τ must be of the general form

$$\tau = \tau[\overline{v}(x',t'), \overline{W}(x',t'); x,t] \quad x' \in D, \ t' \in (-\infty,t). \tag{22}$$

Here it is understood that the explicit functional dependence on $\overline{v}$ in (22)

is frame-indifferent (i.e., does not contain any terms which depend on the motion of the reference frame relative to an inertial framing). Frame-dependence only enters implicitly through the intrinsic mean vorticity $\overline{W}$. Examples of one-point fields that are frame-indifferent functionals of the mean velocity include $\overline{S}$ and $D_c\overline{S}/Dt$; two-point fields that are frame-indifferent include the vorticity difference $\overline{\omega}(x',t) - \overline{\omega}(x,t)$ and its Oldroyd derivative.

Although three-dimensional turbulent closure models can be frame-dependent through the intrinsic mean vorticity tensor $\overline{W}$, it has been shown recently that such models must become frame-indifferent in the limit of two-dimensional turbulence (see Speziale 1981, 1983). By a two-dimensional turbulence we mean a turbulent flow where the fluctuating velocity u is of the plane two-dimensional form

$$u = u_x(x,y,t)\mathbf{i} + u_y(x,y,t)\mathbf{j}. \tag{23}$$

Consistent with this two-dimensional assumption, the angular velocity of the reference frame must be of the form $\Omega = \Omega k$ so that the mean velocity induced by it is comparably two-dimensional. For such a flow, the Coriolis acceleration in the fluctuating momentum Equation (8) is derivable from a scalar potential as follows (see Speziale 1981, 1983)

$$2\Omega \times u = \nabla(2\Omega\psi) \tag{24}$$

since, as a general solution of the two-dimensional continuity equation, the fluctuating velocity can be written in the stream function form

$$\mathbf{u} = \nabla\psi \times \mathbf{k}. \qquad (25)$$

Consequently, the Coriolis acceleration (which constitutes the only non-inertial effect in Equation (8)) can be absorbed into the fluctuating pressure in a two-dimensional turbulence leaving the fluctuating velocity unaffected. Consistent with this result, the dependence of the Reynolds stress tensor on the intrinsic mean vorticity (which characterizes these Coriolis effects) must vanish in the limit of two-dimensional turbulence rendering the model frame-indifferent. Thus, in the two-dimensional limit, turbulence closure models for the Reynolds stress tensor must be of the <u>same form</u>

$$\tau = \tau[\overline{\mathbf{v}}(\mathbf{x}',t'); \mathbf{x},t] \qquad \mathbf{x}' \in D, \quad t' \in (-\infty,t) \qquad (26)$$

independent of whether or not the reference frame is inertial. This invariance property is referred to as <u>the principle of material frame-indifference in the limit of two-dimensional turbulence</u> (see Speziale 1981, 1983).

The limit of two-dimensional turbulence constitutes a real physical limit which can be approached by any statistically steady turbulence, sufficiently far from solid boundaries, in a rapidly rotating framework (a direct consequence of the Taylor-Proudman Theorem; c.f., Tritton 1977). The Taylor-Proudman Theorem in its classical form states that steady inviscid flows in a rapidly rotating framework are two-dimensional, i.e., are independent of the coordinate along the axis of rotation of the fluid. Of course, the Taylor-Proudman Theorem holds in an excellent approximate sense for most laminar viscous flows provided that the flow is sufficiently far removed from solid boundaries where Ekman layers can develop. For a statistically steady

turbulent flow in a rotating frame, the Reynolds equation takes the form

$$\overline{v} \cdot \nabla \overline{v} = -\nabla \overline{P} + \nu \nabla^2 \overline{v} + \nabla \cdot \tau - 2\mathbf{\Omega} \times \overline{v} \tag{27}$$

where the centrifugal acceleration has been absorbed into the modified pressure $\overline{P}$. The associated mean vorticity transport equation, obtained by taking the curl of Equation (27), is given by

$$\overline{v} \cdot \nabla \overline{\omega} = \overline{\omega} \cdot \nabla \overline{v} + \nu \nabla^2 \overline{\omega} + \nabla \times (\nabla \cdot \tau) + 2\mathbf{\Omega} \cdot \nabla \overline{v} \tag{28}$$

where $\overline{\omega} \equiv \nabla \times \overline{v}$ is the local mean vorticity in the rotating frame. If we let $\mathbf{\Omega} = \Omega \mathbf{k}$, Equation (28) can be written in the alternative form

$$\mathbf{k} \cdot \nabla \overline{v} = \frac{1}{2\Omega} (\overline{v} \cdot \nabla \overline{\omega} - \overline{\omega} \cdot \nabla \overline{v} - \nu \nabla^2 \overline{\omega} - \nabla \times \nabla \cdot \tau). \tag{29}$$

In the limit as $\Omega \to \infty$, Equation (29) reduces to

$$\mathbf{k} \cdot \nabla \overline{v} \equiv \frac{\partial \overline{v}}{\partial z} = 0 \tag{30}$$

for a statistically steady turbulence. Sufficiently far from solid boundaries, Equation (30) has the simple solution

$$\overline{v} = \overline{v}(x, y) \tag{31}$$

and, thus, the mean velocity field for a statistically steady turbulence in a rapidly rotating frame must be two-dimensional. If the flow is confined by

boundaries normal to the axis of rotation (at distances sufficiently far removed from the flow region being considered), the mean velocity field will be of the two component form $\bar{v} = \bar{v}_x(x,y)i + \bar{v}_y(x,y)j$ (see Tritton 1977). The same type of two-dimensionalization will hold for the Reynolds stress tensor in an approximate sense since the filtered velocity satisfies an equation of the form of (28) and the large scales of turbulence contain most of the energy. This is the turbulent generalization of the classical Taylor-Proudman Theorem which has been verified experimentally to hold in an excellent approximate sense provided that the Rossby number $Ro \equiv v_0/\Omega \ell_0 \ll 1$ (where ℓ_0 and v_0 are the length and velocity scales of the turbulent fluctuations). Any turbulence model which does not yield such a Taylor-Proudman reorganization in a rapidly rotating framework is fundamentally inconsistent with the non-inertial form of the Navier-Stokes equations.

The last constraint that will be considered in this section involves the application of Rapid Distortion Theory (RDT) to turbulence suddenly subjected to a strong rotation. Since the Taylor-Proudman Theorem serves primarily as a constraint on the large energy containing eddies in a rapidly rotating frame, RDT will be used as a constraint on the small scale turbulence which is not too far removed from isotropy. Hence, RDT for an initially isotropic turbulence which is suddenly subjected to a rapid rotation will be considered. For this problem, the initial Reynolds stress tensor and dissipation rate tensor are of the isotropic form

$$\tau_{ij} = -\frac{2}{3} K_0 \delta_{ij} \tag{32}$$

$$D_{ij} = \frac{2}{3} \varepsilon_0 \delta_{ij} \tag{33}$$

respectively, where K_0 is the initial turbulent kinetic energy and ε_0 is the initial dissipation rate of the turbulence (it should be noted that $D_{ij} = 2 \; \overline{\nu(\partial u_i/\partial x_m)(\partial u_j/\partial x_m)}$ and $\varepsilon = \frac{1}{2} D_{ii}$). Rapid Distortion Theory predicts that the turbulence undergoes an <u>isotropic</u> linearly viscous decay (c.f., Reynolds 1987). More specifically, according to RDT, the Reynolds stress tensor and dissipation rate tensor are of the isotropic form

$$\tau_{ij} = - \frac{2}{3} K(t)\delta_{ij}, \qquad D_{ij} = \frac{2}{3} \varepsilon(t)\delta_{ij} \qquad (34)$$

at a later time $t > 0$ after the turbulence has been subjected to the rapid rotation. Here, the turbulent kinetic energy and dissipation rate are determined from the energy spectrum $E(k,t)$ as follows (c.f., Hinze 1975)

$$K(t) = \int_0^\infty E(k,t)dk \qquad (35)$$

$$\varepsilon(t) = 2\nu \int_0^\infty k^2 E(k,t)dk \qquad (36)$$

which are valid for an isotropic turbulence. Rapid Distortion Theory predicts that the energy spectrum undergoes a linearly viscous decay, and thus at any later time $t > 0$:

$$E(k,t) = E(k,0)\exp(-2\nu k^2 t) \qquad (37)$$

(it should be noted that in the limit of infinite Reynolds numbers, the energy spectrum remains unchanged for finite times $t > 0$; c.f., Reynolds 1987). Of course, RDT is only formally valid for short elapsed times $t \ll K_0/\varepsilon_0$. However, since a rapid rotation destroys the phase coherence needed to cascade

energy from the large scales to the small scales (represented by the energy transfer term), it would appear that RDT could hold for much larger elapsed times for the case of a rapidly rotating isotropic turbulence. This was verified by the results of direct numerical simulations of the Navier-Stokes equations for isotropic turbulence subjected to a rapid rotation (see Speziale, Mansour, and Rogallo 1987). These numerical simulations indicated that the rapid rotation suppresses the energy transfer for long time intervals yielding results in excellent approximate agreement with the RDT solution specified by Equations (34) and (37) for the primary period of the decay (i.e., up to and beyond the point where the turbulent kinetic energy has decayed to 10% of its initial value). The time evolution of the energy spectrum obtained from a 128^3 direct numerical simulation is shown in Figures 2(a)-(b) for a Reynolds number $Re_\lambda = 51$ and a Rossby number $Ro_\lambda = 0.07$ based on the initial turbulent kinetic energy and Taylor microscale. These computed energy spectra were illustrative of a linearly viscous decay during the entire period of the computation (i.e., for $0.1 < K/K_0 < 1.0$). The L_2 norm of the anisotropy tensor was extremely small ($\|b\|_2 < 0.01$) for the entire duration of the computation and, hence, the rotation had no discernible effect on the isotropy of τ. These results demonstrate strong agreement with RDT for large elapsed times. Numerical results for the decay of the turbulent kinetic energy (shown in Figure 3) illustrate that the rapid rotation gives rise to a dramatically reduced turbulence dissipation rate due to the disruption of the energy transfer from large scales to small scales. It is the opinion of the author that these fundamental results (which are important since they capture the essential physical features of the reaction of small scale turbulence to a rapid rotation for long as well as short elapsed times) should serve as a

basic constraint on turbulence models in rotating frames. Models that are in serious violation of these RDT results are likely to give rise to spurious physical results in rotating turbulent flows.

3. INCONSISTENCY OF EXISTING TURBULENCE MODELS

As demonstrated in the previous section, the Navier-Stokes equations in a non-inertial reference frame place the following basic constraints on the allowable form of turbulence models:

(i) Reynolds stress models must be form invariant under arbitrary translational accelerations of the reference frame and should <u>only</u> be affected by rotations of the reference frame through the intrinsic mean vorticity.

(ii) All frame-dependent effects (and thus any dependence on the intrinsic mean vorticity) must vanish in the limit of two-dimensional turbulence — a constraint appropriately named material frame-indifference in the limit of two-dimensional turbulence.

(iii) Reynolds stress models must be consistent with the Taylor-Proudman Theorem for turbulent flows. This requires that a statistically steady turbulence in a rapidly rotating frame (sufficiently far from solid boundaries) be <u>two-dimensional.</u>

(iv) Turbulence models should be consistent with the results of RDT for an initially isotropic turbulence subjected to a rapid rotation. This requires Reynolds stress models to predict that an initially isotropic turbulence undergoes an <u>isotropic linearly viscous decay</u> in a rapidly rotating frame yielding a substantially reduced dissipation rate.

First, we will note the inconsistency of the more empirical Coriolis modified turbulence models with these constraints. These more empirical models are characterized by the introduction of coefficients that depend explicitly on the rotation rate of the reference frame. For example, in the model of Howard, Patankar, and Bordynuik (1980), empirical coefficients in the transport equations for the turbulent kinetic energy and dissipation rate in rotating duct flow were allowed to depend on the Richardson number

$$Ri = \frac{-2\Omega(\frac{\partial \overline{u}}{\partial y} - 2\Omega)}{(\frac{\partial \overline{u}}{\partial y})^2} \tag{38}$$

where Ω is the rotation rate of the duct, $\overline{u}$ is the mean velocity along the axis of the duct, and y is the transverse coordinate. A comparable empirical model, based on the nonlinear algebraic model of Rodi (1976), was proposed recently by Galmes and Lakshminarayana (1983) where an implicit dependence on the Richardson number (given by Equation (38)) was introduced into the Reynolds stresses. Such empirical models (which have also been proposed by other authors) violate constraint (i) and are thus inconsistent with the Navier-Stokes equations. More specifically, rather than an explicit dependence on the rotation rate Ω there should be an implicit dependence on Ω only through the intrinsic mean vorticity (i.e., the quantity $-\partial \overline{u}/\partial y + 2\Omega$ for the rotating channel flow under discussion). The recent large-eddy simulations of Bardina, Ferziger, and Reynolds (1983) for rotating homogeneous shear flow demonstrated that the turbulent Reynolds stresses do not scale with the Richardson number.

Eddy viscosity models form the foundation for most of the turbulence models that are used by scientists and engineers. These models are of the

general form

$$D^\tau_{k\ell} = 2\nu_T \overline{S}_{k\ell} \tag{39}$$

where

$$D^\tau_{k\ell} = \tau_{k\ell} - \frac{1}{3}\tau_{mm}\delta_{k\ell} \tag{40}$$

is the deviatoric part of the Reynolds stress tensor and ν_T is the eddy viscosity in its kinematic form. Equation (39) encompasses a wide variety of turbulence models which, by far, are the most commonly used models for the solution of practical problems. We will now examine the consistency of a variety of popular eddy viscosity models with constraints (i)-(iv) for rotating turbulent flows. The simplest eddy viscosity models are the zero equation models where the turbulent time scale is constructed from the mean velocity gradients and the turbulent length scale is specified algebraically. Two such popular models are the Smagorinsky (1963) Model given by

$$\nu_T = \ell^2 (2\overline{S}_{mn}\overline{S}_{mn})^{1/2} \tag{41}$$

and the Baldwin-Lomax Model (or vorticity model) given by

$$\nu_T = \ell^2 (\overline{\omega}_m \overline{\omega}_m)^{1/2} \tag{42}$$

where ℓ is the turbulent length scale which is usually specified empirically. The Smagorinsky Model, which is the tensorial generalization of

Prandtl's mixing length theory[‡], arose out of the Geophysical Fluid Dynamics community and (as a subgrid scale model) has served as the cornerstone for large-eddy simulations. Since ν_T only depends on $\overline{S}$, it is frame-indifferent for _all_ mean flows and, as such, automatically satisfies constraints (ii)-(iii). However, since it is frame-indifferent in three-dimensions as well as in two-dimensions, the Smagorinsky Model is fundamentally incapable of describing the effects of rotation in retarding the energy transfer process (as described in constraint (iv)) which ultimately has an effect on $_D\tau$. However, such effects are primarily manifested in the large scales and, consequently, the Smagorinsky model would be satisfactory as a subgrid-scale stress model despite the fact that it has undesirable rotational properties as a Reynolds stress model (see Bardina, Ferziger, and Reynolds 1983 and Speziale 1985).

On the other hand, it will now be shown that the Baldwin-Lomax Model is more seriously inconsistent with the Navier-Stokes equations in a rotating frame. It should first be noted that the eddy viscosity (42) is specified for an inertial framing. However, as alluded to earlier, it follows from basic kinematics that (see Speziale 1979)

$$\tau^* = \tau, \quad \overline{S}^* = \overline{S}, \quad \overline{\omega}^* = \overline{\omega} - 2\Omega \qquad (43)$$

where the starred quantities are relative to an arbitrary non-inertial reference frame x^*. Hence, given that (42) is the inertial form of the Baldwin-Lomax Model, it follows that its non-inertial form is given by

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[‡] For a unidirectional turbulent shear flow (with mean velocity $\overline{v} = \overline{u}(y)i$) Equation (41) reduces to $\nu_T = \ell^2 |d\overline{u}/dy|$.

$$\nu_T = \ell^2 (\overline{W}_m \overline{W}_m)^{1/2}. \tag{44}$$

While this model is consistent with constraint (i) (i.e., frame-dependent effects only enter in through the intrinsic mean vorticity), it is in serious violation of constraints (ii)-(iv). To be more specific, in the limit of two-dimensional turbulence, $\overline{W}_m \overline{W}_m = (\overline{\omega} + 2\Omega)^2$ survives and hence there is a violation of material frame-indifference in the limit of two-dimensional turbulence (i.e., in the two-dimensional limit, any dependence on Ω must vanish for there to be consistency with the Navier-Stokes equations). Furthermore, since according to (44), $\nu_T \to \infty$ as $\Omega \to \infty$, the Baldwin-Lomax Model predicts that there is an <u>increase</u> in turbulent dissipation corresponding to an increase in the rotation rate of the framing which violates constraint (iv). This unbounded growth of ν_T as $\Omega \to \infty$ also gives rise to the violation of constraint (iii)--the Taylor-Proudman Theorem. For large Ω, $\nu_T \doteq 2\ell^2 \Omega$ and hence in the limit as $\Omega \to \infty$, (29) reduces to

$$\mathbf{k} \cdot \nabla \overline{\mathbf{v}} = -2\nabla \times [\nabla \cdot (\ell^2 \overline{\mathbf{S}})] \tag{45}$$

with the implication that $\partial \overline{\mathbf{v}}/\partial z$ is not necessarily zero (i.e., $\overline{\mathbf{v}} \neq \overline{\mathbf{v}}(x,y)$ for any statistically steady turbulent flow sufficiently far from solid boundaries) in violation of the Taylor-Proudman Theorem. It is thus clear that vorticity models such as the Baldwin-Lomax Model are likely to yield unphysical results for turbulent flows involving strong rotational strains and, consequently, do not form a general foundation for either a Reynolds stress or subgrid scale stress model.

One-equation models involve the solution of an additional transport equation for the turbulent kinetic energy. The eddy viscosity for such models is of the form

$$\nu_T = K^{1/2} \ell \qquad (46)$$

where K is the turbulent kinetic energy (obtained from its modeled transport equation) and ℓ is an appropriate length scale of turbulence which is usually specified empirically based upon the particular flow geometry under consideration (see Cebeci and Smith 1974 and Rodi 1984 for a survey of such models). Since the transport equation for K is frame-indifferent and since ℓ is usually only specified based on the geometry of the flow configuration, such models are identically frame-indifferent. Due to the fact that they satisfy material frame-indifference in three-dimensional turbulent flows as well as in two-dimensional turbulent flows, they are unable to predict the reduction in turbulence dissipation that results from the application of a strong rotation (i.e., such one-equation models are generally consistent with constraints (i)-(iii) but in serious violation of constraint (iv)). The same precise criticism can be leveled against two-equation turbulence models among which the K-ϵ model has become extremely popular during the past decade. In the K-ϵ model, the eddy viscosity is represented by

$$\nu_T = C_\mu \frac{K^2}{\epsilon} \qquad (47)$$

where $C_\mu = 0.09$ is an empirical constant, K is the turbulent kinetic energy, and ϵ is the turbulent dissipation rate. In the K-ϵ model, K and ϵ are determined from modeled versions of their transport equations

which are usually of the form (see Hanjalic and Launder 1972)

$$\frac{DK}{Dt} = \tau_{ij} \frac{\partial \bar{v}_i}{\partial x_j} + C_1 \frac{\partial}{\partial x_i} \left[\frac{K}{\epsilon} \left(\tau_{jm} \frac{\partial \tau_{ij}}{\partial x_m} - \tau_{ij} \frac{\partial K}{\partial x_j}\right)\right] - \epsilon \tag{48}$$

$$\frac{D\epsilon}{Dt} = -C_2 \frac{\partial}{\partial x_i} \left(\frac{K}{\epsilon} \tau_{ij} \frac{\partial \epsilon}{\partial x_j}\right) + C_3 \frac{\epsilon}{K} \tau_{ij} \frac{\partial \bar{v}_i}{\partial x_j} - C_4 \frac{\epsilon^2}{K} \tag{49}$$

where $D/Dt \equiv \frac{\partial}{\partial t} + \bar{v} \cdot \nabla$ and $C_1 - C_4$ are empirical constants. Equations (48) - (49) are of the same form independent of whether or not the reference frame is inertial. Consequently, the $K-\epsilon$ model is frame-indifferent for all flows thus making it impossible for this model to account for the reduction in dissipation that occurs in rotating isotropic turbulence as well as in other rotating turbulent flows (i.e., the model is in violation of constraint (iv)). Furthermore, the inability of the $K-\epsilon$ model to accurately predict normal Reynolds stress differences in turbulent flows of engineering importance (see Speziale 1987) can be exacerbated further in rotating flows where Coriolis effects usually give rise to stronger such anisotropies.

Problems of a similar nature exist with second-order closure models. All of the commonly used second-order closure models are of the general form (see Speziale 1985)

$$\frac{D_c \tau_{k\ell}}{Dt} = (C_1 - 2)[\tau_{km}\bar{W}_{\ell m} + \tau_{\ell m}\bar{W}_{km}] + \frac{\partial C_{k\ell m}(\tau,\nabla\tau,\ell)}{\partial x_m}$$
$$+ \pi_{k\ell}(\tau,\bar{S},\ell) + \nu\nabla^2\tau_{k\ell} \tag{50}$$

in an arbitrary non-inertial frame, where

$$\frac{D_c \tau_{k\ell}}{Dt} \equiv \frac{\partial \tau_{k\ell}}{\partial t} + \bar{v} \cdot \nabla\tau_{k\ell} - \frac{\partial \bar{v}_k}{\partial x_m} \tau_{m\ell} - \frac{\partial \bar{v}_\ell}{\partial x_m} \tau_{mk} \tag{51}$$

is the frame-indifferent Oldroyd derivative of τ and C_1 is a constant (which arises from the <u>rotationally dependent</u> part of the rapid pressure-strain correlation). Here, $C_{k\ell m}$ is a function of the variables τ, $\nabla\tau$, and ℓ which arises from the modeling of the triple velocity and pressure-diffusion correlations whereas $\pi_{k\ell}$ is a function of τ, S, and ℓ which arises from the irrotational part of the production terms and the modeling of the slow pressure-strain and dissipation rate correlations. This general form (50) encompasses the Launder, Reece, and Rodi (1975) model, the models of Lumley (1978), and the Rotta-Kolmogorov model (see Mellor and Herring 1973). In the former two models, the length scale of turbulence ℓ is taken to be of the form

$$\ell = K^{3/2}/\varepsilon \tag{52}$$

where the dissipation rate ε is determined from a modeled transport equation which is of the same general frame-indifferent form as (49). Analogously, in the Rotta-Kolmogorov model the length scale ℓ is obtained from the transport equation (see Mellor and Herring 1973):

$$\frac{D(K\ell)}{Dt} = \frac{\partial}{\partial x_i} \left[(\nu + \beta_1 K^{1/2}\ell) \frac{\partial}{\partial x_i} (K\ell) + \beta_2 K^{3/2}\ell \frac{\partial \ell}{\partial x_i} \right]$$
$$+ \beta_3 \ell \tau_{ij} \frac{\partial \overline{v}_i}{\partial x_j} - \beta_4 K^{3/2} \tag{53}$$

(where β_1, β_2, β_3, and β_4 are empirical constants) which is of the same form in all frames of reference independent of whether or not they are inertial. Consequently, since $D_c\tau_{k\ell}/Dt$, $\partial C_{k\ell m}/\partial x_m$, $\pi_{k\ell}$, and $\nabla^2\tau_{k\ell}$ are frame-indifferent along with the transport equations for ε and ℓ, it follows

that frame-dependence in the commonly used second-order closures arises exclusively from the term

$$(C_1 - 2)(\tau_{km}\overline{W}_{\ell m} + \tau_{\ell m}\overline{W}_{km}). \tag{54}$$

Since C_1 is a constant that <u>does not</u> equal 2 (in the Launder, Reese, and Rodi model, $C_1 = 0.6$ whereas in the Rotta-Kolmogorov model, $C_1 = 0$) and since $\tau_{km}\overline{W}_{\ell m} + \tau_{\ell m}\overline{W}_{km}$ does not generally vanish in a two-dimensional turbulence, it follows that material frame-indifference in the limit of two-dimensional turbulence is violated. This inconsistent dependence of (50) on **W** also gives rise to a violation of the Taylor-Proudman Theorem in problems of engineering and geophysical interest since the constraint

$$\lim_{\Omega \to \infty} \frac{1}{2\Omega} \, \mathbf{\nabla} \times (\mathbf{\nabla} \cdot \mathbf{\tau}) = 0 \tag{55}$$

(which is a necessary condition for the Taylor-Proudman Theorem) is violated in statistically steady turbulent flows by these second-order closures. Furthermore, since the transport equations for ε and ℓ are frame-indifferent in the commonly used second-order closures, they are unable to account for the reduction in dissipation (and the associated change in length scales) in rotating isotropic turbulence. Thus, for turbulent flows in a rapidly rotating frame, the commonly used second-order closure models are in rather serious violation of the Navier-Stokes equations. Although Launder, Tselepidakis, and Younis (1987) were able to get reasonable correlation with experiments on rotating channel flow using the Launder, Reece, and Rodi second-order closure, it must be noted that only mild rotations with Rossby

numbers Ro ~ 10 were considered (the reader should note that the Rossby number referred to in Launder, Tselepidakis, and Younis 1987 is actually the inverse Rossby number). Had those authors considered more rapid rotations (i.e., Rossby numbers Ro $\leq$ 0.1) serious inconsistences would most likely have arisen as discussed by Speziale (1985).

Recently, a modified transport equation for the turbulence dissipation was proposed by Bardina, Ferziger, and Rogallo (1985) with the purpose of accounting for the reduction in dissipation that occurs when isotropic turbulence is subjected to a rigid body rotation. This model is of the form

$$\frac{D\varepsilon}{Dt} = -C_2 \frac{\partial}{\partial x_i} \left(\frac{K}{\varepsilon} \tau_{ij} \frac{\partial \varepsilon}{\partial x_j}\right) + C_3 \frac{\varepsilon}{K} \tau_{ij} \frac{\partial \overline{v}_i}{\partial x_j}$$
$$-C_4 \frac{\varepsilon^2}{K} - C_5 (\frac{1}{2} \overline{W}_{ij} \overline{W}_{ij})^{1/2} \varepsilon \tag{56}$$

which differs from the more commonly used model (49) by the addition of the last term on the right-hand-side of (56). For isotropic turbulence in a rotating frame, Equation (56) takes the form

$$\frac{d\varepsilon}{dt} = - C_4 \frac{\varepsilon^2}{K} - C_5 \Omega \varepsilon. \tag{57}$$

Bardina, Ferziger, and Rogallo (1985) found that (for C_5 = 0.15) Equation (57) predicted reductions in the dissipation rate that were in fairly good agreement with the experiments of Wigeland and Nagib (1978) for rotating isotropic turbulence. However, several criticisms can be leveled at Equation (56) when applied to anisotropic and inhomogeneous turbulent flows. For example, the dependence on the intrinsic mean vorticity term $\overline{W}_{ij}\overline{W}_{ij}$ does

not vanish in a two-dimensional turbulence, thus, violating material frame-indifference in this limit. Furthermore, Equation (56) was obtained from (57) by an extrapolation which is not unique. More specifically, there are other invariants besides $(\frac{1}{2} \overline{W}_{ij} \overline{W}_{ij})^{1/2}$ which reduce to Ω in a rotating isotropic turbulence (see Speziale, Mansour, and Rogallo 1987). These problems will be addressed in the next section where a hierarchy of consistent models will be developed.

4. IMPROVED TURBULENCE MODELS FOR NON-INERTIAL REFERENCE FRAMES

In this section, improved two-equation turbulence models and second-order closure models that are consistent with the non-inertial constraints derived in Section 2 will be developed. Since rotations can dramatically enhance anisotropic effects and alter the dissipative properties of the turbulence, eddy viscosity models are more likely to yield inaccurate predictions in rotating reference frames. Hence, it is best to base two-equation turbulence models on some suitable nonlinear generalization of the eddy viscosity models when applications to rotating flows are envisioned. Recently, the author developed a nonlinear $K-\ell$ and $K-\varepsilon$ model along these lines (see Speziale 1987) which appears to account for anisotropic effects much more accurately. This model is of the form

$$\tau_{ij} = -\frac{2}{3} K \delta_{ij} + K^{1/2} \ell \overline{S}_{ij} + C_D \ell^2 (\frac{D_c \overline{S}_{ij}}{Dt} + \overline{S}_{im} \overline{S}_{mj} + \frac{1}{3} \overline{S}_{mn} \overline{S}_{mn} \delta_{ij}) \qquad (58)$$

in an arbitrary non-inertial frame where C_D is an empirical constant which was found to assume an approximate value of 1.68 by correlating with turbulent channel flow data. Here, ℓ is the length scale of turbulence which is given by

$$\ell = 2C_\mu \frac{K^{3/2}}{\varepsilon} \qquad (59)$$

for the $K-\varepsilon$ type model. This model constitutes a substantially simplified version of a nonlinear eddy viscosity model recently derived by Yoshizawa (1984) using Kraichnan's DIA formalism; the simplification primarily arises from invoking the constraint of material frame-indifference in the limit of two-dimensional turbulence which Yoshizawa's full nonlinear model violates.

Since (58) represents a quadratic extension of the linear eddy viscosity models which are algebraic in nature, it follows that the satisfaction of frame-indifference in the two-dimensional limit restricts any frame-dependence in three-dimensional turbulence to arise exclusively from changes in the scalar length scale ℓ. Unfortunately, such a weak frame-dependence cannot account for the considerable additional anisotropies that are caused by a moderate system rotation of turbulent shear flows of scientific and engineering interest. The constraint of material frame-indifference in the two-dimensional limit becomes important in rapidly rotating frames where there is a Taylor-Proudman reorganization of the flow—a limit which is largely outside of the domain of applicability of such simplified algebraic models which cannot account for extremely large anisotropies. Hence, we will relax this constraint in favor of another approximation that follows from a simpified analysis of the Reynolds stress transport equation. Bardina, Ferziger, and Reynolds (1983) showed, for homogeneous turbulent flows, that the unmodeled Reynolds stress transport equation yielded the following analogy: the application of a mean strain $\bar{S}$ in a rotating frame is the same as the application of a mean strain $\bar{S}$ and mean rotation 2Ω in an inertial frame of reference. This analogy (which is not a rigorous consequence of the Navier-Stokes equations since rotational effects arising from the higher-order moments were neglected) was shown by Bardina, Ferziger, and Reynolds (1983) to be a relatively good approximation for certain rotating turbulent shear flows and to be consistent with invariance under the Richardson number. The application of this analogy to the derivations in Speziale (1987) yields a non-linear $K-\varepsilon$ model of the form

$$\tau_{ij} = -\frac{2}{3} K\delta_{ij} + K^{1/2}\ell\overline{S}_{ij} - C_D\ell^2(\overset{\circ}{\overline{S}}_{ij} + \overline{S}_{ik}\overline{S}_{kj}$$

$$-\frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij} + 2\overline{W}_{ik}\overline{S}_{kj} + 2\overline{W}_{jk}\overline{S}_{ki})$$

(60)

where

$$\overset{\circ}{\overline{S}}_{ij} = \frac{\partial\overline{S}_{ij}}{\partial t} + \overline{\mathbf{v}}\cdot\nabla\overline{S}_{ij} - \overline{\omega}_{ik}\overline{S}_{kj} - \overline{\omega}_{jk}\overline{S}_{ki}$$

(61)

is the frame-indifferent Jaumann derivative and the length scale is given by $\ell = 2C_\mu K^{3/2}/\varepsilon$. For a homogeneous turbulence with constant mean velocity gradients in an inertial frame or for turbulent channel flow (the two cases considered when the nonlinear K-ε model was first calibrated), both (58) and (60) reduce to the same form

$$\tau_{ij} = -\frac{2}{3} K\delta_{ij} + K^{1/2}\ell\overline{S}_{ij} - C_D\ell^2[\overline{S}_{ik}\overline{S}_{kj}$$

$$-\frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij} + \overline{\omega}_{ik}\overline{S}_{kj} + \overline{\omega}_{jk}\overline{S}_{ki}]$$

(62)

and hence the value of $C_D = 1.68$ will not be altered.[‡] It will now be shown that this new nonlinear K-ε model yields dramatically improved predictions for homogeneous turbulent shear flow in a rotating framework (see Figure 4). Here, the constant C_μ was taken to be 0.055 (the value recommended by Rodi 1984 for homogeneous turbulence where the ratio of the production to dissipation is equal to two) and the traditional transport Equation (49) for ε

[‡] It is interesting to note that Equation (62) bears a resemblance to the nonlinear two-equation models of Pope (1975) and Saffman (1977).

was used with $C_3 = 1.45$ and $C_4 = 1.90$. A closed form equilibrium solution can be obtained which is of the form

$$b_{11} = C_D C_\mu^2 \frac{S^2 K^2}{\varepsilon^2} [\frac{7}{3} - 8 (\frac{\Omega}{S})], \qquad b_{22} = C_D C_\mu^2 \frac{S^2 K^2}{\varepsilon^2} [- \frac{5}{3} + 8 (\frac{\Omega}{S})] \qquad (63)$$

$$b_{12} = - \sqrt{C_\mu \alpha} , \qquad b_{33} = - \frac{2}{3} C_D C_\mu^2 \frac{S^2 K^2}{\varepsilon^2}, \qquad SK/\varepsilon = \sqrt{\alpha/C_\mu} \qquad (64)$$

where the ratio of production to dissipation $\alpha = (C_4 - 1)/(C_3 - 1)$ and the anisotropy tensor $b_{ij} = -(\tau_{ij} + \frac{2}{3} K\delta_{ij})/K$. A comparison of the results obtained from the linear and nonlinear $K-\varepsilon$ models (along with the experiments of Tavoularis and Corrsin (1981) and the large-eddy simulations of Bardina, Ferziger, and Reynolds (1983)) are shown in Table 1. Here, the equilibrium values of the anisotropy tensor obtained from the nonlinear $K-\varepsilon$ model are dramatically improved with respect to its normal components (the reader should note that b_{33} is not shown since it is precisely equal to $-(b_{11} + b_{22})$ due to the fact that b_{ij} is traceless). Unfortunately, no experimental data is available for rotating shear flow and the values of the anisotropies obtained from the large-eddy simulations are somewhat inaccurate due to course resolution and the lack of a good defiltering scheme. However, there is no question that the normal components of the anisotropy tensor predicted by the nonlinear $K-\varepsilon$ model constitute a considerable improvement over their linear counterparts. Both the linear and nonlinear $K-\varepsilon$ models yield the same predictions for the equilibrium values of b_{12} and SK/ε which are the same for all values of Ω/S. This is not consistent with physical and numerical experiments which indicate that b_{12} and SK/ε can vary considerably with Ω/S. In order to predict this dependence, a modi-

fied dissipation rate equation must be developed which properly accounts for rotational strains—a task of considerable difficulty.

Now, a consistent modification of the modeled dissipation rate equation will be developed which can account for the considerable reduction in dissipation which occurs in a rapidly rotating isotropic turbulence. As mentioned in Section 3, the intrinsic mean vorticity invariant $(\frac{1}{2}\overline{W}_{ij}\overline{W}_{ij})^{1/2}$ does not vanish in the limit of two-dimensional turbulence. However, this invariant was arrived at by Bardina, Ferziger, and Rogallo (1985) since it reduces to Ω in a rotating isotropic turbulence (it was Equation (57) that was used to correlate with the experiments of Wigeland and Nagib 1978). Alternatively, there are other invariants that reduce to Ω for rotating isotropic turbulence but vanish in the limit of two-dimensional turbulence. The primary such invariant is

$$I^{(\tau\overline{w})} = |\frac{3}{2}(\overline{W}_{ik}\overline{W}_{jk}\frac{\tau_{ij}}{K} + \overline{W}_{ik}\overline{W}_{ik})|^{1/2}$$ (65)

which was first introduced by Speziale (1985). This gives rise to the alternative modeled transport equation for the dissipation rate

$$\frac{D\varepsilon}{Dt} = -C_2\frac{\partial}{\partial x_i}(\frac{K}{\varepsilon}\tau_{ij}\frac{\partial\varepsilon}{\partial x_j}) + C_3 f_1(\frac{I^{(\tau\overline{w})}K}{\varepsilon})\frac{\varepsilon}{K}\tau_{ij}\frac{\partial\overline{v}_i}{\partial x_j}$$

$$-C_4 f_2(\frac{I^{(\tau\overline{w})}K}{\varepsilon})\frac{\varepsilon^2}{K}$$ (66)

where f_1 and f_2 are sufficiently smooth functions of the dimensionless invariant $I^{(\tau\overline{w})}K/\varepsilon$. For plane homogeneous turbulence, the invariant $I^{(\tau\overline{w})}$ reduces to

$$I^{(\tau\overline{w})} = |\frac{3}{2}\frac{\tau_{33}}{K}\overline{W}_{12}^2|^{1/2}$$ (67)

and, hence, for small anisotropies (where $\tau_{33} \doteq -\frac{2}{3} K$)

$$I^{(\tau\overline{w})} \doteq |\overline{W}_{12}| = (\frac{1}{2}\overline{W}_{ij}\overline{W}_{ij})^{1/2}. \tag{68}$$

If we approximate f_1 and f_2 as linear functions of the form

$$f_1 = 1 - \gamma_1 \frac{I^{(\tau\overline{w})}K}{\varepsilon} \tag{69}$$

$$f_2 = 1 + \gamma_2 \frac{I^{(\tau\overline{w})}K}{\varepsilon} \tag{70}$$

(where γ_1 and γ_2 are dimensionless constants) it follows that (66) is in close approximate agreement with the most recent Bardina modification of the dissipation rate transport equation given that $\gamma_1 \doteq 0.01$ and $\gamma_2 \doteq 0.079$ (this model also reduces to the more simple model given by (57) for rotating isotropic turbulence). Such a model has been shown by Bardina to work reasonably well for rotating isotropic turbulence (at moderate rotation rates) and for simple plane turbulent shear flows subjected to mild rotational strains. However, unlike the Bardina model, this new model for the dissipation rate satisfies material frame-indifference in the limit of two-dimensional turbulence (as can be seen from (67) since $I^{(\tau w)} \to 0$ as $\tau_{33} \to 0$) and allows for more general nonlinear dependence on $\overline{W}$ (the simple linear expressions (69)-(70) break down when a wider variety of flows is considered). Nonlinear generalizations of (69) and (70) should be pursued in future studies.

Finally, the implications that the non-inertial constraints derived in Section 2 have on second-order closure models will be examined. As alluded to

before, the Launder, Reece, and Rodi (1975) model as well as the other common-
ly used second-order closures violate material frame-indifference in the limit
of two-dimensional turbulence and do not give rise to a Taylor-Proudman re-
organization for statistically steady turbulent flows in a rapidly rotating
framework. In fact, Speziale (1985) recently showed that for rotating turbu-
lent channel flow (see Figure 5) these second-order closure models yield the
spurious result of a vanishing Reynolds shear stress

$$\tau_{xy} = 0 \tag{71}$$

in the limit as $\Omega \to \infty$ and do not give rise to a full Taylor-Proudman
reorganization to a two-dimensional state. These problems were not en-
countered in the recent study of Launder, Tselepidakis, and Younis (1987)
since they restricted their attention to flows with Rossby numbers greater
than 10 (a Taylor-Proudman reorganization would only be expected for Rossby
numbers less than 0.1—a value nearly two orders of magnitude smaller than
those considered therein). Complete consistency with the non-inertial con-
straints (i)-(iv) derived herein can be obtained from second-order closures of
the general form

$$\frac{D_c \tau_{k\ell}}{Dt} = \alpha \left[\tau_{km} \overline{W}_{\ell m} + \tau_{\ell m} \overline{W}_{km} - \frac{1}{2K} \left(\tau_{km} \tau_{mn} \overline{W}_{n\ell} \right. \right.$$

$$\left. \left. + \tau_{\ell m} \tau_{mn} \overline{W}_{nk} \right) \right] - \left(\beta \frac{\varepsilon}{k} + \gamma I^{(\tau \overline{w})} \right) \left(\tau_{k\ell} - \frac{1}{3} \tau_{mm} \delta_{k\ell} \right) \tag{72}$$

$$+ \frac{\partial C_{k\ell m}(\tau, \nabla \tau, \varepsilon)}{\partial x_m} + \pi_{k\ell}(\tau, \overline{S}, \varepsilon) + \nu \nabla^2 \tau_{k\ell}$$

where ϵ is obtained from the new modeled transport Equation (66) and α, β, γ are dimensionless functions of $I^{(\tau w)}$ and the invariants of τ (which can be taken to be constants in the first approximation). The first term on the right-hand-side of (72), with the coefficient α, arises from the rotationally dependent part of the rapid pressure-strain correlation. This term was recently derived from a Langevin model by Haworth and Pope (1986) who showed that it vanishes in the limit of two-dimensional turbulence. The second term on the right-hand-side of (72), with the coefficient γ, represents the rotationally dependent part of the return term which, in a rapidly rotating frame, was shown by Speziale (1985) to give rise to a Taylor-Proudman reorganization to a two-dimensional state wherein it then vanishes. Here again, $C_{k\ell m}$ is the third-order diffusion correlation which is frame-indifferent and $\pi_{k\ell}$ accounts for the rotationally independent parts of the production, pressure-strain and dissipation rate correlations. In addition to satisfying material frame-indifference in the limit of two-dimensional turbulence (along with consistency with the Taylor-Proudman Theorem), this new second-order closure also satisfies constraint (iv). To be specific, the rotationally dependent terms in (72) vanish in an isotropic turbulence and the modified dissipation rate equation (66) gives rise to reduced dissipation in a rotating frame consistent with constraint (iv). As a result of the dramatically improved non-inertial properties of (72), spurious physical effects such as (71) (that are predicted by the commonly used second-order closures) can be avoided. Equation (72) represents a hierarchy of second-order closure models whose detailed study represents an extensive research effort that is beyond the scope of the present paper.

5. CONCLUSION

In this paper, several important constraints that turbulence models must satisfy in non-inertial frames of reference were derived as a rigorous consequence of the Navier-Stokes equations. Of particular importance was the constraint that turbulence models should only depend on the frame of reference through the intrinsic mean vorticity tensor and that all such frame-dependent effects must vanish in the limit of two-dimensional turbulence. In addition, it was also shown that Rapid Distortion Theory for an isotropic turbulence suddenly subjected to a strong rotation can serve as an equally important constraint requiring an initially isotropic turbulence to decay isotropically (with a reduced dissipation rate) in a rotating frame. All of the commonly used turbulence models were shown to be in serious violation of these constraints and, thus, inconsistent with the Navier-Stokes equations. An improved two-equation turbulence model was developed which was demonstrated to be substantially superior to the more standardly used $K-\epsilon$ model in the description of homogeneous turbulent shear flow in a rotating frame. Furthermore, a hierarchy of consistent second-order closure models was developed which have dramatically improved properties in rotating frames over the more commonly used second-order closures. A complete calibration and testing of such models is a massive research effort that is beyond the scope of the present study. However, such work is currently underway in collaboration with others.

Finally, it should be mentioned that the results of this study could have important implications in the analysis of curved turbulent flows. As demonstrated herein, once the inertial form of a turbulence model is specified, its non-inertial form is automatically determined by appropriately replacing the

mean vorticity with the intrinsic mean vorticity. Consequently, if a turbulence model exhibits incorrect behavior in a non-inertial frame, this means that the dependence of the inertial form of the model on the mean vorticity is faulty. Since the mean vorticity plays an important role in the description of curved turbulent flows, it is quite likely that the difficulty in describing such flows is a result of the use of models that exhibit physically incorrect non-inertial behavior. A more detailed discussion of this point will be the subject of a future paper.

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	Equilibrium Values	Linear $K\text{-}\varepsilon$ Model	Nonlinear $K\text{-}\varepsilon$ Model	Experiments	Large-Eddy Simulations
$\Omega/S = 0$	b_{11}	0	0.431	0.403	0.606
	b_{22}	0	-0.308	-0.295	-0.530
	b_{12}	-0.332	-0.332	-0.284	-0.288
	$\dfrac{SK}{\varepsilon}$	6.03	6.03	6.08	-
$\Omega/S = 0.25$	b_{11}	0	0.0616	-	0.121
	b_{22}	0	0.0616	-	0.091
	b_{12}	-0.332	-0.332	-	-0.697
	$\dfrac{SK}{\varepsilon}$	6.03	6.03	-	-
$\Omega/S = 0.5$	b_{11}	0	-0.308	-	-0.530
	b_{22}	0	0.431	-	0.500
	b_{12}	-0.332	-0.332	-	-0.197
	$\dfrac{SK}{\varepsilon}$	6.03	6.03	-	-

Table 1. Equilibrium results for homogeneous turbulent shear flow in a rotating frame: Comparison of the predictions of the $K\text{-}\varepsilon$ model with the large eddy simulations of Bardina, Ferziger, and Reynolds (1983) and the experiments of Tavoularis and Corrsin (1981).
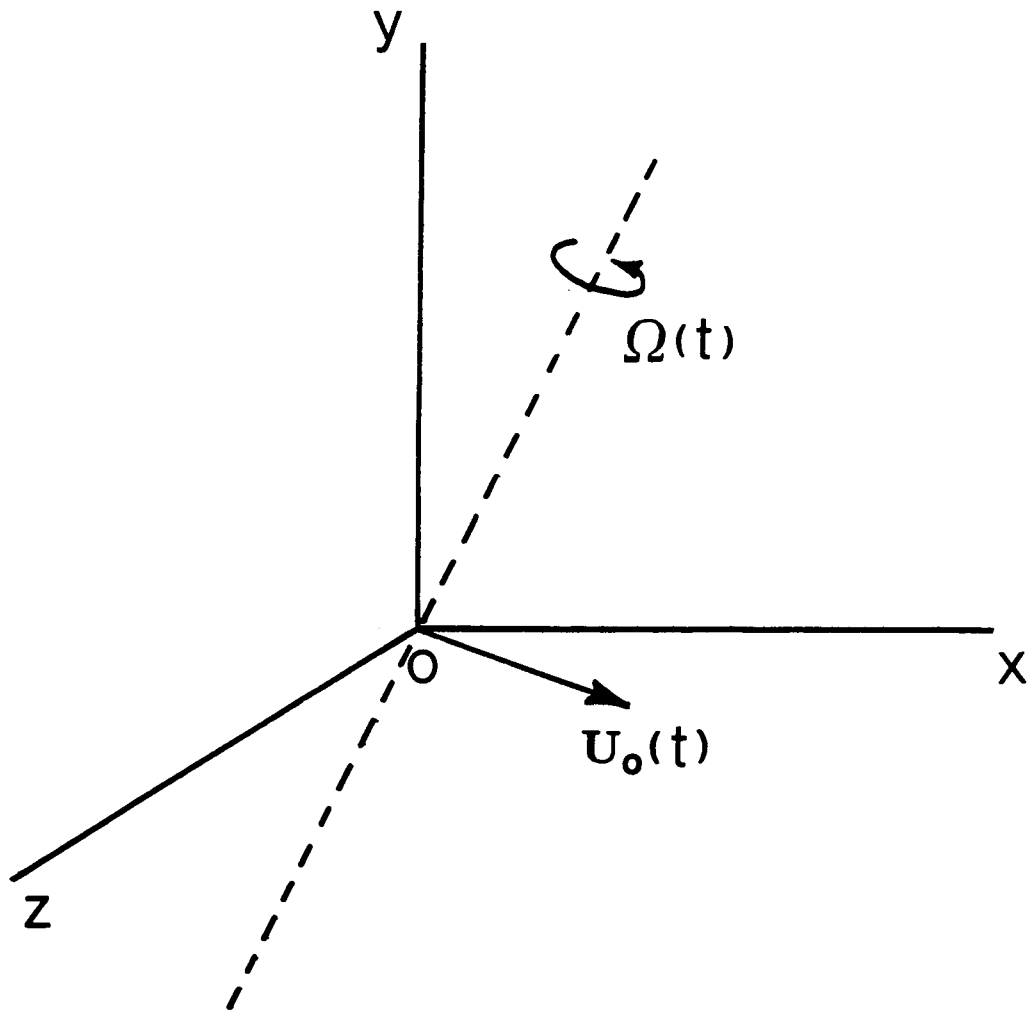
LIST OF FIGURES

Figure 1. Arbitrary non-inertial frame of reference.
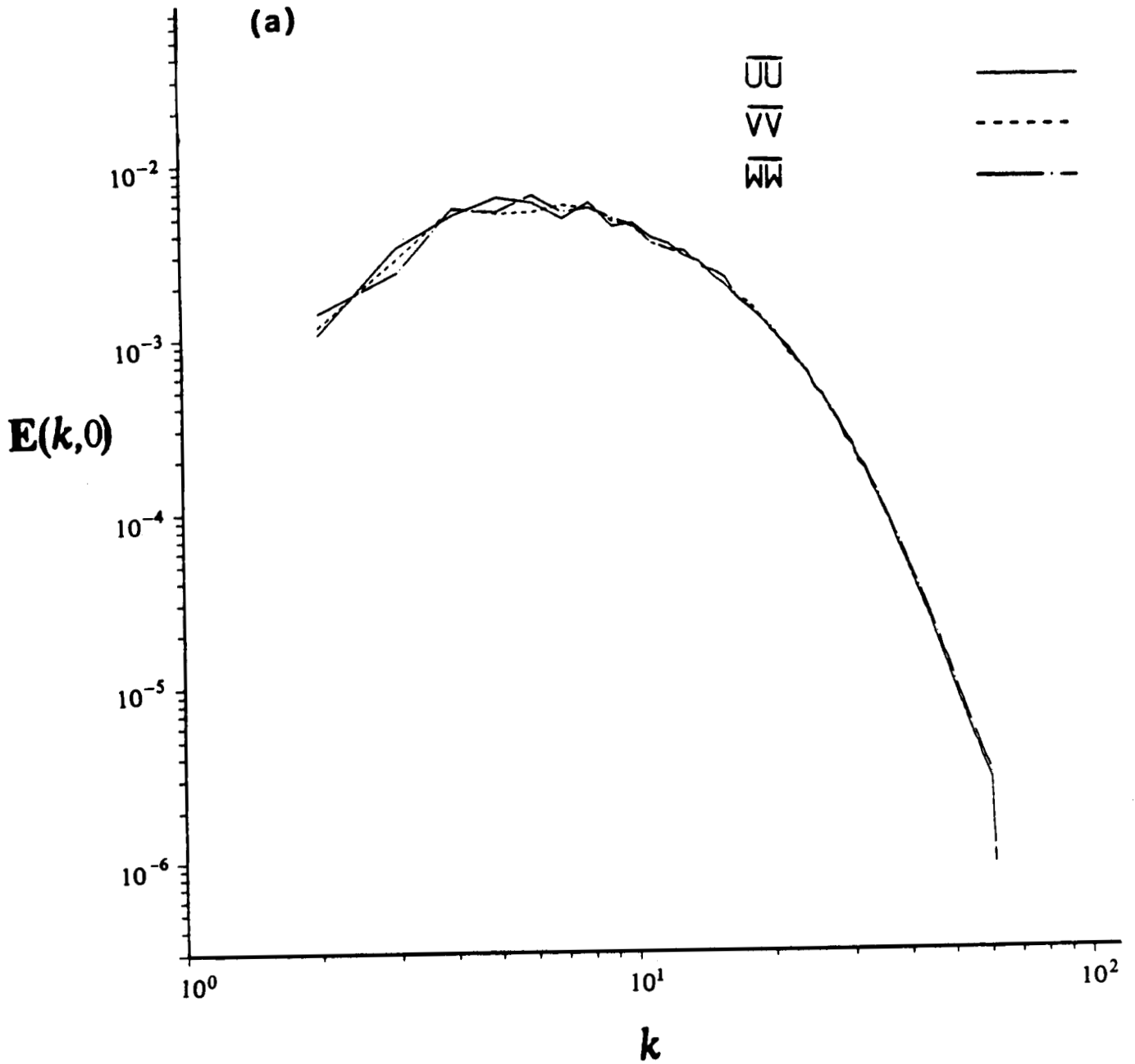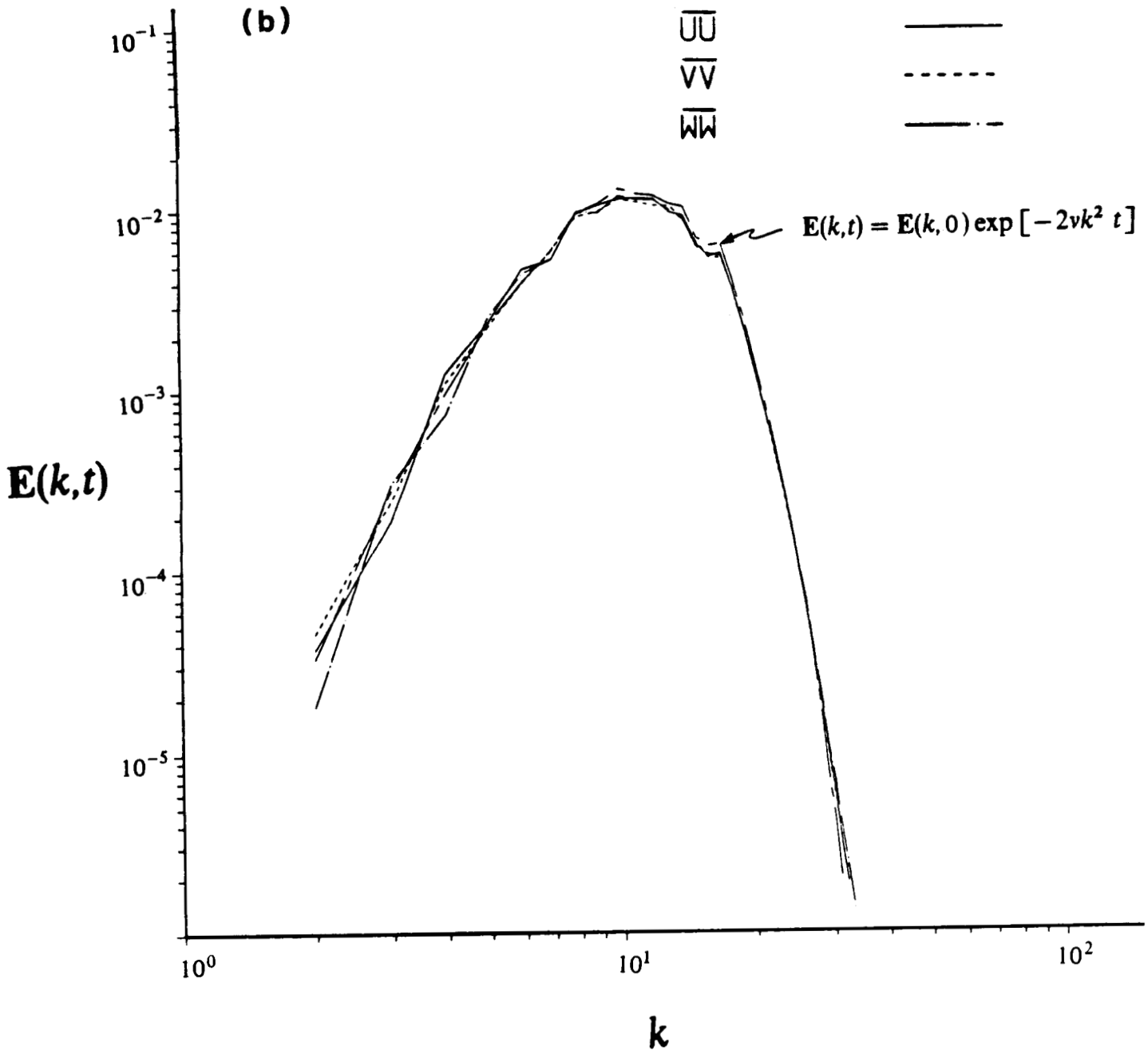
Figure 2. Energy spectrum obtained from a direct numerical simulation of rotating isotropic turbulence: (a) t = 0, (b) t > 0 (K/K₀ ≐ 0.1).

Figure 2. Energy spectrum obtained from a direct numerical simulation of rotating isotropic turbulence: (a) t = 0, (b) t > 0 (K/K$_0$ ≐ 0.1).
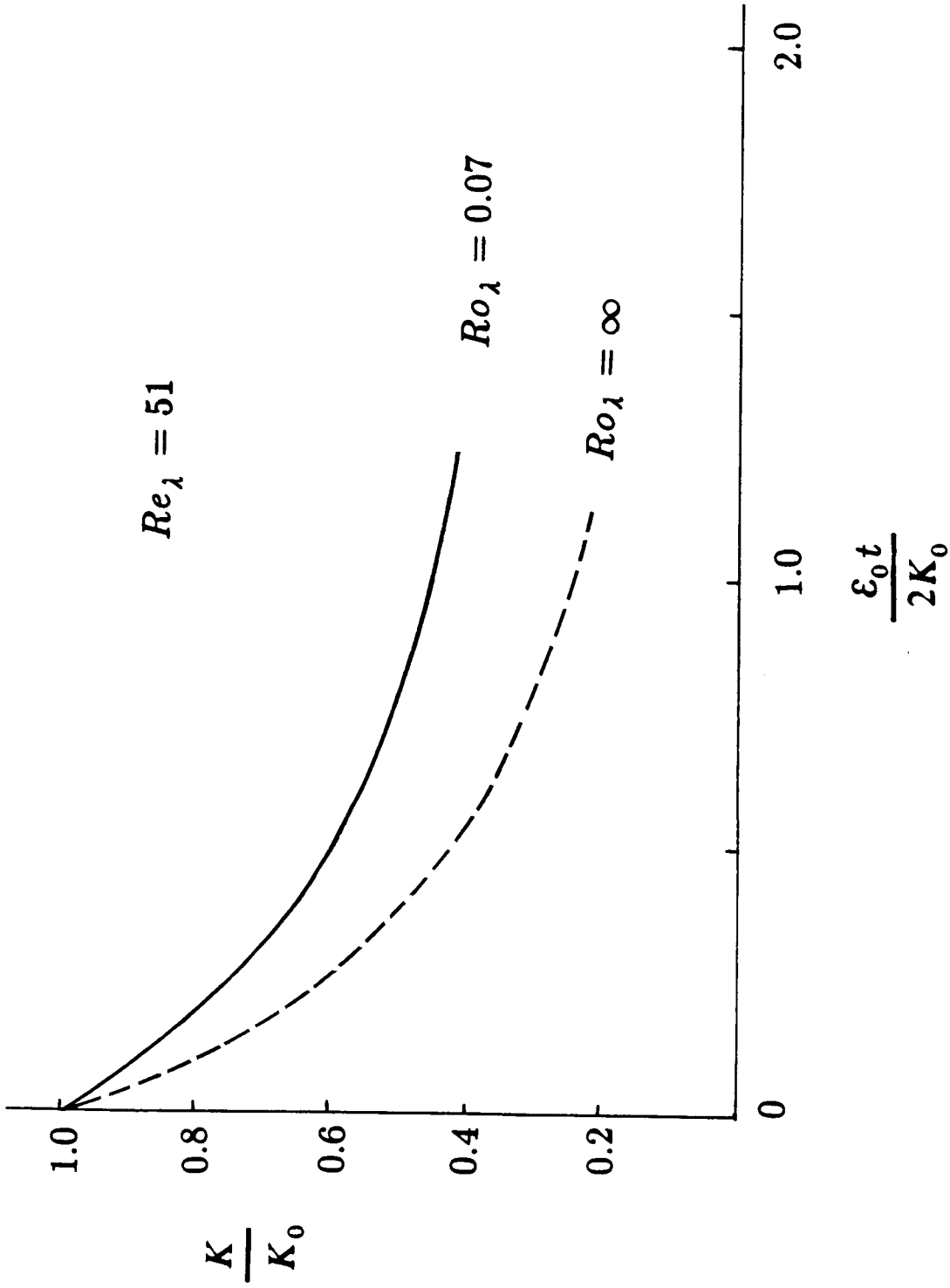
Figure 3. Decay of turbulent kinetic energy in rotating isotropic turbulence obtained from direct numerical simulations.
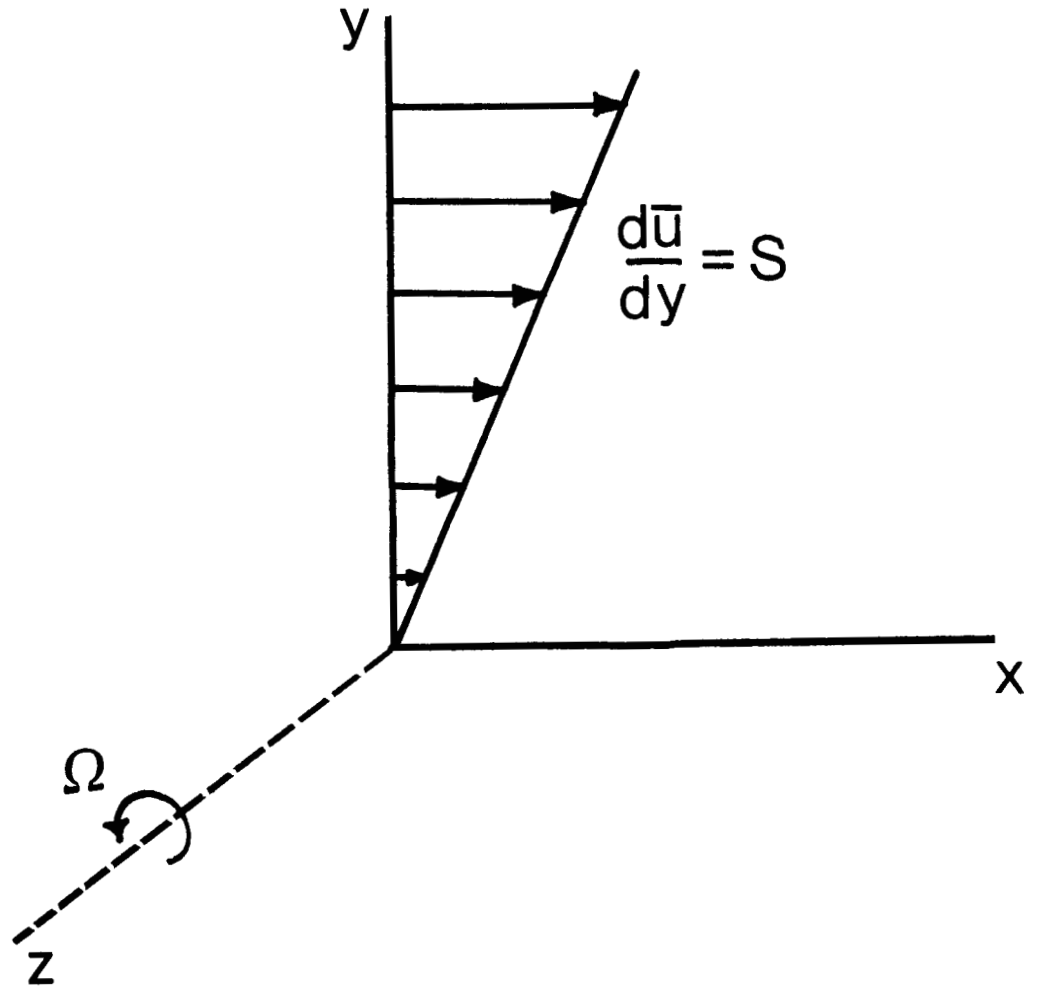
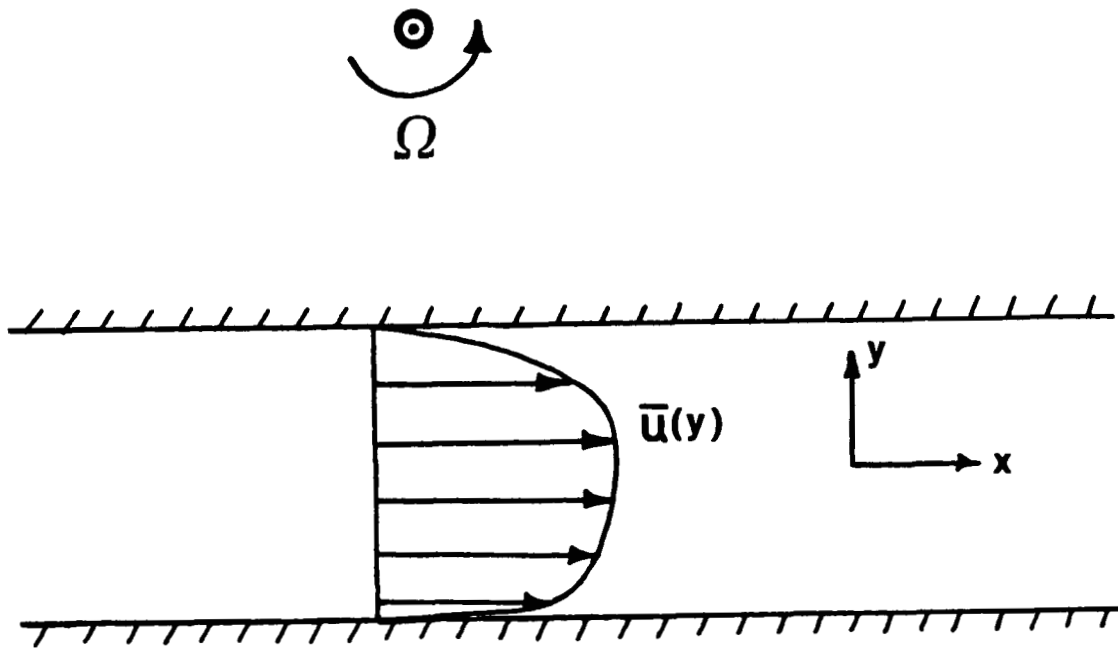Figure 4. Homogeneous turbulent shear flow in a rotating frame.

Figure 5.   Turbulent channel flow in a rotating frame.

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Final Report

16. Abstract

The effect of an arbitrary change of frame on the structure of turbulence models is examined from a fundamental theoretical standpoint. It is proven, as a rigorous consequence of the Navier-Stokes equations, that turbulence models must be form invariant under arbitrary translational acclerations of the reference frame and should only be affected by rotations through the intrinsic mean vorticity. A direct application of this invariance property along with the Taylor-Proudman Theorem, material frame-indifference in the limit of two-dimensional turbulence and Rapid Distortion Theory is shown to yield powerful constraints on the allowable form of turbulence models. Most of the commonly used turbulence models are demonstrated to be in serious violation of these constraints and consequently are inconsistent with the Navier-Stokes equations in non-inertial frames. Alternative models with improved non-inertial properties are developed and some simple applications to rotating turbulent flows are considered.

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