Turbulence of Capillary Waves

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A numerical model for direct simulation of the surface of ideal fluid based on the expansion of the Hamiltonian of the surface up to terms of fourth order is developed. For the case of capillary wave we observe the formation of powerlike spectrum of spatial elevations close to one predicted by weak-turbulent theory $I_k \simeq k^{-19/4}$, which previously was not confirmed either experimentally or numerically.

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An ensemble of weakly interacting waves in a dispersive medium can be described statistically even though it may be very far from the state of thermodynamic equilibrium. Because of the small value of nonlinearity the infinite system of equations for the correlation function in this case can be truncated in a consistent way and reduced to one kinetic equation for "wave numbers" (or wave action):

$$\frac{\partial n_{\vec{k}}}{\partial t} + 2\gamma_{\vec{k}} n_{\vec{k}} = st(n) \tag{1}$$

(see, for instance, [2]). Here $\gamma_{\vec{k}}$ is the wave damping (or the wave pumping if $\gamma_{\vec{k}} < 0$) and $st(n)$ is the "collision term" corresponding to a wave equation.

The collision term describes "cascade" transport of wave energy in K space to the small scale region (direct cascade) and to the large scale region (inverse cascade). The last one exists only if the total number of waves $N = \int n_{\vec{k}}\, d\vec{k}$ is the integral of motion.

The equation

$$st(n_{\vec{k}}) = 0, \tag{2}$$

besides having a trivial thermodynamic solution, has Kolmogorov-type solutions describing cascades. In a medium without a characteristic length they are powerlike functions

$$n_{\vec{k}} \simeq k^{-\alpha}. \tag{3}$$

The theory of weak-turbulent Kolmogorov spectra has advanced far. But direct experimental confirmation of these spectra is very poor. One can consider, more or less, well confirmed existence of the Komogorov spectrum for the direct cascade of gravitational wave on the surface of an incompressible deep fluid

$$I_\omega \simeq \alpha \frac{g v}{\omega^4}.$$

[Here $I(\omega)$ is the spectral density of surface elevations, is the wave frequency, g is the acceleration of gravity, v is the wind velocity, and α is a dimensionless constant.] This spectrum was theoretically derived by Zakharov and Filonenko [3] and experimentally observed by Toba [4]. The confirmation cannot be considered as complete because the Zakharov-Filonenko spectrum is isotropic, while Toba's spectrum is essentially anisotropic.

Another way to check the weak-turbulent theory is by numerical simulation. Some valuable results were obtained by numerical solution of the kinetic equation (1) [5,6]. But the kinetic equation (1) is itself a subject for careful examination. Its derivation assumes that the phases of all interacting waves are random and are in a state of chaotic motion. The validity of this assumption is not clear *a priori.*

The correct way to check weak-turbulent theory and its prediction is by numerical simulation using "first principles," i.e., direct solution of the nonlinear dynamic equation governing propagation and interaction of the waves.

In real cases these equations are of two or three spatial dimensions, and its numerical solution is not a simple problem. It was done so far for the 2D nonlinear Schrödinger equation [8], but in this particular case Kolmogorov spectra do not exist.

In this paper we present results of numerical simulation of capillary waves on the surface of the incompressible infinitely deep fluid. In this case only a direct cascade of energy takes place. The corresponding Kolmogorov spectrum for the surface elevation has the form $n_k \simeq k^{-17/4}$. We will show that this theoretical prediction is confirmed by direct numerical simulation with satisfactory accuracy. The developed numerical approach can be used for solving a wide class of problems pertaining to the interaction of surface waves and—more generally—other types of waves in nonlinear media.

Theoretical background.—We study the potential flow of ideal incompressible deep fluid with the free surface. Let $\eta(\vec{r}, t)$, $\vec{r} = (x, y)$ is the shape of the surface, $\psi(\vec{r}, t)$ is the velocity potential $\Phi = \Phi(\vec{r}, z)$, $\vec{v} = \nabla\Phi$, $\Delta\Phi = 0$, evaluated on the free surface: $\psi(\vec{r}, t) = \Phi(\eta(\vec{r}, t), \vec{r}, t)$. It is known [7] that under these assumption the fluid is a Hamiltonian system;

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \tag{4}$$

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{5}$$

Here H is the total energy of the fluid consisting of the kinetic and the potential components

$$H = H_{\rm pot} + H_{\rm kin}, \tag{6}$$

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where

$$H_{\rm pot} = \sigma \int [1 + (\nabla\eta)^2]^{1/2} - 1\, d\vec r,$$

$$H_{\rm kin} = \frac{1}{2}\int d\vec r_\perp \int_{-\infty}^{\eta} dz (\nabla\Phi)^2.$$

Here σ is a coefficient of surface tension.

Direct numerical simulation of the system (4) provides a solution of the boundary problem for the Laplace equation for every time step. In the full 3D case it is an enormously hard problem. To solve the problem one can use an expansion in powers of nonlinearity. For Fourier transforms this expansion up to the quadratic terms has a form

$$H = H_0 + H_1 + H_2 + \cdots,$$

$$H_0 = \frac{1}{2}\int [|k||\psi_{\vec k}|^2 + \sigma|k|^2|\eta_{\vec k}|^2]\, d\vec k\,,$$

$$H_1 = -\frac{1}{2\times 2\pi}\int L_{\vec k_1 \vec k_2}\psi_{\vec k_1}\psi_{\vec k_2}\eta_{\vec k_3}\delta(\vec k_1 + \vec k_2 + \vec k_3)\, d\vec k_1\, d\vec k_2\, d\vec k_3\,,$$

$$H_2 = \frac{1}{4(2\pi)^2}\int M_{\vec k \vec k_1 \vec k_2 \vec k_3}\psi_{\vec k}\psi_{\vec k_1}\eta_{\vec k_2}\eta_{\vec k_3}\delta(\vec k + \vec k_1 + \vec k_2 + \vec k_3)\, d\vec k\, d\vec k_1\, d\vec k_2\, d\vec k_3\,,$$

$$L_{\vec k_1 \vec k_2} = \vec k_1 \vec k_2 + |\vec k_1||\vec k_2|\,,$$

$$M_{\vec k_1 \vec k_2 \vec k_3 \vec k_4} = |\vec k_1||\vec k_2|\left[\frac{1}{2}(|\vec k_1 + \vec k_3| + |\vec k_1 + \vec k_4| + |\vec k_2 + \vec k_3| + |\vec k_2 + \vec k_4|) - |\vec k_1| - |\vec k_2|\right].$$

The corresponding dynamic equations are

$$\frac{\partial \eta_{\vec r}}{\partial t} = [|\hat k|\psi]_{\vec r} - \operatorname{div}(\eta\nabla\psi) - |\hat k|[[|\hat k|\psi]_{\vec r}\times\eta_{\vec r}]_{\vec r} + |\hat k|[|\hat k|[[|\hat k|\psi]_{\vec r}\times\eta_{\vec r}]_{\vec r}\times\eta_{\vec r}]_{\vec r}$$
$$+ \tfrac{1}{2}\Delta_{\vec r}[[|\hat k|\psi]_{\vec r}\times\eta^2_{\vec r}]_{\vec r} + \tfrac{1}{2}|\hat k|[\Delta_{\vec r}\psi\times\eta^2_{\vec r}] \qquad (7)$$

$$\frac{\partial \psi_{\vec r}}{\partial t} = \sigma\Delta_{\vec r}\eta_{\vec r} + \tfrac{1}{2}[-(\nabla\psi)^2 + [|\hat k|\psi]^2_{\vec r}] - |\hat k|[[|\hat k|\psi]_{\vec r}\times\eta_{\vec r}]_{\vec r}\times[|\hat k|\psi]_{\vec r}$$
$$- \Delta\psi\times[|\hat k|\psi]_{\vec r}\times\eta_{\vec r} + D_{\vec r} + F_{\vec r} \qquad (8)$$

We added to Eq. (8) a phenomenological damping term $D_{\vec r}$ and the external force $F_{\vec r}(t)$.

In the linear approximation Eqs. (7) and (8) describe capillary waves with the dispersion relation

$$\omega_k = (\sigma k^3)^{1/2}.$$

One can introduce normal amplitudes

$$a_{\vec k} = \sqrt{\frac{\sigma k^2}{2\omega_k}}\,\eta_{\vec k} - i\sqrt{\frac{k}{2\omega_k}}\,\psi_{\vec k}\,.$$

According to weak-turbulent theory the pair correlation function

$$\langle a_{\vec k} a^*_{\vec k'}\rangle = n_{\vec k}\delta(\vec k - \vec k')$$

satisfies the kinetic equation (1), where

$$st(n) = \int [R_{\vec k \vec k_1 \vec k_2} - R_{\vec k_1 \vec k \vec k_2} - R_{\vec k_2 \vec k \vec k_1}]\, d\vec k_1\, d\vec k_2\,,$$

$$R_{\vec k \vec k_1 \vec k_2} = 4\pi|V_{\vec k \vec k_1 \vec k_2}|^2\delta(\vec k - \vec k_1 - \vec k_2)\delta(\omega_{\vec k} - \omega_{\vec k_1} - \omega_{\vec k_2})\times[n_{\vec k_1}n_{\vec k_2} - n_{\vec k}n_{\vec k_1} - n_{\vec k}n_{\vec k_2}],$$

$$V_{\vec k \vec k_1 \vec k_2} = \frac{1}{8\pi\sqrt{2\sigma}}(\omega_k\omega_{k_1}\omega_{k_2})^{1/2}\left[\frac{L_{\vec k_1,\vec k_2}}{(k_1k_2)^{1/2}k} - \frac{L_{\vec k,-\vec k_1}}{(kk_1)^{1/2}k_2} - \frac{L_{\vec k,-\vec k_2}}{(kk_2)^{1/2}k_1}\right].$$

In an isotropic medium containing no characteristic length the dispersion relation is a powerlike function

$$\omega_k \simeq k^\alpha$$

as long as $V_{\vec k \vec k_1 \vec k_2}$ is a homogeneous function,

$$V_{\epsilon\vec k,\epsilon\vec k_1,\epsilon\vec k_2} = \epsilon^\beta V_{\vec k \vec k_1 \vec k_2}\,.$$

In this case Eq. (2) has an exact powerlike solution $n_k = CP^{1/2}/k^{\beta+d}$ (d is the dimension of space), which is a Kolmogorov-type spectrum describing the constant flux of energy in K space from large to small scales. P is the value of the energy flux and C is an absolute constant. For capillary waves $\alpha \simeq \frac{3}{2}$, $\beta = \frac{9}{4}$, and $d = 2$. Hence

$$n_k = C\frac{P^{1/2}}{k^{17/4}}\,. \qquad (9)$$

For the correlation function of elevation one gets

$$I_k = \langle |\eta_k|^2 \rangle = \frac{\omega_k}{\sigma k^2} = \frac{C\sigma^{-1/4} p^{1/2}}{k^{19/4}}.$$

This result was obtained first by Zakharov and Filonenko [3]. The solution (9) is linearly stable in the framework of the kinetic equation (1) (see [9]).

The physical interpretation of the spectrum (9) is the following. Assume there a pumping of any type at small wave numbers $k \leq k_p$, while strong damping takes place at large wave numbers $k \geq k_0$, $k_0 \gg k_p$. Then, the spectrum (9) is realized in the inertial interval $k_p \ll k \ll k_0$. A more exact expression for the spectrum in this window is (see [1])

$$I_k = \frac{\sigma^{-1/4} p^{1/2}}{k^{19/4}} \left[1 - C_1 \left(\frac{k_p}{k} \right)^{-9/4} - C_2 \left(\frac{k}{k_0} \right)^{9/4} \right].$$

Here C_1, C_2 are constants.

Numerical simulation.—We have realized a numerical simulation of the system (7) and (8). In spite of the fact that the matrix element of the kinetic equation $V_{\vec{k}\vec{k}_1\vec{k}_2}$ is expressed only through the Hamiltonian coefficients H_0, H_1, we prefer to keep the next term H_2 in the expansion of the Hamiltonian. The reason for keeping the next term of the expansion is the following: it can be shown that the dynamical system generated by the Hamiltonian $H_0 + H_1$ becomes ill posed at very low levels of nonlinearity. Meanwhile, including a consideration of the next term of the expansion essentially improves the situation (details will be published separately). Moreover, the developed scheme after a minor modification can be used for numerical simulation of gravitational waves.

Equations (7) and (8) are not differential in X space. Besides taking derivatives they include taking the operator $|k|[(-\Delta)^{1/2}$ in X space].

As such, the system can be reduced to a set of six partial differential equations for variables interconnected by the consequent application of the operator $(-\Delta)^{1/2}$. This allows us to apply a solution to the system (7) and (8), the spectral code, using the fast Fourier transform for each time step. Omitting the details of this numerical scheme, we reproduce now only the final result of the calculation.

For numerical integration of Eqs. (7) and (8) we used the functions F and D defined in Fourier space through the following relations:

$$F_{\vec{k}} = f_{\vec{k}} e^{i\Omega_{\vec{k}} t},$$

$$D_{\vec{k}} = \gamma_{\vec{k}} \Psi_{\vec{k}},$$

$$\Omega_{\vec{k}} = \omega_{\vec{k}}[1 + R(t)],$$

$$\gamma_{\vec{k}} = \begin{cases} -(|\vec{k}| - |\vec{k}|)^2 & \text{if } k > k_0, \\ 0 & \text{if } 0 \leq k \leq k_0. \end{cases}$$

The pumping force F_k is "almost" in resonance with the local linear frequency ω_k of the corresponding Fourier harmonics; i.e., the frequency Ω_k slightly fluctuates around the exact value of ω_k due to a small random addition in time $R(t)$. The form of the pumping amplitude was chosen to be axially symmetric $f_k = f_0 e^{-[(|\vec{k}|-|\vec{k}_1|)^4/k_2]}$.

The value of k_0 defines a starting point of "hyperviscosity" we used in our experiments to provide a wide enough inertial interval. Calculations were carried out on a 256×256 grid.

The pumping in our experiments was concentrated at small k ($k_1 = 10$, $k_2 = 6$), and so $k_p \simeq 16$. We took $k_0 \simeq 40$, and so the inertial interval was about half a decade.

In the calculation we examined two basic problems. At a low level of nonlinearity $[(H_1 + H_2)/H_0 \leq 10^{-3}]$ we observed no energy flux and formation of KAM-type quasiperiodic regime at $k \leq k_p$. Apparently, this is explained through the discreteness of the wave spectrum caused by periodic boundary conditions: One cannot realize all possible resonance conditions $\omega = \omega_1 + \omega_2$, $\vec{k} = \vec{k}_1 + \vec{k}_2$ on a discrete grid.

At higher levels of nonlinearity we were confronted with a short wave instability similar to that observed by others [10–12]. This instability can be effectively suppressed for moderate nonlinearities $[(H_1 + H_2)/H_0 \leq 5 \times 10^{-2}]$ by low-pass filtering in k space equivalent to the smoothing in the real space or extra damping. The presence of the instability makes, however, the essential increasing of k_0 and the broadening of the inertial interval impossible.

However, at moderate levels of nonlinearity $[(H_1 + H_2)/H_0 \simeq 5 \times 10^{-2}$, $f_0 = 2 \times 10^{-4}]$ we observed fast formation of the stationary wave spectrum carrying a constant flux of energy to the large-k region (see Fig. 1). The observed spectrum was angular isotropic. The plot of the logarithmic derivative (Fig. 2) shows that in the interval $8 < k < 20$ the spectrum can be considered as powerlike ($I_k \simeq k^{-x}$). The exponential varies in the limits $-5.0 < x < -4.8$. The value closest to the theoretical ($x = 4.75$) is reached for $k \simeq 14$.

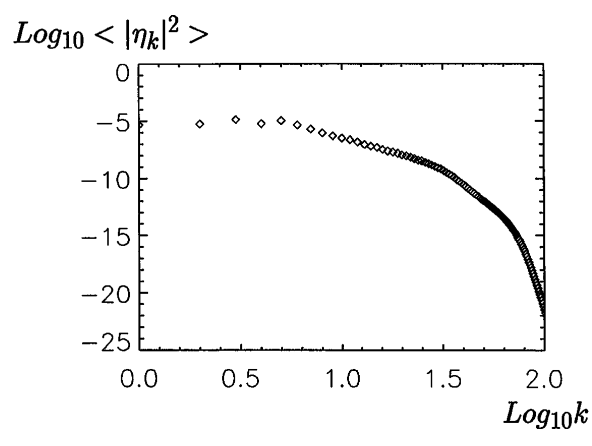

FIG. 1. The logarithm of the spectrum of spatial elevations of the liquid surface as a function of the logarithm of the wave number.

In the neighborhood of this point the spectrum can be presented as ($\sigma = 1$)

$$I_k = qk^{-4.8},$$
$$q = CP^{1/2} \simeq 0.1\,.$$

A direct calculation of the energy flux

$$P = \frac{\partial E}{\partial t} = \int \gamma_k |\vec{k}| |\Psi_k|^2 \, d\vec{k}$$

gives $P \simeq 3 \times 10^{-4}$ which is not a contradiction because the Kolmogorov constant C is so far unknown.

In general, the accuracy achieved in our experiments at the rather narrow (half-decade) interval under the time constraints of the calculation (a typical variant took 30 h of the Cray C-90) can be considered as relatively good.

Summarizing the results we can conclude that the direct numerical simulation of the dynamic nonlinear equation confirms the existence, and important role, of the weak-turbulent Kolmogorov spectra at least in the case of capillary waves. Indirectly this result confirms the validity of the kinetic equation for a description of wave turbulence. We hope that the developed effective approach will allow us to study numerically other types of wave turbulence; first of all, the behavior of a system of wind-driven gravitational waves on the sea surface.

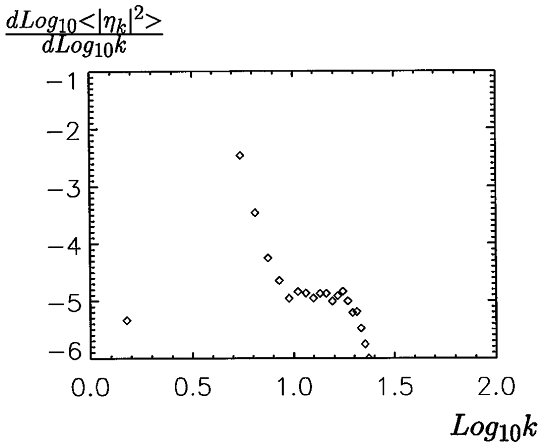

FIG. 2. The derivative of the logarithm of the spectrum of spatial elevations with respect to the logarithm of the wave number as a function of the logarithm of the wave number (the local value of the Kolmogorov index).

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