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TURBULENCE SCALE DEPENDENCE OF THE RICHARDSON CONSTANT IN LAGRANGIAN STOCHASTIC MODELS

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Abstract. We investigate the relative dispersion properties of the well-mixed class of Lagrangian stochastic models. Dimensional analysis shows that, given a model in the class, its properties depend solely on a non-dimensional parameter, which measures the relative weight of Lagrangian-to-Eulerian scales. This parameter is formulated in terms of Kolmogorov constants, and model properties are then studied by modifying its value in a range that contains the experimental variability. Large variations are found for the quantity $g^* = 2gC_0^{-1}$, where g is the Richardson constant.

Keywords: Eulerian turbulence scale, Lagrangian stochastic models, Lagrangian turbulence scale, Relative dispersion, Richardson coefficient.

1. Introduction

Relative dispersion is a process that depends on the combination of the Eulerian and Lagrangian properties of turbulence. If particle separation falls in the inertial subrange, the Eulerian spatial structure affects the dispersion, which can be regarded as a Lagrangian property (Monin and Yaglom, 1975). The combination of these properties requires that both descriptions be considered (see e.g., Boffetta et al., 1999).

Lagrangian Stochastic Modelling (LSM) is one turbulence representation that naturally combines Eulerian spatial structure and Lagrangian temporal correlation. In fact, as formulated by Thomson (1990) using the well-mixed condition (WMC), Lagrangian and Eulerian statistics are accounted for through the second-order Lagrangian structure function and the probability density function (pdf) of Eulerian velocity. Several studies prove that this approach leads to the qualitative reproduction of the main properties, as expected from the Richardson theory (see, Thomson, 1990; Reynolds, 1999; Sawford, 2001, among others).

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Furthermore, recent experimental studies seem to confirm the validity of the Markovian assumption for the velocity (Porta et al., 2001, Renner et al., 2001; Mordant et al., 2001).

However, the intrinsic non-uniqueness of the WMC formulation (see, e.g., Sawford, 1999) and the indeterminacy of the Kolmogorov constants (see, e.g., Sreenivasan, 1995; Anfossi et al., 2000, for reviews) do not allow for a completely reliable representation of the process. In particular, the value of the Richardson constant g predicted by previous studies is not uniquely determined (see, among others, Thomson, 1990; Borgas and Sawford 1994; Kurbanmuradov, 1997; Reynolds, 1999). Whether this indetermination is a result of the different formulation of models, or of the different values of the parameters adopted, is still unclear, and no systematic studies have been performed so far.

However, it is worth noting that variations of the model constants produce significant variability. As an example, Borgas and Sawford (1994, hereafter BS94) present the variation of g with the Lagrangian Kolmogorov constant C_0. Moreover, some of the BS94 results are in disagreement with an upper limit $2C_0$ for g whose existence is claimed by Thomson (1990) and by Borgas and Sawford (1991, hereafter BS91), and which is consistent with the Novikov (1963) arguments.

The aim of the present study is to investigate some general properties of models based on the WMC with regard to inertial subrange relative dispersion features and to shed light on the variability of g with turbulence parameters. In Section 2, the properties of the WMC are highlighted through a dimensional analysis, while the limit for vanishing spatial correlation is studied in Section 3. Subsequently a model formulation is discussed in Section 4, and results analysed in Section 5.

2. The Non-Dimensional Form of the Well-Mixed Condition

Following the logical development of Thomson (1987), Thomson (1990, here-after T90) extended the method for the selection of single particle LSM to models for the evolution of particle-pair statistics. In the latter models, the state of a particle pair is represented by the joint vector of position and velocity $(\mathbf{x}, \mathbf{u}) \equiv (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)})$, where the upper index denotes the particle, whose evolution is given by the set of Langevin type equations (LE) (with implied summation over repeated indices):

$$dx_i = u_i\, dt, \tag{1a}$$

$$du_i = a_i(\mathbf{x}, \mathbf{u}, t)dt + b_{ij}(\mathbf{x}, t)dW_j(t), \tag{1b}$$

where $i, j = 1, \dots, 6$. The coefficients **a** and **b** are determined, as usual, through the well-known WMC (Thomson, 1987) and the consistency with the inertial subrange scaling, respectively. Further details are not given here, in that they are well established and widely used in the literature (see, e.g., Sawford, 2001, for a review). From now on, $b_{ij} = \sqrt{C_0 \varepsilon}\delta_{ij}$, where ε is the mean dissipation rate of turbulent kinetic energy, will be used according to the usual scaling of Lagrangian structure function (Thomson, 1987).

It should be remembered here that the WMC is satisfied by constraining the Fokker–Planck equation associated with Equation (1) (see, e.g., Gardiner, 1990) to be consistent with the Eulerian probability density function of the flow. In the case of particle pairs the considered pdf is the one-time, two-point joint pdf of $\mathbf{x}^{(i)}$ and $\mathbf{u}^{(i)}$, $i = 1, 2$, accounting for the spatial structure of the turbulent flow considered. The open question about the non-uniqueness of the solution in more than one dimension (see, e.g., Sawford, 1999) is not addressed here. However, it will be clear that the following dimensional analysis is independent of the particular solution selected although it cannot be excluded whether quantitative results are also independent.

In order to highlight the effect of turbulence features on the model formulation, characteristic scales for particle-pair motion must be identified. Because the process of relative dispersion has to deal with both Eulerian and Lagrangian properties (see, e.g., Monin and Yaglom, 1975 p. 540), such scales can be defined by considering the second-order Eulerian and Lagrangian structure functions, i.e.,

$$\langle \Delta v_i^2 \rangle = C_K (\varepsilon \Delta x)^{2/3}, \tag{2}$$

for Eulerian velocity $\mathbf{v}$ for a separation $\Delta x = ||\Delta \mathbf{x}||$, according to the standard Kolmogorov (1941) theory (hereinafter K41), and

$$\langle \Delta u_i^2 \rangle = C_0 (\varepsilon \Delta t), \tag{3}$$

for Lagrangian velocity $\mathbf{u}$ (see, e.g., Monin and Yaglom, 1975), where $\Delta v_i = v_i(\mathbf{x} + \Delta \mathbf{x}) - v_i(\mathbf{x})$ and $\Delta u_i = u_i(t + dt) - u_i(t)$. A length scale λ can be defined in the Eulerian frame, so that in the inertial subrange (namely, for $\eta \ll \Delta x \ll \lambda$ where η is the Kolmogorov microscale) the structure function for each component may be written as

$$\langle \Delta v_i^2 \rangle = 2\sigma^2 \left(\frac{\Delta x}{\lambda} \right)^{2/3}, \tag{4}$$

where $\sigma^2 = \langle ||\mathbf{v}||^2 \rangle / 3$, which, together with Equation (2), provides a definition for λ.

A Lagrangian time scale τ can be defined in a similar way using Equation (3) and the Lagrangian version of Equation (4). Thus, for $\tau_\eta \ll t \ll \tau$, one has

$$\langle \Delta u_i^2 \rangle = 2\sigma^2 \frac{\Delta t}{\tau}, \tag{5}$$

from which one can retrieve the known relationship

$$\varepsilon = \frac{2\sigma^2}{C_0 \tau}, \tag{6}$$

suggested by Tennekes (1982). It should be observed that scales for the inertial subrange, at variance with their integral version, can be defined independently of non-homogeneity or unsteadiness, provided that the scales of such variations are sufficiently large to allow an inertial subrange to be identified. As far as the velocity is concerned, σ can be recognised as the appropriate scale of turbulent fluctuations in both descriptions.

The quantities σ, λ and τ form a non-dimensional parameter

$$\beta = \frac{\sigma\tau}{\lambda} = \frac{C_K^{3/2}}{\sqrt{2}C_0}, \tag{7}$$

the last equality being based on the combination of Equations (2) and (3) with (4) and (5). The parameter β can be recognised as a version of the well-known Lagrangian-to-Eulerian scale ratio. The approach adopted here illustrates its connection to fundamental constants of the K41 theory.

Using λ as a scale for spatial variables when an argument of Eulerian functions, the non-dimensional variable $\mathbf{x}_\mathrm{E}$ can be defined. However, $\sigma\tau$ can be considered as a more relevant scale for Lagrangian processes and thus the non-dimensional Lagrangian position will be defined as $\mathbf{x}_\mathrm{L} = \beta^{-1}\mathbf{x}_\mathrm{E}$. Furthermore, the quantities σ and τ will be used to render velocity and time non-dimensional, respectively.

In non-dimensional form, Equation (1) reads

$$dx_{\mathrm{L}_i} = u_i dt, \tag{8a}$$

$$du_i = a_i(\mathbf{x}_\mathrm{E}, \mathbf{u}, t)dt + \sqrt{2}dW_i \tag{8b}$$

where, with a change of notation with respect to Equation (1), all the quantities involved are without physical dimensions.

The associated Fokker–Planck equation is

$$\frac{\partial p_\mathrm{L}}{\partial t} + \beta u_i \frac{\partial p_\mathrm{L}}{\partial x_{\mathrm{E}_i}} + \frac{\partial a_i p_\mathrm{L}}{\partial u_i} = \frac{\partial^2 p_\mathrm{L}}{\partial u_i \partial u_i}, \tag{9}$$

where p_L is the pdf of the Lagrangian process described by Equation (8) for some initial conditions. Using the WMC, $\mathbf{a}$ can be written as

$$a_i = \frac{\partial \ln p_\mathrm{E}}{\partial u_i} + \phi_i, \tag{10}$$

where

$$\frac{\partial \phi_i p_{\mathrm{E}}}{\partial u_i} = -\frac{\partial p_{\mathrm{E}}}{\partial t} - \beta u_i \frac{\partial p_{\mathrm{E}}}{\partial x_{\mathrm{E}_i}}, \tag{11}$$

and p_{E} is the Eulerian one-time, two-point joint pdf of $\mathbf{x}$ and $\mathbf{u}$.

An advantage of this choice of scales emerges clearly in Equation (9). It shows that, given a Eulerian pdf, once the non-uniqueness problem is removed by selecting a suitable solution to Equation (10), or applying a further physical constraint to Equation (11) (Sawford, 1999), any solution of Equation (9) will depend on one parameter only, namely on the Lagrangian-to-Eulerian scale ratio. It can also be observed that this dependence is completely accounted for by the non-homogeneity term, which is an intrinsic property of the particle pair dispersion process in spatially structured velocity fields.

In looking for the universal properties of pair-dispersion in the inertial subrange, it is useful to rewrite the Richardson t^3 law in non-dimensional form, i.e., $\langle \Delta x_{\mathrm{L}}^2 \rangle = g^* t^3$ where $g^* = 2g/C_0$. Equations (8) and (9) show that the numerical value of the 'normalised' Richardson constant g^* depends on β only. This dependence is investigated in the following Sections to highlight the intrinsic properties of the LSM.

3. The Spatial Decorrelation Limit

In the limit $\beta \to \infty$, corresponding to a vanishing Eulerian correlation scale, the WMC solution reduces to an homogeneous process (see Appendix). In particular, selecting a Gaussian pdf will give the Ornstein–Uhlenbeck (OU) process. Although the absence of a spatial correlation contradicts the intimate nature of turbulence, the OU process has sometimes been used to describe relative dispersion processes in turbulent flows, for instance by Gifford (1984), who pioneered the stochastic approach to atmospheric dispersion. The Novikov (1963) model and the NGLS model (Thomson, 1990, p. 124) are simple applications of this concept.

The OU process equivalent to Equation (8) is described by the non-dimensional set of linear LE

$$dx_{\mathrm{L}_i} = u_i\, dt, \tag{12a}$$
$$du_i = -u_i dt + \sqrt{2}\, dW_i, \tag{12b}$$

where $i = 1, \ldots, 6$. The equations for the relative quantities (Δu_i, Δx_{L_i}) can be obtained from the difference between quantities relative to the first ($i = 1, 2, 3$) and second ($i = 4, 5, 6$) particles. The resulting set of equations reads

$$d\Delta x_{\mathrm{L}_i} = \Delta u_i \, dt, \tag{13a}$$

$$d\Delta u_i \;= -\Delta u_i \, dt + 2\, dW_i, \tag{13b}$$

where $i = 1, \ldots, 3$.

Equation (13) can be solved analytically to obtain correlation functions and variances (see e.g., Gardiner, 1990). Some basic results are summarised below (see also Gifford, 1984). The correlation function of velocity difference turns out to be an exponential function dependent on the time interval only

$$\langle \Delta u_i \Delta u_{0i} \rangle = \langle \Delta u_{0i}^2 \rangle \exp(-t), \tag{14}$$

while the displacement variance for a single component is

$$\langle (\Delta x_{\mathrm{L}_i} - \Delta x_{\mathrm{L}_{0i}})^2 \rangle = (\langle \Delta u_{0i}^2 \rangle - 2)(1 - \exp(-t))^2 + 4t \\ -4(1 - \exp(-t)). \tag{15}$$

For short times (expanding Equation (15) to the third power of t), it turns out that

$$\langle (\Delta x_{\mathrm{L}_i} - \Delta x_{\mathrm{L}_{0i}})^2 \rangle \simeq \langle \Delta u_{0i}^2 \rangle t^2 + \left(\tfrac{4}{3} - \langle \Delta u_{0i}^2 \rangle\right) t^3. \tag{16}$$

From Equation (16) it can be observed that, when initial relative velocity Δu_{0i} is distributed in equilibrium with Eulerian turbulence (i.e., $\langle \Delta u_{0i}^2 \rangle = 2$), a t^2 regime takes place with a negative t^3 correction. On the other hand, if $\langle \Delta u_{0i}^2 \rangle = 0$ the short-time regime displays a t^3 growth with a coefficient 4/3, i.e., $2C_0\varepsilon/3$ for the dimensional version (Novikov 1963; Monin and Yaglom 1975; Borgas and Sawford 1991).

4. Model Formulation and Numerical Simulations

In order to proceed with the analysis of the dependence of model features on the parameter β, we select as a possible solution to Equation (10), the expression given by T90 (his Equation (18)) for Gaussian pdf. The spatial structure is accounted for using the Durbin (1980) formula for longitudinal velocity correlation, which is compatible with the 2/3 scaling law in the inertial subrange. Although this form is known not to satisfy completely the inertial subrange requirements (it prescribes a Gaussian distribution for Eulerian velocity differences, while inertial subrange requires a non-zero skewness), it has been successfully used in basic studies (BS94) and applications Reynolds (1999), and provides a useful test case for studying the results shown above.

The stochastic model is formulated for the variable $(\mathbf{x}, \mathbf{u})$ rather than for the (dimensional) variable $(\Delta\mathbf{x}/\sqrt{2}, \Delta\mathbf{u}/\sqrt{2})$ as in Thomson's original formulation. In the present case, assuming homogeneous and isotropic turbulence, the covariance matrix $\mathcal{V}(\mathbf{x})$ of the Eulerian pdf is expressed by

$$\mathcal{V} = \begin{pmatrix} \mathcal{I} & \mathcal{R}^{(1,2)}(\mathbf{x}) \\ \mathcal{R}^{(2,1)}(\mathbf{x}) & \mathcal{I} \end{pmatrix}, \tag{17}$$

where $\mathcal{I}$ is the identity matrix and

$$\mathcal{R}_{ij}^{(p_1,p_2)}(\mathbf{x}) = \langle u_i^{(p_1)} u_j^{(p_2)} \rangle, \tag{18}$$

where $p_1, p_2 = 1, 2$ $(p_1 \neq p_2)$ are the particle indices. The quantity $\langle u_i^{(p_1)} u_j^{(p_2)} \rangle \equiv \langle u_i(\mathbf{x}^{(p_1)}) u_j(\mathbf{x}^{(p_2)}) \rangle$ is the two-point covariance matrix, which is expressed in terms of the longitudinal and transverse functions F and G (see, e.g., Batchelor, 1970) as

$$\mathcal{R}_{ij} = F(\Delta x)\Delta x_i \Delta x_j + G(\Delta x)\delta_{ij}, \tag{19}$$

where

$$F = -\frac{1}{2\Delta x}\frac{\partial f}{\partial \Delta x}, \tag{20}$$

and

$$G = f + \frac{\Delta x}{2}\frac{\partial f}{\partial \Delta x}. \tag{21}$$

It goes without saying that $\mathcal{R}_{ij}^{(p_1,p_2)} = \mathcal{R}_{ij}^{(p_2,p_1)} = \mathcal{R}_{ji}^{(p_1,p_2)}$. As in Durbin (1980), F and G are computed from the parallel velocity correlation

$$f(\Delta x) = 1 - \left(\frac{\Delta x^2}{\Delta x^2 + 1}\right)^{1/3}, \tag{22}$$

which is K41 compliant for $\Delta x \ll 1$.

Using the above formulation, Equation (8) were solved numerically for a number of trajectories large enough to provide reliable statistics for the relevant quantities. Particular attention was paid to the timestep independence of the solution (details are not reported here). It was found that the non-dimensional timestep strongly depends on β because large values of the parameter increase non-homogeneity, which requires greater accuracy. Despite the widespread use of variable timestep algorithms (see, e.g., Thomson, 1990; Schwere et al., 2002) based, in particular, on spatial derivatives, here a fixed timestep short enough for timestep independence of the solution was used throughout the computation.

Simulations were performed for initial velocity differences given according to the second-order Eulerian structure function, i.e., in equilibrium with the flow. The initial condition for the spatial variable was $\Delta x_{\mathrm{L}_i}(0) = 10^{-5}$ for all simulations. It can be noted that this corresponds to different positions in the inertial subrange, i.e., different $\Delta x_{\mathrm{E}_i}(0)$ for different simulations. In other words $\mathcal{V}(\beta \Delta x_L(0))$ differs from case to case. However, this initial separation is chosen small enough to provide a well-defined range of the 'quasy-asymptotic' regime as defined by Batchelor (1952).

The β parameter was varied in the range $[10^{-2} : 10^2]$, well beyond physically meaningful values. In fact, values reported in the literature range from $\mathrm{O}(10^{-1})$ to $\mathrm{O}(10^1)$ (Hinze, 1959; Hanna, 1981; Sato and Yamamoto, 1987; Koeltzsch, 1999) with $\beta = O(1)$ taken as a reference (Corrsin, 1963). This choice was made in order to infer asymptotic properties of the model. Note that, from a numerical point of view, different values of β were obtained by varying the length scale λ, keeping σ, τ and C_0 fixed. In other words, with reference to Equation (7), the variation of β is equivalent to a variation of C_K.

5. Results and Discussion

Figure 1(a–c) show the results of simulations for some values of β in the range defined above. The mean square separation $\delta = \langle \Delta x_{\mathrm{L}}(t)^2 \rangle$ and the mean square separation referred to an inertial system moving with the initial velocity difference, $\delta' = \langle [\Delta x_{\mathrm{L}}(t) - \Delta x_{\mathrm{L}}(0) - \Delta u(0)t]^2 \rangle$ are plotted against the time t. The OU analytical solutions ($\beta = \infty$) are also reported for reference: the solution with initial condition $\langle \Delta u(0)^2 \rangle = \sigma^2$ is the reference for δ while the solution with initial condition $\langle \Delta u(0)^2 \rangle = 0$ is the reference for δ'.

The general behaviour qualitatively fulfils the expectations of the Taylor (1921) and Richardson (1926) theories. It presents an initial regime that differs for the two variables: δ shows a t^2 growth, while δ' grows as t^3 according to Equation (16). After this initial regime there is a transition to an inertial range, the 'quasi-asymptotic' state referred to by Batchelor (1952), in which both δ and δ' grow as t^3. In particular, the time t_1 at which $\delta \simeq \delta'$ can be used to define the starting point of the 'quasi-asymptotic' regime itself, being the time at which memory of the initial conditions is lost.

Particles separate faster as β increases, as expected because the space correlation (two-point correlation) becomes less and less important with respect to that along the trajectories (two-time Lagrangian correlation).

Thus, the time $t_2 > t_1$ at which the process tends to forget any memory of the spatial structure and to behave like an OU process (see Figure 1), decreases as β increases: for $\beta < O(1)$ it results that $t_2 > 1$ while for $\beta > O(1)$ is $t_2 < 1$ (compare Figure 1a with Figure 1c). It goes without saying

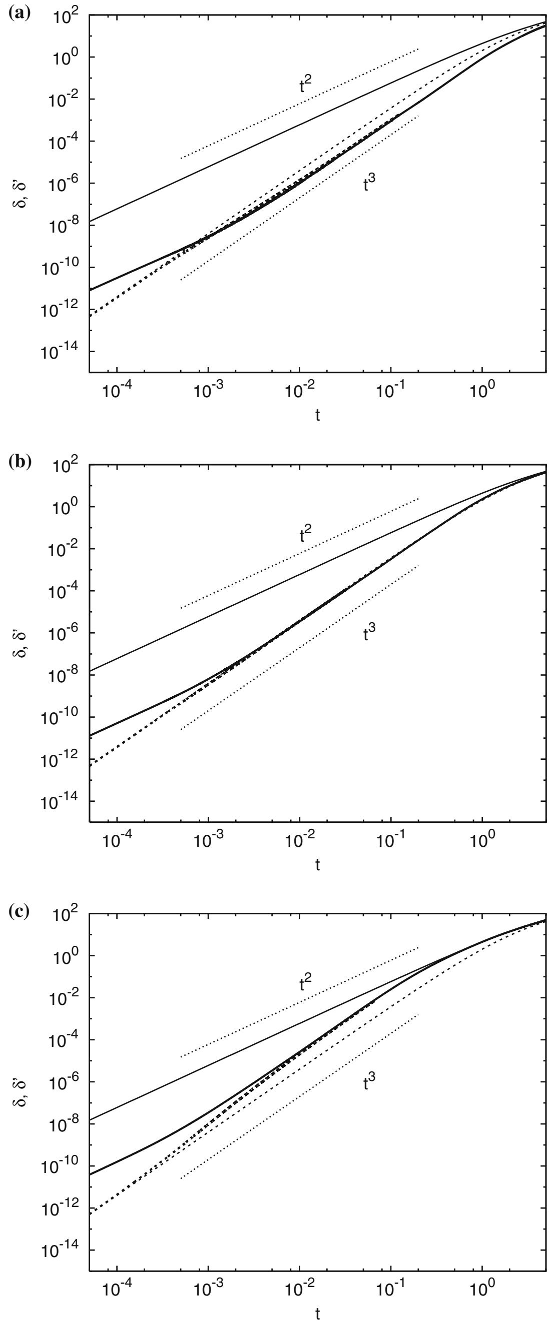

Figure 1. Mean square separation δ and δ' as a function of time for different values of β. Thick lines represent results of present simulations, while the thin lines are the analytical Ornstein–Uhlenbeck solutions (continuous for δ and dashed for δ'). (a) $\beta = 0.5$, (b) $\beta = 1.0$, (c) $\beta = 5.0$.

that the OU process with equilibrium initial condition represents the upper limit for the overall process of two-particle dispersion.

It appears that the t^3 regime begins earlier and lasts for shorter intervals as β increases. An inspection of the variations of t_1 and t_2 confirms that the extension of the inertial regime remains a decreasing function of β, which asymptotically converges to zero.

From the simulations it results that, in the Richardson regime for β larger than a value $O(1)$, the model separation is greater than that predicted by the OU process for the quantity δ'. As already mentioned in the Introduction, this fact violates the BS91 statement for which $g^* < 4$.

Values of g^* as a function of β are shown in Figure 2; in the investigated range, g^* grows monotonically with β. In Figure 2 the results from BS94 and the Kurbanmuradov (1997) Q1D model are also reported. Note that the results take from the literature were studied for varying C_0 but, because of the constancy of ε, variations of C_0 correspond to variations of β. It can be observed that different versions of the WMC model, based on the same Gaussian pdf but on different forms for ϕ, as tested by BS94, give rise to different g^* for the same β. They tested different models against the 'two-to-one reduction' property (see Thomson 1990, p. 123). They used a simple quadratic model (their Equation (4.2a)), the T90 model (their Equation (4.3)) and a modified version according to a 'two-to-one reduction' constaint (their Equation (7.6) with different values of a parameter φ). In particular, it seems that the '4.2' and '7.6 with $\varphi = -0.4$' models of BS94, satisfies the upper limit for g^* in the range studied. However the tendency of the '7.6 with $\varphi = -0.4$' model indicate that it will display $g^* > 4$ for larger values of β. The Q1D closure with a non-Gaussian pdf gives very large values of g^* and violates the BS91 constraint for values of β smaller than for the other models. As a last remark, it can be pointed out that laboratory experiments Ott and Mann(2000) and direct numerical simulations (see, e.g., Boffetta et al., 1999, among the most recent) suggest $g \simeq 0.5$. Using standard values for C_k ($\simeq 2$) and C_0 (from 2 to 6), g^* can be estimated in the range $[0.17:0.5]$ with a variation of β in the range $[0.33:1]$.

6. Conclusions

The dimensional analysis of the WMC, through the non-dimensionalisation of the Fokker–Planck equation has shown that only one parameter plays a role in the determination of two-particle dispersion properties. This parameter is the Lagrangian-to-Eulerian scale ratio β, which can be reliably defined in terms of inertial subrange constants. The dimensional analysis leads to the definition of a normalised Richardson constant g^* whose scale is identified with C_0, as suggested by the comparison of Lagrangian

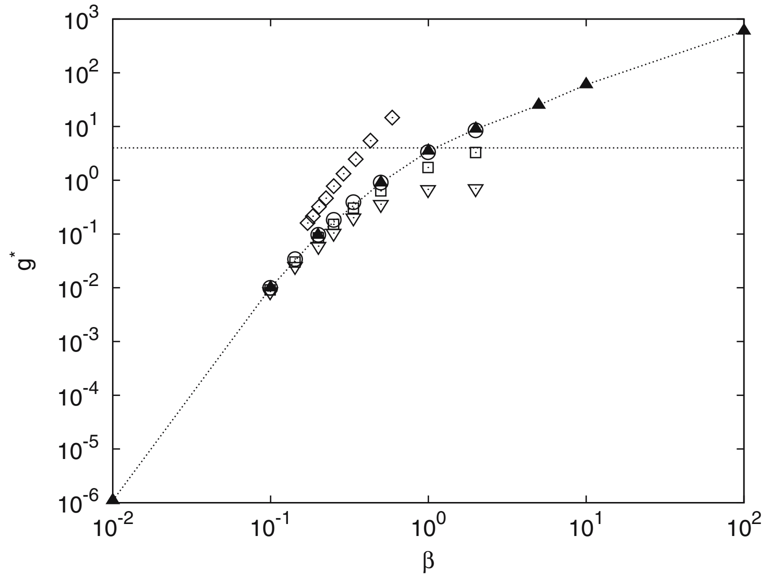

Figure 2. Normalized Richardson coefficient *versus* β. Horizontal line is the theoretical upper limit $g^* = 4$. Present results are represented by ▲ connected with a line; ▽, ◯ and □ are taken from Borgas and Sawford (1994, their symbols) for the '4.2a', '4.3' and '7.6 with $\varphi = -0.4$' models, respectively; ◇ are the results from the Q1D model Kurbanmuradov (1997).

and Eulerian properties. Given a particular model, the numerical value of g^* depends solely on the value of β adopted. This also applies to the duration of the t^3 regime.

Using the T90 formulation, it has been shown that the results of Novikov (1963) are recovered for $\beta \to \infty$, i.e., the spatial structure is negligible with respect to the Lagrangian time correlation. This limit corresponds to the OU process, whose general properties highlight that the observed t^3 growth is actually a correction to the ballistic regime t^2.

There is an inconsistency of the model behaviour in the Richardson regime for large β. The origin of this problem is not understood in detail, but it seems to be attenuated with a proper choice of the ϕ function as shown by the BS94 results, their '4.2a' model. Unfortunately, this does not correspond to the 'best' choice as far as the 'two-to-one reduction' is concerned (BS94).

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Appendix

The stationary structure function of the second order of an homogeneous field, Equation (4), can be generalized to an arbitrary integer order n, in non-dimensional terms as

$$\langle \Delta u^n \rangle = C_{h,n} \Delta r^{hn}, \tag{A1}$$

where $C_{h,n}$ is a proportionality coefficient. The inertial subrange and spatial decorrelation limit are recovered for $h = 1/3$ and $h = 0$, respectively. In particular, when $n = 2$, $C_{0,2} = 2$ denotes the Eulerian equilibrium statistics.

Considering the characteristic function $\hat{p}_E(\Delta w; \Delta r)$ of the stationary Eulerian *pdf* of velocity differences $p_{\mathrm{E}}(\Delta u; \Delta r)$ and using Equation (A1), it turns out that

$$\hat{p}_E(\Delta w; \Delta r) = \sum_{n=0}^{\infty} (i \Delta r^h \Delta w)^n C_{h,n} (n!)^{-1} = \hat{f}(\Delta r^h \Delta w), \tag{A2}$$

with $i = \sqrt{-1}$. From Equation (A2) it follows that

$$p_{\mathrm{E}}(\Delta u; \Delta r) = \frac{1}{\Delta r^h} f\left(\frac{\Delta u}{\Delta r^h}\right), \tag{A3}$$

where the factor Δr^{-h} conserves the normalization and, for the constant values $h = 1/3, 0$, Equation (A3) defines the self similar regimes of the inertial subrange and the spatial decorrelation limit, respectively.

Using the dimensional quantities $\Delta r' = \lambda \Delta r$ and $\Delta u' = \sigma \Delta u$ for the particle separation and the velocity differences, respectively, for any finite Lagrangian correlation time τ and particle separation $\Delta r'$, the following identity holds

$$\lim_{\beta \to \infty} \Phi(\Delta r) \equiv \lim_{\lambda \to 0} \Phi(\Delta r'/\lambda), \tag{A4}$$

where Φ is a generic continuous bounded function. Since continuity is required in the transition from the inertial subrange regime to the equilibrium, the scaling exponent h is assumed to be a monotonic decreasing function of $\Delta r'/\lambda$. Thus

$$\lim_{\lambda \to 0} \lambda^h = 1. \tag{A5}$$

As observed in Section 2, the only term affected by variations of β in Equation (9) is the non-homogeneous one. Therefore for any finite $\Delta r'$ using Equation (A3) and Equation (A5), it turns out that

$$\lim_{\beta \to \infty} \beta \frac{\partial p_{\mathrm{E}}}{\partial r} \sim \lim_{\lambda \to 0} \left\{ \lambda^h \frac{h}{\Delta r'^{h+1}} f\left(\frac{\Delta u' \sigma^{-1}}{(\Delta r' \lambda^{-1})^h}\right)\right.$$

$$+\lambda^{2h}\frac{h}{\Delta r'^{2h+1}}\frac{\Delta u'}{\sigma}f'\left(\frac{\Delta u'\sigma^{-1}}{(\Delta r'\lambda^{-1})^{h}}\right)\Bigg\}\rightarrow 0, \qquad \text{(A6)}$$

which shows that the non-homogeneous term vanishes in this limit.

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