# Turbulent buoyant plumes

Chia-Shun Yih

*Department of Applied Mechanics and Engineering Science, The University of Michigan, Ann Arbor, Michigan 48109*

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The velocity and temperature distributions in turbulent buoyant induced by a line source or point source of heat are calculated by assuming the eddy viscosity and eddy diffusivity to be constant in any cross section of the plume. Two solutions in closed forms are obtained for the two-dimensional plume, corresponding to turbulent Prandtl number $\sigma$ equal to 2/3 and 2. Two such solutions are also obtained for the round plume, corresponding to $\sigma$ equal to 1.1 and 2. The solution for $\sigma = 2/3$ is compared with previous measurements for two-dimensional plumes, and the solution for $\sigma = 1.1$ is compared with previous measurements for the axisymmetric plume. The analytical and experimental results agree well in the two-dimensional case, and satisfactorily in the axisymmetric case.

## I. INTRODUCTION

It is well known that when a flow is turbulent the as yet un-resolved problem of closure prevents any rigorous analytical solution for the velocity field. In attempting to give approximate analytical solutions for turbulent flows, Prandtl[1] gave a mixing-length theory which Tollmien[2] applied to the calculation of the velocity distribution in jets. When compared with the experimental results of Förthmann,[3] Kuethe,[4] and Reichardt,[5] the calculated results of Tollmien for the velocity are invariably less than (although not by much) the measured values over the central part of the jet. A more serious objection to the application of the mixing-length theory to jets is that the curvature of the velocity profile must necessarily be infinite at the center of the jet. In other words, the velocity profile has a cusp there. It is strange that this fact has never been pointed out before. Tollmien,[2] who solved the equation for the velocity distribution in the jet *numerically*, naturally could not have noticed this point.

In 1942, Prandtl[6] proposed a simpler theory for the calculation of the velocity profile in jets and wakes. This was probably suggested by the experimental results of Reichardt.[5] Immediately afterwards, Görtler[7] applied this simpler theory to the calculation of velocity distributions in two-dimensional and round jets. The results of his calculations agree much better with all the experimental results in the core of the jets, but near the edge or edges of the jets Tollmien's calculations with the mixing-length theory seem to give better agreement. This is perhaps not surprising, for it is near walls and other regions (such as the edge of a jet) of significant variation of turbulent-transport coefficients that the mixing-length theory can be expected to give better results. The simpler theory of Prandtl presumes a constant eddy viscosity at each jet cross section, varying only with the longitudinal distance along the jet. The greatest virtue of this simpler theory is that there is no infinite curvature of the velocity profile at the center of the jet, and its greatest advantage is the simplicity with which it can be applied, as so well demonstrated by Görtler.[7]

Confirmation of the *validity* of the simpler theory of Prandtl in the core of jets (indeed in the core of wakes or pipe flows) came from Laufer's measurements,[8] which clearly showed that the eddy viscosity in the core of turbulent flow in a circular pipe is constant, although it varies drastically near the wall of the pipe, and that it varies with the Reynolds number. We now know that Prandtl's simpler theory can be confidently applied to the core of jets, wakes, and flows in pipes and channels, with the expectation that near the edges of jets and wakes and near the walls of pipes and channels the theory cannot be expected to give good results.

In this paper, we shall apply Prandtl's simpler theory to the calculation of velocity and temperature distributions of turbulent buoyant plumes, much as Görtler has applied it to the calculation of the velocity distribution in turbulent jets. After the analyses are given, the calculated results will be compared with the available experimental results of Schmidt,[9] Rouse *et al.*,[10] and Yih.[11,12]

## II. TURBULENT TWO-DIMENSIONAL BUOYANT PLUME

Let $x$ be measured vertically upward from a line source of heat along the center of the buoyant plume above that source. With the location of the source as the origin, $y$ is measured in a horizontal direction normal to the line source. We shall use $u$ and $v$ to denote the velocity components in the directions of increasing $x$ and $y$, respectively, $g$ to denote the gravitational acceleration, $\epsilon$ to denote the kinematic eddy viscosity, $\sigma$ to denote the Prandtl number for turbulent flow, i.e., the ratio of $\epsilon$ to the eddy diffusivity. The ambient atmosphere will be assumed uniform, with a constant specific weight $\gamma_0$, and the difference between the specific weight $\gamma$ in the plume and $\gamma_0$ will be denoted by $\Delta\gamma$. Then, the boundary-layer forms of the equation of motion and the equation of heat diffusion are

$$uu_x + vu_y = \epsilon u_{yy} - g\Delta\gamma/\gamma_0, \tag{1}$$

and

$$u\frac{\partial}{\partial x}\Delta\gamma + v\frac{\partial}{\partial y}\Delta\gamma = \epsilon\sigma^{-1}\frac{\partial^2}{\partial y^2}\Delta\gamma. \tag{2}$$

In (1) the subscripts $x$ and $y$ indicate partial differentiation. In writing (1), it is tacitly assumed that the pressure distribution in the entire atmosphere is hydrostatic. The equation of continuity is

$$u_x + v_y = 0, \tag{3}$$

so that a stream function $\psi$ exists, in terms of which

$$u = \psi_y, \quad v = -\psi_x. \tag{4}$$

The boundary conditions for $u$ and $\Delta\gamma$ are

$$v = u_y = 0 = (\partial/\partial y)\Delta\gamma \text{ at } y = 0, \tag{5}$$

$$\psi \text{ is finite and } \Delta\gamma = 0 \text{ at } y = \pm\infty. \tag{6}$$

Let the strength of the line source per unit length be measured by $G$, so that

$$G = -\int_{-\infty}^{\infty} u\Delta\gamma\, dy, \tag{7}$$

if longitudinal diffusion is neglected, as in (2). Indeed, using (3), (5), and (6), we can obtain (7) by integrating (2).

A dimensional analysis shows that $x(G/\rho)^{1/3}$ has the dimension of kinematic viscosity. Hence, we can assume, in the spirit of Prandtl's simplified theory,

$$\epsilon = \lambda x (G/\rho)^{1/3}, \tag{8}$$

where $\rho$ is the density of the ambient fluid and $\lambda$ is a dimensionless constant to be determined from experimental data. With (8), it can easily be verified that a similarity solution is possible if we make the following transformation:

$$\psi = (G/\rho)^{1/3} x\, f(\eta), \tag{9}$$

$$\Delta\gamma = -x^{-1}(\rho G^2)^{1/3}\theta(\eta), \tag{10}$$

$$\eta = y/x. \tag{11}$$

Equations (4) and (9) give

$$u = (G/\rho)^{1/3} f'(\eta), \quad v = -(G/\rho)^{1/3}(f - \eta f'). \tag{12}$$

Then, (1) and (2) become, after some straightforward calculations,

$$-ff'' = \lambda f''' + \theta, \tag{13}$$

$$-\sigma(\theta f)' = \lambda\theta'', \tag{14}$$

The boundary conditions become

$$f(0) = 0, \quad f''(0) = 0, \quad \theta'(0) = 0. \tag{15}$$

$$f(\pm\infty) \text{ is finite}, \quad \theta(\pm\infty) = 0. \tag{16}$$

Let

$$f = A \tanh B\eta; \tag{17}$$

then, (14) gives

$$\theta = C \operatorname{sech}^m B\eta, \quad m = A\sigma/B\lambda. \tag{18}$$

There are two possible solutions. In case 1, (13) is satisfied if

$$\sigma = 2/3, \quad A = 3B\lambda, \quad m = 2, \quad C = 6B^4\lambda^2. \tag{19}$$

The integral relation (7) now has the form

$$\int_{-\infty}^{\infty} f'\theta\, d\eta = 1. \tag{20}$$

Equations (17), (18), (19), and (20) give, after some calculations,

$$AC = 3/4.$$

Then, (19) gives

$$24B^5\lambda^3 = 1, \tag{21}$$

which determines $B$ in terms of $\lambda$. Then, $A$ and $C$ are also determined in terms of $\lambda$, by (19).

In case 2, a similar calculation gives

$$\sigma = 2, \quad 128B^5\lambda^3 = 15, \quad A = 2B\lambda, \quad C = 15/32B\lambda, \tag{22}$$

by which $A$, $B$, and $C$ are determined in terms of $\lambda$.

It is not known what effect the molecular Prandtl number would have on the "turbulent" Prandtl number, but the latter must be closer to 1 than the former. Indeed, we should be quite contented with a solution with $\sigma = 1$. The closest $\sigma$ for which we have a solution in closed form is 2/3. We shall use the solution to compare with existing data and to determine $\lambda$. Since

$$u = (G/\rho)^{1/3} AB \operatorname{sech}^2 B\eta,$$

the point of inflection of the velocity profile is at

$$B\eta = 0.881.$$

The measurement of Rouse *et al.*[10] gave

$$\left(\frac{\rho}{G}\right)^{1/3} u = 1.80 \exp\left[-\frac{1}{2}\left(\frac{\eta}{0.125}\right)^2\right].$$

$$\left(\frac{x^3}{\rho G^2}\right)^{1/3} \Delta\gamma = -2.6 \exp\left[-\frac{1}{2}\left(\frac{\eta}{0.110}\right)^2\right].$$

Thus, at the point of inflection of the experimental velocity profile $\eta = 0.125$, so that we obtain

$$B = 7.051.$$

Then, for case 1, where $\sigma = \frac{2}{3}$, we obtain from (21) and (19), very closely,

$$\lambda = 0.01337, \quad A = 0.282, \quad AB = 1.99, \quad C = 2.66.$$

Comparing 1.99 with 1.80 and 2.66 with 2.6, we see that it is rather reassuring that the calculation could produce rather good agreement, considering that our $\sigma$ is not 1 but $\frac{2}{3}$. Near the edge of the plume however, the similarity solution gives consistently higher velocities and temperatures than the experimental values of Rouse *et al.*[10]

## III. TURBULENT ROUND BUOYANT PLUME

The point source of heat is now the origin. The $x$ axis is still vertical, but the radial cylindrical coordinate $r$ now replaces $y$, so that the symbol $v$ now denotes the radial velocity component. The meaning of $u$ is unchanged. The equation of motion and the equation of heat diffusion are now, respectively,

$$uu_x + vu_r = \frac{\epsilon}{r}\frac{\partial}{\partial r}(ru_r) - g\frac{\Delta\gamma}{\gamma_0}, \tag{23}$$

$$u\frac{\partial}{\partial x}\Delta\gamma + v\frac{\partial}{\partial r}\Delta\gamma = \frac{\epsilon}{\sigma r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\Delta\gamma\right). \tag{24}$$

The equation of continuity is

$$\frac{\partial}{\partial r}(rv)+\frac{\partial}{\partial x}(ru)=0,$$

which allows the use of Stokes' stream function $\psi$, in terms of which

$$u=(1/r)\psi_r, \quad v=-(1/r)\psi_x. \tag{25}$$

The boundary conditions are

$$u_r=0=v=(\partial/\partial r)\Delta\gamma \text{ at } r=0, \tag{26}$$

$$\psi \text{ is finite and } \Delta\gamma=0 \text{ at } r=\infty. \tag{27}$$

The flux $G$ is now defined by

$$G=-2\pi\int_0^\infty ru\Delta\gamma\,dr\ . \tag{28}$$

A dimensional analysis shows that the eddy viscosity can be taken to be

$$\epsilon=\lambda(Gx^2/\rho)^{1/3} \tag{29}$$

according to Prandtl's simplified theory for jets. As in (2), we have used $\epsilon/\sigma$ for the eddy diffusivity for heat.

With (29), the differential system consisting of (23), (24), (26), and (27) allows a similarity solution if we make the following transformation:

$$\psi=3\lambda(Gx^5/\rho)^{1/3}f(\eta), \tag{30}$$

$$-\Delta\gamma=3\lambda^2(\rho G^2/x^5)^{1/3}\theta(\eta), \tag{31}$$

$$\eta=r/x.$$

Then,

$$u=3\lambda(G/\rho x)^{1/3}f'/\eta, \tag{32a}$$

$$v=\lambda(G/\rho x)^{1/3}(3f'-5f/\eta), \tag{32b}$$

and (23) and (24) become, respectively,

$$(1-5f)(f'/\eta)'-f'^2/\eta=f'''+\eta\theta, \tag{33}$$

$$-5\sigma(f\theta)'=(\eta\theta')'. \tag{34}$$

The boundary conditions become

$$f(0)=f'(0)=\theta'(0)=0, \tag{35}$$

$$f(\infty) \text{ is finite}, \quad \theta(\infty)=0. \tag{36}$$

We try a solution of the form

$$f=B[1-(1+A\eta^2)^{-1}]\ , \tag{37}$$

which gives, by virtue of (34) and the boundary conditions on $\theta$, the results

$$\theta=C/(1+A\eta^2)^m, \quad m=5\sigma B/2. \tag{38}$$

Again, there are two possible solutions. In case 1,

$$B=\tfrac{12}{11}, \quad \sigma=1.1, \quad m=3, \quad C=\frac{1536A^2}{121}. \tag{39}$$

The number $A$ can be related to $\lambda$ through the dimensionless form of (28)

$$18\pi\lambda^3\int_0^\infty f'\theta\,d\eta=1. \tag{40}$$

From (40) we obtain

$$C\lambda^3=11/54\pi, \quad A^2\lambda^3=1331/82942\pi. \tag{41}$$

Thus, only $\lambda$ needs to be determined experimentally.

In case 2,

$$B=\tfrac{4}{5}, \quad \sigma=2, \quad m=4, \quad C=256A^2/25. \tag{42}$$

Use of (40) gives

$$C\lambda^3=25/72\pi, \quad A^2\lambda^3=625/18432\pi. \tag{43}$$

Again, the results for $\sigma=1.1$ (case 1) can be compared with available experimental results. Equation (32a) can now be written as

$$u=\left(\frac{G}{\rho x}\right)^{1/3}\frac{6\lambda AB}{(1+A\eta^2)^2}\,.$$

At the point of inflection of this $u$ profile,

$$\eta=(5A)^{-1/2}.$$

The measurements of Yih[11,12] gave

$$\left(\frac{\rho x}{G}\right)^{1/3}u=4.7\exp\left[-\frac{1}{2}\left(\frac{\eta}{0.072}\right)^2\right]\ , \tag{44}$$

$$-\left(\frac{x^5}{\rho G^2}\right)^{1/3}\Delta\gamma=11.0\exp\left[-\frac{1}{2}\left(\frac{\eta}{0.084}\right)^2\right]\ . \tag{45}$$

Thus,

$$(5A)^{-1/2}=0.072, \quad A=38.58,$$

and (41) and (39) then give, in turn,

$$\lambda=0.0151. \quad 6\lambda AB=3.8.$$

The value 3.8 is somewhat below the 4.7 in (44). It is quite possible that the anemometer used by Yih was too clumsy and not sufficiently accurate, and that the experimental value 0.072 for the $\eta$ at the inflection point is too small. The lack of a computer to take accurate mean values of $u$ was also a source of error. (Yih read the mean by eye.) Perhaps more accurate measurements would give a maximum value for the left-hand side of (44) closer to 4. As to $\Delta\gamma$, we have, from (31),

$$-\Delta\gamma=(\rho G^2/x^5)^{1/3}3\lambda^2C(1+A\eta^2)^{-3}.$$

The value $3\lambda^2C$ can be calculated from the first equation in (41), since $\lambda$ is known, and we have

$$3\lambda^2C=12.88,$$

compared with the experimental value of 11. In spite of the lack of good agreement, it is still rather reassuring that the values are as close as they are. Again, near the edge of the plume the theoretical values for the velocity and the temperature are consistently higher than the experimental values, which are represented by (44) and (45).

It is also satisfying that the $\lambda$ for the two-dimensional plume and the $\lambda$ for the axisymmetric plume are very close to each other.

We now compare the analytic results with Schmidt's experimental data,[9] which can be summarized in the formulae (44) and (45), with 2.7 and 13.7 replacing 4.7 and 11.0, respectively, and 0.105 and 0.099 replacing 0.072 and 0.084, respectively. Thus,

$$(5A)^{-1/2}=0.105, \quad A=18.14\,;$$

and (41) and (39) give

$$\lambda=0.0025, \quad 6\lambda AB=2.96, \quad 3\lambda^2C=7.78,$$

The figure 2.96 is quite near 2.7, but 7.78 is quite far from 13.7. The lack of good agreement may be related to the fact that Schmidt's data do not satisfy the momentum equation obtained by integrating (23) from $r=0$ to $r=\infty$.

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