PSFC/JA-09-27

Turbulent Transport of Toroidal Angular Momentum in Low Flow Gyrokinetics

Felix I. Parra and P.J. Catto

September 2009

Plasma Science and Fusion Center
Massachusetts Institute of Technology
167 Albany Street
Cambridge, MA 02139 USA

This work was supported by the U.S. Department of Energy, Grant No. DE-FG02-91ER-54109 . Reproduction, translation, publication, use and disposal, in whole or in part, by or for the United States government is permitted.

Submitted for publication in Plasma Phys. Control Fusion (September 2009)

Turbulent transport of toroidal angular momentum in low flow gyrokinetics

Felix I Parra and Peter J Catto

Plasma Science and Fusion Center, MIT, Cambridge, MA 02139, USA

E-mail: `fparra@mit.edu`, `catto@psfc.mit.edu`

Abstract. We derive a self-consistent equation for the turbulent transport of toroidal angular momentum in tokamaks in the low flow ordering that only requires solving gyrokinetic Fokker-Planck and quasineutrality equations correct to second order in an expansion on the gyroradius over scale length. We also show that according to our orderings the long wavelength toroidal rotation and the long wavelength radial electric field satisfy the neoclassical relation that gives the toroidal rotation as a function of the radial electric field and the radial gradients of pressure and temperature. Thus, the radial electric field can be solved for once the toroidal rotation is calculated from the transport of toroidal angular momentum. Unfortunately, even though this methodology only requires a gyrokinetic model correct to second order in gyroradius over scale length, current gyrokinetic simulations are only valid to first order. To overcome this difficulty, we exploit the smallish ratio B_p/B, where B is the total magnetic field and B_p is its poloidal component. When B_p/B is small, the usual first order gyrokinetic equation provides solutions that are accurate enough to employ for our expression for the transport of toroidal angular momentum. We show that current δf and full f simulations only need small corrections to achieve this accuracy. Full f simulations, however, are still unable to determine the long wavelength, radial electric field from the quasineutrality equation.

PACS numbers: 52.25.Fi, 52.30.Gz, 52.35.Ra

Submitted to: *Plasma Phys. Control. Fusion*

1. Introduction

The radial electric field in tokamaks has proven an elusive quantity even in the non-turbulent neoclassical limit [1, 2]. The radial electric field has been recently found in the Pfirsch-Schlüter regime [3, 4, 5, 6], and only incomplete results have been obtained for the banana regime [7, 8, 9]. The radial electric field profile and the toroidal rotation in the plasma are uniquely related to each other. Due to axisymmetry, the toroidal rotation is determined exclusively by the radial transport of toroidal angular momentum, contained in the small off-diagonal components of the viscosity and Reynolds stress. Obtaining these small terms makes the calculation extremely challenging.

There is a wealth of published work on the transport of toroidal angular momentum in the high flow ordering [10, 11, 12, 13]. The $\mathbf{E} \times \mathbf{B}$ flow is assumed to be on the order of the ion thermal velocity, and hence much larger than the magnetic drifts and the diamagnetic flow. This assumption simplifies the calculation because the transport of toroidal angular momentum, proportional to the velocity in order of magnitude, becomes larger. We are not going to adopt this approach because in the core of the tokamak and in the absence of neutral beam injection, the average ion velocity is often well below thermal [14, 15]. More importantly, the alternate low flow or drift ordering, in which the $\mathbf{E} \times \mathbf{B}$ flow is assumed of the same order as the diamagnetic flow, contains more physics, including the high flow limit. The high flow ordering neglects the effect of pressure and temperature gradients on the toroidal rotation. The ion velocity has contributions from the radial gradients of pressure and temperature, but these contributions are small by $\rho_{ip}/a \ll 1$ in the high flow ordering, with ρ_{ip} the poloidal ion gyroradius and a the minor radius of the tokamak. For this reason, in the high flow ordering the toroidal rotation depends exclusively on the radial electric field. In the absence of sources of momentum like neutral beams, the toroidal angular momentum will tend to diffuse out of the system. As a result, the rotation slows down and the radial electric field decreases. When the radial electric field is small enough that the contributions of the pressure and temperature gradients to toroidal rotation become important, the transport of momentum becomes dependent on the radial profile of ion temperature, sustained by external heating. In this regime, the equilibrium solution will then be non-zero (or intrinsic) rotation. The high flow assumption orders out the contribution from temperature and will not permit other solutions than zero rotation in the absence of sources of momentum. On the other hand, a low flow or drift ordering contains the contributions of the ion temperature gradient and in addition allows us to explore the high flow limit by varying the relative ordering between the gradients of pressure and temperature and the radial electric field, as we shall see.

In this article, we derive an equation for the turbulent transport of toroidal angular momentum valid in the low flow ordering. The intention is to use it to find the toroidal rotation and then solve for the radial electric field by employing the neoclassical relation between the toroidal rotation and the radial electric field [1, 2]. We have already given arguments in [16] that show that the neoclassical relation for the toroidal flow is applicable at long wavelengths even in turbulent plasmas. We will repeat those arguments in section 2 for completeness.

The transport of toroidal angular momentum is obtained by employing moments of the full Fokker-Planck equation [17] as is done in drift kinetic theory [1, 18]. This approach is valid for the long wavelength transport of toroidal angular momentum. Transport equations obtained from moments of the full Fokker-Planck equation relax the requirements on the accuracy with which the ion distribution function must be determined. The radial transport of toroidal angular momentum is given by the estimate

$$\Pi = M \left\langle \int d^3v\, f_i R(\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})(\mathbf{v} \cdot \nabla\psi) \right\rangle_\psi \sim \delta_i^3 p_i R |\nabla\psi| \tag{1}$$

with $\nabla\psi$ the gradient of the poloidal flux variable ψ, R the major radius, $\hat{\boldsymbol{\zeta}}$ the unit vector in the toroidal direction and $\langle \ldots \rangle_\psi$ the flux surface average. To obtain the order of magnitude of Π we use a simple gyroBohm estimate that gives $\Pi \sim |\nabla\psi| D_{gB} \times \nabla(R n_i M V_i) \sim \delta_i^3 p_i R |\nabla\psi|$, with $\delta_i = \rho_i / a \ll 1$ the ratio between the ion gyroradius ρ_i and the minor radius a, $D_{gB} = \delta_i \rho_i v_i$ the gyroBohm diffusion coefficient, $v_i = \sqrt{2T_i/M}$ the ion thermal speed and $V_i \sim \delta_i v_i$ the ion average velocity in the drift ordering. According to this order of magnitude estimate, calculating Π by direct integration of the ion distribution function f_i requires that f_i be good to order δ_i^3! Fortunately, only the gyrophase dependent piece of f_i is really needed, and at long wavelengths the gyrophase dependent piece of order δ_i^3 depends only on the gyrophase independent pieces up to order δ_i^2. By using moments of the full Fokker-Planck equation we make this relation explicit in the following sections.

In our final expression for the transport of toroidal angular momentum, the neoclassical diffusion is obtained from two integrals of the collision operator, and the turbulent contribution appears as two nonlinear terms that depend on both the electric field and the ion distribution function. In the nonlinear turbulent terms, the short wavelength components of the electric field and the ion distribution function beat to give the long wavelength transport of momentum that determines the toroidal rotation. The turbulent pieces of the distribution function and the electric field must be found employing a gyrokinetic Fokker-Planck equation and a gyrokinetic vorticity or quasineutrality equation correct to order δ_i^2 (in the easier high flow ordering a distribution function good to order δ_i is enough). Most gyrokinetic formulations implemented are only valid to order δ_i. We prove, however, that these formulations only need small modifications to properly transport momentum in the limit $B_p/B \ll 1$, with B_p the poloidal component of the magnetic field, and B the total magnetic field.

The rest of this article is organized as follows. In section 2, we present our orderings and assumptions for the turbulence. To simplify the calculation, we only work with electrostatic turbulence in the gyrokinetic ordering. We carefully study the steady state turbulence in the limit $B_p/B \ll 1$ to show that in this particularly interesting approximation the first order gyrokinetic equation is enough to obtain the largest contributions to the second order corrections to the ion distribution function and potential, of order $(B/B_p)\delta_i^2$. In section 3, the equation for the transport of toroidal angular momentum at long wavelengths is derived in detail. This derivation shows that both the ion distribution and the turbulent electric field must be found to order $(B/B_p)\delta_i^2$ to obtain a physically meaningful result. In section 4, we discuss the minor modifications that the most common gyrokinetic formulations, correct only to order δ_i, need to implement to obtain the ion distribution and the non-axisymmetric piece of the electric field to order $(B/B_p)\delta_i^2$ when $B_p/B \ll 1$. We close with the discussion in section 5.

2. Orderings and assumptions

We follow the orderings and assumptions in [16, 19] for electrostatic gyrokinetics. In addition, we use the extra expansion parameter $B_p/B \ll 1$ to simplify the problem. Since this expansion requires some careful analysis, we have divided this section into three subsections. In subsection 2.1, we present our assumptions for a general magnetic field with $B_p/B \sim 1$, and we remind the reader of some of the gyrokinetic results from [19] that are used in this article. The electrostatic gyrokinetic formalism presented in [19] was derived with great generality, but in reality the steady state solution is more restricted [16]. In subsection 2.2 we justify our orderings for steady state turbulence and we show that the correction to the Maxwellian is small in $\delta_i \ll 1$. Moreover, according to our orderings, the long wavelength, axisymmetric flows remain neoclassical. Then, there is a well-known relation between the radial electric field and the toroidal rotation that we can exploit to solve for the radial electric field once the evolution of the toroidal rotation is calculated. Finally, in subsection 2.3, we show that the short wavelength, turbulent pieces of the ion distribution function scale differently with $B_p/B \ll 1$ than the long wavelength, neoclassical part. This difference allows us to simplify the calculation of the turbulent transport of toroidal angular momentum in the low flow or drift ordering because it implies that the ion distribution function and the turbulent electric field can be found to order $(B/B_p)\delta_i^2$ by employing the usual gyrokinetic equation that is in principle only correct to order δ_i.

2.1. Electrostatic gyrokinetics

We assume that the electric field is electrostatic, $\mathbf{E} = -\nabla\phi$. The magnetic field $\mathbf{B}$ is axisymmetric and constant in time, and its typical length of variation is the major radius R. It can be written as

$$\mathbf{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi, \tag{2}$$

with ζ the toroidal angle and ψ the poloidal flux coordinate. As the third spatial variable we use a poloidal angle θ. The gradient $\nabla\zeta = \hat{\boldsymbol{\zeta}}/R$, where $\hat{\boldsymbol{\zeta}}$ is the unit vector in the toroidal direction, and R is the radial distance to the axis of symmetry. The function $I = R\mathbf{B}\cdot\hat{\boldsymbol{\zeta}}$ depends only on ψ to zeroth order.

To zeroth order, we assume that the ion and electron distribution functions f_i and f_e are stationary Maxwellians, f_{Mi} and f_{Me}, that only depend on ψ. The typical length of variation of f_{Mi} and f_{Me} is the minor radius of the tokamak a. Similarly, the electrostatic potential ϕ depends only on ψ to zeroth order, and $\nabla\phi \sim T_e/ea$, with T_e the electron temperature and e the electron charge magnitude.

The ion and electron distribution functions f_i and f_e and the potential ϕ have short perpendicular wavelengths due to turbulence. The short wavelength pieces are ordered as

$$\frac{f_{i,k}}{f_{Mi}} \sim \frac{f_{e,k}}{f_{Me}} \sim \frac{e\phi_k}{T_e} \sim \frac{1}{k_\perp a} \lesssim 1, \tag{3}$$

with $k_\perp \rho_i \lesssim 1$. We neglect wavelengths shorter than the ion gyroradius to simplify the derivations. The orderings in (3) imply that $\nabla_\perp f_{i,k} \sim k_\perp f_{i,k} \sim f_{Mi}/a \sim \nabla_\perp f_{Mi}$, $\nabla_\perp f_{e,k} \sim f_{Me}/a$ and $\nabla_\perp \phi_k \sim T_e/ea$, making the size of the gradients independent of the wavelength. The electric field $\mathbf{E} = -\nabla\phi \sim T_e/ea$ is in the low flow or drift ordering. The parallel wavelength $k_{||}^{-1}$ is assumed to be much longer than the ion gyroradius.

Under these assumptions, the most convenient formulation to solve for the ion distribution function is gyrokinetics [20]. For the electrons, since we are neglecting wavelengths on the order of or smaller than the electron gyroradius, it is enough to use a drift kinetic equation [18]. For the ions, we employ the higher order gyrokinetic variables derived in [19]: the gyrocenter position $\mathbf{R} = \mathbf{r} + \mathbf{R}_1 + \mathbf{R}_2$, the gyrokinetic kinetic energy $E = E_0 + E_1 + E_2$, the gyrokinetic magnetic moment $\mu = \mu_0 + \mu_1$, and the gyrokinetic gyrophase $\varphi = \varphi_0 + \varphi_1$. The corrections $\mathbf{R}_1$, E_1, μ_1 and φ_1 are first order in the ratio $\delta_i = \rho_i/a \ll 1$, and the corrections $\mathbf{R}_2$ and E_2 are second order. Here, $E_0 = v^2/2$ is the kinetic energy, $\mu_0 = v_\perp^2/2B$ is the lowest order magnetic moment and φ_0 is the gyrophase, defined by $\mathbf{v}_\perp = v_\perp(\hat{\mathbf{e}}_1 \cos\varphi_0 + \hat{\mathbf{e}}_2 \sin\varphi_0)$, with $\mathbf{v}_\perp$ and $v_\perp = |\mathbf{v}_\perp|$ the component of the velocity perpendicular to the magnetic field and its magnitude, and $\hat{\mathbf{e}}_1(\mathbf{r})$ and $\hat{\mathbf{e}}_2(\mathbf{r})$ two unit vectors perpendicular to each other and to $\hat{\mathbf{b}}$ satisfying $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. Notice that we need not calculate the second order corrections to the magnetic moment and the gyrophase because the ion distribution function is a stationary Maxwellian to zeroth order and hence only depends weakly on magnetic moment and gyrophase. The first order correction to the gyrophase, φ_1, and the second order corrections $\mathbf{R}_2$ and E_2, given in [19], are not needed for the rest of the article. The first order corrections $\mathbf{R}_1$, E_1 and μ_1, on the other hand, are necessary and we give them here for completeness;

$$\mathbf{R}_1 = \frac{1}{\Omega_i}\mathbf{v} \times \hat{\mathbf{b}}, \tag{4}$$

$$E_1 = \frac{Ze\widetilde{\phi}}{M} \tag{5}$$

and

$$\mu_1 = \frac{Ze\widetilde{\phi}}{MB(\mathbf{R})} - \frac{v_{||}v_\perp^2}{2B\Omega_i}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} - \frac{v_\perp^2}{2B^2\Omega_i}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla B - \frac{v_{||}^2}{B\Omega_i}\hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}} \cdot (\mathbf{v} \times \hat{\mathbf{b}})$$
$$- \frac{v_{||}}{4B\Omega_i}[\mathbf{v}_\perp(\mathbf{v} \times \hat{\mathbf{b}}) + (\mathbf{v} \times \hat{\mathbf{b}})\mathbf{v}_\perp] : \nabla\hat{\mathbf{b}}, \tag{6}$$

with Ze, M and $\Omega_i = ZeB/Mc$ the ion charge, mass and gyrofrequency, and c the speed of light. The functions $\langle\phi\rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ are derived from the electrostatic potential ϕ. Consequently, they have short wavelength components. Their definitions are

$$\langle\phi\rangle \equiv \langle\phi\rangle(\mathbf{R}, E, \mu, t) = \frac{1}{2\pi}\oint d\varphi\, \phi(\mathbf{r}(\mathbf{R}, E, \mu, \varphi, t), t), \tag{7}$$

$$\widetilde{\phi} \equiv \widetilde{\phi}(\mathbf{R}, E, \mu, \varphi, t) = \phi(\mathbf{r}(\mathbf{R}, E, \mu, \varphi, t), t) - \langle\phi\rangle(\mathbf{R}, E, \mu, t) \tag{8}$$

and

$$\widetilde{\Phi} \equiv \widetilde{\Phi}(\mathbf{R}, E, \mu, \varphi, t) = \int^{\varphi} d\varphi'\, \widetilde{\phi}(\mathbf{R}, E, \mu, \varphi', t) \tag{9}$$

such that $\langle\widetilde{\Phi}\rangle = 0$. Here, $\langle\ldots\rangle$ is the gyroaverage holding $\mathbf{R}$, E, μ and t fixed. It is important to discuss the size of the functions $\langle\phi\rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$. The function $\langle\phi\rangle$ is of the same size as the function ϕ, i.e., $e\langle\phi\rangle/T_e \sim (k_\perp a)^{-1}$ is large for long wavelengths and becomes of the next order in $\delta_i = \rho_i/a \ll 1$ for wavelengths comparable to the ion gyroradius. The functions $\widetilde{\phi}$ and $\widetilde{\Phi}$ are small in δ_i for any wavelength. This ordering is obvious for short wavelengths since ϕ is small as well. For long wavelengths, ϕ is large, but the wavelength is long compared to the ion gyroradius, giving $e[\phi(\mathbf{r})-\phi(\mathbf{R})]/T_e \sim \delta_i$ and hence $e\widetilde{\phi}/T_e \sim e\widetilde{\Phi}/T_e \sim \delta_i$. Importantly, to the order of interest in this article, the functions $\langle\phi\rangle$, $\widetilde{\phi}$ and $\widetilde{\Phi}$ do not depend on the gyrokinetic kinetic energy E because the gyromotion of the particles depends only on $\mathbf{R}$, μ and φ to first order.

Employing the definition of the gyrokinetic variables in [19], the ion distribution function $f_i(\mathbf{R}, E, \mu, t)$ becomes gyrophase independent to order $\delta_i f_{Mi}$, and it must satisfy the gyrokinetic Fokker-Planck equation

$$\left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{R},E,\mu} + \dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}} f_i + \dot{E}\frac{\partial f_i}{\partial E} = \langle C_{ii}\{f_i\}\rangle, \tag{10}$$

where

$$\dot{\mathbf{R}} \simeq \langle\dot{\mathbf{R}}\rangle = u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d \tag{11}$$

and

$$\dot{E} \simeq \langle\dot{E}\rangle = -\frac{Ze}{M}\dot{\mathbf{R}}\cdot\nabla_{\mathbf{R}}\langle\phi\rangle. \tag{12}$$

The gyrocenter velocity $\dot{\mathbf{R}}$ includes the gyrocenter parallel velocity

$$u = \pm\sqrt{2[E - \mu B(\mathbf{R})]}, \tag{13}$$

and the drifts $\mathbf{v}_d = \mathbf{v}_M + \mathbf{v}_E$, composed of the gyrokinetic $\mathbf{E}\times\mathbf{B}$ drift

$$\mathbf{v}_E = -\frac{c}{B(\mathbf{R})}\nabla_{\mathbf{R}}\langle\phi\rangle \times \hat{\mathbf{b}}(\mathbf{R}) \tag{14}$$

and the magnetic drifts

$$\mathbf{v}_M = \frac{\mu}{\Omega_i(\mathbf{R})}\hat{\mathbf{b}}(\mathbf{R})\times\nabla_{\mathbf{R}}B + \frac{u^2}{\Omega_i(\mathbf{R})}\hat{\mathbf{b}}(\mathbf{R})\times\boldsymbol{\kappa}(\mathbf{R}), \tag{15}$$

with $\boldsymbol{\kappa} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$ the curvature of the magnetic field lines. Equation (10) is missing corrections to $\dot{\mathbf{R}}$ and $\dot{E}$ of order $\delta_i^2 v_i$ and $\delta_i^2 v_i^3/a$, respectively. These corrections can be calculated in some simple cases like uniform magnetic fields [27], but for general magnetic geometries they are very complicated and are not implemented in simulations. Thus, the $O(\delta_i^2 f_{Mi})$ correction to the ion distribution function is not self-consistently calculated in general magnetic geometries. We show in subsection 2.3 that the higher order corrections to $\dot{\mathbf{R}}$ and $\dot{E}$ are not really necessary in the limit $B_p/B \ll 1$. The gyrophase dependent part of f_i,

$$\widetilde{f}_i \equiv f_i - \langle f_i \rangle = -\frac{1}{\Omega_i} \int^{\varphi} d\varphi' \left(C_{ii}\{f_i\} - \langle C_{ii}\{f_i\} \rangle \right) \sim \frac{\nu_{ii}}{\Omega_i} \delta_i f_{Mi}, \qquad (16)$$

is comparable to the missing corrections of order $\delta_i^2 f_{Mi}$ and is negligible in the limit $B_p/B \ll 1$.

In equations (10) and (16), $C_{ii}\{f_i\}$ is the ion-ion collision operator. We neglect the ion-electron collision operator because it is small by $\sqrt{m/M}$, with m and M the ion and electron masses. In most of this article we order the ion-ion collision mean free path $\lambda_{ii} \sim v_i/\nu_{ii}$ as comparable to the connection length qR because we want to keep collisions in the formulation. However, the mean free path is usually much longer, $qR\nu_{ii}/v_i \ll 1$, in the plasmas of interest. When necessary we will comment on the possible effects of small collisionality.

2.2. Steady state solution

In steady state, we expect to find turbulent fluctuations for which the time derivative scales as the drift wave frequency $\partial/\partial t \sim \omega_* \sim k_\perp \rho_i v_i/a$, and a much slower turbulent transport across flux surfaces of order $\partial/\partial t \sim D_{gB}/a^2 \sim \delta_i^2 v_i/a$. Then, according to our orderings (3), $\partial f_i/\partial t \lesssim \delta_i f_i v_i/a$, and equation (10) becomes $v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla} f_i = C_{ii}\{f_i\}$ to zeroth order, where $\overline{\nabla}$ is the gradient holding the zeroth order gyrokinetic variables E_0, μ_0 and φ_0 fixed, and we have assumed as usual that $\hat{\mathbf{b}} \cdot \nabla\phi \ll T_e/ea$. Here, we have neglected the higher order corrections to the gyrokinetic variables because we expect the zeroth order solution to be slowly varying in phase space. The solution to equation $v_{||}\hat{\mathbf{b}} \cdot \overline{\nabla} f_i = C_{ii}\{f_i\}$ is a stationary Maxwellian $f_{Mi}(\psi, E_0)$ that only depends on ψ, consistent with our initial assumption. It is important to realize that the condition $\hat{\mathbf{b}} \cdot \overline{\nabla} f_{Mi} = 0$ does not impose any requirements on the radial gradients of the density and temperature in f_{Mi}, and most probably they will have short wavelengths due to turbulence. We assume that these short wavelengths are within the orderings in (3), i.e., $\nabla_\perp n_{i,k} \sim k_\perp n_{i,k} \sim n_{i,k\to 0}/a \sim \nabla n_{i,k\to 0}$, $\nabla_\perp T_{i,k} \sim k_\perp n_{i,k} \sim T_{i,k\to 0}/a \sim \nabla T_{i,k\to 0}$ and $\nabla_\perp \nabla_\perp f_{Mi,k} \sim k_\perp^2 f_{Mi,k} \sim k_\perp f_{Mi,k\to 0}/a \gtrsim f_{Mi,k\to 0}/a^2$. In δf simulations the short wavelength pieces of the Maxwellian are absorbed into the δf turbulent piece.

Continuing the analysis of the steady state solution, we find that equation (10) gives the size of the next order correction $f_{i1}(\mathbf{R}, E, \mu, t) = f_i(\mathbf{R}, E, \mu, t) - f_{Mi}(\psi(\mathbf{R}), E) \sim \delta_i f_{Mi}$ (notice that the Maxwellian distribution function has been written as a function of the gyrokinetic variables). Importantly, this means that $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} f_i \sim \delta_i f_{Mi}/a$ in steady state, a property that we will employ continuously; similarly, $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} f_e \sim \delta_i f_{Me}/a$ and $\hat{\mathbf{b}} \cdot \nabla\phi \sim \delta_i T_e/ea$. Since the average velocity and the gradients along flux surfaces are only due to the next order piece f_{i1}, it is useful to think about the turbulent steady state as is done in δf formulations [21, 22, 23, 24], where the ion distribution function is composed of a slowly varying Maxwellian and a fluctuating turbulent correction. The ion flow $n_i \mathbf{V}_i = \int d^3v f_{i1} \mathbf{v} \sim \delta_i n_e v_i$ is then in the low flow or drift ordering, and the gradients along flux surfaces follow a different ordering than in (3), namely $(\hat{\mathbf{b}} \times \hat{\boldsymbol{\psi}}) \cdot \nabla_{\mathbf{R}} f_i \sim k_\perp f_{i1} \sim k_\perp \rho_i f_{Mi}/a \lesssim \hat{\boldsymbol{\psi}} \cdot \nabla f_i \sim f_{Mi}/a$, with $\hat{\boldsymbol{\psi}} = \nabla\psi/|\nabla\psi|$. Similarly, we expect $(\hat{\mathbf{b}} \times \hat{\boldsymbol{\psi}}) \cdot \nabla f_e \sim k_\perp \rho_i f_{Me}/a \lesssim \hat{\boldsymbol{\psi}} \cdot \nabla f_e \sim f_{Me}/a$ and $(\hat{\mathbf{b}} \times \hat{\boldsymbol{\psi}}) \cdot \nabla \phi \sim k_\perp \rho_i T_e/ea \lesssim \hat{\boldsymbol{\psi}} \cdot \nabla\phi \sim T_e/ea$. Notice that δf simulations include the short wavelength pieces of the Maxwellian in the δf turbulent correction.

Importantly, our orderings require that the long wavelength, axisymmetric flows remain neoclassical [16]. To see this, we examine the equation for the first order correction to the Maxwellian, $f_{i1} \sim \delta_i f_{Mi}$, given by

$$\frac{\partial f_{i1}}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M + \mathbf{v}_E] \cdot \nabla_{\mathbf{R}} f_{i1} - \left\langle C_{ii}^{(\ell)} \left\{ f_{i1} - \frac{Ze\widetilde{\phi}}{T_i} f_{Mi} \right\} \right\rangle = -\mathbf{v}_M \cdot \nabla_{\mathbf{R}} f_{Mi}$$
$$+ \frac{c}{B} (\nabla_{\mathbf{R}} \langle\phi\rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} f_{Mi} - \frac{Ze}{T_i} f_{Mi} [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle\phi\rangle, \tag{17}$$

with $C_{ii}^{(\ell)}\{f_{i1}\}$ the linearized collision operator. To obtain the long wavelength, axisymmetric contribution to equation (17), we use the "transport" or coarse grain average

$$\langle \ldots \rangle_{\mathrm{T}} = \frac{1}{2\pi \Delta t \Delta\psi} \int_{\Delta t} dt \int_{\Delta\psi} d\psi \oint d\zeta \, (\ldots). \tag{18}$$

Here, the integration is over $0 \leq \zeta < 2\pi$, and several turbulence radial correlation lengths and correlation times, $\delta_i \ll \Delta\psi/aRB_p \ll 1$ and $\delta_i^2 \ll \Delta t/t_E \ll 1$, with $t_E \sim a^2/D_{gB} \sim \delta_i^{-2} a/v_i$ the characteristic transport time scale. The "transport" average gives the equation for the long wavelength, axisymmetric first order neoclassical piece $f_{i1}^{\mathrm{nc}}(\psi, \theta, E_0, \mu_0, t) \equiv \langle f_{i1}(\mathbf{R}, E, \mu, t) \rangle_{\mathrm{T}}$, where we have used that at long wavelengths $f_{i1}(\mathbf{R}, E, \mu, t) \simeq f_{i1}(\mathbf{r}, E_0, \mu_0, t)$ to write f_{i1}^{nc} as a function of the lowest order gyrokinetic variables. The neoclassical piece $f_{i1}^{\mathrm{nc}}(\psi, \theta, E_0, \mu_0, t)$ can be found using the "transport" average of equation (17) to obtain

$$v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla} f_{i1}^{\mathrm{nc}} + \langle \mathbf{v}_E^{\mathrm{tb}} \cdot \nabla_{\mathbf{R}} f_{i1}^{\mathrm{tb}} \rangle_{\mathrm{T}} - C_{ii}^{(\ell)}\{f_{i1}^{\mathrm{nc}}\} = -\mathbf{v}_M \cdot \overline{\nabla} f_{Mi}$$
$$- \frac{Ze}{T_i} f_{Mi} (v_{||} \hat{\mathbf{b}} + \mathbf{v}_M) \cdot \nabla(\phi_0 + \phi_1^{\mathrm{nc}}), \tag{19}$$

where $\overline{\nabla}$ is the gradient holding E_0, μ_0, φ_0 and t fixed, $\phi_0(\psi)$ and $\phi_1^{\mathrm{nc}}(\psi, \theta)$ are the zeroth order potential and its first order long wavelength, axisymmetric correction, and f_{i1}^{tb} and $\mathbf{v}_E^{\mathrm{tb}}$ are the short wavelength, turbulent pieces of the ion distribution function and the $\mathbf{E} \times \mathbf{B}$ drift. To obtain equation (19), we have neglected the time derivative due to the time average in $\langle \ldots \rangle_{\mathrm{T}}$, and we have used that at long wavelengths $f_{i1}(\mathbf{R}, E, \mu, t) \simeq f_{i1}(\mathbf{r}, E_0, \mu_0, t)$, $\langle\phi\rangle \simeq \phi$, $\widetilde{\phi} \simeq -\Omega_i^{-1}(\mathbf{v} \times \hat{\mathbf{b}}) \cdot \nabla\phi$ and $\langle C_{ii}^{(\ell)}\{f_{i1}\}\rangle \simeq C_{ii}^{(\ell)}\{f_{i1}\}$. The nonlinear term $\langle \mathbf{v}_E^{\mathrm{tb}} \cdot \nabla_{\mathbf{R}} f_{i1}^{\mathrm{tb}} \rangle_{\mathrm{T}}$ contains short wavelength components, but it happens to be negligible [16]. Notice that $\mathbf{v}_E^{\mathrm{tb}} \cdot \nabla_{\mathbf{R}} f_{i1}^{\mathrm{tb}}$ can be written as $-(c/B)\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} \times [f_{i1}^{\rm tb} \nabla_{\mathbf{R}} \langle \phi^{\rm tb} \rangle] \sim k_\perp f_{i1}^{\rm tb} \delta_i v_i$, with $k_\perp^{-1}$ the perpendicular wavelength of the total nonlinear contribution $(f_{i1}^{\rm tb} \nabla_{\mathbf{R}} \langle \phi^{\rm tb} \rangle)|_{\mathbf{k}_\perp} = \sum f_{i1}^{\rm tb}|_{\mathbf{k}'_\perp} \nabla_{\mathbf{R}} \langle \phi^{\rm tb} \rangle|_{\mathbf{k}''_\perp}$, where the summation is for every $\mathbf{k}'_\perp$ and $\mathbf{k}''_\perp$ such that $\mathbf{k}'_\perp + \mathbf{k}''_\perp = \mathbf{k}_\perp$. Since $f_{i1}^{\rm tb}/f_{Mi} \sim \delta_i$, $\langle \mathbf{v}_E^{\rm tb} \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm tb} \rangle_{\rm T}$ is of order $\delta_i k_\perp \rho_i f_{Mi} v_i/a$. For wavelengths on the order of the minor radius of the machine, $k_\perp a \sim 1$, the nonlinear term $\langle \mathbf{v}_E^{\rm tb} \cdot \overline{\nabla} f_{i1}^{\rm tb} \rangle_{\rm T}$ is of order $\delta_i^2 f_{Mi} v_i/a$ and hence negligible. Then, recalling that $\hat{\mathbf{b}} \cdot \nabla \phi_0 = 0$ and that $\mathbf{v}_M \cdot \nabla \phi_0 \sim v_{||} \hat{\mathbf{b}} \cdot \nabla \phi_1^{\rm nc} \gg \mathbf{v}_M \cdot \nabla \phi_1^{\rm nc}$, equation (19) can be written as

$$v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla} h_{i1}^{\rm nc} - C_{ii}^{(\ell)} \left\{ h_{i1}^{\rm nc} - \frac{I v_{||}}{\Omega_i} \frac{M E_0}{T_i^2} \frac{\partial T_i}{\partial \psi} f_{Mi} \right\} = 0, \tag{20}$$

with

$$h_{i1}^{\rm nc} = f_{i1}^{\rm nc} + \frac{Ze\phi_1^{\rm nc}}{T_i} f_{Mi} + \frac{I v_{||}}{\Omega_i} f_{Mi} \left[\frac{1}{p_i} \frac{\partial p_i}{\partial \psi} + \frac{Ze}{T_i} \frac{\partial \phi}{\partial \psi} + \left(\frac{M E_0}{T_i} - \frac{5}{2} \right) \frac{1}{T_i} \frac{\partial T_i}{\partial \psi} \right]. \tag{21}$$

Equation (20) is the usual neoclassical equation [1, 2] that gives $h_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$. Employing the definition of $h_{i1}^{\rm nc}$ from (21), we find that the long wavelength, axisymmetric flow must be neoclassical, i.e.,

$$n_i \mathbf{V}_i = -\frac{cR}{Ze} \hat{\boldsymbol{\zeta}} \left(\frac{\partial p_i}{\partial \psi} + Ze \frac{\partial \phi}{\partial \psi} \right) + U(\psi) \mathbf{B}, \tag{22}$$

where $U(\psi) = \int d^3v \; h_{i1}^{\rm nc}(v_{||}/B) \propto \partial T_i/\partial \psi$. The correction to the flow in (22) due to turbulence can be estimated by keeping the turbulent contributions in equation (20). For low collisionality, $\partial h_{i1}^{\rm nc}/\partial \theta = 0$ to zeroth order, giving $h_{i1}^{\rm nc} = h_{i1}^{\rm nc}(\psi, E_0, \mu_0)$. Then, taking the bounce/transit average $\langle \ldots \rangle_\tau = [\int d\theta \, (v_{||} \hat{\mathbf{b}} \cdot \nabla \theta)^{-1} (\ldots)]/[\int d\theta \, (v_{||} \hat{\mathbf{b}} \cdot \nabla \theta)^{-1}]$ of equation (19) gives

$$\left\langle C_{ii}^{(\ell)} \left\{ h_{i1}^{\rm nc} - \frac{I v_{||}}{\Omega_i} \frac{M E_0}{T_i^2} \frac{\partial T_i}{\partial \psi} f_{Mi} \right\} \right\rangle_\tau = \frac{\partial f_{Mi}}{\partial t} + \langle \langle (\mathbf{v}_E^{\rm tb} \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm tb} + \ldots) \rangle_{\rm T} \rangle_\tau \,, \tag{23}$$

where we have not written explicitly possible modifications to the turbulent contribution from second order corrections to the drifts. We have kept the time derivative of the Maxwellian because it is of the same order as the turbulent contribution, $\partial f_{Mi}/\partial t \sim (D_{gB}/a^2) f_{Mi} \sim \delta_i^2 f_{Mi} v_i/a$. The correction to the usual neoclassical solution $h_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ due to the turbulence is then of order $\Delta h_{i1}^{\rm nc} \sim (v_i/qR\nu_{ii})\delta_i h_{i1}^{\rm nc} \lesssim h_{i1}^{\rm nc}$. However, we believe that the correction to the flow is an order smaller in δ_i, i.e., it is of order $\delta_i^2 v_i/qR\nu_{ii} \ll 1$ because only the part of the turbulent correction $\Delta h_{i1}^{\rm nc}$ that is odd in $v_{||}$ will contribute to the flows. In an up-down symmetric tokamak, we do not expect the turbulent contribution $\langle \langle (\mathbf{v}_E^{\rm tb} \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm tb} + \ldots) \rangle_{\rm T} \rangle_\tau$ in equation (23) to depend on the sign of the parallel velocity $\sigma = v_{||}/|v_{||}|$ because the short wavelength piece of equation (17) that gives $f_{i1}^{\rm tb}$ does not have a preferred parallel direction. The neoclassical drive, on the other hand, is proportional to $(I v_{||}/\Omega_i)(\partial T_i/\partial \psi)$ and has a preferred parallel direction given by the drift orbits of the particles and the temperature gradient. The term $\mathbf{v}_M \cdot \nabla_{\mathbf{R}} f_{Mi}$ that is the origin of the neoclassical drive at long wavelengths will not contribute coherently to the turbulence even if f_{Mi} has short wavelength components. At short wavelengths $\partial f_{Mi}/\partial \psi$ will change sign with a frequency characteristic of the turbulent processes; a time scale much faster than the long times in which $h_{i1}^{\rm nc}$ evolves. Thus, we expect the turbulent correction to the flow, coming from a turbulent contribution dependent on $\sigma = v_{||}/|v_{||}|$, to be of the next order in δ_i, i.e., $O(\delta_i^2 v_i/qR\nu_{ii}) \ll 1$. This heuristic argument must be checked with computer simulations.

2.3. Solution in the limit $B_p/B \ll 1$

For the rest of this article it will be important to understand the steady state solution in the limit $B_p/B \ll 1$. In this subsection, we examine the first order solution f_{i1}, and we show that the turbulent component $f_{i1}^{\rm tb} \sim \delta_i f_{Mi}$ scales differently with $B_p/B \ll 1$ than the long wavelength, axisymmetric neoclassical piece $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$. Then, employing this difference, we show that the traditional first order gyrokinetic equation (10) is all that is required to obtain the ion distribution function up to order $(B/B_p)\delta_i^2 f_{Mi}$.

It is interesting to analyze the order of magnitude estimate $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ in the low collisionality or banana regime. Neglecting collisions and turbulence in (19), the size of $f_{i1}^{\rm nc}$ seems to be given by the competition between the terms $v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla} f_{i1}^{\rm nc} \sim f_{i1}^{\rm nc} v_i/qR$ and $\mathbf{v}_M \cdot \overline{\nabla} f_{Mi} \sim (\rho_i/R) v_i f_{Mi}/a$, giving $f_{i1}^{\rm nc} \sim q\delta_i f_{Mi} \sim (B/B_p)\epsilon\delta_i f_{Mi}$, with $\epsilon = a/R$ the inverse aspect ratio and $q \sim aB/RB_p$ the safety factor. This simple order of magnitude estimate suggests that $f_{i1}^{\rm nc}$ is larger than $\delta_i f_{Mi}$ near the separatrix, where $q \sim (B/B_p)\epsilon$ is usually large, but becomes comparable near the magnetic axis where $\epsilon \to 0$. Importantly, this simple estimate misses the last terms in (21) and incorrectly predicts the size of the function $h_{i1}^{\rm nc}$ that the collision operator ends up forcing to be $h_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi} \gtrsim q\delta_i f_{Mi}$. Collisions cause part of the momentum carried by the trapped particles to be lost to the passing particles, accelerating them. Trapped particles can only carry toroidal momentum due to the finite radial size of their drift orbits, $\Delta_t \sim q(v_i/v_{||})\rho_i \sim (q/\sqrt{\epsilon})\rho_i$, giving rise to the diamagnetic flow $\Gamma_{i||,t} \sim f_t \Delta_t v_{||}(\partial n_i/\partial r) \sim q\sqrt{\epsilon}\delta_i n_i v_i$. Here $v_{||} \sim v_i\sqrt{\epsilon}$ is the characteristic parallel velocity of the trapped particles and $f_t \sim \sqrt{\epsilon}$ is the fraction of trapped particles. The passing particles, on the other hand, may have an average parallel velocity $V_{i||,p}$ due to the momentum exchange with the trapped. This average velocity is the one that gives the real size of $f_{i1}^{\rm nc}/f_{Mi} \sim V_{i||,p}/v_i$. To obtain the size of $V_{i||,p}$, we balance the collisional momentum loss of trapped particles with the momentum gain of passing particles. The characteristic time between collisions that make a trapped particle become passing is ϵ/ν_{ii} since only a small pitch angle change of order $\sqrt{\epsilon}$ is needed. The opposite process, that is, a collision that makes a passing particle trapped, has a characteristic time $1/(\nu_{ii}\sqrt{\epsilon})$ because there is only a limited volume of velocity space, of order $\sqrt{\epsilon}$, where the particles become trapped. Considering these characteristic times, the momentum balance between passing and trapped is $(\nu_{ii}/\epsilon)\Gamma_{i||,t} \sim \nu_{ii}\sqrt{\epsilon} n_i V_{i||,p}$, leading to $V_{i||,p} \sim (B/B_p)\delta_i v_i$ and $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$.

On the other hand, the time evolution of the short wavelength, turbulent piece $f_{i1}^{\rm tb}$ is described by the short wavelength contribution to equation (17). The self-consistent turbulent potential $\phi^{\rm tb}$ is determined by either a vorticity equation [16] or a quasineutrality equation [25, 26]. The characteristic size of the turbulent pieces is determined by the competition between the nonlinear term $\mathbf{v}_E^{\rm tb} \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm tb} = -(c/B)(\nabla_{\mathbf{R}} \langle \phi^{\rm tb} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm tb}$, and the linear contributions $\mathbf{v}_M \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm tb}$ and $(c/B)(\nabla_{\mathbf{R}} \langle \phi^{\rm tb} \rangle \times \hat{\mathbf{b}}) \cdot \nabla_{\mathbf{R}} f_{Mi}$. Considering that the turbulence usually has wavelengths comparable to the ion gyroradius, $k_\perp \rho_i \sim 1$, and the turbulent potential $\phi^{\rm tb}$ is usually of the order $f_{i1}^{\rm tb}/f_{Mi}$ due to the adiabatic response of the electrons, we find $f_{i1}^{\rm tb}/f_{Mi} \sim \delta_i$ and $e\phi^{\rm tb}/T_e \sim \delta_i$. The size of the turbulent contributions does not depend strongly on B_p/B, and it is smaller in size than the neoclassical piece $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ for $B_p/B \ll 1$. The ratio B_p/B affects the zonal flow residual [28, 29], and plays a role in the parallel structure of the linear stage of the instabilities, but it is unlikely that it increases the size of the turbulent fluctuations.

The difference in size of $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ and $f_{i1}^{\rm tb} \sim \delta_i f_{Mi}$ is important because it simplifies the calculation of the $O(\delta_i^2 f_{Mi} v_i/a)$ gyrokinetic Fokker-Planck equation. We will only keep the terms that are larger by B/B_p. To identify these terms, we let $f_i = f_{Mi} + f_{i1} + f_{i2} + \ldots$ and then write the gyrokinetic equation for the second order perturbation as

$$\frac{\partial f_{i2}}{\partial t} + [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d] \cdot \nabla_{\mathbf{R}} f_{i2} - \langle C_{ii}^{(2)}\{f_i\}\rangle = -\mathbf{v}_d \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm nc} - \dot{\mathbf{R}}^{(2)} \cdot \nabla_{\mathbf{R}} (f_{Mi} + f_{i1}^{\rm tb}) + \frac{Ze}{M}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle \frac{\partial f_{i1}}{\partial E} + \dot{E}^{(2)} \frac{M f_{Mi}}{T_i}, \tag{24}$$

with $\langle C_{ii}^{(2)}\{f_i\}\rangle = \langle C_{ii}\{f_i\}\rangle - \langle C_{ii}^{(\ell)}\{f_{Mi} + f_{i1}\}\rangle$, $\dot{\mathbf{R}}^{(2)} = \langle \dot{\mathbf{R}} \rangle - [u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_d]$ and $\dot{E}^{(2)} = \langle \dot{E} \rangle + (Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle$. Here, $\langle C_{ii}\{f_i\}\rangle$, $\langle \dot{E} \rangle$ and $\langle \dot{\mathbf{R}} \rangle$ are calculated to order $\delta_i^2 \nu_{ii} f_{Mi}$, $\delta_i^2 v_i^3/a$ and $\delta_i^2 v_i$, respectively: an order higher than in equation (10). Notice that the first order correction f_{i1} enters differently depending on its nature. The turbulent short wavelength piece $f_{i1}^{\rm tb}$ has large gradients and it is multiplied by the small quantity $\dot{\mathbf{R}}^{(2)}$, while the gradient of the neoclassical piece $f_{i1}^{\rm nc}$ is small but is multiplied by the lower order term $\mathbf{v}_d \gg \dot{\mathbf{R}}^{(2)}$.

On the right side of equation (24), the dominant terms are $-\mathbf{v}_d \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm nc}$ and $(Ze/M)[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M] \cdot \nabla_{\mathbf{R}} \langle \phi \rangle (\partial f_{i1}^{\rm nc}/\partial E)$ because $f_{i1}^{\rm nc}$ is larger than all other terms by a factor of B/B_p. The higher order corrections $\dot{\mathbf{R}}^{(2)}$ and $\dot{E}^{(2)}$ are finite gyroradius correction that do not contain any B/B_p factors. Since $f_{i1}^{\rm nc}$ determines the parallel velocity and the parallel heat flow, the term $\mathbf{v}_d \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm nc}$ represents the effect of the gradient of the parallel velocity and parallel heat flow on turbulence. All the terms that contain $\dot{\mathbf{R}}^{(2)}$ and $\dot{E}^{(2)}$ may be neglected, and the resulting equation will give a solution for $f_{i2} \sim (B/B_p)\delta_i^2 f_{Mi}$. Therefore, equation (10), that does not include the higher order corrections to $\dot{\mathbf{R}}$ and $\dot{E}$, is enough to determine the ion distribution function up to order $(B/B_p)\delta_i^2 f_{Mi}$! Moreover, the second order corrections to the gyrokinetic variables, $\mathbf{R}_2$ and E_2, are also negligible. When $f_i(\mathbf{R}, E, \mu, t)$ is expanded about $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, E_0, and μ_0, the contributions of $\mathbf{R}_2$ and E_2, $\mathbf{R}_2 \cdot \nabla_{\mathbf{R}_g} f_i$ and $E_2(\partial f_i/\partial E_0)$, are of order $\delta_i^2 f_{Mi}$ and hence negligible compared to the second order correction $f_{i2} \sim (B/B_p)\delta_i^2 f_{Mi}$. The gyrophase dependent correction from (16) is also negligible because $\widetilde{f}_i \sim (\nu_{ii}/\Omega_i)\delta_i f_{Mi} \lesssim \delta_i^2 f_{Mi} \ll f_{i2} \sim (B/B_p)\delta_i^2 f_{Mi}$. For all these estimates to work, we need the turbulence to have reached a steady state, and a converged solution of the neoclassical contribution $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$.

Finally, the function f_{i2} has a turbulent piece $f_{i2}^{\rm tb}$, and a neoclassical piece $f_{i2}^{\rm nc}$. The turbulent piece is given by the balance between the drifts $\mathbf{v}_d \cdot \nabla_{\mathbf{R}} f_{i2}^{\rm tb} \sim \delta_i v_i k_\perp f_{i2}^{\rm tb}$ and the driving term $\mathbf{v}_d \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm nc} \sim (B/B_p)\delta_i^2 v_i f_{Mi}/a$, giving $f_{i2}^{\rm tb} \sim (B/B_p)\delta_i^2 f_{Mi}$ for $k_\perp \rho_i \sim 1$. The neoclassical piece is a result of a balance between the parallel streaming term $u\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}} f_{i2}^{\rm nc} \sim (v_i/qR) f_{i2}^{\rm nc}$ and the magnetic drift term $\mathbf{v}_M \cdot \nabla_{\mathbf{R}} f_{i1}^{\rm nc} \sim (\rho_i/R) v_i (B/B_p)\delta_i f_{Mi}/a$, leading to $f_{i2}^{\rm nc} \sim (B/B_p)^2 \delta_i^2 f_{Mi}$. Here we have ignored possible factors of $\epsilon = a/R$ that have to be sorted out in the future by correctly evaluating the effect of the collision operator on $f_{i2}^{\rm nc}$.

3. Transport of toroidal angular momentum

In this section, we obtain a conservation equation for the transport of toroidal angular momentum that only requires the ion distribution function f_i and potential ϕ correct to order $\delta_i^2 f_{Mi}$ and $\delta_i^2 T_e/e$ to calculate the toroidal rotation in the low flow ordering. Equation (1) indicates that we need Π to order $\delta_i^3 p_i R|\nabla\psi|$, and we keep all the terms to that order for a general magnetic geometry with $B_p/B \sim 1$. We then refine the estimate of the size of the different terms with the limit $B_p/B \ll 1$ in mind. We finish this section by arguing that in up-down symmetric tokamaks with $B_p/B \ll 1$ the transport of toroidal angular momentum is at the gyroBohm level and that in this case the ion distribution function and the electrostatic potential need only be found to order $(B/B_p)\delta_i^2 f_{Mi}$ and $(B/B_p)\delta_i^2 T_e/e$, respectively. Up-down asymmetry complicates the treatment and is left for future work, but we expect the asymmetry required to modify the results to be severe.

The total momentum conservation equation is given by

$$\frac{\partial}{\partial t}(n_i M \mathbf{V}_i) = -\nabla \cdot \left[\overleftrightarrow{\mathbf{P}}_i + p_{e\perp}(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + p_{e||}\hat{\mathbf{b}}\hat{\mathbf{b}}\right] + \frac{1}{c}\mathbf{J} \times \mathbf{B}, \tag{25}$$

where $p_{e||} = \int d^3v\, f_e m v_{||}^2$ and $p_{e\perp} = \int d^3v\, f_e m v_\perp^2/2$ are the parallel and perpendicular electron pressures, $\mathbf{J} = Ze\int d^3v\, f_i \mathbf{v} - e\int d^3v\, f_e \mathbf{v}$ is the current density and $\overleftrightarrow{\mathbf{P}}_i = M\int d^3v\, f_i \mathbf{v}\mathbf{v}$ is the ion stress tensor. Here, we have neglected the electron inertial terms and the electron gyroviscosity and perpendicular viscosity pieces of the stress tensor because they are small by m/M. Multiplying equation by $R\hat{\boldsymbol{\zeta}}$ and flux surface averaging, we find that

$$\frac{\partial}{\partial t}\langle R n_i M \mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}}\rangle_\psi = -\frac{1}{V'}\frac{\partial}{\partial \psi}(V'\Pi), \tag{26}$$

with $\langle \ldots \rangle_\psi = (V')^{-1}\int d\theta\, d\zeta\, (\mathbf{B}\cdot\nabla\theta)^{-1}(\ldots)$ the flux surface average, $V' = \int d\theta\, d\zeta\, (\mathbf{B} \cdot \nabla\theta)^{-1}$ the flux surface volume and Π the radial flux of toroidal angular momentum given in (1). To obtain equation (26) we have used that $\langle R\hat{\boldsymbol{\zeta}} \cdot (\mathbf{J}\times\mathbf{B})\rangle_\psi = \langle \mathbf{J}\cdot\nabla\psi\rangle_\psi$ according to (2). Employing $\nabla \cdot \mathbf{J} = 0$, it is easy to see that $\langle \mathbf{J}\cdot\nabla\psi\rangle_\psi$ necessarily vanishes. Consequently, the Lorentz force $c^{-1}\mathbf{J}\times\mathbf{B}$ does not enter in the determination of the toroidal rotation.

Equation (26) proves that only $\Pi = M\langle \int d^3v\, f_i R(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})(\mathbf{v}\cdot\nabla\psi)\rangle_\psi$ is needed to find the toroidal rotation. To avoid evaluating the ion viscosity by direct integration of f_i, we propose using moments of the full ion Fokker-Planck equation

$$\frac{df_i}{dt} \equiv \left.\frac{\partial f_i}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \mathbf{v}\cdot\nabla f_i + \left(-\frac{Ze}{M}\nabla\phi + \Omega_i \mathbf{v}\times\hat{\mathbf{b}}\right)\cdot\nabla_v f_i = C_{ii}\{f_i\}. \quad (27)$$

This moment approach is followed in drift kinetics [17] and to formulate a hybrid gyrokinetic-fluid description [30]. In this section, we use two moments of (27). The $M\mathbf{vv}$ moment gives a form for Π requiring a less accurate f_i. In this new equation for Π, there is a term that contains a component of the tensor $M\int d^3v\, f_i\mathbf{vvv}$, and the $M\mathbf{vvv}$ moment of (27) allows us to solve for it.

The transport of toroidal angular momentum Π is evaluated from the $M\mathbf{vv}$ moment of the full ion Fokker-Planck equation (27), given by

$$\Omega_i(\overleftrightarrow{\mathbf{P}}_i\times\hat{\mathbf{b}} - \hat{\mathbf{b}}\times\overleftrightarrow{\mathbf{P}}_i) = \overleftrightarrow{\mathbf{K}}, \quad (28)$$

with

$$\overleftrightarrow{\mathbf{K}} = \frac{\partial \overleftrightarrow{\mathbf{P}}_i}{\partial t} + \nabla\cdot\left(M\int d^3v\, f_i\mathbf{vvv}\right) + Zen_i(\nabla\phi\mathbf{V}_i + \mathbf{V}_i\nabla\phi) - M\int d^3v\, C_{ii}\{f_i\}\mathbf{vv}. \quad (29)$$

From the moment equation (28), the off-diagonal elements of $\overleftrightarrow{\mathbf{P}}_i$ can be evaluated as a function of $\overleftrightarrow{\mathbf{K}}$. Additionally, equation (28) contains the energy conservation equation, Trace$(\overleftrightarrow{\mathbf{K}}) = 0$, and the parallel pressure equation, $\hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{K}}\cdot\hat{\mathbf{b}} = 0$.

To solve for the toroidal-radial component $R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\nabla\psi$, pre-multiply and post-multiply equation (28) by $R\hat{\boldsymbol{\zeta}}$ to find

$$R\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\nabla\psi = \frac{Mc}{2Ze}\frac{\partial}{\partial t}(R^2\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\hat{\boldsymbol{\zeta}}) + \frac{M^2c}{2Ze}\nabla\cdot\left[\int d^3v\,\mathbf{v}f_iR^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right]$$
$$+c\frac{\partial\phi}{\partial\zeta}Rn_iM(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}) - \frac{M^2c}{2Ze}\int d^3v\, C_{ii}\{f_i\}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2, \quad (30)$$

where we use $R(\hat{\mathbf{b}}\times\hat{\boldsymbol{\zeta}}) = \nabla\psi/B$ from (2) and $\nabla(R\hat{\boldsymbol{\zeta}}) = (\nabla R)\hat{\boldsymbol{\zeta}} - \hat{\boldsymbol{\zeta}}(\nabla R)$. Flux surface averaging this expression gives

$$\Pi = \frac{Mc}{2Ze}\frac{\partial}{\partial t}\langle R^2\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi + \frac{M^2c}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\, f_i(\mathbf{v}\cdot\nabla\psi)R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi$$
$$+\left\langle c\frac{\partial\phi}{\partial\zeta}Rn_iM(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi - \frac{M^2c}{2Ze}\left\langle\int d^3v\, C_{ii}\{f_i\}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi. \quad (31)$$

In equation (26), we are only interested in the evolution of the long wavelength toroidal rotation on long transport time scales. Even if we average out the short wavelengths in (31), there may still be uninteresting fast time scale variations over the long time scale irreversible transport of momentum. Thus, we must average over both ψ and time, like we did in the "transport" or coarse grain averaging $\langle \ldots \rangle_{\rm T}$ from (18). Applying the "transport" average to (31), we find

$$\langle \Pi \rangle_{\rm T} = \frac{Mc}{2Ze} \frac{\partial}{\partial t} \langle R^2 \hat{\boldsymbol{\zeta}} \cdot \langle \overleftrightarrow{\mathbf{P}}_i \rangle_{\rm T} \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi + \frac{M^2 c}{2Ze} \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle \int d^3v \, \langle f_i \rangle_{\rm T} (\mathbf{v} \cdot \nabla \psi) R^2 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi$$
$$+ \left\langle \left\langle c \frac{\partial \phi}{\partial \zeta} R n_i M (\mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}}) \right\rangle_\psi \right\rangle_{\rm T} - \frac{M^2 c}{2Ze} \left\langle \int d^3v \, \langle C_{ii}\{f_i\} \rangle_{\rm T} R^2 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi . \quad (32)$$

The first term contains a time derivative. Since the "transport" average $\langle \ldots \rangle_{\rm T}$ includes a time average over an intermediate time between the very fast turbulence time a/v_i and the slow transport time $\delta_i^{-2} a/v_i$, only the slow transport time scale evolution of the ion pressure is large enough to contribute to (32), giving

$$\frac{Mc}{2Ze} \frac{\partial}{\partial t} \langle R^2 \hat{\boldsymbol{\zeta}} \cdot \langle \overleftrightarrow{\mathbf{P}}_i \rangle_{\rm T} \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi \simeq \frac{Mc}{2Ze} \langle R^2 \rangle_\psi \frac{\partial p_i}{\partial t} \sim \delta_i^3 p_i R |\nabla \psi|. \quad (33)$$

The second term in equation (32) only depends on the gyrophase dependent piece of the ion distribution function. For this reason, it can be evaluated by employing the $M\mathbf{vvv}$ moment of the Fokker-Planck equation, given by

$$\Omega_i \int d^3v \, f_i M [(\mathbf{v} \times \hat{\mathbf{b}}) \mathbf{vv} + \mathbf{v} (\mathbf{v} \times \hat{\mathbf{b}}) \mathbf{v} + \mathbf{vv} (\mathbf{v} \times \hat{\mathbf{b}})] = \frac{\partial}{\partial t} \left(\int d^3v \, f_i M \mathbf{vvv} \right)$$
$$+ \nabla \cdot \left(\int d^3v \, f_i M \mathbf{vvvv} \right) + Ze \int d^3v \, f_i (\nabla \phi \mathbf{vv} + \mathbf{v} \nabla \phi \mathbf{v} + \mathbf{vv} \nabla \phi)$$
$$- \int d^3v \, C_{ii}\{f_i\} M \mathbf{vvv}. \quad (34)$$

Multiplying every index in this tensor by $R\hat{\boldsymbol{\zeta}}$, employing $R(\hat{\mathbf{b}} \times \hat{\boldsymbol{\zeta}}) = \nabla \psi / B$, and flux surface and "transport" averaging gives

$$\left\langle M \int d^3v \, \langle f_i \rangle_{\rm T} (\mathbf{v} \cdot \nabla \psi) R^2 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi = \frac{M^2 c}{3Ze} \frac{\partial}{\partial t} \left\langle \int d^3v \, \langle f_i \rangle_{\rm T} R^3 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi$$
$$+ \frac{M^2 c}{3Ze} \frac{1}{V'} \frac{\partial}{\partial \psi} V' \left\langle \int d^3v \, \langle f_i \rangle_{\rm T} (\mathbf{v} \cdot \nabla \psi) R^3 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi + \left\langle \left\langle c \frac{\partial \phi}{\partial \zeta} R^2 (\hat{\boldsymbol{\zeta}} \cdot \overleftrightarrow{\mathbf{P}}_i \cdot \hat{\boldsymbol{\zeta}}) \right\rangle_\psi \right\rangle_{\rm T}$$
$$- \frac{M^2 c}{3Ze} \left\langle \int d^3v \, \langle C_{ii}\{f_i\} \rangle_{\rm T} R^3 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^3 \right\rangle_\psi . \quad (35)$$

This equation has to be evaluated to order $\delta_i^2 p_i v_i R^2 |\nabla \psi|$ to give terms of order $\delta_i^3 p_i R |\nabla \psi|$ in (32). We prove now that the first two terms are of higher order and hence negligible. The first term has a time derivative and in addition $\mathbf{vvv}$ is composed of terms either odd in $v_{||}$ or $\mathbf{v}_\perp$. With turbulence that has reached statistical equilibrium and after "transport" averaging, the time derivative becomes of the order of the transport time scale, i.e., $\partial/\partial t \sim D_{gB}/a^2 \sim \delta_i^2 v_i / a$, and the ion distribution function is even in $v_{||}$ and $\mathbf{v}_\perp$ to lowest order. Consequently the contribution of the first term in (35) is negligible since $f_{i1} \sim \delta_i f_{Mi}$ gives a term of order $\delta_i^4 p_i v_i R^2 |\nabla \psi|$. In the second term of (35), only the long wavelength gyrophase dependent piece of the distribution function contributes because $\overline{(\mathbf{v}\cdot\nabla\psi)(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3} = 0$. Here $\overline{(\ldots)}$ is the gyroaverage holding $\mathbf{r}$, E_0, μ_0 and t fixed. To order $\delta_i f_{Mi}$, the long wavelength gyrophase dependent piece of f_i is given by

$$\langle f_i\rangle_{\mathrm{T}} - \langle\overline{f}_i\rangle_{\mathrm{T}} \simeq \frac{1}{\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\left[\frac{\nabla p_i}{p_i} + \frac{Ze}{T_i}\nabla\phi + \left(\frac{ME_0}{T_i} - \frac{5}{2}\right)\frac{\nabla T_i}{T_i}\right] f_{Mi} \sim \delta_i f_{Mi}. \qquad (36)$$

The integral over velocity of the first order piece of $(\langle f_i\rangle_{\mathrm{T}} - \langle\overline{f}_i\rangle_{\mathrm{T}})(\mathbf{v}\cdot\nabla\psi)(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3$ vanishes because it is odd in $\mathbf{v}$. Thus, the second term in (35) is of higher order than $\delta_i^2 p_i v_i R^2|\nabla\psi|$ and hence negligible. Finally, substituting relations (33) and (35) into equation (32), and using that the first two terms in (35) are negligible gives

$$\begin{aligned}\langle\Pi\rangle_{\mathrm{T}} = {} & \left\langle\left\langle c\frac{\partial\phi}{\partial\zeta}Rn_iM(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi\right\rangle_{\mathrm{T}} + \frac{Mc^2}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\left\langle\frac{\partial\phi}{\partial\zeta}R^2(\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi\right\rangle_{\mathrm{T}} \\ & + \frac{Mc}{2Ze}\langle R^2\rangle_\psi\frac{\partial p_i}{\partial t} - \frac{M^2c}{2Ze}\left\langle\int d^3v\,\langle C_{ii}\{f_i\}\rangle_{\mathrm{T}}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi \\ & - \frac{M^3c^2}{6Z^2e^2}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\,\langle C_{ii}\{f_i\}\rangle_{\mathrm{T}}R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3\right\rangle_\psi. \qquad (37)\end{aligned}$$

There are five contributions to the transport of toroidal angular momentum in the low flow or drift ordering. The first and second terms in (37) are turbulent contributions where the short wavelength, turbulent pieces of the potential and the ion distribution function beat together to give a long wavelength contribution. The third term contributes if the energy transport has not reached steady state. In general it must be kept. The fourth and fifth terms are collisional and account for the neoclassical transport of momentum.

We now estimate the order of magnitude of the different terms in (37) in the limit $B_p/B \ll 1$. The third term in (37) scales as

$$\frac{Mc}{2Ze}\langle R^2\rangle_\psi\frac{\partial p_i}{\partial t} \sim \frac{B}{B_p}\delta_i^3 p_i R|\nabla\psi|, \qquad (38)$$

where we have used $|\nabla\psi| = RB_p$ to obtain the order of magnitude estimate. We can estimate the size of the neoclassical contributions by employing $f_{i1}^{\mathrm{nc}} = \langle f_{i1}\rangle_{\mathrm{T}} \sim (B/B_p)\delta_i f_{Mi}$ and $f_{i2}^{\mathrm{nc}} = \langle f_{i2}\rangle_{\mathrm{T}} \sim (B/B_p)^2\delta_i^2 f_{Mi}$ (see subsection 2.3) to find

$$-\frac{M^2c}{2Ze}\left\langle\int d^3v\,\langle C_{ii}\{f_i\}\rangle_{\mathrm{T}}R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi \sim \frac{B}{B_p}\frac{qR\nu_{ii}}{v_i}\delta_i^2 p_i R|\nabla\psi| \qquad (39)$$

and

$$-\frac{M^3c^2}{6Z^2e^2}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\,\langle C_{ii}\{f_i\}\rangle_{\mathrm{T}}R^3(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^3\right\rangle_\psi \sim \left(\frac{B}{B_p}\right)^2\frac{qR\nu_{ii}}{v_i}\delta_i^3 p_i R|\nabla\psi|. \qquad (40)$$

The formal estimate in (39) would seem to indicate that the gyroBohm estimate in (1) is incorrect. However, the zeroth order contribution to (39) is small by the collision frequency, and in addition it exactly cancels to order $(qR\nu_{ii}/v_i)\delta_i^2 p_i R|\nabla\psi|$ in an up-down symmetric tokamak (see appendix A). To next order and for $B_p/B \ll 1$, we find that (39) becomes

$$-\frac{M^2c}{2Ze}\left\langle\int d^3v\ \left(C_{ii}^{(\ell)}\{f_{i2}^{\rm nc}\}+C_{ii}^{(n\ell)}\{f_{i1}^{\rm nc},f_{i1}^{\rm nc}\}\right)R^2(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})^2\right\rangle_\psi$$
$$\sim\left(\frac{B}{B_p}\right)^2\frac{qR\nu_{ii}}{v_i}\delta_i^3p_iR|\nabla\psi|, \qquad (41)$$

where $C_{ii}^{(n\ell)}$ is the nonlinear, quadratic collision operator. In equation (41) we have neglected the second order gyrophase dependent piece, obtained from Taylor expanding the gyrokinetic solution $f_i(\mathbf{R},E,\mu,t)$ around $\mathbf{r}$, E_0 and μ_0. The largest contribution to the second order gyrophase dependent piece is $\mathbf{R}_1\cdot\overline{\nabla}f_{i1}^{\rm nc}+E_1(\partial f_{i1}^{\rm nc}/\partial E_0)+\mu_1(\partial f_{i1}^{\rm nc}/\partial\mu_0)\sim(B/B_p)\delta_i^2f_{Mi}$ and thus smaller than $f_{i2}^{\rm nc}\sim(B/B_p)^2\delta_i^2f_{Mi}$. According to equations (40) and (41), in an up-down symmetric tokamak the neoclassical transport of toroidal angular momentum is of order $(B/B_p)^2(qR\nu_{ii}/v_i)\delta_i^3p_iR|\nabla\psi|$.

The turbulent contribution given by the first and second terms in (37) is more interesting. Considering that in steady state the pieces of the distribution function that have $\partial/\partial\zeta\neq0$ are of order δ_if_{Mi}, we find that

$$\left\langle\left\langle c\frac{\partial\phi}{\partial\zeta}Rn_iM(\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi\right\rangle_{\rm T}\sim\delta_i^2p_iR|\nabla\psi| \qquad (42)$$

and

$$\frac{Mc^2}{2Ze}\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\left\langle\frac{\partial\phi}{\partial\zeta}R^2(\hat{\boldsymbol{\zeta}}\cdot\overleftrightarrow{\mathbf{P}}_i\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi\right\rangle_{\rm T}\sim\frac{B}{B_p}\delta_i^3p_iR|\nabla\psi|, \qquad (43)$$

where we have used that $(\partial\phi/\partial\zeta)=R\hat{\boldsymbol{\zeta}}\cdot\nabla\phi\sim R\hat{\boldsymbol{\zeta}}\cdot\mathbf{k}_\perp\phi_k\sim R(B_p/B)k_\perp\phi_k\sim(B_p/B)(R/a)(T_e/e)$ because the component of $\hat{\boldsymbol{\zeta}}$ perpendicular to $\hat{\mathbf{b}}$ is $|\hat{\mathbf{b}}\times\hat{\boldsymbol{\zeta}}|=|\nabla\psi|/RB=B_p/B$. The term (42) is formally larger than the gyroBohm estimate in (1). It is plausible that the Reynolds stress in (42) averaged over time is almost zero. If this is the case, the turbulent contribution to order $\delta_i^2p_iR|\nabla\psi|$ does not determine the evolution of the long wavelength toroidal rotation on transport time scales. This possibility does not conflict with fast growth and evolution of zonal flow structure, that happens in relatively short times, but does not transport angular momentum through large distances.

It is difficult to prove unarguably that the Reynolds stress in (42) must vanish to order $\delta_i^2p_iR|\nabla\psi|$. In δf flux tube codes [21, 22, 24], only the gradients of density and temperature enter the equation for the turbulent correction to the Maxwellian $f_{i1}^{\rm tb}$. The gradient of the toroidal rotation is ordered out because the average velocity in the plasma is assumed to be small by δ_i. If in addition the tokamak is up-down symmetric, the system does not have a preferred direction and it is unlikely that there is any transport of angular momentum. Quasilinear calculations suggest that in up-down symmetric tokamaks, δf flux tube formulations must give zero transport [31]. If the average velocity is ordered as large as the thermal velocity, the symmetry in the flux tube is broken and there is a net radial momentum transport [32], but such a description is not relevant in many tokamaks.

It seems reasonable to assume that, at least in a time averaged sense, the Reynolds stress in (42) vanishes to order $\delta_i^2 p_i R|\nabla\psi|$. Consequently, we take

$$\left\langle\left\langle c\frac{\partial\phi_1^{\rm tb}}{\partial\zeta}\int d^3v\left(f_{i1}^{\rm tb}-\frac{Ze\widetilde{\phi}_1^{\rm tb}}{T_i}f_{Mi}\right)RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi\right\rangle_{\rm T}=0, \quad (44)$$

with $f_{i1}^{\rm tb}/f_{Mi}\sim e\phi_1^{\rm tb}/T_e\sim\delta_i$. To the next order, equation (42) becomes

$$\left\langle\left\langle c\frac{\partial\phi_1^{\rm tb}}{\partial\zeta}\int d^3v\left[f_{i2}^{\rm tb}-\frac{Ze\widetilde{\phi}_2^{\rm tb}}{T_i}f_{Mi}+\frac{Ze\widetilde{\phi}_1^{\rm tb}}{M}\left(\frac{\partial f_{i1}^{\rm nc}}{\partial E_0}+\frac{1}{B}\frac{\partial f_{i1}^{\rm nc}}{\partial\mu_0}\right)\right]RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})\right.\right.$$
$$\left.\left.+c\frac{\partial\phi_2^{\rm tb}}{\partial\zeta}\int d^3v\left(f_{i1}^{\rm tb}-\frac{Ze\widetilde{\phi}_1^{\rm tb}}{T_i}f_{Mi}\right)RM(\mathbf{v}\cdot\hat{\boldsymbol{\zeta}})\right\rangle_\psi\right\rangle_{\rm T}\sim\frac{B}{B_p}\delta_i^3p_iR|\nabla\psi|, \quad (45)$$

where we have used that according to subsection 2.3 $f_{i2}^{\rm tb}/f_{Mi}\sim e\phi_2^{\rm tb}/T_e\sim(B/B_p)\delta_i^2$. Thus, according to (43) and (45), the fast time averaged turbulent contribution to the transport of toroidal angular momentum in an up-down symmetric tokamak is of order $(B/B_p)\delta_i^3 p_i R|\nabla\psi|$. This estimate corresponds to the size of the gyroBohm transport of toroidal angular momentum. To see this, recall (1) and the discussion below it. To obtain the correct scaling with B/B_p, notice that the estimate for the toroidal velocity is $\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\sim(B/B_p)\delta_i v_i$ instead of $\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\sim\delta_i v_i$.

To summarize, equation (37) gives the transport of toroidal angular momentum in the low flow ordering up to order $(B/B_p)\delta_i^3 p_i R|\nabla\psi|$. In an up-down symmetric tokamak, we have shown that the size of the different contributions is given by (38), (40), (41), (43) and (45), i.e., the transport of toroidal angular momentum is indeed at the gyroBohm level. To evaluate (37) we need to obtain the ion distribution function and the turbulent electrostatic potential to order $(B/B_p)\delta_i^2 f_{Mi}$ and $(B/B_p)\delta_i^2 T_e/e$, respectively. These small corrections enter in equations (41) and (45). In subsection 2.3 we already showed that the lower order gyrokinetic equation (10) is good enough to obtain these higher order corrections if $B_p/B\ll 1$. In section 4 we discuss how these corrections can be obtained in practice.

4. Distribution function and potential to second order

To evaluate (37), the ion distribution function and the potential have to be found to order $(B/B_p)\delta_i^2 f_{Mi}$ and $(B/B_p)\delta_i^2 T_e/e$, respectively. In subsection 2.3 we argued that the first order gyrokinetic equation (10) was enough to obtain the ion distribution function to this order. In this section, we show that minor modifications to existing δf gyrokinetic codes [21, 22, 23, 24] provide the necessary higher order corrections to the ion distribution. We also discuss briefly the implication for full f simulations.

To find the electrostatic potential to high enough order, we can either use a gyrokinetic quasineutrality equation [25, 26] or employ a vorticity equation [16]. For δf flux tube simulations both choices should give consistent results since the long wavelength radial electric field is not retained in the equations for turbulence and can be independently determined by the equations for transport of toroidal angular momentum

(26) and (37), and the neoclassical relation (22). For global simulations, however, the gyrokinetic quasineutrality equation becomes a problem. The long wavelength radial electric field obtained from the lower order quasineutrality equation differs from the radial electric field that corresponds to the toroidal rotation. This discrepancy can be overcome by employing a consistent vorticity equation that ensures that the toroidal rotation does not change with time at short time scales. Such a vorticity equation was found in [16], but it was only valid to first order, giving $e\phi_1^{\rm tb}/T_e \sim \delta_i$. In this section, we extend that calculation to obtain a vorticity equation accurate enough to give $e\phi_2^{\rm tb}/T_e \sim (B/B_p)\delta_i^2$.

The rest of this section is organized as follows. In subsection 4.1, we extend our conclusions of subsection 2.3 for $B_p/B \ll 1$ to write δf equations for the short wavelength, turbulent pieces of distribution function, and we comment on the requirements that a full f simulation must satisfy to obtain the ion distribution function to order $(B/B_p)\delta_i^2 f_{Mi}$. We also discuss how the high flow limit can be explored in this formalism. Finally, in subsection 4.2, we obtain two vorticity equations correct to order $(B/B_p)\delta_i^2 e n_e v_i/a$. The details of the calculation are relegated to appendices B-F.

4.1. Higher order ion distribution function

In subsection 2.3 we argued that the first order gyrokinetic equation (10) is able to provide the ion distribution function up to order $(B/B_p)\delta_i^2 f_{Mi}$. In this subsection we streamline the procedure for δf simulations and we comment on the implications for full f codes.

For δf simulations, we modify slightly the arguments employed in subsection 2.3. The Maxwellian distribution function f_{Mi} is now slowly varying in space, and the axisymmetric, short wavelength structure in density and temperature is absorbed into the correction to the Maxwellian. This correction is written as $\delta f_i = f_{i1}^{\rm nc} + f_{i1}^{\rm tb} + f_{i2}^{\rm nc} + f_{i2}^{\rm tb}$, where $f_{i1}^{\rm tb}$ and $f_{i2}^{\rm tb}$ contain the axisymmetric, short wavelength corrections to the Maxwellian. We now proceed to describe how to find the pieces $f_{i1}^{\rm nc}$, $f_{i1}^{\rm tb}$ and $f_{i2}^{\rm tb}$. These are the only pieces needed to obtain the turbulent transport of toroidal angular momentum, given by the first two terms in (37). The second order correction $f_{i2}^{\rm nc}$ is not necessary for turbulent transport of toroidal angular momentum. An explicit evaluation requires some care in the low collisionality or banana regime, which we leave for future work.

The first order neoclassical correction $f_{i1}^{\rm nc}$ is determined by equations (20) and (21), and it depends on the particular collision operator. In the banana regime and for a momentum-conserving pitch angle scattering operator

$$C_{ii}^{(\ell)}\{f_{i1}\} = \nu_{ii}(v)\nabla_v \cdot \left[(v^2 \overleftrightarrow{\mathbf{I}} - \mathbf{v}\mathbf{v}) \cdot \nabla_v \left(f_{i1} - \frac{M\mathbf{v}\cdot\mathbf{u}_i}{T_i} f_{Mi} \right) \right], \qquad (46)$$

with $\mathbf{u}_i = [\int d^3v\, \nu_{ii}(v)\mathbf{v} f_{i1}]/[\int d^3v\, \nu_{ii}(v)(Mv^2/3T_i) f_{Mi}]$, the result may be approximated by [2]

$$f_{i1}^{\mathrm{nc}}(\psi(\mathbf{R}),E,\mu) = -\frac{Iu}{\Omega_i} f_{Mi}\left[\frac{1}{p_i}\frac{\partial p_i}{\partial \psi} + \frac{Ze}{T_i}\frac{\partial \phi}{\partial \psi} + \left(\frac{ME}{T_i} - \frac{5}{2}\right)\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}\right] + \frac{I\overline{V}_{||}}{\Omega_{i0}} f_{Mi}\left(\frac{ME}{T_i} - \Sigma\right)\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}, \quad (47)$$

where $u = \pm\sqrt{2[E - \mu B(\mathbf{R})]}$ is the gyrokinetic parallel velocity, $\Omega_{i0} = ZeB_0/Mc$ and $B_0 = \sqrt{\langle B^2 \rangle_\psi}$ are some conveniently averaged gyrofrequency and magnetic field, and

$$\Sigma = \frac{\int dv\, \nu_{ii}(v)(Mv^2/2T_i)^3 \exp(-Mv^2/2T_i)}{\int dv\, \nu_{ii}(v)(Mv^2/2T_i)^2 \exp(-Mv^2/2T_i)} \quad (48)$$

is a constant calculated so that $C_{ii}^{(\ell)}\{f_{i1}\}$ conserves momentum. The function

$$\overline{V}_{||}(\psi(\mathbf{R}),E,\mu) = \sigma H(B_0/B_{\max} - \mu B_0/E)\sqrt{\frac{E}{2}}\int_{\mu B_0/E}^{B_0/B_{\max}} \frac{d\lambda}{\langle \sqrt{1-\lambda B/B_0}\rangle_\psi} \quad (49)$$

goes smoothly from zero for trapped particles, $B_0/B_{\max} < \mu B_0/E < B_0/B_{\min}$, to $u = \pm\sqrt{2[E - \mu B(\mathbf{R})]}$ for passing particles, $0 < \mu B_0/E < B_0/B_{\max}$. In equation (49), $\sigma = u/|u|$ is the sign of the parallel velocity, $H(x)$ is the Heaviside step function, and $B_{\max}$ and $B_{\min}$ are the maximum and minimum values of B in a flux surface, located in the midplane in an up-down symmetric tokamak. Solution (47) is schematic, and including other collisional effects like energy diffusion will modify the result, but it has the advantage of showing the main features of the solution, in particular, its dependence on $v_{||}$. We use it here as an example.

For δf codes, we write equations for f_{i1}^{tb} and f_{i2}^{tb} that differ slightly from (17) and (24). There are two reasons for the differences. On the one hand, we assume that f_{Mi} is slowly varying in space, as already noted. On the other hand, we are going to split the turbulent potential into two pieces, namely $e\phi_1^{\mathrm{tb}}/T_e \sim \delta_i$ and $e\phi_2^{\mathrm{tb}}/T_e \sim (B/B_p)\delta_i^2$. Taking these differences into consideration, the equation for f_{i1}^{tb} is the short wavelength contribution to (10) up to order $\delta_i f_{Mi} v_i/a$, given by

$$\frac{\partial f_{i1}^{\mathrm{tb}}}{\partial t} + \left[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M - \frac{c}{B}\nabla_{\mathbf{R}}\langle\phi_1^{\mathrm{tb}}\rangle \times \hat{\mathbf{b}}\right]\cdot\nabla_{\mathbf{R}} f_{i1}^{\mathrm{tb}} - \left\langle C_{ii}^{(\ell)}\left\{f_{i1}^{\mathrm{tb}} - \frac{Ze\widetilde{\phi}_1^{\mathrm{tb}}}{T_i} f_{Mi}\right\}\right\rangle = \frac{c}{B}(\nabla_{\mathbf{R}}\langle\phi_1^{\mathrm{tb}}\rangle \times \hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}} f_{Mi} - \frac{Ze}{T_i} f_{Mi}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M]\cdot\nabla_{\mathbf{R}}\langle\phi_1^{\mathrm{tb}}\rangle. \quad (50)$$

The equation for $f_{i2}^{\mathrm{tb}} \sim (B/B_p)\delta_i^2 f_{Mi}$ can be found from the short wavelength contribution to equation (10) of order $(B_p/B)\delta_i^2 f_{Mi} v_i/a$ that gives

$$\frac{\partial f_{i2}^{\mathrm{tb}}}{\partial t} + \left[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M - \frac{c}{B}\nabla_{\mathbf{R}}\langle\phi_1^{\mathrm{tb}}\rangle \times \hat{\mathbf{b}}\right]\cdot\nabla_{\mathbf{R}} f_{i2}^{\mathrm{tb}} - \left\langle C_{ii}^{(\ell)}\left\{f_{i2}^{\mathrm{tb}} - \frac{Ze\widetilde{\phi}_2^{\mathrm{tb}}}{T_i} f_{Mi} + \frac{Ze\widetilde{\phi}_1^{\mathrm{tb}}}{M}\left(\frac{\partial f_{i1}^{\mathrm{nc}}}{\partial E_0} + \frac{1}{B}\frac{\partial f_{i1}^{\mathrm{nc}}}{\partial \mu_0}\right)\right\}\right\rangle = \frac{c}{B}(\nabla_{\mathbf{R}}\langle\phi_1^{\mathrm{tb}}\rangle \times \hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}} f_{i1}^{\mathrm{nc}} + \frac{c}{B}(\nabla_{\mathbf{R}}\langle\phi_2^{\mathrm{tb}}\rangle \times \hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}} f_{Mi} + \frac{Ze}{M}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M]\cdot\nabla_{\mathbf{R}}\langle\phi_1^{\mathrm{tb}}\rangle\frac{\partial f_{i1}^{\mathrm{nc}}}{\partial E} - \frac{Ze}{T_i} f_{Mi}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M]\cdot\nabla_{\mathbf{R}}\langle\phi_2^{\mathrm{tb}}\rangle. \quad (51)$$

Here, we have neglected terms like $C_{ii}^{(n\ell)}\{f_{i1}^{\rm tb}, f_{i1}^{\rm nc}\} \sim (qR\nu_{ii}/v_i)\delta_i^2 f_{Mi} v_i/a \lesssim \delta_i^2 f_{Mi} v_i/a$ because they are smaller than $(B/B_p)\delta_i^2 f_{Mi} v_i/a$. In equation (51), the neoclassical correction $f_{i1}^{\rm nc}$ enters in the same place as the density and temperature gradients do in the first order δf equation (50), and in the linearized collision operator, the $\widetilde{\phi}$ term from μ_1 in (6) must be retained in the expansion of f_i about μ_0. The neoclassical solution (47) is a simplified example of possible input for $f_{i1}^{\rm nc}$. The dependence of $f_{i1}^{\rm nc}$ on $v_{||}$ breaks the symmetry in the parallel velocity.

Splitting the gyrokinetic equation into the two contributions (50) and (51) has some advantages. Notice that equation (51) is now linear in $f_{i2}^{\rm tb}$, making it easy to evolve the second order turbulent correction in time. Additionally, we can assume that the $O(\delta_i^2 p_i R|\nabla\psi|)$ contribution to the piece of transport of toroidal angular momentum given in (44) vanishes, and use directly the expression in (45), of order $(B/B_p)\delta_i^3 p_i R|\nabla\psi|$.

Equations (50) and (51) are, however, not very flexible because they decouple $f_{i1}^{\rm tb}$ and $f_{i2}^{\rm tb}$. To avoid this, we can add both equations and define $F_i = f_{Mi} + f_{i1}^{\rm nc}$, $f_i^{\rm tb} = f_{i1}^{\rm tb} + f_{i2}^{\rm tb}$ and $\phi^{\rm tb} = \phi_1^{\rm tb} + \phi_2^{\rm tb}$ to obtain

$$\frac{\partial f_i^{\rm tb}}{\partial t} + \left[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M - \frac{c}{B}\nabla_{\mathbf{R}}\langle\phi^{\rm tb}\rangle \times \hat{\mathbf{b}}\right]\cdot\nabla_{\mathbf{R}} f_i^{\rm tb} - \left\langle C_{ii}^{(\ell)}\left\{f_i^{\rm tb} + \frac{Ze\widetilde{\phi}^{\rm tb}}{M}\left(\frac{\partial F_i}{\partial E_0} + \frac{1}{B}\frac{\partial F_i}{\partial \mu_0}\right)\right\}\right\rangle = \frac{c}{B}(\nabla_{\mathbf{R}}\langle\phi^{\rm tb}\rangle\times\hat{\mathbf{b}})\cdot\nabla_{\mathbf{R}} F_i + \frac{Ze}{M}[u\hat{\mathbf{b}}(\mathbf{R}) + \mathbf{v}_M]\cdot\nabla_{\mathbf{R}}\langle\phi^{\rm tb}\rangle\frac{\partial F_i}{\partial E}. \quad (52)$$

Then, to obtain the higher order correction to the ion distribution function in a δf simulation it is enough to replace f_{Mi} by $F_i = f_{Mi} + f_{i1}^{\rm nc}$! The replacement in the collision operator is probably the most involved, but it is also the least important and could be ignored in preliminary calculations.

Solving for $f_i^{\rm tb} = f_{i1}^{\rm tb} + f_{i2}^{\rm tb}$ complicates somewhat the evaluation of the turbulent transport of toroidal angular momentum. The term (42) is formally of order $\delta_i^2 p_i R|\nabla\psi|$, much larger than the gyroBohm estimate in (1). In section 3 we argued that the time average contribution to that order, given by (44), vanishes, and only the higher order piece (45) is important. However, to evaluate (45) we need to split $f_i^{\rm tb}$ and $\phi^{\rm tb}$ into their first and second order pieces. When we use $f_i^{\rm tb}$ to evaluate (42), we must ensure that the time average is over a period of time that is long enough for (44) to hold. Otherwise, spurious transport of toroidal angular momentum is introduced. We believe that this disadvantage of (52) is outweighed by its flexibility.

Importantly, equation (52) allows us to explore the high flow regime. The high flow limit is characterized by $Zen_i(\partial\phi/\partial\psi) \gg \partial p_i/\partial\psi, n_i(\partial T_i/\partial\psi)$. Employing this ordering in equations (20) and (21) (or the particular solution (47)) we find that to zeroth order $f_{i1}^{\rm nc}$ becomes $(Mv_{||}V_{i||}^{\rm hf}/T_i)f_{Mi}$, with $V_{i||}^{\rm hf} = -(cI/B)(\partial\phi/\partial\psi)$ the ion parallel velocity in the high flow limit. We now show that this correction is an adequate approximation for moderate Mach numbers $M_i = V_i/v_i \sim 0.4$ in the $B_p/B \ll 1$ limit. The derivation of the gyrokinetic equation (10) in [19] is valid if the electric field satisfies $|\mathbf{E}| = |\nabla\phi| \ll T_e/e\rho_i$. The size of the electric field is estimated from

$V_{i||} = -(cI/B)(\partial\phi/\partial\psi)$, giving $\mathbf{E} = -\nabla\phi \sim V_{i||}|\nabla\psi|/cR \sim (B_p/B)M_{i||}(T_e/e\rho_i)$, with $M_{i||} = V_{i||}/v_i$ the parallel Mach number. Thus, the gyrokinetic equation is valid even for $M_{i||} \sim 1$ if $B_p/B \ll 1$. The only modification to the gyrokinetic formalism is then that the zeroth order distribution function is no longer a stationary Maxwellian, but instead has an average parallel velocity, i.e.,

$$F_i = n_i \left(\frac{M}{2\pi T_i}\right)^{3/2} \exp\left(-\frac{M(\mathbf{v} - V_{i||}\hat{\mathbf{b}})^2}{2T_i}\right) \simeq n_i \left(\frac{M}{2\pi T_i}\right)^{3/2} \exp\left(-\frac{ME}{T_i} + \frac{MuV_{i||}}{T_i} - \frac{MV_{i||}^2}{2T_i}\right). \quad (53)$$

In the second equality we have written the Maxwellian as a function of the gyrokinetic variables. There has been already some work in this high flow limit with $B_p/B \ll 1$ [32]. Importantly, the first order term in an expansion in $M_{i||}$ in (53) is equal to the correction obtained by taking the limit $Zen_i(\partial\phi/\partial\psi) \gg \partial p_i/\partial\psi, n_i(\partial T_i/\partial\psi)$ in $f_{i1}^{\rm nc}$, i.e., $f_{i1}^{\rm nc} \simeq (Mv_{||}V_{i||}/T_i)f_{Mi}$. The next order term, of order $M_{i||}^2$, is only a 10% correction for $M_{i||} = 0.4$, and it is irrelevant for the transport of toroidal angular momentum because it is even in $v_{||}$ and does not break the symmetry of the parallel velocity. The next correction odd in $v_{||}$ is of order $M_{i||}^3$, clearly negligible for $M_{i||} = 0.4$. Thus, equation (52) is reasonably good even for moderate parallel Mach numbers in the $B_p/B \ll 1$ limit.

Finally, any analysis performed for δf formulations is valid in full f codes. However, it is important to realize that the collision operator becomes crucial in full f simulations. It is necessary because it drives the long wavelength piece of the distribution function towards the solution $F_i = f_{Mi} + f_{i1}^{\rm nc}$. Thus, any full f simulation must run for longer than the characteristic time for the relaxation to the neoclassical solution, given by $1/(\sqrt{\epsilon}\nu_{ii})$ [29]. Moreover, it is probably very convenient to initialize the simulation employing F_i plus some small short wavelength contributions as the initial condition.

4.2. Higher order electrostatic potential

We have already argued in [16] that having a gyrokinetic vorticity equation is desirable. A vorticity equation shows explicitly the connection between the radial electric field and the transport of toroidal angular momentum, and unlike the quasineutrality equation, it can be modified to include higher order terms. In this subsection we construct a higher order vorticity equation employing a technique similar to the method developed in [16]. We find that the vorticity equations (77) and (85) of reference [16] become valid to order $(B/B_p)\delta_i^2 en_e v_i/a$ by replacing the Maxwellian f_{Mi} by the neoclassical solution $F_i = f_{Mi} + f_{i1}^{\rm nc}$ and slightly modifying the definition of the polarization density. This similarity indicates that both vorticity equations have the desired properties, i.e., they keep the long wavelength toroidal rotation constant for short time scales and could be extended to higher order in the future.

The rest of the subsection is organized as follows. In section 4.2.1, we write the gyrokinetic equation in (10) as a function of the "physical" phase space, i.e., as a function of the variables $\mathbf{r}$, E_0, μ_0, φ_0 and t. In these variables, the real space and the velocity space are not mixed as they were in the gyrokinetic variables, and we can integrate in velocity space to obtain moment equations. We write a general equation for the moment $\int d^3v\, f_i G(\mathbf{r}, \mathbf{v}, t)$ in section 4.2.2, and then use it to obtain the conservation of ion number and ion perpendicular momentum in subsections 4.2.3 and 4.2.4. Finally combining these equations and the conservation of electron number, we obtain two equivalent vorticity equations valid up to order $(B/B_p)\delta_i^2 e n_e v_i/a$ in subsection 4.2.5.

4.2.1. Gyrokinetics in "physical" phase space. We write the gyrokinetic equation (10) as a function of the "physical" phase space variables $\mathbf{r}$, E_0, μ_0, φ_0 and t. We loosely follow the procedure in [16], but we present it in a more convenient form.

The gyrokinetic equation is valid to order $(B/B_p)\delta_i^2 f_{Mi} v_i/a$, and the expansions are performed to that order. We expand $f_i(\mathbf{R}, E, \mu, t)$ around $\mathbf{R}_g = \mathbf{r} + \Omega_i^{-1}\mathbf{v} \times \hat{\mathbf{b}}$, E_0 and μ_0 to obtain

$$f_i(\mathbf{R}, E, \mu, t) = f_{ig} + E_1 \frac{\partial F_i}{\partial E_0} + \mu_1 \frac{\partial F_i}{\partial \mu_0} + O(\delta_i^2 f_{Mi}), \tag{54}$$

where E_1 and μ_1 are given in (5) and (6), the function $f_{ig} \equiv f_i(\mathbf{R}_g, E_0, \mu_0, t)$ is obtained by replacing $\mathbf{R}$, E and μ by $\mathbf{R}_g$, E_0 and μ_0 in f_i, and we have already defined $F_i = f_{Mi} + f_{i1}^{\rm nc}$. Notice that we continue to neglect corrections of order $\delta_i^2 f_{Mi}$ as small compared to $f_{i2}^{\rm tb} \sim (B/B_p)\delta_i^2 f_{Mi}$. The function $f_{ig} \equiv f_i(\mathbf{R}_g, E_0, \mu_0, t)$ is convenient to obtain moment equations from the gyrokinetic equation (10). Replacing $\mathbf{R}$, E and μ by $\mathbf{R}_g$, E_0 and μ_0 in (10) gives

$$\left.\frac{\partial f_{ig}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + [u_g \hat{\mathbf{b}}(\mathbf{R}_g) + \mathbf{v}_{Mg} + \mathbf{v}_{Eg}] \cdot \left(\nabla_{\mathbf{R}_g} f_{ig} - \frac{Ze}{M} \nabla_{\mathbf{R}_g} \langle \phi \rangle \frac{\partial F_i}{\partial E_0} \right) = \langle C_{ii}\{f_i\} \rangle_g. \tag{55}$$

Here u_g, $\mathbf{v}_{Mg}$, $\mathbf{v}_{Eg}$ and $\langle C_{ii}\{f_i\}\rangle_g$ are obtained by replacing $\mathbf{R}$, E and μ by $\mathbf{R}_g$, E_0 and μ_0 in $u = \pm\sqrt{2[E - \mu B(\mathbf{R})]}$, $\mathbf{v}_M$, $\mathbf{v}_E$ and $\langle C_{ii}\{f_i\}\rangle$. Notice that the time derivative holding $\mathbf{R}_g$, E_0, μ_0 and φ_0 fixed is equivalent to the time derivative holding $\mathbf{r}$ and $\mathbf{v}$ because the magnetic field is constant in time.

In steady state $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}_g} f_{ig} \sim \delta_i f_{Mi}/a$ and $\hat{\mathbf{b}} \cdot \nabla\phi \sim \delta_i T_e/ea$. This is true even for the neoclassical piece since $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}_g} \sim 1/qR$ and $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$, giving $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}_g} f_{i1}^{\rm nc} \sim \delta_i f_{Mi}/a$. Using that $\hat{\mathbf{b}} \cdot \nabla_{\mathbf{R}_g} f_{ig} \sim \delta_i f_{Mi}/a$ and $\hat{\mathbf{b}} \cdot \nabla\phi \sim \delta_i T_e/ea$, we can show (see appendix B) that equation (55) is to the order of interest

$$\left.\frac{\partial f_{ig}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + [v_{||}\hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1] \cdot \left(\overline{\nabla} f_{ig} - \frac{Ze}{M} \overline{\nabla} \langle \phi \rangle \frac{\partial F_i}{\partial E_0} \right) = \langle C_{ii}\{f_i\} \rangle_g, \tag{56}$$

with

$$\mathbf{v}_{M0} = \frac{v_\perp^2}{2B\Omega_i} \hat{\mathbf{b}} \times \nabla B + \frac{v_{||}^2}{\Omega_i} \hat{\mathbf{b}} \times \boldsymbol{\kappa}, \tag{57}$$

$$\mathbf{v}_{E0} = -\frac{c}{B} \overline{\nabla} \langle \phi \rangle \times \hat{\mathbf{b}} \tag{58}$$

and

$$\tilde{\mathbf{v}}_1 = \frac{v_{||}}{\Omega_i}\overline{\nabla} \times \mathbf{v}_\perp. \tag{59}$$

Here, the gradient $\overline{\nabla}$ is with respect to $\mathbf{r}$ holding E_0, μ_0, φ_0 and t fixed.

4.2.2. General gyrokinetic moment equation. The integral over velocity space $\{E_0, \mu_0, \varphi_0\}$ of equation (56) multiplied by the general function $G(\mathbf{r}, \mathbf{v}, t)$ will help us obtain the transport of density and momentum. However, equation (56) must be first written in conservative form. To do so, we use

$$\overline{\nabla} \cdot \left(\frac{B}{v_{||}} \mathbf{v}_{\mathrm{GK}} \right) - \frac{Ze}{M} \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} \mathbf{v}_{\mathrm{GK}} \cdot \overline{\nabla} \langle \phi \rangle \right) = 0, \tag{60}$$

with

$$\mathbf{v}_{\mathrm{GK}} = \left(v_{||} + \frac{v_{||}^2}{\Omega_i} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \right) \hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1 = \\ v_{||}\hat{\mathbf{b}} + \frac{v_{||}}{\Omega_i} \overline{\nabla} \times (v_{||}\hat{\mathbf{b}}) + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1. \tag{61}$$

To obtain the second equality we use $\nabla \times \hat{\mathbf{b}} = \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \boldsymbol{\kappa}$. With the second equality of (61) and the fact that $\partial\langle\phi\rangle/\partial E_0 = 0$, proving (60) becomes trivial.

Equation (60) is useful because equation (56) is to order $(B/B_p)\delta_i^2 v_i/a$

$$\left.\frac{\partial f_{ig}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \mathbf{v}_{\mathrm{GK}} \cdot \left(\overline{\nabla} f_{ig} - \frac{Ze}{M} \overline{\nabla} \langle \phi \rangle \frac{\partial F_i}{\partial E_0} \right) = \langle C_{ii}\{f_i\} \rangle_g, \tag{62}$$

where the contribution of $(v_{||}^2/\Omega_i)\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \sim \delta_i v_i$ in (61) is negligible because in steady state $\hat{\mathbf{b}} \cdot \overline{\nabla} f_{ig} \sim \delta_i f_{Mi}/a$ and $\hat{\mathbf{b}} \cdot \overline{\nabla}\langle\phi\rangle \sim \delta_i T_e/ea$. Combining equations (60) and (62) gives

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{r},\mathbf{v}} \left(\frac{B}{v_{||}} f_{ig} \right) + \overline{\nabla} \cdot \left(\frac{B}{v_{||}} f_{ig} \mathbf{v}_{\mathrm{GK}} \right) - \frac{Ze}{M} \frac{\partial}{\partial E_0} \left(\frac{B}{v_{||}} f_{ig} \mathbf{v}_{\mathrm{GK}} \cdot \overline{\nabla} \langle \phi \rangle \right) = \frac{B}{v_{||}} \langle C_{ii}\{f_i\} \rangle_g, \tag{63}$$

where we have employed that $\partial f_{ig}/\partial E_0 \simeq \partial F_i/\partial E_0$.

Equation (63) is in conservative form and can be used to obtain moment equations. Multiplying by a general function $G(\mathbf{r}, \mathbf{v}, t)$ and integrating in velocity space gives

$$\frac{\partial}{\partial t} \left(\int d^3v\, f_{ig} G \right) + \nabla \cdot \left(\int d^3v\, f_{ig} \mathbf{v}_{\mathrm{GK}} G \right) = \int d^3v\, f_{ig} K\{G\} + \int d^3v\, G \langle C_{ii}\{f_i\} \rangle_g, \tag{64}$$

where

$$K\{G\} = \left.\frac{\partial G}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + \mathbf{v}_{\mathrm{GK}} \cdot \left(\overline{\nabla} G - \frac{Ze}{M} \overline{\nabla} \langle \phi \rangle \frac{\partial G}{\partial E_0} \right). \tag{65}$$

4.2.3. Gyrokinetic conservation of number. In this section, we find the time derivatives of the ion and electron densities up to order $(B/B_p)\delta_i^2 n_e v_i/a$. In section 4.2.5, we will use these derivatives to calculate $\partial(Zn_i - n_e)/\partial t$, and by imposing that this derivative vanishes we find the turbulent, short wavelength piece of the electric field.

The electrons are drift kinetic in our model and the electron number conservation can be written as

$$\frac{\partial n_e}{\partial t} + \nabla\cdot\left(n_e V_{e||}\hat{\mathbf{b}} + n_e \mathbf{V}_{ed} - \frac{cn_e}{B}\nabla\phi\times\hat{\mathbf{b}}\right) = 0, \tag{66}$$

where the parallel flow is $n_e V_{e||} = \int d^3v\, f_e v_{||}$, and the magnetic drifts give

$$n_e\mathbf{V}_{ed} = -\frac{cp_{e\perp}}{eB}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} - \frac{cp_{e\perp}}{eB^2}\hat{\mathbf{b}}\times\nabla B - \frac{cp_{e||}}{B}\hat{\mathbf{b}}\times\boldsymbol{\kappa}. \tag{67}$$

To obtain these expressions, we have kept the electron pressure anisotropy $p_{e||} - p_{e\perp} \sim (B/B_p)\delta_e p_e \sim (B/B_p)\sqrt{m/M}\delta_i p_e$ because it gives a contribution of order $(B/B_p)\delta_i^2 n_e v_i/a$ for $\sqrt{m/M}\sim\delta_i$.

For the ion number conservation equation, we use equation (64) with $G = 1$ up to order $(B/B_p)\delta_i^2 n_e v_i/a$ (see appendix C) to find

$$\frac{\partial}{\partial t}(n_i - n_{ip}) + \nabla\cdot(n_i V_{ig||}\hat{\mathbf{b}} + n_i\mathbf{V}_{igd} + n_i\mathbf{V}_{igE} + n_i\tilde{\mathbf{V}}_i + n_i\mathbf{V}_{iC}) = 0, \tag{68}$$

where the ion polarization density is given by

$$n_{ip} = \int d^3v\left(E_1\frac{\partial F_i}{\partial E_0} + \mu_1\frac{\partial F_i}{\partial\mu_0}\right) \simeq \int d^3v\,\frac{Ze\tilde{\phi}}{M}\left(\frac{\partial F_i}{\partial E_0} + \frac{1}{B}\frac{\partial F_i}{\partial\mu_0}\right), \tag{69}$$

the ion gyrokinetic parallel flow is

$$n_i V_{ig||} = \int d^3v\, f_{ig}v_{||}, \tag{70}$$

finite gyroradius effects lead to the flow

$$n_i\tilde{\mathbf{V}}_i = \int d^3v\, f_{ig}\tilde{\mathbf{v}}_1 = \int d^3v\, f_{ig}\frac{v_{||}}{\Omega_i}\overline{\nabla}\times\mathbf{v}_\perp, \tag{71}$$

and the $\mathbf{E}\times\mathbf{B}$ and magnetic drifts give the flows

$$n_i\mathbf{V}_{igE} = \int d^3v\, f_{ig}\mathbf{v}_{E0} = -\frac{c}{B}\int d^3v\, f_{ig}\overline{\nabla}\langle\phi\rangle\times\hat{\mathbf{b}}. \tag{72}$$

and

$$n_i\mathbf{V}_{igd} = \int d^3v\, f_{ig}\left(\frac{v_{||}^2}{\Omega_i}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \mathbf{v}_{M0}\right) \simeq$$
$$\frac{cp_{ig\perp}}{ZeB}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \frac{cp_{ig\perp}}{ZeB^2}\hat{\mathbf{b}}\times\nabla B + \frac{cp_{ig||}}{ZeB}\hat{\mathbf{b}}\times\boldsymbol{\kappa} \tag{73}$$

Here, $p_{ig\perp} = \int d^3v\, f_{ig}Mv_\perp^2/2$ and $p_{ig||} = \int d^3v\, f_{ig}Mv_{||}^2$. The flow $n_i\mathbf{V}_{iC}$ is due to finite gyroradius effects on ion-ion collisions. It is calculated in appendix D to be

$$n_i\mathbf{V}_{iC} = -\frac{\gamma_{ii}}{\Omega_i}\int d^3v\left(\langle\boldsymbol{\Gamma}_{ii}\rangle\times\hat{\mathbf{b}} - \frac{1}{v_\perp^2}\langle\boldsymbol{\Gamma}_{ii}\cdot\mathbf{v}_\perp\rangle\mathbf{v}\times\hat{\mathbf{b}}\right), \tag{74}$$

with $\gamma_{ii} = 2\pi Z^4 e^4\ln\Lambda/M^2$ and

$$\boldsymbol{\Gamma}_{ii} = \int d^3v'\,\nabla_g\nabla_g g\cdot(f_i'\nabla_v f_i - f_i\nabla_{v'}f_i'). \tag{75}$$

Here, $f_i = f_i(\mathbf{v})$, $f_i' = f_i(\mathbf{v}')$, $\mathbf{g} = \mathbf{v} - \mathbf{v}'$, $g = |\mathbf{g}|$ and $\nabla_g\nabla_g g = (g^2\overleftrightarrow{\mathbf{I}} - \mathbf{gg})/g^3$.

4.2.4. Gyrokinetic conservation of momentum. Employing $G = M\mathbf{v}_\perp$ in equation (64) and using appendix E, we find that the conservation of ion perpendicular momentum is, up to order $(B/B_p)\delta_i^2 p_i/a$,

$$\frac{\partial}{\partial t}(n_i M \mathbf{V}_{ig\perp}) + \nabla\cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times} = \mathbf{F}_{iB\perp}^{\mathrm{nc}} + \mathbf{F}_{iB\perp}^{\mathrm{tb}} + \mathbf{F}_{iC\perp}, \tag{76}$$

where $n_i\mathbf{V}_{ig\perp} = \int d^3v\, f_{ig}\mathbf{v}_\perp$ is the perpendicular gyrocenter flow; the tensor $\overleftrightarrow{\boldsymbol{\pi}}_{ig\times}$ gives the transport of perpendicular momentum due to the parallel velocity and the drifts,

$$\overleftrightarrow{\boldsymbol{\pi}}_{ig\times} = \int d^3v\, f_{ig}(v_{||}\hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1)M\mathbf{v}_\perp; \tag{77}$$

the vectors $\mathbf{F}_{iB\perp}^{\mathrm{nc}}$ and $\mathbf{F}_{iB\perp}^{\mathrm{tb}}$ account for the change in perpendicular velocity as the particle drifts in a spatially varying magnetic field,

$$\mathbf{F}_{iB\perp}^{\mathrm{nc}} = M\int d^3v\, F_i\tilde{\mathbf{v}}_1\cdot\overline{\nabla}\mathbf{v}_\perp \tag{78}$$

and

$$\mathbf{F}_{iB\perp}^{\mathrm{tb}} = M\int d^3v\, f_{ig}(v_{||}\hat{\mathbf{b}} + \mathbf{v}_{E0})\cdot\overline{\nabla}\mathbf{v}_\perp; \tag{79}$$

and the force $\mathbf{F}_{iC\perp}$ is due to finite gyroradius effects on the ion-ion collisions. It is calculated in appendix D and is given by

$$\begin{aligned}\mathbf{F}_{iC\perp} = {} & -M\gamma_{ii}\int d^3v\,\frac{1}{v_\perp^2}\mathbf{v}_\perp\langle\boldsymbol{\Gamma}_{ii}\cdot\mathbf{v}_\perp\rangle \\ & +\nabla\cdot\left\{\frac{M\gamma_{ii}}{\Omega_i}\int d^3v\left[\langle\boldsymbol{\Gamma}_{ii}\rangle\times\hat{\mathbf{b}} - \frac{1}{v_\perp^2}(\mathbf{v}\times\hat{\mathbf{b}})\langle\boldsymbol{\Gamma}_{ii}\cdot\mathbf{v}_\perp\rangle\right]\mathbf{v}_\perp\right\}.\end{aligned} \tag{80}$$

4.2.5. Gyrokinetic vorticity equations. Finally, we use equations (66), (68) and (76) to obtain two equivalent vorticity equations. These vorticity equations are the extension to order $(B/B_p)\delta_i^2 e n_e v_i/a$ of the equations found in [16].

The first vorticity equation is obtained by subtracting (66) from Z times (68) to find

$$\frac{\partial}{\partial t}(Zen_{ip}) = \nabla\cdot(J_{g||}\hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_p + Zen_i\tilde{\mathbf{V}}_i + Zen_i\mathbf{V}_{iC}) = 0, \tag{81}$$

with n_{ip} the polarization density given in (69), $n_i\tilde{\mathbf{V}}_i$ the finite gyroradius correction in (71), and $n_i\mathbf{V}_{iC}$ the collisional drift from (74). The parallel gyrocenter current is

$$J_{g||} = Zen_iV_{ig||} - en_eV_{e||} = Ze\int d^3v\, f_{ig}v_{||} - e\int d^3v\, f_e v_{||}; \tag{82}$$

the current due to the magnetic drifts is

$$\mathbf{J}_{gd} = Zen_i\mathbf{V}_{igd} - en_e\mathbf{V}_{ed} = \frac{cp_{g\perp}}{B}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}} + \frac{cp_{g\perp}}{B^2}\hat{\mathbf{b}}\times\nabla B + \frac{cp_{g||}}{B}\hat{\mathbf{b}}\times\boldsymbol{\kappa}; \tag{83}$$

with $p_{g\perp} = p_{ig\perp} + p_{e\perp}$ and $p_{g||} = p_{ig||} + p_{e||}$; and finally, there is a polarization current density due to the difference between the $\mathbf{E}\times\mathbf{B}$ drifts of ions and electrons given by

$$\tilde{\mathbf{J}}_p = \frac{Zec}{B}\int d^3v\left[f_i(\nabla\phi\times\hat{\mathbf{b}}) - f_{ig}(\overline{\nabla}\langle\phi\rangle\times\hat{\mathbf{b}})\right]. \tag{84}$$

The second vorticity equation can be found by adding $\nabla\cdot[(c/B)\hat{\mathbf{b}}\times(\text{equation (76)})]$ to equation (81). Since equation (76) is zero for any solution of (10), the new vorticity equation is equivalent to (81). Adding $\nabla\cdot[(c/B)\hat{\mathbf{b}}\times(\text{equation (76)})]$ to equation (81) gives

$$\frac{\partial\varpi_G}{\partial t} = \nabla\cdot\left[J_{g||}\hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_\phi + \frac{c}{B}\hat{\mathbf{b}}\times(\nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_{iG}) + Zen_i\mathbf{V}_{iC} - \frac{c}{B}\hat{\mathbf{b}}\times\mathbf{F}_{iC}\right], \tag{85}$$

where we have neglected $(c/B)\hat{\mathbf{b}}\times\mathbf{F}^{\mathrm{nc}}_{iB\perp}$, and combined $\tilde{\mathbf{J}}_p$, $Zen_i\tilde{\mathbf{V}}_i$, $(c/B)\hat{\mathbf{b}}\times(\nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_{ig\times})$ and $(c/B)\hat{\mathbf{b}}\times\mathbf{F}^{\mathrm{tb}}_{iB\perp}$ to obtain $\tilde{\mathbf{J}}_\phi$, $(c/B)\hat{\mathbf{b}}\times(\nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_{iG})$ and a term with vanishing divergence. The details of the derivation are in appendix F. We have defined a new gyrokinetic vorticity

$$\varpi_G = Zen_{ip} + \nabla\cdot\left(\frac{Ze}{\Omega_i}n_i\mathbf{V}_{ig}\times\hat{\mathbf{b}}\right), \tag{86}$$

a new polarization current

$$\tilde{\mathbf{J}}_\phi = \tilde{\mathbf{J}}_p - \frac{Ze}{\Omega_i}\hat{\mathbf{b}}\times\left(\int d^3v\, F_i\mathbf{v}_{E0}\cdot\overline{\nabla}\mathbf{v}_\perp\right) \tag{87}$$

and a new viscosity

$$\overleftrightarrow{\boldsymbol{\pi}}_{iG} = \overleftrightarrow{\boldsymbol{\pi}}_{ig\times} + M\int d^3v\, f_{ig}\mathbf{v}_\perp v_{||}\hat{\mathbf{b}} = M\int d^3v\, f_{ig}[v_{||}(\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp\hat{\mathbf{b}}) + (\mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1)\mathbf{v}_\perp]. \tag{88}$$

Notice that the vorticity equations (81) and (85) are very similar to their lower order versions, equations (77) and (85) of reference [16]. The differences are that f_{Mi} has been replaced by $F_i = f_{Mi} + f^{\mathrm{nc}}_{i1}$, and that the polarization density, n_{ip}, is now given by the higher order expression (69), and not by $-\int d^3v\,(Ze\widetilde{\phi}/T_i)f_{Mi}$. The similarities between the new vorticity equations (81) and (85), and the lower order equations (77) and (85) of reference [16] make obvious that the same properties hold for both of them. Moreover, these equations are equivalent to each other to order $\delta_i en_e v_i/a$. In particular, it is possible to prove, as was done in [16], that the flux surface averages of both (81) and (85) give, at long wavelengths, $\partial\langle Rn_iM\mathbf{V}_i\cdot\hat{\boldsymbol{\zeta}}\rangle_\psi/\partial t \sim \delta_i k_\perp\rho_i p_i$, and consequently the time derivative of the toroidal rotation becomes smaller as we go to longer wavelengths. Thus, for the typical short time scales of turbulence, equations (81) and (85) keep the global toroidal rotation constant. The physical evolution of the long wavelength toroidal rotation can only be obtained from (26).

Finally, equation (85) is a perfect candidate to be extended to even higher order to retain the long wavelength terms of (37). It is written in a form that makes the relation with the transport of momentum more transparent than the gyrokinetic quasineutrality or the vorticity equation (81). Thus, one could employ the transport of toroidal angular momentum given in (37), accurate enough to calculate the radial electric field, substitute it into the exact vorticity equation (22) of reference [16], and compare the result with the long wavelength limit of (85). Then, the differences between both equations are the higher order terms that equation (85) is missing to obtain the long wavelength radial electric field. As a result, it is possible that we could correct for these differences.

5. Discussion

We have obtained expression (37) for the transport of toroidal angular momentum. This expression is valid up to order $(B/B_p)\delta_i^3 p_i R|\nabla\psi|$, and it only requires the ion distribution function and the non-axisymmetric piece of the electric field up to order $(B/B_p)\delta_i^2 f_{Mi}$ and $(B/B_p)\delta_i^2 T_e/ea$, respectively. Equation (37) is then enough to self-consistently calculate the toroidal rotation profile in the low flow ordering if the turbulent, short wavelength piece of the ion distribution function is obtained from the ion gyrokinetic Fokker-Planck equation (10) (or its δf version (52)), and the turbulent, short wavelength piece of the electrostatic potential is calculated using either one of the vorticity equations (81) and (85) (in flux tube δf simulations, it is also acceptable to use the gyrokinetic quasineutrality equation to solve for the short wavelength pieces of the potential). Once the toroidal rotation profile is obtained, the long wavelength, radial electric field is solved from the neoclassical relation (22) that we have shown to hold at long wavelengths in our ordering.

The other options to calculate the long wavelength, radial electric field are direct integration of the lower order expression for the toroidal angular momentum (1) or solving directly for the radial electric field using quasineutrality. Integrating (1) requires a third order ion distribution function. On the other hand, using the quasineutrality condition to obtain the radial electric field is equivalent to forcing the radial current to vanish. Multiplying equation (25) by $R\hat{\boldsymbol{\zeta}}$ and flux surface averaging, we find that the radial current is

$$\langle \mathbf{J} \cdot \nabla\psi \rangle_\psi = c \left[\frac{\partial}{\partial t} \langle R n_i M \mathbf{V}_i \cdot \hat{\boldsymbol{\zeta}} \rangle_\psi + \frac{1}{V'} \frac{\partial}{\partial \psi} (V' \Pi) \right]. \tag{89}$$

(Notice that we have used this equation and $\langle \mathbf{J} \cdot \nabla\psi \rangle_\psi = 0$ to obtain (26)). This relation implies that for transport time scales, the average radial current density $\langle \mathbf{J} \cdot \nabla\psi \rangle_\psi / |\nabla\psi|$ is identically zero up to order $c(\partial\Pi/\partial\psi)/|\nabla\psi| \sim (B/B_p)^2 \delta_i^4 e n_e v_i$. Then, the current density must be obtained self-consistently to that order, and we would need a gyrokinetic equation good to fourth order! Using equations (26) and (37) to obtain the toroidal rotation and (22) to solve for the radial electric field is clearly the most convenient method because it requires the lowest order gyrokinetic equation.

We have exploited the extra expansion parameter $B_p/B \ll 1$ because by doing so the lowest order gyrokinetic equation (10) is able to provide the ion distribution function up to order $(B/B_p)\delta_i^2 f_{Mi}$. In subsection 4.1 we explain how current δf codes should be modified to achieve the higher accuracy, and we propose a simple δf equation (52) in which the long wavelength, background Maxwellian f_{Mi} is replaced by the distribution function $F_i = f_{Mi} + f_{i1}^{\mathrm{nc}}$ that contains the first order neoclassical correction. This modification should be relatively simple to implement, and in a first approach to the problem the simplified neoclassical solution in (47) is probably enough. Importantly, equation (52) allows us to explore the high flow limit even for Mach numbers around 0.4.

Finally, to determine the non-axisymmetric pieces of the potential we need to solve for it from either a vorticity equation or the quasineutrality condition. The quasineutrality condition works for local, flux tube codes in which the problematic long wavelength components of the potential are intentionally ignored. In global simulation, however, the long wavelength components of the potential are included in the simulation. In [16, 19] we argued that the long wavelength radial electric field cannot be determined self-consistently from the lower order quasineutrality equations typically used in gyrokinetic simulations. A promising alternative is a gyrokinetic vorticity equation. We presented two of them in [16] and we proved that they have very desirable properties, namely they keep the toroidal rotation constant at short time scales and they could be extended to higher order. Here, we have extended both gyrokinetic vorticity equations to order $(B/B_p)\delta_i^2 en_e v_i/a$ to obtain the short wavelength, turbulent potential up to order $(B/B_p)\delta_i^2 T_e/e$ self-consistently. The new higher order vorticity equations are given in (81) and (85). They are obviously equivalent to the equations found in [16] to order $\delta_i en_e v_i/a$ and they have the same properties. Both of these vorticity equations can be used in global simulations for short time scales. The transport time scale evolution of the radial electric field must be obtained from the transport of toroidal angular momentum given by (26) and (37). It might be possible to extend equation (85) to even higher order to obtain the physical long wavelength radial electric field.

Acknowledgments

The authors are indebted to Bill Dorland of University of Maryland for his support, and to Grigory Kagan of MIT for many helpful discussions.

This research was supported by the U.S. Department of Energy Grant No. DE-FG02-91ER-54109 at the Plasma Science and Fusion Center of the Massachusetts Institute of Technology and by the Center for Multiscale Plasma Dynamics of University of Maryland.

Appendix A. Collisional contribution (39) in up-down symmetric tokamaks

The pieces of order $(B/B_p)(qR\nu_{ii}/v_i)\delta_i^2 p_i R|\nabla\psi|$ and $(qR\nu_{ii}/v_i)\delta_i^2 p_i R|\nabla\psi|$ of integral (39) vanish for up-down symmetric tokamaks. To lowest order, the long wavelength, axisymmetric piece of the distribution function is $f_i \simeq f_{Mi} + f_{i1}^{\rm nc} + (\langle f_i \rangle_{\rm T} - \langle \overline{f}_i \rangle_{\rm T})$, with $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ and $\langle f_i \rangle_{\rm T} - \langle \overline{f}_i \rangle_{\rm T} \sim \delta_i f_{Mi}$ given in (36). The Maxwellian piece f_{Mi} does not contribute because makes the collision operator vanish. The gyrophase dependent piece $\langle f_i \rangle_{\rm T} - \langle \overline{f}_i \rangle_{\rm T}$ also vanishes because the integrand of (39) becomes a summation of terms that are either odd in $v_{||}$ or $\mathbf{v}_\perp$. Employing the function $h_{i1}^{\rm nc}$, related to the first order neoclassical piece $f_{i1}^{\rm nc}$ by equation (21), we find that the difference $f_{i1}^{\rm nc} - h_{i1}^{\rm nc}$ vanishes because some of the terms make the collision operator zero and others are odd in $v_{||}$. Thus, equation (39) becomes

$$-\frac{M^2 c}{2Ze} \left\langle \int d^3v\, C_{ii}\{f_i\} R^2 (\mathbf{v} \cdot \hat{\boldsymbol{\zeta}})^2 \right\rangle_\psi =$$

$$-\frac{M}{2B\Omega_i}\left\langle \int d^3v\, C_{ii}^{(\ell)}\{h_{i1}^{\rm nc}\}\left(|\nabla\psi|^2\frac{v_\perp^2}{2}+I^2v_{||}^2\right)\right\rangle_\psi, \qquad \text{(A.1)}$$

where we have used the gyroaverage $\overline{\mathbf{v}\mathbf{v}} = (v_\perp^2/2)(\overleftrightarrow{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + v_{||}^2\hat{\mathbf{b}}\hat{\mathbf{b}}$.

Finally, to prove that integral (A.1) vanishes, we employ the neoclassical drift kinetic equation (20) for $h_{i1}^{\rm nc}$. In equation (20), replacing θ by $-\theta$, $v_{||}$ by $-v_{||}$ and $h_{i1}^{\rm nc}$ by $-h_{i1}^{\rm nc}$ does not change the equation since $\hat{\mathbf{b}}\cdot\nabla\theta$ does not change sign. Thus, $h_{i1}^{\rm nc}$ changes sign if both θ and $v_{||}$ do. Due to this property, the collisional integral in (A.1) vanishes. In the contributions to this integral, the piece of the distribution function with positive $v_{||}$ in the upper half ($\theta > 0$) of the tokamak cancels the piece of the distribution function with negative $v_{||}$ in the lower half ($\theta < 0$). Similarly, the piece with negative $v_{||}$ in the upper half cancels the piece with positive $v_{||}$ in the lower half.

Appendix B. Gyrokinetic equation in "physical" phase space

In this appendix we explain how to obtain equation (56) from equation (55). To do so, we neglect terms that are of order $\delta_i^2 f_{Mi}v_i/a$ in equation (56) because it is enough to obtain the equation up to order $(B/B_p)\delta_i^2 f_{Mi}v_i/a$.

In equation (55), the magnetic and $\mathbf{E}\times\mathbf{B}$ drifts $\mathbf{v}_{Mg}$ and $\mathbf{v}_{Eg}$ are to lowest order $\mathbf{v}_{Mg} = \mathbf{v}_{M0} + O(\delta_i^2 v_i)$ and $\mathbf{v}_{Eg} = \mathbf{v}_{E0} + O(\delta_i^2 v_i)$, with $\mathbf{v}_{M0}$ and $\mathbf{v}_{E0}$ given in (57) and (58). Since any term of order $\delta_i^2 f_{Mi}v_i/a$ is negligible, the drifts can be approximated by $\mathbf{v}_{Mg} \simeq \mathbf{v}_{M0}$ and $\mathbf{v}_{Eg} \simeq \mathbf{v}_{E0}$. Moreover, considering that in steady state the parallel gradients of f_{ig} and $\langle\phi\rangle$ must be of order $\delta_i f_{Mi}/a$ and $\delta_i T_e/ea$, the difference $u_g - v_{||} = O(\delta_i v_i)$ is also neglected, giving

$$\left.\frac{\partial f_{ig}}{\partial t}\right|_{\mathbf{r},\mathbf{v}} + [v_{||}\hat{\mathbf{b}}(\mathbf{R}_g) + \mathbf{v}_{M0} + \mathbf{v}_{E0}]\cdot\nabla_{\mathbf{R}_g}\mathbf{r}\cdot\left(\overline{\nabla} f_{ig} - \frac{Ze}{M}\overline{\nabla}\langle\phi\rangle\frac{\partial F_i}{\partial E_0}\right) = \langle C_{ii}\{f_i\}\rangle_g, \qquad \text{(B.1)}$$

where we have also used that $\nabla_{\mathbf{R}_g} f_{ig} = \nabla_{\mathbf{R}_g}\mathbf{r}\cdot\overline{\nabla} f_{ig}$ and $\nabla_{\mathbf{R}_g}\langle\phi\rangle = \nabla_{\mathbf{R}_g}\mathbf{r}\cdot\overline{\nabla}\langle\phi\rangle$. Employing that $\nabla_{\mathbf{R}_g}\mathbf{r} = \overleftrightarrow{\mathbf{I}} - \nabla_{\mathbf{R}_g}(\Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}) = \overleftrightarrow{\mathbf{I}} - \overline{\nabla}(\Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}) + O(\delta_i^2)$, we can write

$$v_{||}\hat{\mathbf{b}}(\mathbf{R}_g)\cdot\nabla_{\mathbf{R}_g}\mathbf{r} \simeq v_{||}\hat{\mathbf{b}} + \frac{v_{||}}{\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}} - v_{||}\hat{\mathbf{b}}\cdot\overline{\nabla}\left(\frac{1}{\Omega_i}\mathbf{v}\times\hat{\mathbf{b}}\right) =$$
$$\left[v_{||} - \frac{v_{||}}{\Omega_i}\overline{\nabla}\cdot(\mathbf{v}\times\hat{\mathbf{b}})\right]\hat{\mathbf{b}} + \frac{v_{||}}{\Omega_i}\overline{\nabla}\times\mathbf{v}_\perp, \qquad \text{(B.2)}$$

where we have neglected terms of order $\delta_i^2 v_i$ and we have used $(\mathbf{v}\times\hat{\mathbf{b}})\cdot\nabla\hat{\mathbf{b}} = \overline{\nabla}\cdot[(\mathbf{v}\times\hat{\mathbf{b}})\hat{\mathbf{b}}] - [\overline{\nabla}\cdot(\mathbf{v}\times\hat{\mathbf{b}})]\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\cdot\overline{\nabla}(\Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}) = \Omega_i^{-1}\overline{\nabla}\cdot[\hat{\mathbf{b}}(\mathbf{v}\times\hat{\mathbf{b}})]$ and $\overline{\nabla}\cdot[(\mathbf{v}\times\hat{\mathbf{b}})\hat{\mathbf{b}}] - \overline{\nabla}\cdot[\hat{\mathbf{b}}(\mathbf{v}\times\hat{\mathbf{b}})] = \overline{\nabla}\times[\hat{\mathbf{b}}\times(\mathbf{v}\times\hat{\mathbf{b}})] = \overline{\nabla}\times\mathbf{v}_\perp$. Substituting relation (B.2) into (B.1) and neglecting terms of order $\delta_i^2 f_{Mi}v_i/a$ finally gives (56).

Appendix C. Gyrokinetic conservation of ion number

In this appendix, we show how to obtain (68) from (64) with $G = 1$. The integral $\int d^3v\,\langle C_{ii}\{f_i\}\rangle_g$ gives $-\nabla\cdot(n_i\mathbf{V}_{iC})$, as show in appendix D. The rest of the terms are almost trivial.

We have to comment on two terms in which we have neglected terms of order $\delta_i^2 n_e v_i/a$ as small compared to terms of order $(B/B_p)\delta_i^2 n_e v_i/a$. First we treat the ion polarization density, given by

$$n_{ip} = n_i - \int d^3v\, f_{ig} = \int d^3v\, \left(E_1 \frac{\partial F_i}{\partial E_0} + \mu_1 \frac{\partial F_i}{\partial \mu_0} \right), \tag{C.1}$$

where we have employed equation (54). In equation (C.1), several terms in μ_1 from (6) gyroaverage to zero, giving

$$n_{ip} \simeq \int d^3v\, \frac{Ze\widetilde{\phi}}{M} \left(\frac{\partial F_i}{\partial E_0} + \frac{1}{B}\frac{\partial F_i}{\partial \mu_0} \right) - \int d^3v\, \frac{v_{||} v_\perp^2}{2B\Omega_i} \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \frac{\partial F_i}{\partial \mu_0}. \tag{C.2}$$

The last term in (C.2) is also small. The integral over the neoclassical piece $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ would seem to be large enough to contribute, but n_{ip} only enters through its time derivative, and the time derivative can only be of order $\partial f_{i1}^{\rm nc}/\partial t \sim \mathbf{v}_M \cdot \overline{\nabla} f_{Mi} \sim \delta_i f_{Mi} v_i/a$, making the time derivative of the last term in (C.2) of order $\delta_i^2 n_e v_i/a$ and hence negligible.

The other term that needs some explanation is the parallel flow $\int d^3v\, f_{ig} (v_{||}^2/\Omega_i) \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}$ in $\nabla \cdot (\int d^3v\, f_{ig} \mathbf{v}_{\rm GK})$, with $\mathbf{v}_{\rm GK}$ from (61). We find that the divergence of this parallel flow is

$$\nabla \cdot \left(\int d^3v\, f_{ig} \frac{v_{||}^2}{\Omega_i} \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \right) \simeq \nabla \cdot \left(\int d^3v\, f_{Mi} \frac{v_{||}^2}{\Omega_i} \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \right), \tag{C.3}$$

where we have neglected the integrals over $f_{i1}^{\rm nc}$ and $f_{i1}^{\rm tb}$ because they are of order $\delta_i^2 n_e v_i/a$. For the integral over $f_{i1}^{\rm nc} \sim (B/B_p)\delta_i f_{Mi}$ it is important to realize that $\hat{\mathbf{b}} \cdot \nabla \sim 1/qR \sim (B_p/B)(1/a)$, giving $\nabla \cdot [\int d^3v\, f_{i1}^{\rm nc} (v_{||}^2/\Omega_i) \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}] \sim \delta_i^2 n_e v_i/a$. Since the integral in (C.3) is over the stationary Maxwellian f_{Mi} we have that $p_{i||} = p_i = p_{i\perp} \simeq p_{ig\perp}$. We choose to write it as $p_{ig\perp}$ so that it is similar to the more familiar form of the parallel drift [16].

Appendix D. Gyrokinetic collision operator

In this appendix we extend the work on the gyrokinetic collision operator presented in appendix D of [16]. The gyroaveraged collision operator is now calculated up to order $(B/B_p)(qR\nu_{ii}/v_i)\delta_i^2 f_{Mi} v_i/a$ in the $B_p/B \ll 1$ limit.

The ion-ion collision operator is

$$C_{ii}\{f_i\} = \gamma_{ii} \nabla_v \cdot \mathbf{\Gamma}_{ii}, \tag{D.1}$$

where $\gamma_{ii} = 2\pi Z^4 e^4 \ln\Lambda/M^2$ and $\mathbf{\Gamma}_{ii}$ is given by (75). In appendix D of [16] we used the usual expression for the divergence in a new reference system $\{y_j\}$,

$$\nabla_x \cdot \mathbf{\Gamma} = \frac{1}{J_y} \sum_j \frac{\partial}{\partial y_j} (J_y \mathbf{\Gamma} \cdot \nabla_x y_j) = \frac{1}{J_y} \sum_j \frac{\partial}{\partial y_j} (J_y \Gamma_{y_j}), \tag{D.2}$$

with ∇_x the gradient in the reference system $\{x_i\}$, $J_y = \partial(x_i)/\partial(y_j)$ the Jacobian of the transformation between $\{x_i\}$ and $\{y_j\}$, and $\Gamma_{y_j} = \mathbf{\Gamma} \cdot \nabla_x y_j$. Employing (D.2), we showed that the gyroaveraged of $C_{ii}\{f_i\}$ holding $\mathbf{R}$, E, μ and t fixed is

$$\langle C_{ii}\{f_i\}\rangle = \frac{\gamma_{ii}}{J}\left[\frac{\partial}{\partial E}\left(J\langle \mathbf{\Gamma}_{ii}\cdot\nabla_v E\rangle\right) + \frac{\partial}{\partial \mu}\left(J\langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mu\rangle\right) + \nabla_{\mathbf{R}}\cdot\left(J\langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mathbf{R}\rangle\right)\right], \quad \text{(D.3)}$$

where $J = \partial(\mathbf{r},\mathbf{v})/\partial(\mathbf{R},E,\mu,\varphi)$ is the Jacobian of the gyrokinetic transformation. To write (D.3) in "physical" phase space, we use rule (D.2) to transform divergences from one reference system $\{y_j\}$ to another $\{z_k\}$, given by

$$\frac{1}{J_y}\sum_j \frac{\partial}{\partial y_j}\left(J_y \Gamma_{y_j}\right) = \frac{1}{J_z}\sum_k \frac{\partial}{\partial z_k}\left(J_z \sum_j \Gamma_{y_j}\frac{\partial z_k}{\partial y_j}\right), \quad \text{(D.4)}$$

with $J_z = \partial(x_i)/\partial(z_k)$ the Jacobian of the transformation between $\{x_i\}$ and $\{z_k\}$. Using this relation, equation (D.3) becomes

$$\langle C_{ii}\{f_i\}\rangle = \gamma_{ii}\frac{v_{||}}{B}\left[\frac{\partial}{\partial E_0}\left(\frac{B}{v_{||}}\Gamma_{E_0}\right) + \frac{\partial}{\partial \mu_0}\left(\frac{B}{v_{||}}\Gamma_{\mu_0}\right) + \frac{\partial}{\partial \varphi_0}\left(\frac{B}{v_{||}}\Gamma_{\varphi_0}\right) + \overline{\nabla}\cdot\left(\frac{B}{v_{||}}\mathbf{\Gamma}_{\mathbf{r}}\right)\right]. \text{(D.5)}$$

where $B/v_{||} = \partial(\mathbf{v})/\partial(E_0,\mu_0,\varphi_0)$,

$$\Gamma_{E_0} = \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v E\rangle\frac{\partial E_0}{\partial E} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mu\rangle\frac{\partial E_0}{\partial \mu} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mathbf{R}\rangle\cdot\nabla_{\mathbf{R}}E_0, \quad \text{(D.6)}$$

$$\Gamma_{\mu_0} = \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v E\rangle\frac{\partial \mu_0}{\partial E} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mu\rangle\frac{\partial \mu_0}{\partial \mu} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mathbf{R}\rangle\cdot\nabla_{\mathbf{R}}\mu_0, \quad \text{(D.7)}$$

$$\Gamma_{\varphi_0} = \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v E\rangle\frac{\partial \varphi_0}{\partial E} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mu\rangle\frac{\partial \varphi_0}{\partial \mu} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mathbf{R}\rangle\cdot\nabla_{\mathbf{R}}\varphi_0. \quad \text{(D.8)}$$

and

$$\mathbf{\Gamma}_{\mathbf{r}} = \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v E\rangle\frac{\partial \mathbf{r}}{\partial E} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mu\rangle\frac{\partial \mathbf{r}}{\partial \mu} + \langle \mathbf{\Gamma}_{ii}\cdot\nabla_v \mathbf{R}\rangle\cdot\nabla_{\mathbf{R}}\mathbf{r}. \quad \text{(D.9)}$$

To zeroth order, we can use $E \simeq E_0$, $\mu \simeq \mu_0$, $\varphi \simeq \varphi_0$ and $\mathbf{R} \simeq \mathbf{r} + \Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}}$ to find that the functions (D.6), (D.7), (D.8) and (D.9) are

$$\Gamma_{E_0} \simeq \langle \mathbf{\Gamma}_{ii}\cdot\mathbf{v}\rangle, \quad \text{(D.10)}$$

$$\Gamma_{\mu_0} \simeq \frac{1}{B}\langle \mathbf{\Gamma}_{ii}\cdot\mathbf{v}_\perp\rangle, \quad \text{(D.11)}$$

$$\Gamma_{\varphi_0} \simeq 0. \quad \text{(D.12)}$$

and

$$\mathbf{\Gamma}_{\mathbf{r}} \simeq -\frac{1}{v_\perp^2\Omega_i}(\mathbf{v}\times\hat{\mathbf{b}})\langle \mathbf{\Gamma}_{ii}\cdot\mathbf{v}_\perp\rangle + \frac{1}{\Omega_i}\langle \mathbf{\Gamma}_{ii}\rangle\times\hat{\mathbf{b}}, \quad \text{(D.13)}$$

where to obtain (D.13) we have used $\nabla_v\mathbf{R} \simeq \nabla_v\mathbf{R}_1 = \Omega_i^{-1}\overleftrightarrow{\mathbf{I}}\times\hat{\mathbf{b}}$ and $\partial\mathbf{r}/\partial\mu \simeq -\partial\mathbf{R}_1/\partial\mu_0 = -(2\mu_0\Omega_i)^{-1}\mathbf{v}\times\hat{\mathbf{b}}$. Notice that we have kept the higher order correction $\mathbf{R}_1$ only in (D.13) because the large perpendicular gradients make this small correction important.

The zeroth order functions (D.10), (D.11), (D.12) and (D.13) are of the order of the largest term in (D.5), given by $\langle C_{ii}^{(\ell)}\{f_{i1}^{\rm nc}\}\rangle \sim (qR\nu_{ii}/v_i)\delta_i f_{Mi} v_i/a$. The next order corrections to this term have two origins. On the one hand, we must keep higher order corrections to f_i in $\mathbf{\Gamma}_{ii}$ in (75), finding linear and nonlinear terms like $\langle C_{ii}^{(\ell)}\{f_{i2}^{\rm nc}\}\rangle$ or $\langle C_{ii}^{(n\ell)}\{f_{i1}^{\rm nc}, f_{i1}^{\rm nc}\}\rangle$, both of order $(B/B_p)(qR\nu_{ii}/v_i)\delta_i^2 f_{Mi} v_i/a$. On the other hand, we must consider the higher order corrections to the gyrokinetic variables $\mathbf{R} = \mathbf{r} + \Omega_i^{-1}\mathbf{v}\times\hat{\mathbf{b}} + \ldots$, $E = E_0 + \ldots$, $\mu = \mu_0 + \ldots$ and $\varphi = \varphi_0 + \ldots$. According to (D.6), (D.7), (D.8) and (D.9), these corrections give contributions of order $(qR\nu_{ii}/v_i)\delta_i^2 f_{Mi} v_i/a$, and hence, are negligible for us. Then, we can use the lower order expressions (D.10), (D.11), (D.12) and (D.13), but inside $\mathbf{\Gamma}_{ii}$ we must keep the higher order corrections to the distribution function.

In the main text there are two integrals of the gyroaveraged collision operator, namely $\nabla\cdot(n_i\mathbf{V}_{iC}) = -\int d^3v\,\langle C_{ii}\{f_i\}\rangle_g$, given in (74), and $\mathbf{F}_{iC\perp} = M\int d^3v\,\mathbf{v}_\perp\langle C_{ii}\{f_i\}\rangle_g$, given in (80). To obtain the final expressions in equations (74) and (80), we use (D.5) and the lower order results (D.10), (D.11), (D.12) and (D.13). In addition, in equation (80) we employ $\partial\mathbf{v}_\perp/\partial E_0 = 0$ and $\partial\mathbf{v}_\perp/\partial\mu_0 = (B/v_\perp^2)\mathbf{v}_\perp$, and we neglect $M\gamma_{ii}\int d^3v\,\mathbf{\Gamma}_{\mathbf{r}}\cdot\overline{\nabla}\mathbf{v}_\perp \sim (qR\nu_{ii}/v_i)\delta_i^2 p_i/a \ll (B/B_p)(qR\nu_{ii}/v_i)\delta_i^2 p_i/a$. Notice that in $\nabla\cdot(n_i\mathbf{V}_{iC}) = -\int d^3v\,\langle C_{ii}\{f_i\}\rangle_g$ and $\mathbf{F}_{iC\perp} = M\int d^3v\,\mathbf{v}_\perp\langle C_{ii}\{f_i\}\rangle_g$, we have neglected the difference between $\langle C_{ii}\{f_i\}\rangle$ and $\langle C_{ii}\{f_i\}\rangle_g$, where the subindex g indicates that the gyrokinetic variables $\mathbf{R}$, E and μ have been replaced by $\mathbf{R}_g$, E_0 and μ_0. Since we have shown that to order $(B/B_p)(qR\nu_{ii}/v_i)\delta_i^2 f_{Mi} v_i/a$ the difference due to replacing $\mathbf{R}$, E and μ by $\mathbf{R}_g$, E_0 and μ_0 is negligible, we can safely use $\langle C_{ii}\{f_i\}\rangle_g \simeq \langle C_{ii}\{f_i\}\rangle$.

Appendix E. Gyrokinetic conservation of ion momentum

In this appendix we show how to obtain equation (76) from (64) with $G = M\mathbf{v}_\perp$. We only keep terms up to order $(B/B_p)\delta_i^2 p_i/a$. The term $\mathbf{F}_{iC\perp}$ is obtained from $M\int d^3v\,\mathbf{v}_\perp\langle C_{ii}\{f_i\}\rangle_g$ as show in appendix D. Equation (64) becomes

$$\frac{\partial}{\partial t}(n_i M\mathbf{V}_{ig\perp}) + \nabla\cdot\overleftrightarrow{\boldsymbol{\pi}}_{ig\times} = M\int d^3v\, f_{ig}(v_{||}\hat{\mathbf{b}} + \mathbf{v}_{M0} + \mathbf{v}_{E0} + \tilde{\mathbf{v}}_1)\cdot\overline{\nabla}\mathbf{v}_\perp, \tag{E.1}$$

where the integrals $\nabla\cdot[M\int d^3v\, f_{ig}(v_{||}^2/\Omega_i)(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})\hat{\mathbf{b}}\mathbf{v}_\perp]$ and $M\int d^3v\, f_{ig}(v_{||}^2/\Omega_i)(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})\hat{\mathbf{b}}\cdot\overline{\nabla}\mathbf{v}_\perp$ have been neglected because the integrals over the gyrophase independent piece $F_i = f_{Mi} + f_{i1}^{\rm nc}$ vanish, leaving only the contribution of $f_{i1}^{\rm tb}$, of order $\delta_i^2 p_i/a$ and hence negligible. To obtain (76) from (E.1), we use the definitions $\mathbf{F}_{iB\perp}^{\rm tb} = M\int d^3v\, f_{ig}(v_{||}\hat{\mathbf{b}} + \mathbf{v}_{E0})\cdot\overline{\nabla}\mathbf{v}_\perp$ and $\mathbf{F}_{iB\perp}^{\rm nc} = M\int d^3v\, f_{ig}(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)\cdot\overline{\nabla}\mathbf{v}_\perp$. The integral $\mathbf{F}_{iB\perp}^{\rm nc} = M\int d^3v\, f_{ig}(\mathbf{v}_{M0} + \tilde{\mathbf{v}}_1)\cdot\overline{\nabla}\mathbf{v}_\perp$ gives the result in (78) because only the neoclassical piece of the distribution function $f_{i1}^{\rm nc}$ is large enough to be important. Since $f_{i1}^{\rm nc}$ is gyrophase independent to the requisite order, the integral in velocity space of $F_i\mathbf{v}_{M0}\cdot\overline{\nabla}\mathbf{v}_\perp$ vanishes and $\mathbf{F}_{iB\perp}^{\rm nc} \simeq M\int d^3v\, F_i\tilde{\mathbf{v}}_1\cdot\overline{\nabla}\mathbf{v}_\perp$.

Appendix F. Gyrokinetic vorticity equation (85)

In this appendix we show how to obtain (85) by adding $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times (\text{equation (76)})]$ to equation (81). This operation gives

$$\frac{\partial \varpi_G}{\partial t} = \nabla \cdot \left[J_{g||}\hat{\mathbf{b}} + \mathbf{J}_{gd} + \tilde{\mathbf{J}}_p + Zen_i\tilde{\mathbf{V}}_i + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times}) - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}^{\mathrm{nc}}_{iB\perp} - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}^{\mathrm{tb}}_{iB\perp} + Zen_i\mathbf{V}_{iC} - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}_{iC} \right], \tag{F.1}$$

In this equation we will only keep terms up to order $(B/B_p)\delta_i^2 en_e v_i/a$. The term $\nabla \cdot [(c/B)\hat{\mathbf{b}} \times \mathbf{F}^{\mathrm{nc}}_{iB\perp}]$ is of order $(B/B_p)\delta_i^3 en_e v_i/a$ because the function $\mathbf{F}^{\mathrm{nc}}_{iB\perp}$ from (78) is slowly varying in space and its gradient is of order $1/a$.

To simplify equation (F.1) we employ the same procedure as in appendix F of [16]. We combine $\tilde{\mathbf{J}}_p$, $Zen_i\tilde{\mathbf{V}}_i$, $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times})$ and $(c/B)\hat{\mathbf{b}} \times \mathbf{F}^{\mathrm{tb}}_{iB\perp}$ to obtain $\tilde{\mathbf{J}}_\phi$, $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG})$ and a term with vanishing divergence. First, we rewrite in the perpendicular component of $Zen_i\tilde{\mathbf{V}}_i$ in a convenient form. Using that $(\overline{\nabla} \times \mathbf{v}_\perp) \times \hat{\mathbf{b}} = \hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp - \overline{\nabla}\mathbf{v}_\perp \cdot \hat{\mathbf{b}}$ and $\overline{\nabla}\mathbf{v}_\perp \cdot \hat{\mathbf{b}} = -\nabla\hat{\mathbf{b}} \cdot \mathbf{v}_\perp = -\mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} - \mathbf{v}_\perp \times (\nabla \times \hat{\mathbf{b}})$, the perpendicular component of $\tilde{\mathbf{v}}_1$ from (59) is written as

$$\tilde{\mathbf{v}}_{1\perp} = \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \times [(\overline{\nabla} \times \mathbf{v}_\perp) \times \hat{\mathbf{b}}] = \frac{v_{||}}{\Omega_i}\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp + \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}}) + \frac{v_{||}}{\Omega_i}\mathbf{v}_\perp(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}), \tag{F.2}$$

where we have used $\hat{\mathbf{b}} \times [\mathbf{v}_\perp \times (\nabla \times \hat{\mathbf{b}})] = \mathbf{v}_\perp(\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}})$. Integrating in velocity space gives the perpendicular component of $Zen_i\tilde{\mathbf{V}}_i$,

$$Zen_i\tilde{\mathbf{V}}_{i\perp} = \frac{Mc}{B}\hat{\mathbf{b}} \times \left(\int d^3v\, f_{ig} v_{||} \hat{\mathbf{b}} \cdot \overline{\nabla}\mathbf{v}_\perp + \int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} \right) + \frac{Mc}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp. \tag{F.3}$$

One of the terms in this expression cancels one of the terms in $(c/B)\hat{\mathbf{b}} \times \mathbf{F}^{\mathrm{tb}}_{iB\perp}$, with $\mathbf{F}^{\mathrm{tb}}_{iB\perp}$ given by (79). Then, we obtain

$$Zen_i\tilde{\mathbf{V}}_{i\perp} - \frac{c}{B}\hat{\mathbf{b}} \times \mathbf{F}^{\mathrm{tb}}_{iB\perp} = \frac{Mc}{B}\hat{\mathbf{b}} \times \left(\int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}} - \int d^3v\, F_i \mathbf{v}_{E0} \cdot \overline{\nabla}\mathbf{v}_\perp \right) + \frac{Mc}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp. \tag{F.4}$$

The integral $(Mc/B)\hat{\mathbf{b}} \times (\int d^3v\, F_i \mathbf{v}_{E0} \cdot \overline{\nabla}\mathbf{v}_\perp)$ is included in the definition of $\tilde{\mathbf{J}}_\phi$, given by (87). Moreover, we can write $(Mc/B)\hat{\mathbf{b}} \times (\int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp \cdot \nabla\hat{\mathbf{b}})$ as $(c/B)\hat{\mathbf{b}} \times \nabla \cdot (M \int d^3v\, f_{ig} \mathbf{v}_\perp v_{||} \hat{\mathbf{b}})$, where $M \int d^3v\, f_{ig} \mathbf{v}_\perp v_{||} \hat{\mathbf{b}}$ is part of the definition of $\overleftrightarrow{\boldsymbol{\pi}}_{iG}$ in (88). Considering this, we add $\tilde{\mathbf{J}}_p$ and $(c/B)\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times})$ to equation (F.4) to obtain

$$Zen_i\tilde{\mathbf{V}}_{i\perp} + \tilde{\mathbf{J}}_p + \frac{c}{B}\hat{\mathbf{b}} \times \left(\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{ig\times} - \mathbf{F}^{\mathrm{tb}}_{iB\perp} \right) = \tilde{\mathbf{J}}_\phi + \frac{c}{B}\hat{\mathbf{b}} \times (\nabla \cdot \overleftrightarrow{\boldsymbol{\pi}}_{iG}) + \frac{Mc}{B}\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp. \tag{F.5}$$

Substituting this expression into (F.1) we recover (85). It is important to realize that the divergence of the flows $Zen_i\tilde{V}_{i||}\hat{\mathbf{b}}$ and $(Mc/B)\hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}} \int d^3v\, f_{ig} v_{||} \mathbf{v}_\perp$ vanish to the order of interest. The divergence of $Zen_i\tilde{V}_{i||}\hat{\mathbf{b}}$ is negligible because the integral over the gyrophase independent piece $F_i = f_{Mi} + f_{i1}^{\rm nc}$ vanishes and only $f_{i1}^{\rm tb}$ contributes, giving $\nabla\cdot(Zen_i\tilde{V}_{i||}\hat{\mathbf{b}}) \sim \delta_i^2 en_e v_i/a \ll (B/B_p)\delta_i^2 en_e v_i/a$. The divergence of $(Mc/B)\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp$ is given by

$$\nabla\cdot\left(\frac{Mc}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp\right) = \frac{Mc}{B}\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}\int d^3v\, f_{ig}v_{||}\mathbf{v}_\perp\cdot\overline{\nabla}f_{ig} + \frac{Mc}{B}\int d^3v\, f_{ig}v_{||}\overline{\nabla}\cdot[\mathbf{v}_\perp(\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}})]. \qquad \text{(F.6)}$$

The only contribution that could give a term of order $(B/B_p)\delta_i^2 en_e v_i/a$ in the second term of (F.6) is the integral over the neoclassical piece $f_{i1}^{\rm nc}$, but this integral vanishes because $f_{i1}^{\rm nc}$ is gyrophase independent. The first integral in (F.6) vanishes because the only gyrophase dependence of f_{ig} is through $\mathbf{R}_g$, giving $\mathbf{v}_\perp\cdot\overline{\nabla}f_{ig} = \mathbf{v}_\perp\cdot\nabla_{\mathbf{R}_g}f_{ig} + O(\delta_i^2 f_{Mi}) = \Omega_i(\partial f_{ig}/\partial\varphi_0) + O(\delta_i^2 f_{Mi})$. Using this form for $\mathbf{v}_\perp\cdot\overline{\nabla}f_{ig}$, it is obvious that the first term in (F.6) vanishes to the order of interest due to the integration over gyrophase.

References

[1] Hinton F L and Hazeltine R D 1976 *Rev. Mod. Phys.* **48** 239
[2] Helander P and Sigmar D J 2002 *Collisional Transport in Magnetized Plasmas* (*Cambridge Monographs on Plasma Physics*) ed Haines M G *et al* (Cambridge, UK: Cambridge University Press)
[3] Catto P J and Simakov A N 2005 *Phys. Plasmas* **12** 012501
[4] Catto P J and Simakov A N 2005 *Phys. Plasmas* **12** 114503
Catto P J and Simakov A N 2005 *Phys. Plasmas* **13** 129901
[5] Wong S K and Chan V S 2007 *Phys. Plasmas* **14** 112505
[6] Catto P J and Simakov A N 2009 *Phys. Plasmas* **16** 044509
[7] Rosenbluth M N, Rutherford P H, Taylor J B, Frieman E A and Kovrizhnikh L M 1971 *Plasma Phys. Controlled Nucl. Fusion Res.* **1** 495
[8] Wong S K and Chan V S 2005 *Physi. Plasmas* **12** 092513
Wong S K and Chan V S 2009 *Phys. Plasmas* **16** 089901
[9] Wong S K and Chan V S 2007 *Phys. Plasmas* **14** 122501
[10] Hinton F L and Wong S K 1985 *Phys. Fluids* **28** 3082
[11] Connor J W, Cowley S C, Hastie R J and Pan L R 1987 *Plasma Phys. Control. Fusion* **29** 919
[12] Catto P J, Bernstein I B and Tessarotto M 1987 *Phys. Fluids* **30** 2784
[13] Newton S L, Helander P and Catto P J 2007 *Phys. Plasmas* **14** 062301
[14] Greenwald M *et al* 2005 *Nucl. Fusion* **45** S109
[15] deGrassie J S, Rice J E, Burrell K H, Groebner R J and Solomon W M 2007 *Phys. Plasmas* **14** 056115
[16] Parra F I and Catto P J 2009 *Plasma Phys. Control. Fusion* **51** 095008
[17] Simakov A N and Catto P J 2007 *Plasma Phys. Control. Fusion* **49** 729
[18] Hazeltine R D 1973 *Plasma Phys.* **15** 77
[19] Parra F I and Catto P J 2008 *Plasma Phys. Control. Fusion* **50** 065014
[20] Catto P J 1978 *Plasma Physics* **20** 719
[21] Dorland W, Jenko F, Kotschenreuther M and Rogers B N 2000 *Phys. Rev. Lett.* **85** 5579
[22] Dannert T and Jenko F 2005 *Phys. Plasmas* **12** 072309
[23] Candy J 2005 *Phys. Plasmas* **12** 072307

[24] Chen Y and Parker S E 2003 *J. Comput. Phys.* **189** 463
[25] Lee W W 1983 *Phys. Fluids* **26** 556
[26] Dubin D H E, Krommes J A, Oberman C and Lee W W 1983 *Phys. Fluids* **26** 3524
[27] Parra F I and Catto P J 2009 *Plasma Phys. Control. Fusion* **51** 065002
[28] Rosenbluth M N and Hinton F L 1998 *Phys. Rev. Lett.* **80** 724
[29] Hinton F L and Rosenbluth M N 1999 *Plasma Phys. Control. Fusion* **41** A653
[30] Catto P J, Simakov A N, Parra F I and Kagan G 2008 *Plasma Phys. Control. Fusion* **50** 115006
[31] Peeters A G and Angioni C 2005 *Phys. Plasmas* **12** 072515
[32] Waltz R E, Staebler G M, Candy J and Hinton F L 2007 *Phys. Plasmas* **14** 122507