Turing Progressions and their well-orders

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Abstract. We see how Turing progressions are closely related to the closed fragment of GLP, polymodal provability logic. Next we study natural well-orders in GLP that characterize certain aspects of these Turing progressions.

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1 Turing progressions and modal logic

Gödel's Second Incompleteness Theorem tells us that any sound recursive theory that is strong enough to code syntax will not prove its own consistency. Thus, adding $\mathsf{Con}(T)$ to such a theory T will yield a strictly stronger theory. Turing took up this idea in his seminal paper [6] to consider recursive ordinal progressions of some recursive sound base theory T:

$$\begin{array}{ll} T_0 & := T; \\ T_{\alpha+1} & := T_\alpha + \mathsf{Con}(T_\alpha); \\ T_\lambda & := \bigcup_{\alpha<\lambda} T_\alpha \qquad \text{for limit } \lambda. \end{array}$$

As we shall see, poly-modal provability logics turn out to be suitably well equipped to talk about Turing progressions. These logics have modalities $[n]$ that shall be interpreted as "provable in EA using all true Π_n sentences" abbreviated $[n]_{\mathrm{EA}}$. By EA we denote *Elementary Arithmetic* which is a formal arithmetic theory axiomatized by the recursive equations for successor, addition and multiplication, by open induction together with an axiom stating the totality of exponentiation. Often we shall not distinguish a modal formula from its arithmetical interpretation.

We recall ([5]) that the provability logic of any Σ_1-sound theory extending EA is Gödel Löb's provability logic **GL** as defined below. Various mathematical statements can be expressed within **GL** like Gödel's Second Incompleteness Theorem: $\Diamond\top \to \Diamond\Box\top$. It is also not hard to see that finite Turing progressions are definable in **GL** as T_n is provably equivalent to $T + \Diamond_T^n\top$. Transfinite progressions are not expressible in the modal language with just one modal operator. However, using stronger provability predicates provides a way out (see [3]):

Proposition 1. *$T + \langle n+1\rangle_T\top$ is a Π_{n+1} conservative extension of $T + \{\langle n\rangle_T^k\top \mid k \in \omega\}$.*

The provability behavior of these mixed modalities is fully described by what we call GLP_ω. We give a more general definition GLP_Λ for any ordinal Λ.

Definition 1. *For Λ an ordinal, the logic GLP_Λ is the propositional normal modal logic that has for each $\alpha < \Lambda$ a modality $[\alpha]$ and is axiomatized by the following schemata:*

$$\begin{array}{ll} [\alpha](\chi \to \psi) \to ([\alpha]\chi \to [\alpha]\psi) & \\ [\alpha]([\alpha]\chi \to \chi) \to [\alpha]\chi & \\ \langle\alpha\rangle\psi \to [\beta]\langle\alpha\rangle\psi & \textit{for } \alpha < \beta, \\ [\alpha]\psi \to [\beta]\psi & \textit{for } \alpha \leq \beta. \end{array}$$

The rules of inference are Modus Ponens and necessitation for each modality: $\frac{\psi}{[\alpha]\psi}$. By GLP we denote the class-size logic that has a modality $[\alpha]$ for each ordinal α and all the corresponding axioms and rules. **GL** *refers to GLP_1.*

As suggested by Proposition 1, for the sake of Turing progressions a particular interest lies in GLP^0_Λ, the closed fragment – that is, the modal formulas that have no propositional variables but rather just $\bot$ and $\top$– of these GLP_Λ. We shall call iterated consistency statements in this closed fragments *worms* in reference to the heroic worm-battle, a variant of the Hydra battle (see [1]).

Definition 2 (Worms, S, S_α). *By S we denote the set of* worms *of GLP which is inductively defined as $\top \in S$ and $A \in S \Rightarrow \langle\alpha\rangle A \in S$. Similarly, we inductively define for each ordinal α the set of worms S_α where all ordinals are at least α as $\top \in S_\alpha$ and $A \in S_\alpha \wedge \beta \geq \alpha \Rightarrow \langle\beta\rangle A \in S$.*

We shall denote worms by roman uppercase letters like $A, B, \ldots$. The next easy lemma ([2], [4]) is the basis of most of our reasoning and we shall use it often in the remainder of this paper without explicit mention.

Lemma 1.
1. $\mathsf{GLP} \vdash AB \to A$
2. *For a closed formula ϕ and a worm B, if $\beta < \alpha$, then* $\mathsf{GLP} \vdash (\langle\alpha\rangle\phi \wedge \langle\beta\rangle B) \leftrightarrow \langle\alpha\rangle(\phi \wedge \langle\beta\rangle B)$;
3. *For a closed formula ϕ and a worm B, if $\beta < \alpha$, then* $\mathsf{GLP} \vdash (\langle\alpha\rangle\varphi \wedge [\beta]B) \leftrightarrow \langle\alpha\rangle(\varphi \wedge [\beta]B)$;
4. *If $A \in S_{\alpha+1}$, then* $\mathsf{GLP} \vdash A \wedge \langle\alpha\rangle B \leftrightarrow A\alpha B$;
5. *If $A, B \in S_\alpha$ and $\mathsf{GLP} \vdash A \leftrightarrow B$, then* $\mathsf{GLP} \vdash A\alpha C \leftrightarrow B\alpha C$.

Worms can be conceived as the backbone of GLP^0. It is known that each closed formula of GLP is equivalent a Boolean combination of worms. Moreover, a decision procedure for GLP^0 proceeds via a reduction to worms. Also, there are various important generalizations of Proposition 1 in terms of worms. In particular ([3]):

Proposition 2. *For each ordinal $\alpha < \epsilon_0$ there is some GLP_ω-worm A such that $T + A$ is Π_1 equivalent to T_α.*

To get generalizations of this lemma beyond ϵ_0 one should consider more than ω modalities. In particular $[\alpha]$ should be read as "provable in EA together with all true hyperarithmetical sentences of level α". This paper focusses on the modal calculus involved in such generalizations.

2 Omega sequences

We define an order $<_\alpha$ on S_α by $A <_\alpha B :\Leftrightarrow \mathsf{GLP} \vdash B \to \langle\alpha\rangle A$. In [2] and [4] it is shown that $<_\alpha$ defines a well-order on S_α modulo provably equivalence. We can consider the ordering $<_\alpha$ also on the full class S. As we shall see $<_\alpha$ is no longer linear on S. However, we shall see that it is still well-founded. Anticipating this, we can define $\Omega_\alpha(A)$, the generalized $<_\alpha$ order-type of a worm A.

Definition 3. *Given an ordinal ξ and a worm A, we define a new ordinal $\Omega_\xi(A)$ by induction on $<_\xi$ by $\Omega_\xi(A) = \sup_{B<_\xi A} \Omega_\xi(B)$. Likewise, we define $o_\xi(A) = \sup\{o_\xi(B) \mid B \in S_\xi \wedge B <_\xi A\}$.*

With this, we can assign to each worm A a sequence of order-types $\boldsymbol{\Omega}(A)$ for the sequence $(\Omega_\xi(A))_{\xi\in\mathsf{On}}$.

To link back to the Turing progressions we mention here that the $\Omega_0(A)$ corresponds to the α from Proposition 2 and the $\Omega_\xi(A)$ for $\xi > 0$ correspond to further generalizations of Proposition 2.

In further sections we shall show how to calculate $\Omega_\xi(A)$ for given ξ and A. In the current section we shall see how questions about Ω_ξ can be recursively reduced to questions about o_ζ. For this reduction we need the syntactical definitions of *head* and *remainder*.

Definition 4. *Let A be a word. By $h_\xi(A)$ we denote the ξ-*head *of A. Recursively: $h_\xi(\epsilon) = \epsilon$, and $h_\xi(\zeta \star A) = \zeta \star h_\xi(A)$ if $\zeta \geq \xi$ and $h_\xi(\zeta \star A) = \epsilon$ if $\zeta < \xi$. Likewise, by $r_\xi(A)$ we denote the ξ-*remainder *of A: $r_\xi(\epsilon) = \epsilon$, and $r_\xi(\zeta \star A) = r_\xi(A)$ if $\zeta \geq \xi$ and $r_\xi(\zeta \star A) = \zeta \star A$ if $\zeta < \xi$.*

We obviously have $A = h_\xi(A) \star r_\xi(A)$ for all ξ and A and, clearly, over GLP, $h_\xi(A) \star r_\xi(A)$ and $h_\xi(A) \wedge r_\xi(A)$ are equivalent as the first symbol of $r_\xi(A)$ is less than ξ and $h_\xi(A) \in S_\xi$ (see Lemma 1).

Lemma 2. $(A \to \langle\xi\rangle\, B) \Leftrightarrow [(h_\xi(A) \to \langle\xi\rangle\, h_\xi(B)) \wedge (A \to r_\xi(B))]$.

Proof. "$\Rightarrow$" Clearly, $B \leftrightarrow h_\xi(B) \wedge r_\xi(B)$ whence $A \to r_\xi(B)$. Likewise $A \leftrightarrow h_\xi(A) \wedge r_\xi(A)$. As $h_\xi(A), h_\xi(B) \in S_\xi$ we know that either $h_\xi(A) = h_\xi(B)$, $h_\xi(B) \to \langle\xi\rangle\, h_\xi(A)$, or $h_\xi(A) \to \langle\xi\rangle\, h_\xi(B)$. By assumption $A \to \langle\xi\rangle\, B$ whence $A \to \langle\xi\rangle\, h_\xi(B) \wedge r_\xi(B)$.

Suppose now $h_\xi(A) = h_\xi(B)$. Then, $h_\xi(A) \wedge r_\xi(A) \to \langle\xi\rangle\, h_\xi(A) \wedge r_\xi(A)$ whence also

$$h_\xi(A) \wedge r_\xi(A) \to \langle\xi\rangle\, (h_\xi(A) \wedge r_\xi(A)).$$

The latter is equivalent to $A \to \langle\xi\rangle\, A$ which contradicts the ireflexivity of $<_\xi$.

Likewise, the assumption that $h_\xi(B) \to \langle\xi\rangle\, h_\xi(A)$ contradicts the reflexivity of $<_\xi$ and we conclude that $h_\xi(A) \to \langle\xi\rangle\, h_\xi(B)$.

"$\Leftarrow$" This is the easier direction.

$$\begin{aligned} A &\leftrightarrow h_\xi(A) \wedge r_\xi(A) \\ &\to \langle\xi\rangle\, h_\xi(B) \wedge r_\xi(B) \\ &\to \langle\xi\rangle\, (h_\xi(B) \wedge r_\xi(B)) \\ &\to \langle\xi\rangle\, B. \end{aligned}$$

Note that this lemma indeed recursively reduces the general $<_\xi$ question between to words, to the $<_\xi$ questions between words in S_ξ. Note that $<_\xi$ is not tree-like. For example, we see that both $011 <_1 10111 <_1 1111$ and $011 <_1 11011 <_1 1111$ while 10111 and 11011 are $<_1$ incomparable.

3 Basic properties of Omega sequences

In these final sections we give a full characterization of the sequences $\boldsymbol{\Omega}(A)$. That is, we shall determine for given A each of the values $\Omega_\xi(A)$ and shall classify at what coordinates ξ the $\boldsymbol{\Omega}(A)$ sequence changes value. Clearly, $\boldsymbol{\Omega}(A)$ defines a weakly decreasing sequence of ordinals.

Lemma 3. *For $\xi < \zeta$ we have that $\Omega_\xi(A) \geq \Omega_\zeta(A)$.*

Proof. In general we have for $\xi < \zeta$ that $A \to \langle\zeta\rangle B$ implies $A \to \langle\xi\rangle B$. Thus, any $<_\zeta$ sequence is automatically also a $<_\xi$ sequence.

Lemma 4. $\Omega_\xi(A) = o_\xi(h_\xi(A))$

Proof. Suppose $\alpha_0 <_\xi \alpha_1 <_\xi \ldots <_\xi \alpha$, then

$$h_\xi(\alpha_0) <_\xi h_\xi(\alpha_1) <_\xi \ldots <_\xi h_\xi(A)$$

by Lemma 2 whence $\Omega_\xi(h_\xi(A)) \leq o_\xi(A)$.

On the other hand, if $B <_\xi h_\xi(A)$, then $h_\xi(A) \to \langle\xi\rangle B$. But as $A \leftrightarrow h_\xi(A) \wedge r_\xi(A)$ we also have $A \to \langle\xi\rangle B$. Consequently $o_\xi(A) \leq \Omega_\xi(h_\xi(A))$.

Corollary 1. *The $\boldsymbol{\Omega}(A)$ sequence has a maximal non-zero coordinate. In particular, the maximal non-zero coordinate is given by $\Omega_{\mathsf{First}(A)}(A)$, where $\mathsf{First}(A)$ is the first element of the word A.*

Proof. For $A \in S$, we denote by $\mathsf{First}(A)$ the first element of A, that is, $\mathsf{First}(\epsilon) = \epsilon$, and $\mathsf{First}(\xi{\star}B) = \xi$. Clearly, $h_{\mathsf{First(A)}}(A) \neq \epsilon$ whence by Lemma 4, $\Omega_{\mathsf{First}(A)}(A) \neq 0$. On the other hand, for $\xi > \mathsf{First}(A)$, clearly $h_\xi(A) = \epsilon$ whence $\Omega_\xi(A) = 0$.

It is good to have reduced $\Omega_\xi(A)$ to $o_\xi(A)$ as in [2], and [4] a full calculus for it is given. Let $e_0\alpha := -1 + \omega^\alpha$ and let e_α denote the Veblen progression based on e_0. That is, each e_α enumerates the ordinals which are simultaneous fixed points of all the e_β for $\beta < \alpha$. We define $e^0\alpha := \alpha$ and $e^{\omega^\xi + \zeta} = e_\xi \circ e^\zeta$ whenever $\zeta < \omega^\xi + \zeta$. Further, we define $\xi{\uparrow}\zeta := \xi + \zeta$ and for $\xi < \zeta$ we define $\xi{\downarrow}\zeta$ to be the unique ordinal such that $\xi{\uparrow}(\xi{\downarrow}\zeta) = \zeta$. These operations are naturally extended to worms by simultaneously applying them to all occurrences of ordinals.

Lemma 5.

1. $o(0^n) = n$;
2. If $A = A_1 0 \dots 0 A_n$, then $o(A) = \omega^{o_1(A_n)} + \dots + \omega^{o_1(A_1)}$;
3. For $A \in S_\xi$ we have $o_\xi(A) = o(\xi \downarrow A)$;
4. $o(\xi \uparrow A) = e^\xi o(A)$.

4 Successor coordinates

First, let us compute $\Omega_{\xi+1}(A)$ in terms of $\Omega_\xi(A)$. By $\ell\alpha$ we denote the unique β such that $\alpha = \alpha' + \omega^\beta$ for $\alpha > 0$. We define $\ell 0 := 0$.

Lemma 6. *Given an ordinal* ξ *and a worm* A,

$$o_{\xi+1} h_{\xi+1} h_\xi(A) = \ell o_\xi h_\xi(A).$$

Proof. We write $h_\xi(A)$ as $A_0 \xi \dots \xi A_n$. Clearly, $h_{\xi+1}(h_\xi(A)) = A_0$. We shall now see that $\ell(o_\xi(h_\xi(A))) = o_{\xi+1}(A_0)$.

To this end, we observe that

$$\begin{aligned} o_\xi h_\xi(A) &= o_\xi(A_0 \xi \dots \xi A_n) \\ &= o\Big((\xi{\downarrow}A_0)0 \dots 0(\xi{\downarrow}A_n)\Big) \\ &= \omega^{o_1(\xi\downarrow A_n)} + \dots + \omega^{o_1(\xi\downarrow A_0)} \\ &= \omega^{o_{\xi+1}(A_n)} + \dots + \omega^{o_{\xi+1}(A_0)} \end{aligned}$$

Consequently $\ell o_\xi h_\xi(A) = o_{\xi+1}(A_0)$, as desired.

Now we are ready to describe the relation between successor coordinates of the $\boldsymbol{\Omega}(A)$ sequence.

Theorem 1. $\Omega_{\xi+1}(A) = \ell\Omega_\xi(A)$

Proof.

$$\begin{aligned} \Omega_{\xi+1}(A) &= o_{\xi+1} h_{\xi+1}(A) && \text{by Lemma 2} \\ &= o_{\xi+1} h_{\xi+1} h_\xi(A) && \text{by Lemma 6} \\ &= \ell o_\xi h_\xi(A) \\ &= \ell \Omega_\xi(A) && \text{by Lemma 2.} \end{aligned}$$

Theorem 1 tells us what the relation between successor coordinates of $\boldsymbol{\Omega}(A)$ is. However, it does not directly tell us when successor coordinates are different. If $\Omega_\xi(A)$ is a fixed point of $\zeta \mapsto \omega^\zeta$ then $\Omega_\xi(A) = \Omega_{\xi+1}(A)$.

5 Equal coordinates

Theorem 2 below gives us a characterization of when different coordinates attain different or equal values. Before we can state and prove this theorem we need some notation and back-ground on Cantor Normal Forms (CNFs).

For $\alpha \in \mathsf{On}$ we define N_α and the syntactic operation $\mathsf{CNF}(\alpha) := \sum_{i=1}^{N_\alpha} \omega^{\xi_i}$ to be the unique CNF expression of α. Next, we define for an ordinal α the set of its *Cantor Normal Form Approximation* as the set of partial sums of $\mathsf{CNF}(\alpha)$, that is, if

$$\mathsf{CNF}(\alpha) = \sum_{i=1}^{N_\alpha} \omega^{\xi_i},$$

then

$$\mathsf{CNA}(\alpha) \ := \ \left\{ \sum_{i=1}^{k} \omega^{\xi_i} : 0 \le k \le N_\alpha \right\}.$$

We also define the *Cantor Normal Form Projection* of some ordinal ζ on another ordinal ξ as follows:

$$\mathsf{CNP}(\zeta, \xi) \ \ := \ \ \max\{\xi' {\in} \mathsf{CNA}(\xi) \mid \xi' \le \zeta\}.$$

Note that $\mathsf{CNP}(\zeta, \xi)$ is defined for all $\zeta, \xi \in \mathsf{On}$ by setting $\max \varnothing = 0$.

For $\alpha, \beta, \gamma \in \mathsf{On}$ we define

$$\alpha \sim_\gamma \beta \ \ :\Leftrightarrow \ \ \mathsf{CNA}(\alpha, \gamma) = \mathsf{CNA}(\beta, \gamma).$$

In words, $\alpha \sim_\gamma \beta$ whenever there is no partial sum of the CNF of γ that falls in between α and β.

The just-defined notions of $\mathsf{CNA}(\xi)$, $\mathsf{CNP}(\zeta, \xi)$ and $\alpha \sim_\gamma \beta$ are needed to characterize the $\xi{\downarrow}\zeta$ operation.

Lemma 7.

1. $\forall \zeta {\le} \xi \ \zeta{\downarrow}\xi = \mathsf{CNP}(\zeta, \xi){\downarrow}\xi$;
2. $\forall \zeta {\le} \xi \exists ! \eta {\in} \mathsf{CNA}(\xi) \ \zeta{\downarrow}\xi = \eta{\downarrow}\xi$;
3. *For* $\xi, \zeta \le \eta$, *we have*
$$\xi{\downarrow}\eta = \zeta{\downarrow}\eta \ \Leftrightarrow \ \xi \sim_\eta \zeta.$$

Proof. (1.) We consider $\zeta \le \xi$. Now let η be shorthand for $\max\{\eta' \in \mathsf{CNA}(\xi) \mid \eta' \le \zeta\} = \mathsf{CNP}(\zeta, \xi)$. The claim is that $\zeta{\downarrow}\xi = \eta{\downarrow}\xi$. Let $\mathsf{CNF}(\xi) = \sum_{i=1}^{N_\xi} \omega^{\xi_i}$.

As $\eta = \sum_{i=1}^{k} \omega^{\xi_i}$ for some $k \le N_\xi$, we see that

$$\eta{\downarrow}\xi = \sum_{i=k+1}^{N_\xi} \omega^{\xi_i}$$

for $k < N_\xi$ and $\eta{\downarrow}\xi = 0$ for $k = N_\xi$. We now claim that $\zeta + (\eta{\downarrow}\xi) = \xi$ so that $\zeta{\downarrow}\xi = \eta{\downarrow}\xi$ follows from the fact that

$$\forall \, \zeta {<} \xi \, \exists ! \delta \ \zeta + \delta = \xi.$$

We may assume $\zeta > \eta$ otherwise $\zeta + (\eta{\downarrow}\xi) = \xi$ is trivial.

Thus,

$$\eta = \sum_{i=1}^{k} \omega^{\xi_i} < \zeta \leq \sum_{i=1}^{k+1} \omega^{\xi_i}.$$

As by the definition of η we see that $\zeta \leq \sum_{i=1}^{k+1} \omega^{\xi_i}$ cannot be an equality whence

$$\eta = \sum_{i=1}^{k} \omega^{\xi_i} < \zeta < \sum_{i=1}^{k+1} \omega^{\xi_i}.$$

Thus, $\eta \in CNA(\zeta)$ and $\zeta + \sum_{i=k+1}^{N_\xi} \omega^{\xi_i} = \xi$, whence

$$\sum_{i=k+1}^{N_\xi} \omega^{\xi_i} = \zeta{\downarrow}\xi = \sum_{i=1}^{k} \omega^{\xi_i} = \eta{\downarrow}\xi.$$

(2.) Follows from (1.) once we realize that for different η and η' both in $\mathsf{CNA}(\xi)$ we have $\eta{\downarrow}\xi \neq \eta'{\downarrow}\xi$.

(3). From the proof of (1.) we see that

$$\xi{\downarrow}\eta = \zeta{\downarrow}\eta \;\Leftrightarrow\; \max\{\eta'{\in}\mathsf{CNA}(\eta) \mid \eta' \leq \xi\} = \max\{\eta'{\in}\mathsf{CNA}(\eta) \mid \eta' \leq \zeta\}$$

where the latter is precisely the definition of $\xi \sim_\eta \zeta$.

Once we have this lemma to characterize the $\xi{\downarrow}\zeta$ operation, we are armed to prove a characterization for when two coordinates in $\boldsymbol{\Omega}(A)$ are equal. But first we need a definition of when a worm A is in Beklemishev Normal Form (BNF). Recursively we say that the empty worm $\epsilon \in$ BNF, and if $A_i \in S_{\xi+1}\cap$BNF, with $A_i \geq_{\xi+1} A_{i+1}$, then $A_n\alpha \dots \alpha A_1 \in$BNF. It is easy to see that if a worm is in BNF, then so are its head and remainder. From [2], and [4] we know that the set $S\cap$BNF is well-ordered by $<_0$ and that o_0 provides an isomorphism between $\langle S, <_0\rangle$ and $\langle \mathsf{On}, <\rangle$.

Theorem 2. *For $A \in$BNF, the following five conditions are equivalent.*

1. $\Omega_\xi(A) = \Omega_\zeta(A)$
2. $o_\xi h_\xi(A) = o_\zeta h_\zeta(A)$
3. $\xi{\downarrow}h_\xi(A) = \zeta{\downarrow}h_\zeta(A)$
4. $h_\xi(A) = h_\zeta(A)$ *and* $\xi{\downarrow}h_\xi(A) = \zeta{\downarrow}h_\zeta(A)$
5. $h_\xi(A) = h_\zeta(A)$ *and* $\forall\eta \in h_\xi(A),\ \xi \sim_\eta \zeta$

Proof. (1.) $\Leftrightarrow$ (2.) is just Lemma 4.

(2.) $\Leftrightarrow$ (3.): Observe that $o_\xi(h_\xi(A)) = o(\xi{\downarrow}h_\xi(A))$ and $o_\zeta(h_\zeta(A)) = o(\zeta{\downarrow}h_\zeta(A))$. As o defines an isomorphism between S and On, we obtain

$$o_\xi h_\xi(A) = o_\zeta h_\zeta(A) \;\Leftrightarrow\; \xi{\downarrow}h_\xi(A) = \zeta{\downarrow}h_\zeta(A).$$

(3.) $\Leftrightarrow$ (4.): Suppose $h_\xi(A) \neq h_\zeta(A)$. W.l.o.g. we may assume that $\zeta < \xi$ whence

$$\mathsf{Length}(h_\xi(A)) < \mathsf{Length}(h_\zeta(A))$$

and also

$$\mathsf{Length}(h_\xi(A)) = \mathsf{Length}(\xi{\downarrow}h_\xi(A)) < \mathsf{Length}(\zeta{\downarrow}h_\zeta(A)) = \mathsf{Length}(h_\zeta(A)).$$

As $A \in$ BNF, also $h_\xi(A)$ and $h_\zeta(A)$ are in BNF, whence also $\xi{\downarrow}h_\xi(A)$, $\zeta{\downarrow}h_\zeta(A)$ are in BNF. We know that normal forms are graphically unique so that $\xi{\downarrow}h_\xi(A) \neq \zeta{\downarrow}h_\zeta(A)$ whence $o(\xi{\downarrow}h_\xi(A)) \neq o(\zeta{\downarrow}h_\zeta(A))$.

(4.) $\Leftrightarrow$ (5.):

$$\begin{array}{ll} h_\xi(A) = h_\zeta(A) \wedge \xi{\downarrow}h_\xi(A) = \zeta{\downarrow}h_\zeta(A) & \Leftrightarrow \\ h_\xi(A) = h_\zeta(A) \wedge \forall\, \eta{\in}h_\xi(A)\ \xi{\downarrow}\eta = \zeta{\downarrow}\eta & \Leftrightarrow \text{ by Lemma 7.3} \\ h_\xi(A) = h_\zeta(A) \wedge \forall\, \eta{\in}h_\xi(A)\ \xi \sim_\eta \zeta & \end{array}$$

6 Limit coordinates

The results so far have already provided us with quite some insight about what the sequences $\boldsymbol{\Omega}(A)$ look like. By Lemma 3 we know that the set of values that occur in $\boldsymbol{\Omega}(A)$ is finite. Moreover, by Theorem 1 we know exactly the values at successor coordinates. In particular, we know that if the value of $\boldsymbol{\Omega}(A)$ at ξ is the same as at the successor coordinate, then it remains the same for all further successors.

The question remains what happens at limit ordinals coordinates. In this section we shall determine at what limit ordinals a new value can be attained and how the new value relates to previous values. Let us start out the analysis by formulating a negative version of Theorem 2.

Lemma 8. *For $\xi < \zeta$ we have that*

$$\Omega_\xi(A) > \Omega_\zeta(A) \;\Leftrightarrow\; (\exists\, \eta{\in}h_\xi(A)\ \xi{\leq}\eta{<}\zeta) \;\vee\; (\exists\, \eta{\in}h_\xi(A)\ \mathsf{CNP}(\xi,\eta){<}\mathsf{CNP}(\zeta,\eta)).$$

Proof. By contraposing equivalence (1.) $\Leftrightarrow$ (5.) of Theorem 2 we get

$$\Omega_\zeta(A) \neq \Omega_\xi(A) \;\Leftrightarrow\; h_\xi(A){\neq}h_\zeta(A) \;\vee\; \exists\, \eta{\in}h_\zeta(A)\ \xi \not\sim_\eta \zeta.$$

But, as $\zeta < \xi$ we see

$$h_\xi(A){\neq}h_\zeta(A) \;\Leftrightarrow\; \exists\, \eta{\in}h_\zeta(A)\ \zeta \leq \eta < \xi.$$

Likewise,

$$\exists\, \eta{\in}h_\zeta(A)\ \xi \not\sim_\eta \zeta \;\Leftrightarrow\; \exists\, \eta{\in}h_\zeta(A)\ \mathsf{CNP}(\zeta,\eta){\neq}\mathsf{CNP}(\xi,\eta).$$

As $\zeta < \xi$ we have

$$\mathsf{CNP}(\zeta,\eta){\neq}\mathsf{CNP}(\xi,\eta) \;\Leftrightarrow\; \mathsf{CNP}(\zeta,\eta){<}\mathsf{CNP}(\xi,\eta).$$

The first question to ask is at which limit coordinates the sequence $\boldsymbol{\Omega}(A)$ can change. Let us first write precisely what it means for the sequence $\boldsymbol{\Omega}(A)$ to change at some coordinate ζ. We express this by the formula

$$\mathsf{Change}(\zeta, A) \;\;:=\;\; \exists\, \xi{<}\zeta\ (\Omega_\xi(A){>}\Omega_\zeta(A) \;\wedge\; \forall\eta\ (\xi{\leq}\eta{<}\zeta \rightarrow \Omega_\xi(A){=}\Omega_\zeta(A))).$$

The next lemma gives an alternative characterization of $\mathsf{Change}(\zeta, A)$.

Lemma 9. $\mathsf{Change}(\zeta, A) \quad \Leftrightarrow \quad \forall\, \xi{<}\zeta \; \Omega_\xi(A){>}\Omega_\zeta(A)$

Proof. For $\zeta \in \mathsf{Succ}$ this is clear. If $\zeta \in \mathsf{Lim}$, then $\{\Omega_\xi(A) \mid \xi < \zeta\}$ is a finite set as all the $\Omega_\xi(A) \in \mathsf{On}$ and these are weakly decreasing. Thus, at some point below ζ the sequence must stabilize.

We can now characterize at what limit ordinals the sequence $\boldsymbol{\Omega}(A)$ can change.

Theorem 3. *For* $\zeta \in \mathsf{Lim}$*:* $\quad \mathsf{Change}(\zeta, A) \quad \Leftrightarrow \quad \exists\, \xi{\in}h_\zeta(A) \; \zeta{\in}\mathsf{CNA}(\xi)$

Proof. For $\zeta \in \mathsf{Lim}$ we reason:

$$\begin{array}{c}
\mathsf{Change}(\zeta, A) \qquad\qquad \Leftrightarrow \text{By Lemma 9} \\
\forall\, \xi{<}\zeta \; \Omega_\xi(A){>}\Omega_\zeta(A) \Leftrightarrow \text{By Lemma 8} \\
\forall\, \xi{<}\zeta \; (\exists\, \eta{\in}h_\xi(A) \; \xi{\leq}\eta < \zeta \;\vee\; \exists\, \eta{\in}h_\xi(A) \; \mathsf{CNP}(\xi,\eta){<}\mathsf{CNP}(\zeta,\eta)) \Leftrightarrow \\
\forall \xi \; (\xi_0{<}\xi{<}\zeta \to \exists\, \eta{\in}h_\zeta(A) \; \mathsf{CNP}(\xi,\eta){<}\mathsf{CNP}(\zeta,\eta))
\end{array}$$

where $\xi_0 \; := \; \sup\{\xi' \in A \mid \xi' < \zeta\}$. Note that for these ξ, indeed, we have $h_\xi(A) = h_\zeta(A)$. We now claim that the latter is equivalent to $\exists\, \eta{\in}h_\zeta(A) \; \zeta{\in}\mathsf{CNA}(\eta)$. Clearly, if $\zeta \in \mathsf{CNA}(\eta)$ for some $\eta \in h_\zeta(A)$, then $\xi{\downarrow}\eta < \zeta{\downarrow}\eta$ for each $\xi < \zeta$.

For the converse direction, suppose $\zeta \notin \mathsf{CNA}(\eta)$ for all $\eta \in h_\zeta(A)$. Then, for all ξ' with

$$\sup\{\xi \mid \exists\, \eta{\in}h_\zeta(A) \; (\xi{\in}\mathsf{CNA}(\eta) \;\wedge\; \xi < \zeta)\} < \xi' < \zeta$$

we have $\xi' \sim_\eta \zeta$ for all $z \in h_\zeta(A)$, whence by Theorem 2 $\Omega_{\xi'}(A) = \Omega_\zeta(A)$.

Now that we have fully determined at which limit coordinates a change can occur the only thing left to establish is the size of the change. In other words, if $\mathsf{Change}(\zeta, A)$ for some $\zeta \in \mathsf{Lim}$, how does $\Omega_\zeta(A)$ relate to $\Omega_\xi(A)$ for $\xi < \zeta$. In order to answer this question, we need a generalization of Lemma 5.4.

Lemma 10. $o_\xi(\zeta{\uparrow}A) = e^\zeta o_\xi(A)$ *for* $A \in S_\xi$*.*

Proof. We claim that for $B \in S_\xi$ we have that $\xi{\downarrow}(\zeta{\uparrow}B) = \zeta{\uparrow}(\xi{\downarrow}B)$. From this claim the statement follows easily from Lemma 5.4.

$$o_\xi(\zeta{\uparrow}A) = o(\xi{\downarrow}(\zeta{\uparrow}A)) = o(\zeta{\uparrow}(\xi{\downarrow}A)) = e^\zeta o(\xi{\downarrow}A) = e^\zeta o_\xi(A).$$

Thus, we only need to prove our claim. Clearly, it suffices to show the claim for any ordinal $\eta \geq \xi$ instead of for any word in S_ξ. By definition, $\xi{\downarrow}(\zeta{\uparrow}z){=}\delta \;\Leftrightarrow\; \xi + \delta = \eta + \zeta$. Likewise, $\xi{\downarrow}\eta = \delta' \;\Leftrightarrow\; \xi + \delta' = \eta$. As $\zeta{\uparrow}(\xi{\downarrow}\eta) = \xi{\downarrow}\eta + \zeta = \delta' + \zeta$ we obtain $\xi + \delta' = \eta \;\Rightarrow\; (\xi + \delta') + \zeta = \eta + \zeta$ and by associativity of ordinal addition also $\xi + (\delta' + \zeta) = \eta + \zeta$. We conclude that $\delta' + \zeta = \delta$ which translates exactly to $\zeta{\uparrow}(\xi{\downarrow}\eta) = \xi{\downarrow}(\zeta{\uparrow}\eta)$, quod erat demostrandum.

With this technical lemma at hand we are ready to prove the concluding theorem of this section.

Theorem 4. *Let* $\zeta \in \mathsf{Lim}$*, and let* $\xi < \zeta$ *be such that, whenever* $\xi' \in [\xi, \zeta)$*, it follows that* $\Omega_\xi(A) = \Omega_{\xi'}(A)$*. Then,*

$$\Omega_\xi(A) = e^{-\xi+\zeta}\Omega_\zeta(A) = e_{\ell\zeta}\Omega_\zeta(A).$$

Proof. As the values of $\Omega_{\xi'}(A)$ do not change for $\xi \leq \xi' < \zeta$ we know in particular by Theorem 2 that

$$h_\xi(A) \;=\; h_\zeta(A). \tag{1}$$

Likewise by Theorem 2 we see that $-\xi + \zeta = \omega^{\ell\zeta}$. Let $\delta = -\xi + \zeta$.

Then,

$$\Omega_\zeta(A) = o_\zeta h_\zeta(A) = o_{\xi+(\xi\downarrow\zeta)}(h_\zeta(A)) = o_{\xi+\delta}(h_\zeta(A)) = o_\xi(\delta\downarrow h_\zeta(A)). \tag{2}$$

Thus,

$$\begin{array}{ll}
\Omega_\xi(A) & = \\
o_\xi(h_\xi(A)) & = \text{ By (1)} \\
o_\xi(h_\zeta(A)) & = \\
o_\xi(\delta\uparrow(\delta\downarrow h_\zeta(A))) & = \text{By Lemma 10} \\
e^\delta o_\xi(\delta\downarrow h_\zeta(A)) & = \text{By (2)} \\
e^\delta \Omega_\zeta(A) & = \text{By definition of } e^\alpha \\
e_{\ell\zeta}\Omega_\zeta(A). &
\end{array}$$

Note that this theorem establishes the size of limit coordinates both in case a change does occur and in case no change occurs. The latter case can only be so when $\Omega_\zeta(A)$ is a fixed point of $e_{\ell\zeta}$.

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