

TUTTE AND JONES POLYNOMIALS OF LINK FAMILIES

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Abstract

This article contains general formulas for the Tutte and Jones polynomials of families of knots and links given in Conway notation and the corresponding plots of zeroes for the Jones polynomials.

1 Introduction

Knots and links (or shortly *KLs*) will be given in Conway notation [1, 2, 3, 4].

Definition 1. For a link or knot L given in an unreduced¹ Conway notation $C(L)$ denote by S a set of numbers in the Conway symbol excluding numbers denoting basic polyhedron and zeros (determining the position of tangles in the vertices of polyhedron) and let $\tilde{S} = \{a_1, a_2, \dots, a_k\}$ be a non-empty subset of S . Family $F_{\tilde{S}}(L)$ of knots or links derived from L consists of all knots or links L' whose Conway symbol is obtained by substituting all $a_i \neq \pm 1$, by $\text{sgn}(a_i)|a_i + k_{a_i}|$, $|a_i + k_{a_i}| > 1$, $k_{a_i} \in \mathbb{Z}$. [4].

An infinite subset of a family is called a *subfamily*. If all k_{a_i} are even integers, the number of components is preserved within the corresponding subfamilies, i.e., adding full-twists preserves the number of components inside the subfamilies.

Definition 2. A link given by Conway symbol containing only tangles ± 1 and ± 2 is called a *source link*.

*Project 144032D: CL5 supported by the Ministry of Science and Technological Development of the Republic Serbia.

†Research is supported by the Ministry of Science and Technological Development of the Republic Serbia - Grant No. TR 12010.

2010 *Mathematics Subject Classifications*. 05C31, 57M25, 57M27.

Key words and Phrases. Conway notation; Tutte polynomial; Jones polynomial; link family.

Received: January 14, 2010

Communicated by Dragan Stevanović

¹The Conway notation is called unreduced if in symbols of polyhedral links elementary tangles 1 in single vertices are not omitted.

A *graph* is defined as a pair (V, E) , where V is the *vertex set* and $E \subseteq V \times V$ the *edge set*. We consider only *undirected* graphs, meaning (x, y) is the same as (y, x) for $x, y \in V$ and $(x, y) \in E$. A *loop* is an edge (x, x) between the same vertex, and a *bridge* is an edge whose removal disconnects two or more vertices (i.e. there is no longer a path between them) [5, 8, 11].

Two operations are essential to understanding the Tutte polynomial definition. These are: *edge deletion* denoted by $G - e$, and *edge contraction* G/e . The latter involves first deleting e , and then merging its endpoints as follows [5]:

Definition 3. *The Tutte polynomial of a graph $G(V, E)$ is a two-variable polynomial defined as follows:*

$$T(G) = \begin{cases} 1 & E(\emptyset) \\ xT(G/e) & e \in E \text{ and } e \text{ is a bridge} \\ yT(G - e) & e \in E \text{ and } e \text{ is a loop} \\ T(G - e) + T(G/e) & e \in E \text{ and } e \text{ is not a loop or a bridge} \end{cases}$$

The definition of a Tutte polynomial outlines a simple recursive procedure for computing it, but the order in which rules are applied is not fixed.

According to Thistlethwaite's Theorem, the Jones polynomial of an alternating link, up to a factor, can be obtained from the Tutte polynomial by substitutions: $x \rightarrow -x$ and $y \rightarrow -\frac{1}{x}$ [6, 7, 8, 9]. Moreover, from general formulas for the Tutte polynomials with negative values of parameters we obtain Tutte polynomials expressed as Laurent polynomials. Using the substitutions above, we obtain, up to a factor, Jones polynomials of non-alternating links.

A *cut-vertex* (or articulation vertex) of a connected graph is a vertex whose removal disconnects the graph [10]. In general, a cut-vertex is a vertex of a graph whose removal increases the number of components [11]. A *block* is a maximal biconnected subgraph of a given graph.

One-point union or *block sum* of two (disjoint) graphs G_1 and G_2 , neither of which is a vertex graph, and which we shall denote as $G_1 * G_2$ is of particular interest. This one-point union is such that the intersection of G_1 and G_2 can only consist of a vertex [12].

Decomposition of a graph G into a finite number of blocks G_1, \dots, G_n , denoted by

$$G = G_1 * G_2 * \dots * G_n$$

is called the *block sum* of G_1, \dots, G_n . The following formula holds for the Tutte polynomial of the block sum:

$$T(G_1 * G_2 * \dots * G_n) = T(G_1)T(G_2) \dots T(G_n).$$

A dual graph \overline{G} of a given planar graph G is a graph which has a vertex for each plane region of G , and an edge for each edge in G joining two neighboring regions, for a certain embedding of G . The Tutte polynomial of \overline{G} can be obtained from $T(G)$ by replacements $x \rightarrow y$, $y \rightarrow x$, i.e. $T(\overline{G})(x, y) = T(G)(y, x)$.

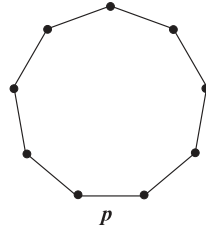
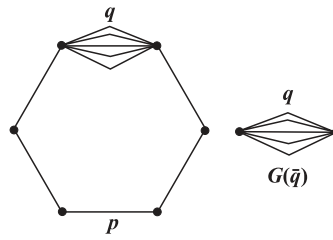
There is a nice bijective correspondence between KL s and graphs: to obtain a graph from a projection of KL , first color every other region of the KL diagram black or white, so that the infinite outermost region is black. In the *checker-board coloring* (or *Tait coloring*) of the plane obtained, put a vertex at the center of each white region. Two vertices of a graph are connected by an edge if there was a crossing between corresponding regions in a KL diagram. In addition, to each edge of a graph we can assign the sign of its corresponding vertex of the KL diagram. Family of graphs corresponding to a family of link diagrams L will be denoted by $G(L)$.

Tutte polynomials were known for the following special families of graphs corresponding to the knots and links: circuit graphs C_p which correspond to the link family p , graphs corresponding to the link family $p2$, wheel graphs $Wh(n+1)$ corresponding to the class of antiprismatic basic polyhedra $(2n)^*$ (6^* , 8^* , 10^* , ...), so-called "hammock" graphs corresponding to the pretzel links $2, 2, \dots, 2$ where tangle 2 occurs n times ($n \geq 3$), and graphs corresponding to the links of the form $(2n)^* : 20 : 20 \dots : 20$, where tangle 2 occurs n times ($6^*20 : 20 : 20$, $8^*20 : 20 : 20 : 20$, ...) [9]. Recursive formulas for the computation of Tutte polynomials corresponding to the link families 313 , 31213 , 3121213 , 312121213 , ..., 212 , 21212 , 2121212 , 212121212 , ..., and the family of polyhedral links $.2 : 2$, $9^*2 : 2$, ... are derived by F. Emmert-Streib [13]. Moreover, general formulas for the Jones polynomial are known for some other particular classes of knots and links, e.g., torus knots, or repeating chain knot with $3n$ crossings, proposed by L. Kauffman, given by Conway symbols of the form $3, 211, 2$; $(3, 2)1, 211, 2$; $((3, 2)1, 2)1, 211, 2$; $((((3, 2)1, 2)1, 2)1, 2)1, 211, 2$; $(((((3, 2)1, 2)1, 2)1, 2)1, 2)1, 211, 2$; ... [14]. Zeros of the Jones polynomial are analyzed by X-S. Lin [15], S. Chang and R. Shrock [9], F. Wu and J. Wang [14], and X. Jin and F. Zhang [16]. Experimental observations of many authors who have studied the distribution of roots of Jones polynomials for various families of knots and links are explained by A. Champanarekar and I. Kofman [17]. Numerous knot invariants can be considered in the context of knot and link families (see papers [18, 19, 20] and the book [4]). In particular, we describe the behavior of Jones and Tutte polynomials for families of KL s and their associated graphs with explicit formulas and "portraits of families".

2 Tutte polynomials of KL families

2.1 Family p

The first family we consider is the family p ($p \geq 1$), which consists of the knots and links $1_1, 2_1^2, 3_1, 4_1^2, 5_1, \dots$ [9]. Graphs corresponding to links of this family are cycles of length p , which we can denote by $G(p)$. By deleting one edge $G(p)$ gives the chain of edges of the length $p-1$ with the Tutte polynomial x^{p-1} , and by contraction it gives $G(p-1)$. Hence, $T(G(p)) - T(G(p-1)) = x^{p-1}$, and $T(G(1)) = y$, so the general formula for the Tutte polynomial of the graph $G(p)$ is

Figure 1: Cycle graph $G(p)$ ($p = 9$).Figure 2: Graphs $G(pq)$ ($p = 5, q = 5$) and $G(\bar{q})$.

$$T(G(p)) = \frac{x^p - 1}{x - 1} + y - 1.$$

2.2 Family pq

The link family pq gives the family of graphs, shown on Fig. 2, satisfying the relations

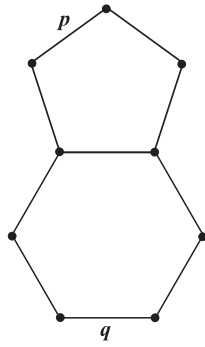
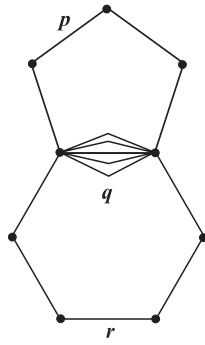
$$T(G(pq)) - T(G((p-1)q)) = x^{p-1}T(G(\bar{q})),$$

where $G(\bar{q})$ is the dual of the graph $G(q)$. Since the Tutte polynomial of the graph $G(0q)$ is $T(G(0q)) = y^q$, the general formula for the Tutte polynomial of the graphs $G(pq)$ is

$$T(G(pq)) = \frac{(x^p - 1)(y^q - 1)}{(x - 1)(y - 1)} + x^p + y^q - 1.$$

2.3 Family $p1q$

The family of graphs (Fig. 3) corresponds to the link family $p1q$. Since

Figure 3: Graph $G(p1q)$ ($p = 4$, $q = 5$).Figure 4: Graph $G(pqr)$ ($p = 4$, $q = 5$, $r = 5$).

$$T(G(p1q)) - T(G(p1(q-1))) = x^{q-1}T(G(p+1)),$$

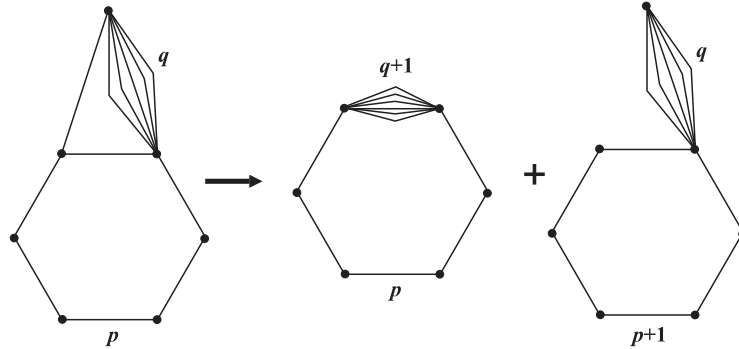
the general formula for the Tutte polynomial of the graphs $G(p1q)$ is

$$T(G(p1q)) = \frac{x(x^p - 1)(x^q - 1)}{(x - 1)^2} + \frac{(x^p + x^q + xy - x - y - 1)y}{(x - 1)}.$$

2.4 Family pqr

The family of graphs (Fig. 4) corresponds to the family of rational links pqr and their Tutte polynomials satisfy the following relations

$$T(G(pqr)) - T(G((p-1)qr)) = x^{p-1}T(G(rq)).$$

Figure 5: Graph $G(p 1 1 q)$ ($p = 5, q = 5$).

The general formula for the Tutte polynomial of the graphs $G(p q r)$ is

$$T(G(p q r)) = \frac{(x+y)(x^p-1)(x^r-1)}{(x-1)^2} + \frac{y^q(x^{r+1}+x^p-x-1)}{(x-1)} \\ + \frac{(x^p-1)(x^r-1)(y^q-y^2)}{(x-1)^2(y-1)} - (x^r-y)y^q.$$

2.5 Family $p 1 1 q$

The graph of the link family $p 1 1 q$, see Fig. 5, resolves into the graph $G(p(q+1))$ and the block sum of the graphs $G(p+1)$ and $G(\bar{q})$. The general formula for the Tutte polynomial of the graphs $G(p 1 1 q)$ is

$$T(G(p 1 1 q)) = T(G(p(q+1))) + \left(\frac{x^{p+1}-1}{x-1} + y - 1\right) \left(\frac{y^q-1}{y-1} + x - 1\right).$$

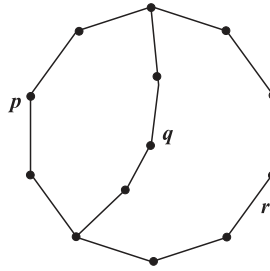
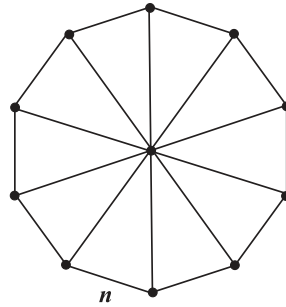
2.6 Family p, q, r

The graph family from Fig. 6 corresponds to the family of pretzel links p, q, r , whose Tutte polynomials satisfy the relations

$$T(G(p, q, r)) - T(G(p-1, q, r)) = x^{p-1}T(G(q+r))$$

The general formula for the Tutte polynomial of the graphs $G(p, q, r)$ isⁱⁱ

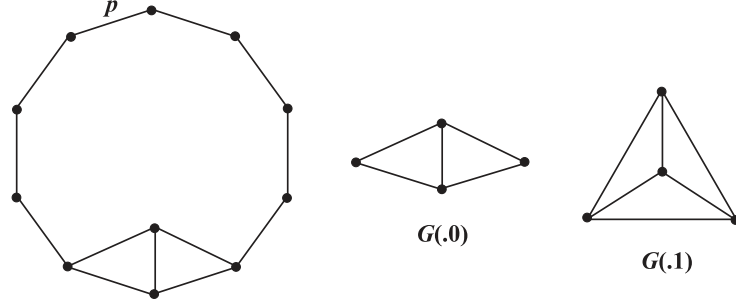
ⁱⁱIn this paper we derived the general formula for Tutte polynomial of pretzel links with three parameters, which can be easily generalized to general formulas for pretzel links with an arbitrary number of parameters, by using the relation $T(G(p_1, \dots, p_n)) = \frac{x^{p_n}-1}{x-1}T(G(p_1), \dots, G(p_{n-1})) + T(G(p_1)) \dots T(G(p_{n-1}))$ ($n \geq 3$).

Figure 6: Graph $G(p, q, r)$ ($p = 4, q = 4, r = 6$).Figure 7: Wheel graph $Wh(n+1)$ ($n = 10$).

$$T(G(p, q, r)) = \frac{x^{p+q+r} + (x^{p+1} + x^{q+1} + x^{r+1})(y-1) - (x^p + x^q + x^r)y}{(x-1)^2} + \frac{(xy - x - y)(xy - x - y - 1)}{(x-1)^2}.$$

2.7 Antiprismatic basic polyhedra $(2n)^*$

Basic polyhedron 6^* is the first member of the class of antiprismatic basic polyhedra $(2n)^*$ ($n \geq 3$): $6^*, 8^*, 10^*, \dots$. The corresponding graphs are wheel graphs (Fig. 7) denoted by $Wh(n+1)$. Their Tutte polynomials are given by the general formula [9]:

Figure 8: Graphs $G(.p)$ ($p = 8$).

$$T(G((2n)^*)) = T(Wh(n+1)) = \left[\frac{1}{2}[(1+x+y) + [(1+x+y)^2 - 4xy]^{1/2}] \right]^n +$$

$$\left[\frac{1}{2}[(1+x+y) - [(1+x+y)^2 - 4xy]^{1/2}] \right]^n + xy - x - y - 1.$$

2.8 Family . p

The graphs from Fig. 8. correspond to the link family $.p$. The Tutte polynomials of this graph family satisfy the relations

$$T(G(.p)) - T(G(.p-1)) = x^{p-1}T(G(.0)),$$

$T(G(.0)) = x + 2x^2 + x^3 + y + 2xy + y^2$ and $T(G(.1)) = 2x + 3x^2 + x^3 + 2y + 4xy + 3y^2 + y^3$. The general formula for the Tutte polynomial of the graphs $G(.p)$ is

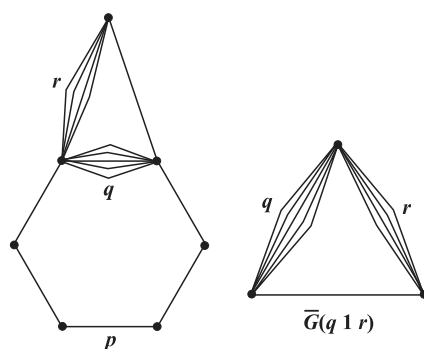
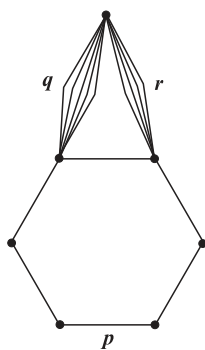
$$T(G(.p)) = \frac{x^p(2x^3 + 2x^2 + y^2 + 2xy + y) - 2x^2 - 2x - y^2 - y - 2xy}{x-1}$$

$$- x^{p+1}(x+1) + y + 2xy + 2y^2 + y^3.$$

2.9 Family $p q 1 r$

The graphs illustrated in Fig. 9 correspond to the link family $p q 1 r$. In order to obtain formula for the Tutte polynomial we use the relations

$$T(G(p q 1 r)) - T(G((p-1) q 1 r)) = x^{p-1}T(\overline{G}(q 1 r)),$$

Figure 9: Graphs $G(pq1r)$ ($p = 5, q = 5, r = 4$) and $\overline{G}(q1r)$.Figure 10: Graph $G(p1,q,r)$ ($p = 5, q = 5, r = 4$).

where $\overline{G}(q1r)$ denotes the graph from Fig. 9. Since its Tutte polynomial is

$$T(\overline{G}(q1r)) = \left(\frac{y^{r+1}-1}{y-1} + x - 1\right) \frac{y^q - 1}{y-1} + x \left(\frac{y^r - 1}{y-1} + x - 1\right),$$

for the Tutte polynomial of the graphs $G(pq1r)$ we obtain the general formula

$$T(G(pq1r)) = \left(\left(\frac{y^{r+1}-1}{y-1} + x - 1\right) \frac{y^q - 1}{y-1} + x \left(\frac{y^r - 1}{y-1} + x - 1\right)\right) \frac{x^p - 1}{x-1} + y^q \left(\frac{y^{r+1}-1}{y-1} + x - 1\right).$$

2.10 Family $p1,q,r$

The graphs on Fig. 10 correspond to the link family $p1,q,r$. In order to obtain general formula for the Tutte polynomial we use the relations

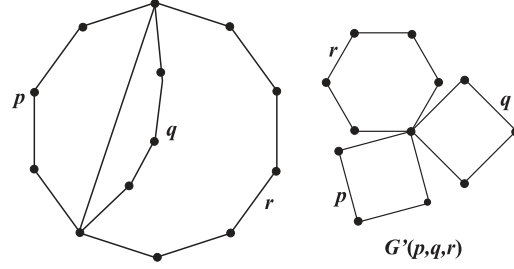


Figure 11: Graphs $G(p, q, r+)$ ($p = 4, q = 4, r = 6$) and $G'(p, q, r)$.

$$T(G(p1, q, r)) - T(G((p-1)1, q, r)) = x^{p-1}T(\overline{G}(q1r)),$$

where $\overline{G}(q1r)$ denotes the graph from Fig. 9, which now occupies a different position with regards to the chain of edges p . The general formula for the Tutte polynomial of the graphs $G(p1, q, r)$ is

$$T(G(p1, qr)) = \left(\left(\frac{y^{r+1} - 1}{y - 1} + x - 1\right)\frac{y^q - 1}{y - 1} + x\left(\frac{y^r - 1}{y - 1} + x - 1\right)\right)\frac{x^p - 1}{x - 1} + y\left(\frac{y^{q+r} - 1}{y - 1} + x - 1\right).$$

2.11 Family $p, q, r+$

The Tutte polynomial of the graphs corresponding to the link family $p, q, r+$ (Fig. 11) we obtain from the Tutte polynomial of the graphs $G(p, q, r)$, where the additional term is the Tutte polynomial of the graphs $G'(p, q, r)$. The general formula for the Tutte polynomial of the graphs $G(p, q, r+)$ is

$$T(G(p, q, r+)) = T(G(p, q, r)) + \left(\frac{x^p - 1}{x - 1} + y - 1\right)\left(\frac{x^q - 1}{x - 1} + y - 1\right)\left(\frac{x^r - 1}{x - 1} + y - 1\right).$$

2.12 Family $p111q$

By resolving the graph $G(p111q)$ into the graphs $G(p2q)$ and $G'(p+1, q+1)$ (Fig. 12) we obtain the general formula for the Tutte polynomial of the graphs $G(p111q)$

$$T(G(p111q)) = T(G(p2q)) + \left(\frac{x^{p+1} - 1}{x - 1} + y - 1\right)\left(\frac{x^{q+1} - 1}{x - 1} + y - 1\right).$$

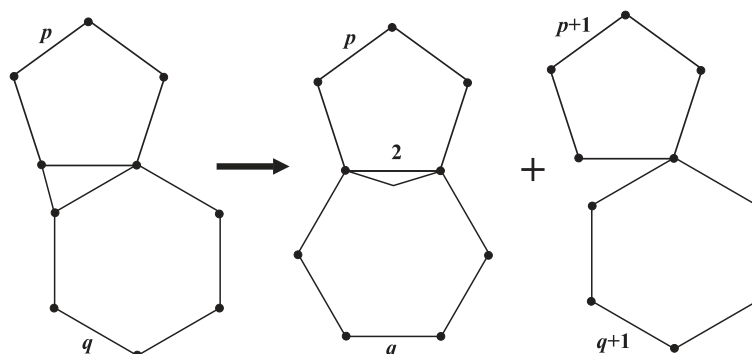


Figure 12: Resolving the graph $G(p\ 1\ 1\ 1\ q)$ ($p = 4, q = 5$) into graphs $G(p\ 2\ q)$ and $G'(p+1, q+1)$.

3 Conclusion

In summary, we provide explicit formulae for the Tutte polynomial of all families of KL s derived from source links with at most $n = 7$ crossings. The Jones polynomials of all alternating and non-alternating knots and links which belong to all families derived from source links up to 7 crossings can be obtained by substituting $x \rightarrow -x$ and $y \rightarrow -\frac{1}{x}$ [8, 9]. Complete results for all families of KL s derived from source links with at most 10 crossings are given in the extended version of this paper [21].

Computational results in this paper were obtained by *Mathematica* package which can be downloaded from the address

<http://www.mi.sanu.ac.rs/vismath/Tutte.htm>

It contains a function for computing graph corresponding to a KL called **GraphFam** and functions **TutteFam** and **JonesFam** for computing the Tutte and Jones polynomials of the graphs or links, respectively. Additional functions provide information about zeros of the Jones polynomial: **Zeros** computes zeros of the Jones polynomial of a particular KL , **ZeroSum** outputs the sum of absolute values of zeroes, and **Portrait** plots zeros of the Jones polynomials of a given family. All functions are based on general formulas, so there are no restrictions on the number of crossings of KL s except for the hardware ones.

Obtained results can be used for studying the behavior of the Tutte and Jones polynomials of KL families. For example, the leading coefficient of the Tutte polynomial is equal to 1 for all alternating algebraic KL s, and greater than 1 for non-algebraic ones. In particular, it is invariant for the Tutte polynomials of graphs corresponding to knots and links inside a family derived from a basic polyhedron. Additionally they can be used for studying zeros of the Jones polynomials for the

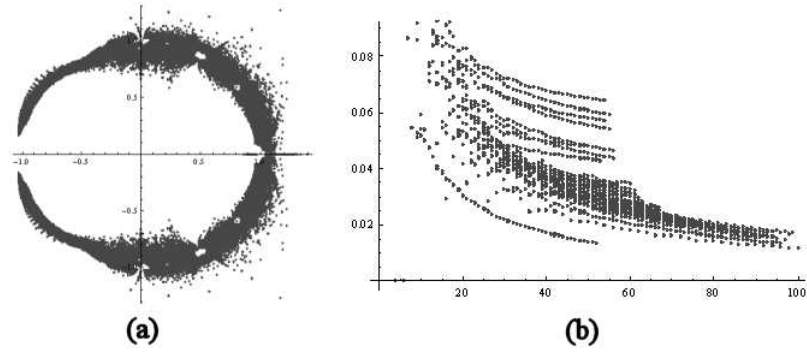


Figure 13: (a) Plot of zeros of Jones polynomials for the family pq ; (b) plot of defects ($2 \leq p \leq 50$, $2 \leq q \leq 50$).

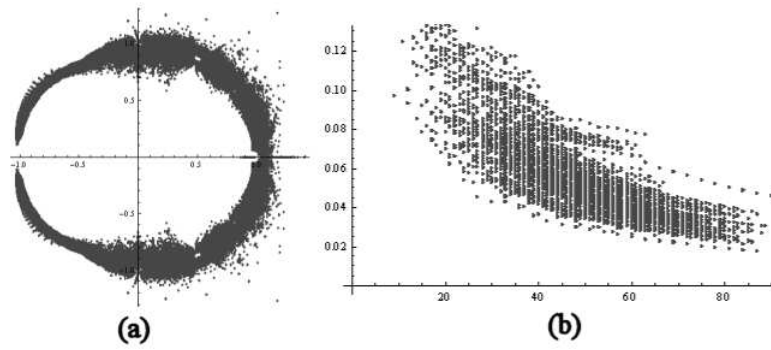


Figure 14: (a) Plot of zeros of Jones polynomials for the family of alternating pretzel KLs p, q, r ; (b) plot of defects ($2 \leq p \leq 30$, $2 \leq q \leq 30$, $2 \leq r \leq 30$).

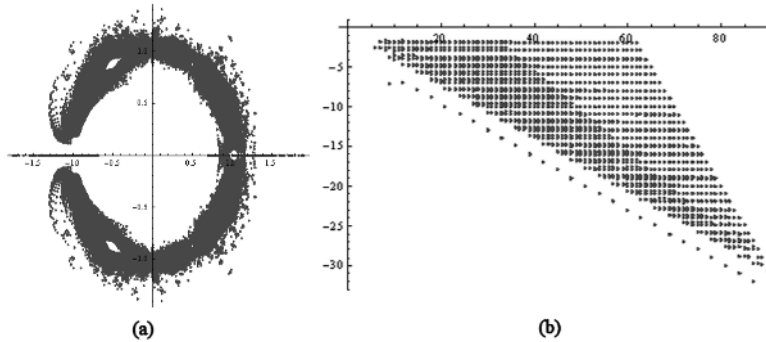


Figure 15: (a) Plot of zeros of Jones polynomials for the family of non-alternating pretzel KL s $p, q, -r$; (b) plot of defects ($2 \leq p \leq 30$, $2 \leq q \leq 30$, $2 \leq r \leq 30$).

whole family. Portraits, plots of zeroes of the Jones polynomials, of the alternating link family pq ($2 \leq p \leq 50$, $2 \leq q \leq 50$), alternating pretzel link family p, q, r ($2 \leq p \leq 30$, $2 \leq q \leq 30$, $2 \leq r \leq 30$), and non-alternating pretzel link family $p, q, -r$ ($2 \leq p \leq 30$, $2 \leq q \leq 30$, $2 \leq r \leq 30$) are shown in Figures 13a, 14a and 15a, respectively.

We interpret our plots in the light of results in [9, 14, 15, 16, 17, 22].

From Figures 13a, 14a, 15a we can see that almost all of the roots of the Jones polynomials approach the unit circle under twistingⁱⁱⁱ [17], they are dense in the whole unit circle [22], and critical points are the third [9, 14] and sixth roots of unity. Hence, the plot on Figure 15a must correspond to a non-alternating KL family, because some of the zeros are real and negative [23].

Let $\lambda_1, \dots, \lambda_n$ denote zeroes of the Jones polynomial $V_L(x)$ of a link L and $\|L\| = |\lambda_1| + \dots + |\lambda_n|$ the sum of their absolute values. According to Lemma [15, page 17], for every alternating knot or a link L with the crossing number N , $\|L\| \geq N$, while in the case of non-alternating links, according to the experimental results, $\|L\| < N$. The difference $d = \|L\| - N$ will be called the defect of the link L .

The plots of defects for the link families pq ($2 \leq p \leq 50$, $2 \leq q \leq 50$), p, q, r ($2 \leq p \leq 30$, $2 \leq q \leq 30$, $2 \leq r \leq 30$), and $p, q, -r$ ($2 \leq p \leq 30$, $2 \leq q \leq 30$, $2 \leq r \leq 30$) are shown in the Figures 13b, 14b, and 15b, respectively (x -coordinate states the numbers of crossings and y -coordinates for defects). Plots in the first two figures illustrate that the minimal defects of alternating KL families pq and p, q, r tend to 0 when $p, q, r \rightarrow \infty$ while the last one hints that the minimal defects of the non-alternating KL family $p, q, -r$ diverge and tend to $-\infty$. In general, for all alternating KL families under twisting over all parameters, minimal defects tend to zero, i.e., $\|L\| \rightarrow N$, and we expect that for non-alternating KL families they tend to a negative integer k , or diverge.

ⁱⁱⁱAdding a twist changes the corresponding parameter in a Conway symbol by ± 1 .

More detailed results of this kind will be given in the forthcoming paper.

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