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Twist-bending vibrations of horizontally circular curved beams considering effects of transverse shear, rotatory inertia and warping

Bor-Tsung Hsiao
University of New Hampshire, Durham

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U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

TWIST–BENDING VIBRATIONS OF HORIZONTALLY CIRCULAR CURVED BEAMS CONSIDERING EFFECTS OF TRANSVERSE SHEAR, ROTATORY INERTIA AND WARPING

BY

BOR–TSUNG HSIAO
B. Sc., Tamkang University, Taipei, Taiwan, R. O. C., 1972
M. Sc., Southern Methodist University, 1983

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This dissertation has been examined and approved.

Dissertation Director,
Tung-Ming Wang, Professor of Civil Engineering

Paul J. Ossenbruggen, Associate Professor of Civil Engineering

Charles H. Goodspeed, Associate Professor of Civil Engineering

Shan S. Kuo, Professor of Computer Science

Loren D. Meeker, Professor of Mathematics

May 6, 1988
Date

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NOMENCLATURE

a_n	n^{th} constant of integration.
a'_n	n^{th} constant of integration.
A	cross-sectional area of beam element.
$\underline{\mathbf{A}}, \underline{\mathbf{A}}_1, \underline{\mathbf{A}}_2$ $\underline{\mathbf{A}}_3, \underline{\mathbf{A}}_4$	matrices relating the displacement amplitudes to the constants of integration.
$\underline{\mathbf{A}}^{-1}, \underline{\mathbf{A}}_1^{-1}, \underline{\mathbf{A}}_2^{-1}$ $\underline{\mathbf{A}}_3^{-1}, \underline{\mathbf{A}}_4^{-1}$	an inverse matrix of $\underline{\mathbf{A}}, \underline{\mathbf{A}}_1, \underline{\mathbf{A}}_2, \underline{\mathbf{A}}_3, \underline{\mathbf{A}}_4$, respectively.
b	natural frequency parameter.
$\bar{b}$	forcing frequency parameter.
b_n	n^{th} constant of integration.
b'_n	n^{th} constant of integration.
B	normal function of $\beta(\theta, t)$.
c_n	n^{th} constant of integration.

c'_n — n^{th} constant of integration.

C — torsional rigidity as same as $G \cdot J$.

C_w — warping rigidity as same as $E \cdot \Gamma$.

$\underline{\mathbf{D}}, \underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2$
$\underline{\mathbf{D}}_3, \underline{\mathbf{D}}_4, \underline{\mathbf{DD}}_1$ — displacement matrices.

e — base of Naperian logarithms.

E — modulus of elasticity.

f_n, f'_n — n^{th} constant of integration.

$\underline{\mathbf{F}}, \underline{\mathbf{F}}_1, \underline{\mathbf{F}}_2$
$\underline{\mathbf{F}}_3, \underline{\mathbf{F}}_4$ — force matrices.

$\tilde{\underline{\mathbf{F}}}_3, \tilde{\underline{\mathbf{F}}}_4$ — fixed-end reaction force matrices.

$\underline{\mathbf{FF}}, \underline{\mathbf{FF}}_2$ — a vector of displacement matrices of $\underline{\mathbf{D}}_3, \underline{\mathbf{D}}_4$, respectively.

$\underline{\mathbf{F}}_{22}$ — fixed–end moment matrix.

G — shear modulus.

h	height of the beam element.
$\underline{\mathbf{H}}, \underline{\mathbf{H}}_1, \underline{\mathbf{H}}_2$ $\underline{\mathbf{H}}_3, \underline{\mathbf{H}}_4$	matrices relating the force amplitudes to the constants of integration.
i	$\sqrt{-1}$.
I	moment of inertia about principal axis.
I_p	the polar moment of inertia of the principal pole.
J	torsional constant.
k	cross-sectioal shape factor.
k_1, k_2, k_3, k_4, k_5 $k'_1, k'_2, k'_3, k'_4, k'_5$	coefficienct of characteristic equation.
L	distance of a beam element.
m_n	n^{th} constant of integration.
$[m], [m \cdot e^{\lambda\alpha}]$ $[mm], [mm \cdot e^{\lambda\alpha}]$	constant matrices defined in Appendix B.

$\overline{M}(\theta, t)$	bending moment as function of θ and t.
$M(\theta)$	normal function of $\overline{M}(\theta, t)$.
M_{ab}, M_{ba}	bending moments at ends A and B respectively.
M_{fab}, M_{fba}	fixed-end bending moments at ends A and B respectively.
$\overline{q}(t)$	distributed load as functions of time t.
$[q], [q \cdot e^{\lambda\alpha}]$	constant matrix defined in Appendix B.
Q	normal function of $\overline{q}(t)$.
r	rotatory inertia parameter.
R	radius of circular curved beam.
s	shear deformation parameter.
S	length of circular curved beam.
$\underline{S}, \underline{S}_1, \underline{S}_2$ $\underline{S}_3, \underline{S}_4, \underline{SS}_1$	dynamic stiffness matrices.

t	thickness of beam element, or time.
$[t]$, $[t \cdot e^{\lambda\alpha}]$ $[t\,t]$, $[t\,t \cdot e^{\lambda\alpha}]$	constant matrix defined in Appendix B.
$\overline{T}(\theta, t)$	twisting moment as function of θ and t.
T	normal function of $\overline{T}(\theta, t)$.
T_{ab}, T_{ba}	twisting moments at both ends A and B of the beam element, respectively.
T_{fab}, T_{fba}	fixed-end twisting moments at both ends A and B of the beam element, respectively.
u_n, u'_n	constants of integration.
$[v]$, $[v \cdot e^{\lambda\alpha}]$ $[v\,v]$, $[v\,v \cdot e^{\lambda\alpha}]$	constant matrices defined in Appendix B.
$\overline{V}(\theta, t)$	transverse shearing force as function of θ and t.
V	normal function of $\overline{V}(\theta, t)$.
V_{ab}, V_{ba}	shear forces at ends A and B respectively.

V_{fab}, V_{fba} fixed-end shear forces at ends A and B respectively.

w warping parameter.

$\underline{X}$, $\underline{X}_1$, $\underline{X}_2$

$\underline{X}_3$, $\underline{X}_4$ vectors of constants of integration.

$y(\theta, t)$ vertical displacement as function of θ and t.

Y normal function of $y(\theta, t)$

$BM(\theta)$ warping moment as function of θ.

BM_{ab}, BM_{ba} warping moments at both ends a and b of the beam element, respectively.

BM_{fab}, BM_{fba} fixed-end warping moments at both ends a and b respectively.

α open angle.

$\beta(\theta, t)$ angle of twist as function of θ and t.

γ	mass per unit volume.
Γ	warping function.
η	moment inertia parameter.
ρ	stiffness parameter.
θ	angular coordinate.
ϕ	angle of shear.
Ω	natural frequency.
$\overline{\Omega}$	forcing frequency.
$\tau(\theta)$	warpage as function of θ.
$\psi(\theta, t)$	bending slope of beam element as function of θ and t.
Ψ	normal function of $\psi(\theta, t)$.

ABSTRACT

TWIST–BENDING VIBRATIONS OF HORIZONTALLY CIRCULAR CURVED BEAMS CONSIDERING EFFECTS OF TRANSVERSE SHEAR, ROTATORY INERTIA AND WARPING

by

BOR–TSUNG HSIAO
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This thesis is devoted to the dynamic analysis of horizontally circular curved beams. The direct stiffness method is used to derive the dynamic stiffness matrix for finding the natural frequencies and joint moments of curved beams having different rectangular cross–sections. Four examples are presented to illustrate the application of the proposed method and to show the effects of rotatory inertias, shear deformation, warping and opening angle of the arc on the beam. First three examples are for the free vibration of the beam. In these examples, beams with different thickness are used for finding effects of warping. In each example, there are three cases; case (a) consider rotatory inertias, shear deformation and warping effects; case (b) consider flexural rotatory inertia, shear deformation and warping effects; and case (c) consider rotatory inertias and shear deformation effects. Example four is for the forced vibration of the beam subjected to a uniformly distributed harmonic load. The results of the last example show the effects of cases (a), (b) and (c) on the joint moment of the beam.

CHAPTER 1

INTRODUCTION

The first study on horizontally curved beams was made by Saint–Venant [1] who investigated a circular curved cantilever bar under the action of a load applied at the end of the beam and normal to the plane of initial curvature. He also presented a solution for the deformations and internal forces of the bar. Since then, many investigators have used different approaches to solve curved beam problems; such as: the methods of virtual work, stiffness, flexibility and conjugate beam [2]. Vlasov [3] derived the general differential equations governing the behavior of horizontally curved girders and obtained the closed form solutions.

The first dynamic analysis of the transverse vibrations of a ring with arbitrary cross section were treated by Love [1]. Ojalvo [5] carried out a mathematical study of the free vibrations of elastic rings employing the equations of motion given by Love.

Den Hartog [6] used the Rayleigh–Ritz technique for finding the lowest natural frequency of circular arcs. His work was extended by Volterra and Morrell [7] for vibrations of arcs having center lines in the form of circles, catenaries, cycloids, and parabolas. Culver [8] investigated the free vibrations of simple horizontally curved beams with fixed and hinged ends. Culver and Oestel [9] applied the Rayleigh–Ritz method to find the natural frequencies of a two–span curved beam. Chen [10] developed the dynamic three–moment equation for finding the natural frequencies of multispan beams on rigid, nontwisting supports. Recently, Wang, Nettleton and Keita [11] analyzed the free vibration of continuous circular curved beams by means of the dynamic slope–deflection equations.

The classical Bernoulli–Euler theory of flexural motions of beams has been known to be inadequate for the vibration of higher modes. It is also known to be inadequate for those beams when the effect of the cross–sectional dimensions cannot be neglected. The effect of rotatory inertia on the cross–section of beams was first considered by Rayleigh [12]. His work was extended by Timoshenko [13,14] to include the effect of transverse–shear deformation. However the values of the shear coefficient obtained by Timoshenko are less accurate in the low–frequency range than in the high–frequency range [15]. Cowper [15] inproved Timoshenko's beam theory and derived a new formula for the shear coefficient. Numerical values of the shear coefficient obtained by Cowper are satisfactory for high–frequency as well as low–frequency deformations of beams.

In 1971 Rao [16] investigated the coupled twist–bending vibrations of complete and incomplete rings considering the effects of shear and rotatory inertia. More recently, Wang, Laskey and Ahmad [17] presented the dynamic stiffness matrix for analyzing out–of–plane free vibrations of continuous curved beams including both shear and rotatory inertia effects.

There have been many investegations of curved members applied to bridge structures since 1960s. In 1968 Tan and Shore [18] analyzed a simply–supported horizontally curved bridge under the passage of moving vehicles considering the effect of warping of the cross–section. Four years later, Shore and Shaudhuri [19] investigated the free vibration of horizontally curved beams with the effects of shearing deformation and flectural rotatory inertia being considered. The free vibration of horizontally curved girder bridges on the basis of the theory of orthotropic plates with the effects of warping of the cross–section neglected was studied by Yonezawa [20]. Resently, Just and Walley [21] studied the torsion of rectangular beams and found the importance of the warping effect on the torsional response.

The objective of this study is to present a general method for the dynamic analysis of circular curved beams, single or continuous, including the effects of transverse–shear deformation, rotatory inertia, associated with coupling of bending and twisting and warping. In the present investigation, the general dynamic stiffness matrix which includes the effects of a dynamic distributed load in terms of rotations, angles of twist and vertical deflections has been derived. A three–span circular curved beam undergoing out–of–plane free and forced vibrations is provided to illustrate the application of the proposed method. Numerical results are given to show the effects of shear, rotatory inertia, warping and opening angle of the arc on beams for a wide range of vibration modes.

CHAPTER 2

GENERAL DERIVATION

2.1. Basic Assumptions

1. Assume linear stress-strain relations.

2. The vibrations of the circular curved member are considered small. As a result, the effect of high order differentials are neglected.

3. The mass of the member is assumed uniformly distributed across the span.

4. Cross-section plane of curved members do not remain plane after they have been twisted.

5. Neglect axial forces and damping effect.

6. Concentrated loads and/or uniformly distributed loads will be considered.

7. Assumed free and forced harmonic vibrations.

2.2. Derivation of Equations of Motion

Figure 1a. shows the out-of-plane small vibration of a horizontally circular curved element subjected to a dynamic uniformly distributed load $\bar{q}(t)$ with the effects of shear deformation, rotatory inertia and warping being considered.

The expressions for the bending moment, $\overline{M}$, and twisting moment, $\overline{T}$, of a curved member can be written as [3]

$$\overline{M}(\theta, t) = \frac{EI}{R}\left(\beta - \frac{\partial\psi}{\partial\theta}\right) \tag{1}$$

$$\overline{T}(\theta, t) = \frac{C}{R}\left(\frac{\partial\beta}{\partial\theta} + \psi\right) - \frac{C_w}{R^3}\left(\frac{\partial^3\beta}{\partial\theta^3} + \frac{\partial^2\psi}{\partial\theta^2}\right) \tag{2}$$

where **EI** is the flexural rigidity ,**C** the torsional rigidity, **C**$_w$ the warping rigidity, ψ the bending slope, β the angle of twist, θ the angular coordinates, **R** the radius of a circular member and **t** the time.

The total angle between the deformed and undeformed center lines of the member is [13]

$$\frac{1}{R}\frac{\partial y}{\partial\theta} = \psi + \phi \tag{3}$$

where ϕ is the angular shear deformation, and y the vertical displacement.

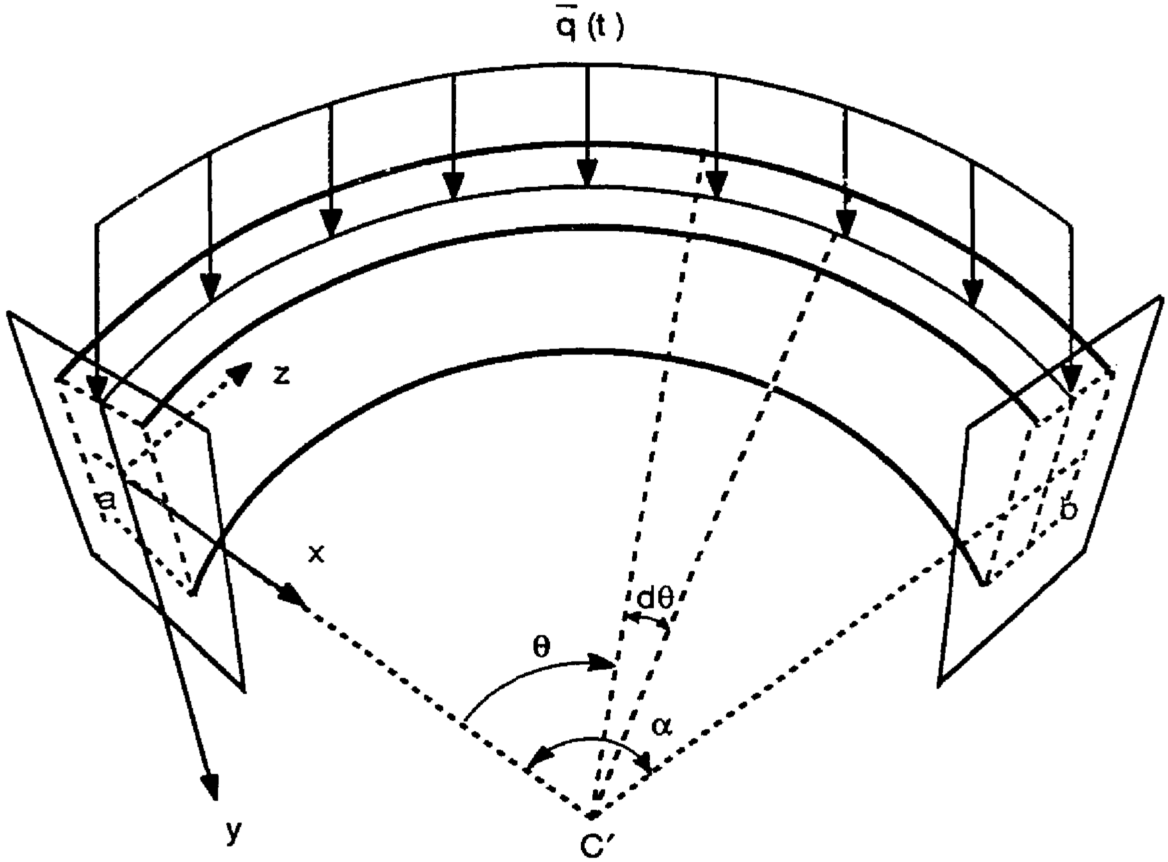

Figure 1a. A horizontally circular curved member subjected to a uniformly distributed load.

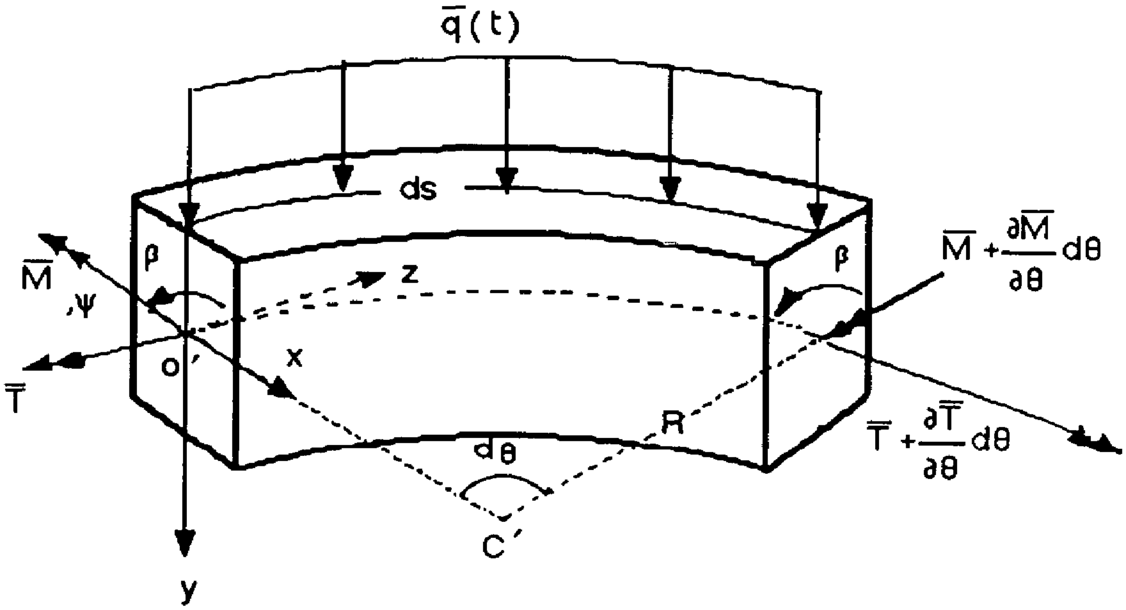

Figure 1b. Element of a horizontally curved member subjected to moments and load.

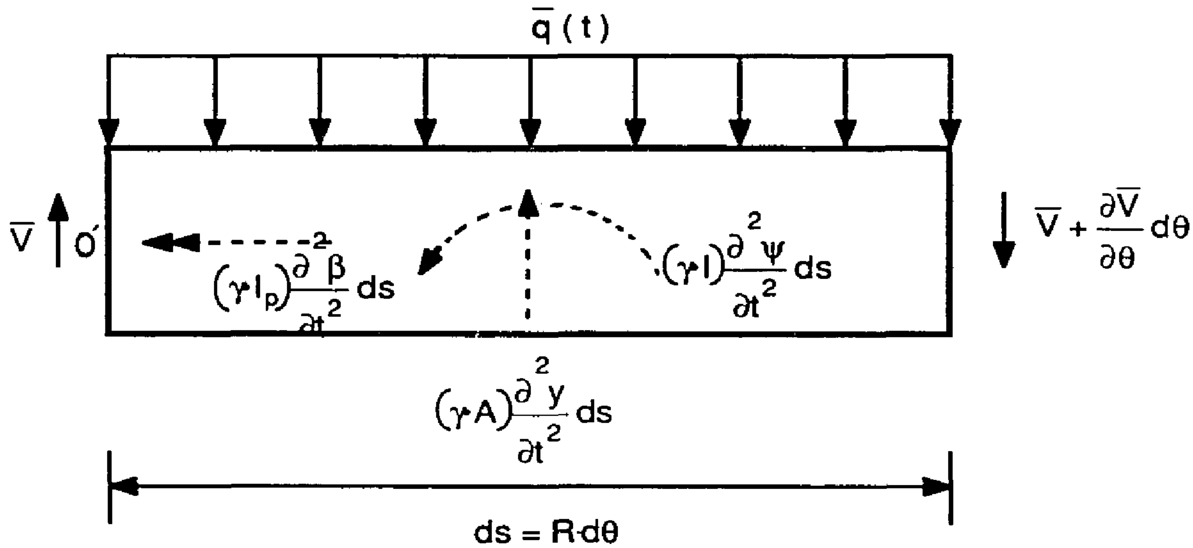

Figure 2. Element of a horizontally curved member subjected to forces and moments.

The transverse shearing force may be expressed as [13]

$$\overline{V}(\theta, t) = k\phi AG = kAG\left(\frac{1}{R}\frac{\partial y}{\partial \theta} - \psi\right) \tag{4}$$

where k is the cross-sectional shape factor , G the modulus of rigidity, and A the cross-sectional area.

Figure 1.b and Figure 2., the conditions for equilibrium of the curved element give

$\sum F_y = 0,$

$$-\overline{V} + \left(\overline{V} + \frac{\partial \overline{V}}{\partial \theta} d\theta\right) - \gamma AR\frac{\partial^2 y}{\partial t^2} d\theta + \overline{q}\, Rd\theta = 0 \tag{5}$$

$\sum \overline{M} = 0,$

$$-\overline{M}\cos d\theta + \overline{T}\sin d\theta - (\overline{V}R)\sin d\theta + (\overline{q}d\theta)\, R^2\sin\frac{d\theta}{2} + \left(\gamma IR\frac{\partial^2 \psi}{\partial t^2} d\theta\right)\cos\frac{d\theta}{2}$$

$$-\left(\gamma AR\frac{\partial^2 y}{\partial t^2} d\theta\right)R\sin\frac{d\theta}{2} + \left(\gamma I_p R\frac{\partial^2 \beta}{\partial t^2}\, d\theta\right)\sin\frac{d\theta}{2} + \left(\overline{M} + \frac{\partial \overline{M}}{\partial \theta}\, d\theta\right) = 0 \tag{6}$$

$\sum \overline{T} = 0,$

$$-\overline{T}\cos d\theta - \overline{M}\sin d\theta - \overline{V}R\left(1 - \cos d\theta\right) + (\overline{q}\, d\theta)R^2\left(1 - \cos\frac{d\theta}{2}\right)$$

$$+\left(\gamma IR\frac{\partial^2 \psi}{\partial t^2} d\theta\right)\sin\frac{d\theta}{2} - \left(\gamma A\frac{\partial^2 y}{\partial t^2} d\theta\right)R^2\left(1 - \cos\frac{d\theta}{2}\right)$$

$$-\left(\gamma I_p R\frac{\partial^2 \beta}{\partial t^2} d\theta\right)\cos\frac{d\theta}{2} + \left(\overline{T} + \frac{\partial \overline{T}}{\partial \theta}\, d\theta\right) = 0 \tag{7}$$

Factoring and/or neglecting higher orders, equations (5), (6), and (7) become

$$\frac{\partial \overline{V}}{\partial \theta} - \gamma AR \frac{\partial^2 y}{\partial t^2} + \overline{q}R = 0 \tag{8}$$

$$\frac{\partial \overline{M}}{\partial \theta} - \overline{V}R + \overline{T} + \gamma IR \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{9}$$

$$\overline{M} - \frac{\partial \overline{T}}{\partial \theta} + \gamma I_p R \frac{\partial^2 \beta}{\partial t^2} = 0 \tag{10}$$

When equation (4) is substituted into equation (8) one has

$$kAG\frac{\partial \psi}{\partial \theta} - \frac{kAG}{R}\frac{\partial^2 y}{\partial \theta^2} + \gamma AR\frac{\partial^2 y}{\partial t^2} = \overline{q}R \tag{11}$$

Introducing equations (1), (2) and (4) into equation (9) yields

$$kAG\left(\frac{\partial y}{\partial \theta}\right) - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{\partial \beta}{\partial \theta} + \frac{C_w}{R^3}\frac{\partial^3 \beta}{\partial \theta^3} - \left(kAGR + \frac{C}{R}\right)\psi + \left(\frac{EI}{R} + \frac{C_w}{R^3}\right)\frac{\partial^2 \psi}{\partial \theta^2}$$
$$- \gamma IR\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{12}$$

Finally, substituting equations (1) and (2) into equation (10) one obtains

$$\frac{EI}{R}\beta - \frac{C}{R}\frac{\partial^2\beta}{\partial\theta^2} + \gamma I_p R\frac{\partial^2\beta}{\partial t^2} + \frac{C_w}{R^3}\frac{\partial^4\beta}{\partial\theta^4} - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{\partial\psi}{\partial\theta} + \frac{C_w}{R^3}\frac{\partial^3\psi}{\partial\theta^3} = 0 \tag{13}$$

2.2.1.For free vibrations.

Setting $\overline{q}(t) = 0$ in equation (11), gives

$$kAG\frac{\partial\psi}{\partial\theta} - \frac{kAG}{R}\frac{\partial^2 y}{\partial\theta^2} + \gamma AR\frac{\partial^2 y}{\partial t^2} = 0 \tag{14}$$

Assuming the beam is excited with a natural frequency Ω and letting

$$\left.\begin{aligned} y(\theta, t) &= Y(\theta)e^{i\Omega t} \\ \psi(\theta, t) &= \Psi(\theta)e^{i\Omega t} \\ \beta(\theta, t) &= B(\theta)e^{i\Omega t} \end{aligned}\right\} \tag{15}$$

where Y, Ψ and B are normal functions of y, ψ and β, respectively.

According to equations (15) , equations (14), (12) and (13) can be written as

$$R\frac{d\Psi}{d\theta} = \frac{d^2Y}{d\theta^2} + \frac{\gamma R^2\Omega^2}{kG}Y \tag{16}$$

$$kAG\left(\frac{dY}{d\theta}\right) - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{dB}{d\theta} + \frac{Cw}{R^3}\frac{d^3B}{d\theta^3} - \left(kAGR + \frac{C}{R}\right)\Psi + \left(\frac{EI}{R} + \frac{Cw}{R^3}\right)\frac{d^2\Psi}{d\theta^2} + \gamma IR\Omega^2\Psi = 0 \tag{17}$$

$$\frac{EI}{R}B - \frac{C}{R}\frac{\partial^2 B}{\partial\theta^2} - \gamma I_p R\Omega^2 B + \frac{Cw}{R^3}\frac{d^4B}{d\theta^4} - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{d\Psi}{d\theta} + \frac{Cw}{R^3}\frac{d^3\Psi}{d\theta^3} = 0 \tag{18}$$

By using the linear operator method [4], equations (16), (17) and (18) may be expressed as :

$$\left(D^2 + b^2s^2\right)Y - RD\Psi = 0 \tag{19}$$

$$(D)\,Y + Rs^2\left(-\frac{1}{s^2} - \rho + b^2r^2 + D^2 + wD^2\right)\Psi + Rs^2\left(-D - \rho D + WD^3\right)B = 0$$

------(20)

$$\left(-D - \rho D + wD^3\right)\Psi + \left(1. - \rho D^2 - \eta b^2r^2 + wD^4\right)B = 0 \tag{21}$$

where

$$D = \frac{d}{d\theta},\quad b^2 = \frac{\gamma AR^4\Omega^2}{EI},\quad s^2 = \frac{EI}{kAGR^2},\quad r^2 = \frac{I}{AR^2},\quad \rho = \frac{C}{EI},\quad w = \frac{C_w}{EIR^2},\quad \eta = \frac{I_p}{I}$$

With the operators defined as follows :

$$\left.\begin{aligned}
&L_1 = D^2 + b^2 s^2 , \qquad L_2 = -RD , \qquad L_3 = 0 \\
&L_4 = D , \quad L_5 = Rs^2\left(-\frac{1.}{s^2} - \rho + b^2 r^2 + D^2 + wD^2\right) , \quad L_6 = Rs^2\left(-D - \rho D + WD^3\right) , \\
&L_7 = 0, \qquad L_8 = -D - \rho D + wD^3 , \qquad L_9 = 1. - \rho D^2 - \eta b^2 r^2 + wD^4
\end{aligned}\right\}$$

------(22)

Equations (19), (20) and (21) can now be written as

$$\left.\begin{aligned}
&L_1 Y + L_2 \Psi + L_3 B = 0 \\
&L_4 Y + L_5 \Psi + L_6 B = 0 \\
&L_7 Y + L_8 \Psi + L_9 B = 0
\end{aligned}\right\} \tag{23}$$

where the L's are linear differential operators in θ. The unknown functions Y, Ψ and B are to be determined as functions of θ. The operators involved are commutative, all unknown functions except one can be eliminated successively from the given set of linear differential equations to give a new set of equations, each involving only one unknown function.

For non-trivial solutions, equations (23) can be shown [4] to imply that

$$\Delta Y = 0 , \quad \Delta \Psi = 0 , \quad \Delta B = 0 \tag{24}$$

where the determinantal operator Δ is defined as

$$\Delta = \begin{vmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{vmatrix} \tag{25}$$

Expanding expression (25) one has

$$\Delta = L_1 \ (L_5 L_9 - L_6 L_8) - L_2 \ (L_4 L_9 - L_6 L_7) + L_3 \ (L_4 L_8 - L_5 L_7) \tag{26}$$

Introducing equation (22) into equation (26) gives

$$\begin{aligned}\Delta \ = \{ & wD^8 + (wb^2 r^2 - \rho + wb^2 s^2 + 2w)D^6 \\ & + (-wb^2 - \rho b^2 r^2 + wb^4 r^2 s^2 - \rho b^2 s^2 - \eta b^2 r^2 + w - \eta w b^2 r^2 - 2\rho \\ & + 2wb^2 s^2)D^4 + (\rho b^2 - \rho + \rho\eta b^2 r^2 + b^2 r^2 - \rho b^4 r^2 s^2 - \eta b^4 r^4 - \eta b^4 r^2 s^2 \\ & + wb^2 s^2 - \eta w b^4 r^2 s^2 - 2\rho b^2 s^2)D^2 + (-b^2 + \eta b^4 r^2 - \rho b^2 s^2 + \rho\eta b^4 r^2 s^2 \\ & + b^4 r^2 s^2 - \eta b^6 r^4 s^2)\}\end{aligned} \tag{27}$$

The details of derivation of equation (27) are shown in Appendix A.

In order to study the effects of torsional inertia and warping on the beam, the following cases are considered :

Case (a) consider the effects of rotatory inertia r, shear deformation s, torsional inertia η and warping w considered ($r \neq 0$, $s \neq 0$, $\eta \neq 0$, $w \neq 0$). Substituting equation (27) into $\Delta Y = 0$ of equations (24) one obtains

$$\frac{d^8Y}{d\theta^8} + k_1\frac{d^6Y}{d\theta^6} + k_2\frac{d^4Y}{d\theta^4} + k_3\frac{d^2Y}{d\theta^2} + k_4Y = 0 \tag{28}$$

where

$$k_1 = 2. + b^2r^2 + b^2s^2 - \frac{\rho}{w}$$

$$k_2 = 1. + 2\,b^2s^2 + b^4r^2s^2 - b^2 - \eta b^2r^2 - \left(2\rho + \rho b^2r^2 + \rho b^2s^2 + \eta b^2r^2\right)\frac{1}{w}$$

$$k_3 = b^2s^2 - \eta b^4r^2s^2 + \left(\rho\eta b^2r^2 + b^2r^2 + \rho b^2 - \rho - 2\rho b^2s^2 - \rho b^4r^2s^2 - \eta b^4r^4 - \eta b^4r^2s^2\right)\frac{1.}{w}$$

$$k_4 = \left(b^4r^2s^2 + \eta b^4r^2 + \rho\eta b^4r^2s^2 - b^2 - \rho b^2s^2 - \eta b^6r^4s^2\right)\frac{1.}{w}$$

The solution of equation (28) may be expressed as

$$Y(\theta) = \sum_{n=1}^{8} a_n e^{\lambda_n \theta} \tag{29}$$

where the a_n are constants to be determined from boundary conditions and λ_n are the roots of the characteristic equation as shown below.

$$\lambda^8 + k_1\lambda^6 + k_2\lambda^4 + k_3\lambda^2 + k_4 = 0 \tag{30}$$

Similarly, for $\Delta \Psi = 0$ and $\Delta B = 0$, one has the same form as equation (28) and their solutions are, respectively

$$R\Psi(\theta) = \sum_{n=1}^{8} b_n e^{\lambda_n \theta} \tag{31}$$

$$RB(\theta) = \sum_{n=1}^{8} c_n e^{\lambda_n \theta} \tag{32}$$

Substituting equations (29) and (31) into equation(19) yields

$$b_n = f_n a_n \tag{33}$$

where

$$f_n = \frac{\left(\lambda_n^2 + b^2 s^2\right)}{\lambda_n}$$

Similarly, introducing equations (31) and (32) into equation (21) one has

$$c_n = u_n a_n \tag{34}$$

where

$$u_n = \frac{\left(+1. + \rho - w\lambda_n^2\right)\left(\lambda_n^2 + b^2 s^2\right)}{\left(1. - \rho\lambda_n^2 - \eta b^2 r^2 + w\lambda_n^4\right)}$$

Thus, equations (31) and (32) become respectively

$$R\Psi(\theta) = \sum_{n=1}^{8} f_n a_n e^{\lambda_n \theta} \tag{35}$$

$$RB(\theta) = \sum_{n=1}^{8} u_n a_n e^{\lambda_n \theta} \tag{36}$$

If the individual effects of transverse shear s, rotatory inertia r, torsional inertia η and warping w are to be determined, the following specializations of equation (28) can be used.

Case (b) Effect of torsional inertia neglected ($\eta = 0, r \neq 0, s \neq 0, w \neq 0$), equation (28) becomes :

$$\frac{d^8Y}{d\theta^8} + k_1^{(2)} \frac{d^6Y}{d\theta^6} + k_2^{(2)} \frac{d^4Y}{d\theta^4} + k_3^{(2)} \frac{d^2Y}{d\theta^2} + k_4^{(2)} Y = 0 \tag{37}$$

where

$$k_1^{(2)} = 2. + b^2 r^2 + b^2 s^2 - \frac{\rho}{w}$$

$$k_2^{(2)} = 1. + 2 b^2 s^2 + b^4 r^2 s^2 - b^2 - \frac{2\rho}{w} - \frac{\rho b^2 r^2}{w} - \frac{\rho b^2 s^2}{w}$$

$$k_3^{(2)} = b^2 s^2 + \frac{b^2 r^2}{w} + \frac{\rho b^2}{w} - \frac{\rho}{w} - 2\frac{\rho b^2 s^2}{w} - \frac{\rho b^4 r^2 s^2}{w}$$

$$k_4^{(2)} = \left(b^4 r^2 s^2 - b^2 - \rho b^2 s^2\right)\frac{1.}{w}$$

Case (c) Effect of warping neglected ($w = 0$, $r \neq 0$, $s \neq 0$, $\eta \neq 0$), in this case equation (28) becomes :

$$\frac{d^6 Y}{d\theta^6} + k_1^{(3)}\frac{d^4 Y}{d\theta^4} + k_2^{(3)}\frac{d^2 Y}{d\theta^2} + k_3^{(3)} Y = 0 \tag{38}$$

where

$$k_1^{(3)} = 2. + b^2 r^2 + b^2 s^2 + \frac{\eta b^2 r^2}{\rho}$$

$$k_2^{(3)} = 1. + 2\, b^2 s^2 + b^4 r^2 s^2 - b^2 - \frac{b^2 r^2}{\rho} + \frac{\eta b^4 r^4}{\rho} - \eta b^2 r^2 + \frac{\eta b^4 r^2 s^2}{\rho}$$

$$k_3^{(3)} = b^2 s^2 + \frac{b^2}{\rho} - \frac{b^4 r^2 s^2}{\rho} + \frac{\eta b^6 r^4 s^2}{\rho} - \frac{\eta b^4 r^2}{\rho} - \eta b^4 r^2 s^2$$

The solution of equation (38) takes the form of

$$\mathbf{Y}(\theta) = \sum_{n=1}^{6} \mathbf{a}'_n \mathbf{e}^{\lambda_n \theta} \tag{39}$$

where the constants a'_n are to be determined from boundary conditions and the λ_n are the roots of the following auxiliary equation

$$\lambda^6 + k'_1 \lambda^4 + k'_2 \lambda^2 + k'_3 = 0 \tag{40}$$

From equations (29), (31) and (32) it can be seen that the solution for $\Psi(\theta)$ and $B(\theta)$ will have the same form as equation (39). Thus

$$R\Psi(\theta) = \sum_{n=1}^{6} b'_n e^{\lambda_n \theta} \tag{41}$$

$$RB(\theta) = \sum_{n=1}^{6} c'_n e^{\lambda_n \theta} \tag{42}$$

In order to find the relation among a'_n, b'_n and c'_n equations (19) and (21) are rewritten as follows :

$$R\frac{d\Psi}{d\theta} = \frac{d^2Y}{d\theta^2} + b^2 s^2 Y \tag{43}$$

$$(1.+\rho)\frac{d\Psi}{d\theta} = \left(1.-\eta b^2 r^2\right)B - \rho\frac{d^2B}{d\theta^2} \tag{44}$$

Substituting equations (39), (41) and (42) into eqautions (43) and (44) will yield

$$b'_n = f'_n a'_n \tag{45}$$

$$c'_n = u'_n a'_n \tag{46}$$

where

$$f'_n = \frac{\left(\lambda_n^2 + b^2 s^2\right)}{\lambda_n}$$

and

$$u'_n = \frac{\left(+1. + \rho\right)\left(\lambda_n^2 + b^2 s^2\right)}{\left(1. - \rho\lambda_n^2 - \eta b^2 r^2\right)}$$

2.2.2. For Forced Vibrations

Consider again a curved beam subjected to a uniformly distributed dynamic load $\bar{q}$ (t) acting normal to the horizontal plane and undergoing out-of-plane vibrations as shown in Figure 1a.

Following the same procedure as given in the previous section, the same form of equations (11), (12) and (13) for coupled bending-twisting vibrations can be obtained as :

$$kAG\frac{\partial \psi}{\partial \theta} - \frac{kAG}{R}\frac{\partial^2 y}{\partial \theta^2} + \gamma AR\frac{\partial^2 y}{\partial t^2} = \bar{q}R \tag{47}$$

$$kAG\left(\frac{\partial y}{\partial \theta}\right) - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{\partial \beta}{\partial \theta} + \frac{C_w}{R^3}\frac{\partial^3 \beta}{\partial \theta^3} - \left(kAGR + \frac{C}{R}\right)\psi + \left(\frac{EI}{R} + \frac{C_w}{R^3}\right)\frac{\partial^2 \psi}{\partial \theta^2}$$
$$- \gamma IR\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{48}$$

$$\frac{EI}{R}\beta - \frac{C}{R}\frac{\partial^2 \beta}{\partial \theta^2} + \gamma I_p R\frac{\partial^2 \beta}{\partial t^2} + \frac{C_w}{R^3}\frac{\partial^4 \beta}{\partial \theta^4} - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{\partial \psi}{\partial \theta} + \frac{C_w}{R^3}\frac{\partial^3 \psi}{\partial \theta^3} = 0 \tag{49}$$

Assuming the beam is excited with a forcing frequency $\bar{\Omega}$ and letting

$$\left.\begin{aligned} y(\theta,t) &= Y(\theta)e^{i\bar{\Omega}t} \\ \psi(\theta,t) &= \Psi(\theta)e^{i\bar{\Omega}t} \\ \beta(\theta,t) &= B(\theta)e^{i\bar{\Omega}t} \\ \bar{q}(t) &= Q\,e^{i\bar{\Omega}t} \end{aligned}\right\} \tag{50}$$

where, Y, Ψ, B and Q are normal functions of y, ψ, β, and $\bar{q}$, respectively.

Substituting equations (50) into equations (47), (48) and (49) gives

$$R\frac{d\Psi}{d\theta} = \frac{d^2Y}{d\theta^2} + \frac{\gamma R^2\bar{\Omega}^2}{kG}Y + \frac{R^2}{kAG}Q \tag{51}$$

$$kAG\left(\frac{dY}{d\theta}\right) - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{dB}{d\theta} + \frac{C_w}{R^3}\frac{d^3B}{d\theta^3} - \left(kAGR + \frac{C}{R}\right)\Psi + \left(\frac{EI}{R} + \frac{C_w}{R^3}\right)\frac{d^2\Psi}{d\theta^2}$$
$$+ \gamma IR\bar{\Omega}^2\Psi = 0 \tag{52}$$

$$\frac{EI}{R}B - \frac{C}{R}\frac{d^2B}{d\theta^2} + \gamma I_p R\bar{\Omega}^2 B + \frac{C_w}{R^3}\frac{d^4B}{d\theta^4} - \left(\frac{EI}{R} + \frac{C}{R}\right)\frac{d\Psi}{d\theta} + \frac{C_w}{R^3}\frac{d^3\Psi}{d\theta^3} = 0 \tag{53}$$

In operational form [4] , equations (51), (52) and (53) become

$$\left(D^2+\bar{b}^2 s^2\right)Y - RD\Psi = -\left(s^2\right)\frac{QR^4}{EI} \tag{54}$$

$$(D)\,Y + Rs^2\left(-\frac{1}{s^2} - \rho + \bar{b}^2 r^2 + D^2 + wD^2\right)\Psi + Rs^2\left(-D - \rho D + WD^3\right)B = 0$$

------(55)

$$\left(-D - \rho D + wD^3\right)\Psi + \left(1. - \rho D^2 - \eta \bar{b}^2 r^2 + wD^4\right)B = 0 \tag{56}$$

where

$$D = \frac{d}{d\theta},\quad \bar{b}^2 = \frac{\gamma A R^4 \bar{\Omega}^2}{EI},\quad s^2 = \frac{EI}{kAGR^2},\quad r^2 = \frac{I}{AR^2},\quad \rho = \frac{C}{EI},\quad w = \frac{C_w}{EIR^2},\quad \eta = \frac{I_p}{I}$$

Defining the following operators as

$$\left.\begin{array}{l} L_1 = D^2 + \bar{b}^2 s^2,\qquad L_2 = -RD,\qquad L_3 = 0,\qquad L_{10} = -\left(s^2\right)\frac{QR^4}{EI} \\ L_4 = D,\ L_5 = Rs^2\left(-\frac{1.}{s^2} - \rho + \bar{b}^2 r^2 + D^2 + wD^2\right),\ L_6 = Rs^2\left(-D - \rho D + WD^3\right) \\ L_7 = 0,\qquad L_8 = -D - \rho D + wD^3,\qquad L_9 = 1. - \rho D^2 - \eta \bar{b}^2 r^2 + wD^4 \end{array}\right\} \tag{57}$$

equations (54), (55) and (56) can be written as

$$L_1 Y + L_2 \Psi + L_3 B = L_{10} \tag{58a}$$

$$L_4 Y + L_5 \Psi + L_6 B = 0 \tag{58b}$$

$$L_7 Y + L_8 \Psi + L_9 B = 0 \tag{58c}$$

The system of equations (58) implies that

$$\Delta Y = -\left(s^2\right)\frac{QR^4}{EI} \ , \quad \Delta\Psi = 0 \ , \quad \Delta B = 0 \tag{59}$$

where the determinantal operator Δ is again defined as

$$\Delta = \begin{vmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{vmatrix} \tag{60}$$

For case (a) effects of rotatory inertia r, shear deformation s, torsional inertia η and warping w considered ($r \neq 0$, $s \neq 0$, $\eta \neq 0$, $w \neq 0$).

Substituting equations (57) into equations (58) yields

$$\frac{d^8Y}{d\theta^8} + k_1\frac{d^6Y}{d\theta^6} + k_2\frac{d^4Y}{d\theta^4} + k_3\frac{d^2Y}{d\theta^2} + k_4 Y = k_5\frac{QR^4}{EI} \tag{61}$$

$$\frac{d^8\Psi}{d\theta^8} + k_1\frac{d^6\Psi}{d\theta^6} + k_2\frac{d^4\Psi}{d\theta^4} + k_3\frac{d^2\Psi}{d\theta^2} + k_4\Psi = 0 \tag{62}$$

$$\frac{d^8B}{d\theta^8} + k_1\frac{d^6B}{d\theta^6} + k_2\frac{d^4B}{d\theta^4} + k_3\frac{d^2B}{d\theta^2} + k_4B = 0 \tag{63}$$

where

$$k_1 = 2. + \bar{b}^2 r^2 + \bar{b}^2 s^2 - \frac{\rho}{w}$$

$$k_2 = 1. + 2\,\bar{b}^2 s^2 + \bar{b}^4 r^2 s^2 - \bar{b}^2 - \frac{2\rho}{w} - \frac{\rho\bar{b}^2 r^2}{w} - \frac{\rho\bar{b}^2 s^2}{w} - \frac{\eta\bar{b}^2 r^2}{w} - \eta\bar{b}^2 r^2$$

$$k_3 = \bar{b}^2 s^2 - \eta\bar{b}^4 r^2 s^2 + \frac{\rho\eta\bar{b}^2 r^2}{w} + \frac{\bar{b}^2 r^2}{w} + \frac{\rho\bar{b}^2}{w} - \frac{\rho}{w} - 2\frac{\rho\bar{b}^2 s^2}{w} - \frac{\rho\bar{b}^4 r^2 s^2}{w} - \frac{\eta\bar{b}^4 r^4}{w} - \frac{\eta\bar{b}^4 r^2 s^2}{w}$$

$$k_4 = \left(\bar{b}^4 r^2 s^2 + \eta\bar{b}^4 r^2 + \rho\eta\bar{b}^4 r^2 s^2 - \bar{b}^2 - \rho\bar{b}^2 s^2 - \eta\bar{b}^6 r^4 s^2\right)\left(\frac{1.}{w}\right)$$

$$k_5 = \left(\frac{-1.}{\bar{b}^2}\right)k_4$$

The general solution for Y(θ) may be written as

$$Y(\theta) = Y_h(\theta) + Y_p \qquad (64)$$

where $Y_h(\theta)$ is the homogeneous solution given by expression (29) and Y_p is the particular solution for equation (61).

By undetermined coefficient method, it can be found that

$$Y_p = \left(\frac{k_5}{k_4}\right)\frac{QR^4}{EI}, \quad \Psi_p = 0\,, \; B_p = 0 \qquad (65)$$

Substituting equations (65) into equation (58) one obtains

$$k_5 = -\frac{1.}{\bar{b}^2} k_4 \qquad (66)$$

Therefore, the particular solution Y_p can be written as :

$$Y_p = -\frac{1.}{\bar{b}^2}\frac{QR^4}{EI} \qquad (67)$$

Thus equation (64) becomes

$$Y(\theta) = \sum_{n=1}^{8} a_n e^{\lambda_n \theta} - \frac{1.}{\bar{b}^2}\frac{QR^4}{EI} \tag{68}$$

where a_n are defined already and λ_n are the roots of the following characteristic equation :

$$\lambda^8 + k_1\lambda^6 + k_2\lambda^4 + k_3\lambda^2 + k_4 = 0 \tag{69}$$

Similarly, for $\Delta\Psi = 0$ and $\Delta B = 0$,the following equations which have the same forms as equations (35) and (36) can be obtained:

$$R\Psi(\theta) = \sum_{n=1}^{8} f_n a_n e^{\lambda_n \theta} \tag{70}$$

$$RB(\theta) = \sum_{n=1}^{8} u_n a_n e^{\lambda_n \theta} \tag{71}$$

where

$$f_n = \frac{\left(\lambda_n^2 + \bar{b}^2 s^2\right)}{\lambda_n}$$

and

$$u_n = \frac{\left(+1. + \rho - w\lambda_n^2\right)\left(\lambda_n^2 + \bar{b}^2 s^2\right)}{\left(1. - \rho\lambda_n^2 - \eta\bar{b}^2 r^2 + w\lambda_n^4\right)}$$

Case (b) Effect of torsional inertia neglected ($\eta = 0$, $r \neq 0$, $s \neq 0$, $w \neq 0$),

$$\frac{d^8Y}{d\theta^8} + k_1^{(2)}\frac{d^6Y}{d\theta^6} + k_2^{(2)}\frac{d^4Y}{d\theta^4} + k_3^{(2)}\frac{d^2Y}{d\theta^2} + k_4^{(2)}Y = k_5^{(2)}\frac{QR^4}{EI} \tag{72}$$

where

$$k_1^{(2)} = 2. + \bar{b}^2 r^2 + \bar{b}^2 s^2 - \frac{\rho}{w}$$

$$k_2^{(2)} = 1. + 2\,\bar{b}^2 s^2 + \bar{b}^4 r^2 s^2 - \bar{b}^2 - \frac{2\rho}{w} - \frac{\rho \bar{b}^2 r^2}{w} - \frac{\rho \bar{b}^2 s^2}{w}$$

$$k_3^{(2)} = \bar{b}^2 s^2 + \frac{\bar{b}^2 r^2}{w} + \frac{\rho \bar{b}^2}{w} - \frac{\rho}{w} - 2\frac{\rho \bar{b}^2 s^2}{w} - \frac{\rho \bar{b}^4 r^2 s^2}{w}$$

$$k_4^{(2)} = \left(\bar{b}^4 r^2 s^2 - \bar{b}^2 - \rho \bar{b}^2 s^2\right)\frac{1.}{w}$$

$$k_5^{(2)} = \frac{-1.}{\bar{b}^2}\,k_4^{(2)}$$

Case (c) Effect of warping neglected ($w = 0$, $r \neq 0$, $s \neq 0$, $\eta \neq 0$),

$$\frac{d^6Y}{d\theta^6} + k_1^{(3)}\frac{d^4Y}{d\theta^4} + k_2^{(3)}\frac{d^2Y}{d\theta^2} + k_3^{(3)}Y = k_4^{(3)}\frac{QR^4}{EI} \tag{73}$$

where

$$k_1^{(3)} = 2. + \bar{b}^2 r^2 + \bar{b}^2 s^2 + \frac{\eta \bar{b}^2 r^2}{\rho}$$

$$k_2^{(3)} = 1. + 2\,\bar{b}^2 s^2 + \bar{b}^4 r^2 s^2 - \bar{b}^2 - \frac{\bar{b}^2 r^2}{\rho} + \frac{\eta \bar{b}^4 r^4}{\rho} - \eta \bar{b}^2 r^2 + \frac{\eta \bar{b}^4 r^2 s^2}{\rho}$$

$$k_3^{(3)} = \bar{b}^2 s^2 + \frac{\bar{b}^2}{\rho} - \frac{\bar{b}^4 r^2 s^2}{\rho} + \frac{\eta \bar{b}^6 r^4 s^2}{\rho} - \frac{\eta \bar{b}^4 r^2}{\rho} - \eta \bar{b}^4 r^2 s^2$$

$$k_4^{(3)} = \frac{-1.}{\bar{b}^2}\,k_3^{(3)}$$

The solution of equation (73) takes the form of

$$\mathbf{Y}(\theta) = \sum_{n=1}^{6} \mathbf{a}'_n \mathbf{e}^{\lambda_n \theta} - \frac{1.}{\bar{b}^2}\frac{QR^4}{EI} \tag{74}$$

Where the constants a'_n are to be determined from boundary conditions and the λ_n are the roots of the following auxiliary equation :

$$\lambda^6 + k'_1\lambda^4 + k'_2\lambda^2 + k'_3 = 0 \tag{75}$$

Again the solution for $\Psi(\theta)$ and $B(\theta)$ will have the same form as equations (41) and (42). Thus

$$R\Psi(\theta) = \sum_{n=1}^{6} b'_n e^{\lambda_n \theta} \tag{76}$$

$$RB(\theta) = \sum_{n=1}^{6} c'_n e^{\lambda_n \theta} \tag{77}$$

In case w = 0, rewritting equations (54) and (56) as follows :

$$R\frac{d\Psi}{d\theta} = \frac{d^2Y}{d\theta^2} + \bar{b}^2 s^2 Y + \left(s^2\right)\frac{QR^4}{EI} \tag{78}$$

$$\left(1.+\rho\right)\frac{d\Psi}{d\theta} = \left(1.-\eta\bar{b}^2 r^2\right)B - \rho\frac{d^2B}{d\theta^2} \tag{79}$$

When equations (74), (76) and (77) are substituted into eqautions (78) and (79), the following relations among a'_n, b'_n and c'_n are obtained :

$$b'_n = f'_n a'_n \tag{80}$$

$$C'_n = u'_n a'_n \tag{81}$$

where

$$f'_n = \frac{\left(\lambda_n^2 + \bar{b}^2 s^2\right)}{\lambda_n}$$

and

$$u'_n = \frac{(1. + \rho)\left(\lambda_n^2 + \bar{b}^2 s^2\right)}{\left(1. - \rho\lambda_n^2 - \eta\bar{b}^2 r^2\right)}$$

CHAPTER 3

DERIVATION OF GENERAL DYNAMIC STIFFNESS MATRIX

Consider a horizontally circular curved member of constant cross-section subjected to harmonic displacements ,linear , rotational, twisting and warping, at the two ends a and b as shown in Figures 3 and 4.

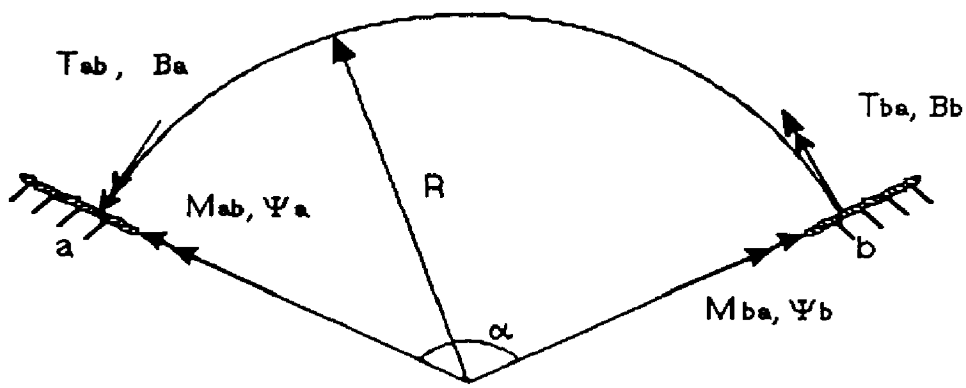

Figure 3 Positive moments of a circular curved beam.

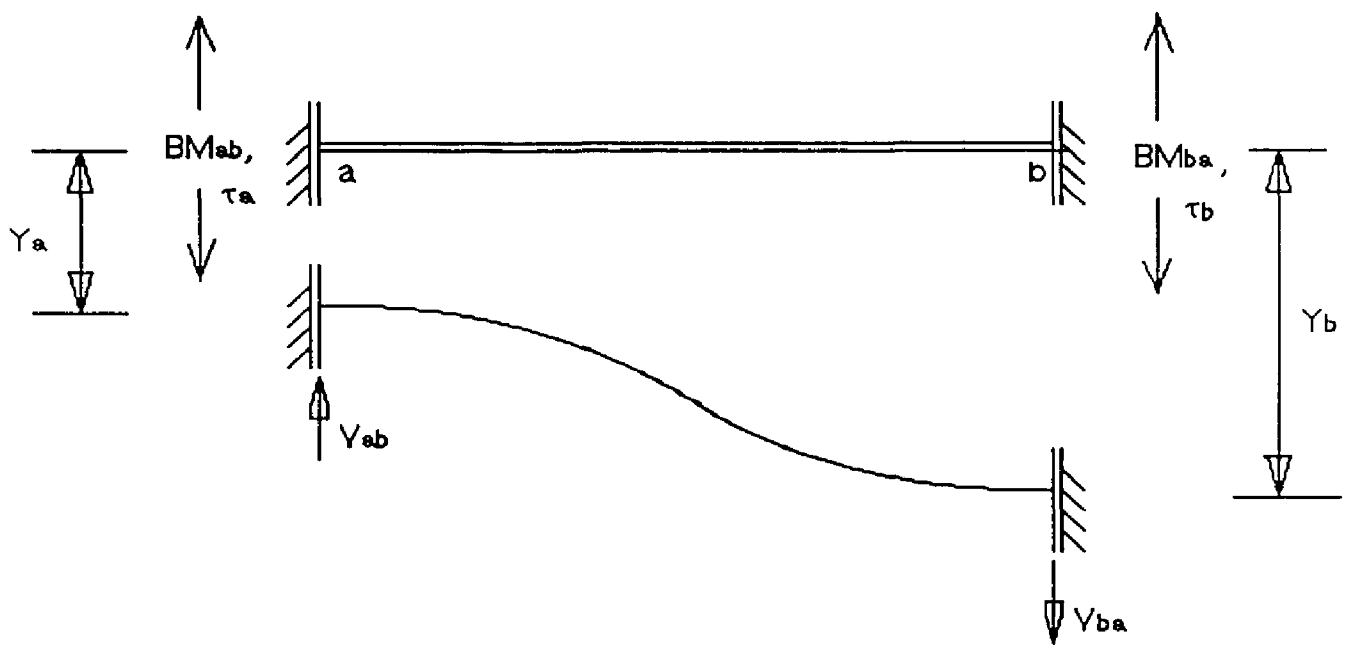

Figure 4. Positive warping moments and vertical forces of a circular curved beam.

For harmonic vibrations, let

$$\left.\begin{aligned}\overline{M}(\theta, t) &= M(\theta)\, e^{i\Omega t}\\ \overline{T}(\theta, t) &= T(\theta)\, e^{i\Omega t}\\ \overline{V}(\theta, t) &= V(\theta)\, e^{i\Omega t}\end{aligned}\right\} \tag{82}$$

where M, T, V are the normal functions for $\overline{M}$, $\overline{T}$, $\overline{V}$, respectively.

Introducing equations (15) and (82) into equations (1), (2), (4) and omitting the common term $e^{i\Omega t}$ gives

$$M(\theta) = \frac{EI}{R}\left[B(\theta) - \Psi'(\theta)\right] \tag{83}$$

$$T(\theta) = \frac{C}{R}\left[B'(\theta) + \Psi(\theta)\right] - \frac{C_w}{R^3}\left[B'''(\theta) + \Psi''(\theta)\right] \tag{84}$$

$$V(\theta) = kAG\left[\frac{1}{R}Y'(\theta) - \Psi(\theta)\right] \tag{85}$$

where the primes for Y, Ψ and B represent differentiation with respect to θ respectively.

Consider the effect of warping due to torsion, the warping moment BM is given by [3]

$$BM(\theta) = -\frac{C_w}{R^2}\tau'(\theta) = -\frac{C_w}{R^2}\left[B''(\theta) + \Psi'(\theta)\right] \tag{86}$$

where the warpage $\tau(\theta)$ is given by

$$\tau(\theta) = \frac{1.}{R^2}\left[B'(\theta) + \Psi(\theta)\right] \tag{87}$$

Substituting equations (29), (35) and (36) into equations (83)–(86) one obtains the following equations :

$$M(\theta) = \frac{EI}{R^2}\sum_{n=1}^{8} m_n a_n e^{\lambda_n \theta} \tag{88}$$

$$T(\theta) = \frac{EI}{R^2}\sum_{n=1}^{8} t_n a_n e^{\lambda_n \theta} \tag{89}$$

$$V(\theta) = \frac{EI}{R^3}\sum_{n=1}^{8} v_n a_n e^{\lambda_n \theta} \tag{90}$$

$$BM(\theta) = -\frac{C_w}{R^3}\sum_{n=1}^{8} z_n a_n e^{\lambda_n \theta} \tag{91}$$

where

$$m_n = u_n - \lambda_n f_n$$

$$t_n = \rho\left(\lambda_n c_n + b_n\right) - w\left(\lambda_n^3 c_n + \lambda_n^2 b_n\right)$$

$$v_n = \frac{\lambda_n - f_n}{s^2}$$

$$z_n = \lambda_n^2 u_n + \lambda_n f_n$$

Referring again Figure 3, the geometric boundary conditions at the beam ends are

$$\left.\begin{array}{ll} \Psi_a = \Psi(0), & \Psi_b = \Psi(\alpha) \\ B_a = B(0), & B_b = B(\alpha) \\ Y_a = Y(0), & Y_b = Y(\alpha) \\ \tau_a = \tau(0), & \tau_b = \tau(\alpha) \end{array}\right\} \quad (92)$$

and the force boundary conditions are

$$\left.\begin{aligned} &M_{ab} = M(0)\,,\ M_{ba} = -M(\alpha) \\ &T_{ab} = T(0)\,,\quad T_{ba} = -T(\alpha) \\ &V_{ab} = V(0)\,,\quad V_{ba} = V(\alpha) \\ &BM_{ab} = -BM(0)\,,\quad BM_{ba} = BM(\alpha) \end{aligned}\right\} \tag{93}$$

3.1 Free Vibrations of Beams.

The same cases discussed in the previous chapter will again be considered.

Cases (a) and (b) with warping effect being included.

Substituting equations (29), (35), (36), (87) and (88)–(91) into equations (92) and (93) yield the following equations in matrix forms

$$\underline{\mathbf{D}}_1 = \underline{\mathbf{A}}_1 \cdot \underline{\mathbf{X}}_1 \tag{94}$$

$$\underline{\mathbf{F}}_1 = \left(\frac{EI_x}{R^3}\right) \underline{\mathbf{H}}_1 \cdot \underline{\mathbf{X}}_1 \tag{95}$$

in which

$$\underline{\mathbf{F}}_1 = \left[\ \frac{M_{ab}}{R} \quad \frac{M_{ba}}{R} \quad \frac{T_{ab}}{R} \quad \frac{T_{ba}}{R} \quad V_{ab} \quad V_{ba} \quad \frac{BM_{ab}}{R^2} \frac{BM_{ba}}{R^2}\ \right]^T \tag{96}$$

$$\underline{\mathbf{D}}_1 = \left[R\Psi_a \; R\Psi_b \; RB_a \; RB_b \; Y_a \; Y_b \; R^2\tau_a R^2\tau_b \right]^T \tag{97}$$

$$\underline{\mathbf{X}}_1 = \left[a_1 \; a_2 \; a_3 \; a_4 \; a_5 \; a_6 \; a_7 \; a_8 \right]^T \tag{98}$$

$$\underline{\mathbf{A}}_1 = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \\ f_1 e^{\lambda_1 \alpha} & f_2 e^{\lambda_2 \alpha} & f_3 e^{\lambda_3 \alpha} & f_4 e^{\lambda_4 \alpha} & f_5 e^{\lambda_5 \alpha} & f_6 e^{\lambda_6 \alpha} & f_7 e^{\lambda_7 \alpha} & f_8 e^{\lambda_8 \alpha} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ u_1 e^{\lambda_1 \alpha} & u_2 e^{\lambda_2 \alpha} & u_3 e^{\lambda_3 \alpha} & u_4 e^{\lambda_4 \alpha} & u_5 e^{\lambda_5 \alpha} & u_6 e^{\lambda_6 \alpha} & u_7 e^{\lambda_7 \alpha} & u_8 e^{\lambda_8 \alpha} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ e^{\lambda_1 \alpha} & e^{\lambda_2 \alpha} & e^{\lambda_3 \alpha} & e^{\lambda_4 \alpha} & e^{\lambda_5 \alpha} & e^{\lambda_6 \alpha} & e^{\lambda_7 \alpha} & e^{\lambda_8 \alpha} \\ p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ p_1 e^{\lambda_1 \alpha} & p_2 e^{\lambda_2 \alpha} & p_3 e^{\lambda_3 \alpha} & p_4 e^{\lambda_4 \alpha} & p_5 e^{\lambda_5 \alpha} & p_6 e^{\lambda_6 \alpha} & p_7 e^{\lambda_7 \alpha} & p_8 e^{\lambda_8 \alpha} \end{bmatrix} \tag{99}$$

$$\underline{\mathbf{H}}_1 = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 \\ -m_1e^{\lambda_1\alpha} & -m_2e^{\lambda_2\alpha} & -m_3e^{\lambda_3\alpha} & -m_4e^{\lambda_4\alpha} & -m_5e^{\lambda_5\alpha} & -m_6e^{\lambda_6\alpha} & -m_7e^{\lambda_7\alpha} & -m_8e^{\lambda_8\alpha} \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\ -t_1e^{\lambda_1\alpha} & -t_2e^{\lambda_2\alpha} & -t_3e^{\lambda_3\alpha} & -t_4e^{\lambda_4\alpha} & -t_5e^{\lambda_5\alpha} & -t_6e^{\lambda_6\alpha} & -t_7e^{\lambda_7\alpha} & -t_8e^{\lambda_8\alpha} \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\ v_1e^{\lambda_1\alpha} & v_2e^{\lambda_2\alpha} & v_3e^{\lambda_3\alpha} & v_4e^{\lambda_4\alpha} & v_5e^{\lambda_5\alpha} & v_6e^{\lambda_6\alpha} & v_7e^{\lambda_7\alpha} & v_8e^{\lambda_8\alpha} \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 & q_8 \\ -q_1e^{\lambda_1\alpha} & -q_2e^{\lambda_2\alpha} & -q_3e^{\lambda_3\alpha} & -q_4e^{\lambda_4\alpha} & -q_5e^{\lambda_5\alpha} & -q_6e^{\lambda_6\alpha} & -q_7e^{\lambda_7\alpha} & -q_8e^{\lambda_8\alpha} \end{bmatrix} \tag{100}$$

and where $\underline{D}_1$ is the displacement matrix, $\underline{F}_1$ the force matrix, $\underline{A}_1$ the shape matrix for displacements, $\underline{H}_1$ the shape matrix for forces, and $\underline{X}_1$ the amplitude matrix.

Eliminating $\underline{X}_1$ from equations (94) and (95) yields

$$\underline{F}_1 = \frac{EI}{R^3}\underline{H}_1 \cdot \underline{A}_1^{-1}\underline{D}_1 = \underline{\mathbf{S}}_1 \cdot \underline{\mathbf{D}}_1 \tag{101}$$

where $\underline{\mathbf{S}}_1$, the dynamic stiffness matrix for a horizontally circular curved beam member, is given by

$$\underline{\mathbf{S}}_1 = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} \\ S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} \\ S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} \end{bmatrix} \tag{102}$$

Case (c) with warping effect being neglected.

Substitutung equation (39), (41) and (42) into equations (83)–(85) will yield:

$$M(\theta) = \frac{EI}{R^2} \sum_{n=1}^{8} m_n' a_n' e^{\lambda_n \theta} \tag{103}$$

$$T(\theta) = \frac{EI}{R^2} \sum_{n=1}^{8} t_n' a_n' e^{\lambda_n \theta} \tag{104}$$

$$V(\theta) = \frac{EI}{R^3} \sum_{n=1}^{8} v_n' a_n' e^{\lambda_n \theta} \tag{105}$$

where

$$m_n' = u_n' - \lambda_n' u_n' \tag{106}$$

$$t_n' = \rho\left(\lambda_n' u_n' + f_n'\right) \tag{107}$$

$$v_n' = \left(\frac{\lambda_n' - f_n'}{s^2}\right) \tag{108}$$

The geometric boundary conditions are

$$\left.\begin{array}{ll} \Psi_a = \Psi(0), & \Psi_b = \Psi(\alpha) \\ B_a = B(0), & B_b = B(\alpha) \\ Y_a = Y(0), & Y_b = Y(\alpha) \end{array}\right\} \tag{109}$$

and the force boundary conditions are

$$\left.\begin{array}{ll} M_{ab} = M(0), & M_{ba} = -M(\alpha) \\ T_{ab} = T(0), & T_{ba} = -T(\alpha) \\ V_{ab} = V(0), & V_{ba} = V(\alpha) \end{array}\right\} \tag{110}$$

Equations (96)–(102) can be written as

$$\underline{\mathbf{F}}_2 = \left[\frac{M_{ab}}{R} \ \frac{M_{ba}}{R} \ \frac{T_{ab}}{R} \ \frac{T_{ba}}{R} \ V_{ab} \ \ V_{ba} \right]^T \tag{111}$$

$$\underline{\mathbf{D}}_2 = \left[R\Psi_a R\Psi_b RB_a RB_b \ Y_a \ \ Y_b \right]^T \tag{112}$$

$$\underline{\mathbf{X}}_2 = \left[a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \right]^T \tag{113}$$

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ f_1 e^{\lambda_1 \alpha} & f_2 e^{\lambda_2 \alpha} & f_3 e^{\lambda_3 \alpha} & f_4 e^{\lambda_4 \alpha} & f_5 e^{\lambda_5 \alpha} & f_6 e^{\lambda_6 \alpha} \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ u_1 e^{\lambda_1 \alpha} & u_2 e^{\lambda_2 \alpha} & u_3 e^{\lambda_3 \alpha} & u_4 e^{\lambda_4 \alpha} & u_5 e^{\lambda_5 \alpha} & u_6 e^{\lambda_6 \alpha} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ e^{\lambda_1 \alpha} & e^{\lambda_2 \alpha} & e^{\lambda_3 \alpha} & e^{\lambda_4 \alpha} & e^{\lambda_5 \alpha} & e^{\lambda_6 \alpha} \end{bmatrix} \tag{114}$$

$$\underline{H}_2 = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ -m_1 e^{\lambda_1 \alpha} & -m_2 e^{\lambda_2 \alpha} & -m_3 e^{\lambda_3 \alpha} & -m_4 e^{\lambda_4 \alpha} & -m_5 e^{\lambda_5 \alpha} & -m_6 e^{\lambda_6 \alpha} \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ -t_1 e^{\lambda_1 \alpha} & -t_2 e^{\lambda_2 \alpha} & -t_3 e^{\lambda_3 \alpha} & -t_4 e^{\lambda_4 \alpha} & -t_5 e^{\lambda_5 \alpha} & -t_6 e^{\lambda_6 \alpha} \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_1 e^{\lambda_1 \alpha} & v_2 e^{\lambda_2 \alpha} & v_3 e^{\lambda_3 \alpha} & v_4 e^{\lambda_4 \alpha} & v_5 e^{\lambda_5 \alpha} & v_6 e^{\lambda_6 \alpha} \end{bmatrix} \tag{115}$$

Where

$\underline{D}_2$, $\underline{F}_2$, $\underline{A}_2$, $\underline{H}_2$ and $\underline{X}_2$ are defined the same as those of cases (a) and (b).

and

$$\underline{F}_2 = \underline{S}_2 \cdot \underline{D}_2 \tag{116}$$

The dynamic stiffness matrix for case (c) is given by

$$\underline{S}_2 = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \tag{117}$$

3.2. Forced Vibrations of Beams.

Consider a horizontally circular curved beam under the action of a harmonic uniform load $\bar{q}(t)$ as shown in Figure 1a.

Cases (a) and (b) with warping effect being included.

The geometric boundary conditions for both ends a and b being fixed are

$$\left.\begin{aligned} &\Psi_a = \Psi(0) = 0, && \Psi_b = \Psi(\alpha) = 0 \\ &B_a = B(0) = 0, && B_b = B(\alpha) = 0 \\ &Y_a = Y(0) = 0, && Y_b = Y(\alpha) = 0 \\ &\tau_a = \tau(0) = 0, && \tau_b = \tau(\alpha) = 0 \end{aligned}\right\} \tag{118}$$

Substituting equations (68), (70), (71) and (87) into equations (118) gives

$$\underline{\mathbf{D}}_3 = \underline{\mathbf{A}}_3 \cdot \underline{\mathbf{X}}_3 - \frac{1.}{\bar{b}^2} \frac{QR^4}{EI} \underline{\mathbf{FF}} = \underline{\mathbf{0}} \tag{119}$$

where $\underline{FF} = [\,0\;0\;1\;0\;0\;0\;1\;0\,]^T$

Premultiplying both sides of equation (119) by an $\underline{A}_3^{-1}$ yields

$$\mathbf{X}_3 = \frac{1.}{\bar{b}^2}\frac{QR^4}{EI}\underline{\mathbf{A}}_3^{-1}\cdot\underline{\mathbf{FF}} \tag{120}$$

Introducing equation (120) into equations (88)–(91) one obtains the fixed-end moments and forces as follows :

$$M_{fab} = \frac{1.}{\bar{b}^2}QR^2[m]\,\underline{\mathbf{A}}_3^{-1}\underline{\mathbf{FF}} \tag{121}$$

$$M_{fba} = \frac{1.}{\bar{b}^2}QR^2\left[m\cdot e^{\lambda\alpha}\right]\underline{\mathbf{A}}_3^{-1}\underline{\mathbf{FF}} \tag{122}$$

$$T_{fab} = \frac{1.}{\bar{b}^2}QR^2[t]\,\underline{\mathbf{A}}_3^{-1}\underline{\mathbf{FF}} \tag{123}$$

$$T_{fba} = \frac{1.}{\bar{b}^2}QR^2\left[t\cdot e^{\lambda\alpha}\right]\underline{\mathbf{A}}_3^{-1}\underline{\mathbf{FF}} \tag{124}$$

$$V_{fab} = \frac{1.}{\bar{b}^2}QR\,[v]\,\underline{\mathbf{A}}_3^{-1}\underline{\mathbf{FF}} \tag{125}$$

$$V_{fba} = \frac{1.}{\bar{b}^2}QR\left[v\cdot e^{\lambda\alpha}\right]\underline{\mathbf{A}}_3^{-1}\underline{\mathbf{FF}} \tag{126}$$

$$BM_{fab} = \frac{-1.}{\bar{b}^2} wQR^3 \left[q\right] \mathbf{A}_3^{-1} \underline{\mathbf{FF}} \tag{127}$$

$$BM_{fba} = \frac{-1.}{\bar{b}^2} wQR^3 \left[q \cdot e^{\lambda\alpha}\right] \mathbf{A}_3^{-1} \underline{\mathbf{FF}} \tag{128}$$

where [m] , [m $e^{\lambda\alpha}$] , [t] , [t $e^{\lambda\alpha}$], [v] , [$ve^{\lambda\alpha}$] , [q] and [$qe^{\lambda\alpha}$] are given in Appendix B.

The general solutions for the bending, twisting, and warping moments and for the vertical shears can be obtained by combining equations (93) and (121)–(128) and the results are

$$M_{ab} = M(0) + M_{fab} \tag{129}$$

$$M_{ba} = M(\alpha) + M_{fba} \tag{130}$$

$$T_{ab} = T(0) + T_{fab} \tag{131}$$

$$T_{ba} = T(\alpha) + T_{fba} \tag{132}$$

$$V_{ab} = V(0) + V_{fab} \tag{133}$$

$$V_{ba} = V(\alpha) + V_{fba} \tag{134}$$

$$BM_{ab} = BM(0) + BM_{fab} \tag{135}$$

$$BM_{ba} = BM(\alpha) + BM_{fba} \tag{136}$$

or in the following form

$$\begin{aligned} \mathbf{F}_3 &= \left(\frac{EI}{R^3}\right)\mathbf{H}_3 \cdot \mathbf{X}_3 + \tilde{\mathbf{F}}_3 = \left(\frac{EI}{R^3}\right)\mathbf{H}_3 \cdot \mathbf{A}_3^{-1} \cdot \mathbf{D}_3 + \tilde{\mathbf{F}}_3 \\ &= \mathbf{S}_3 \cdot \mathbf{D}_3 + \tilde{\mathbf{F}}_3 \end{aligned} \tag{137}$$

where

$$\tilde{\mathbf{F}}_3 = \begin{bmatrix} M_{fab} & M_{fba} & T_{fab} & T_{fba} & V_{fab} & V_{fba} & BM_{fab} & BM_{fba} \end{bmatrix}^T$$

and $\mathbf{D}_3$, $\mathbf{A}_3$, $\mathbf{X}_3$, $\mathbf{F}_3$, $\mathbf{H}_3$, $\mathbf{S}_3$ have the same forms as those for free vibration cases (a) and (b).

Case (c) with warping effect being neglected.

The geometric boundary conditions for both ends a and b being fixed are

$$\left.\begin{aligned} &\Psi_a = \Psi(0) = 0, \qquad \Psi_b = \Psi(\alpha) = 0 \\ &B_a = B(0) = 0, \qquad B_b = B(\alpha) = 0 \\ &Y_a = Y(0) = 0, \qquad Y_b = Y(\alpha) = 0 \end{aligned}\right\} \tag{138}$$

Substituting equations (74), (76) and (77) into equations (138) one obtains

$$\underline{\mathbf{D}}_4 = \underline{\mathbf{A}}_4 \cdot \underline{\mathbf{X}}_4 - \frac{1.}{\bar{b}^2}\frac{QR^4}{EI}\underline{\mathbf{FF}}_2 = \underline{\mathbf{0}} \tag{139}$$

where $\underline{FF}_2 = [\,0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 1\,]^T$

Premultiplying both sides of equation (139) by an $\underline{A}_4^{-1}$ yields

$$\underline{\mathbf{X}}_4 = \frac{1.}{\bar{b}^2}\frac{QR^4}{EI}\underline{\mathbf{A}}_4^{-1} \cdot \underline{\mathbf{FF}}_2 \tag{140}$$

Introducing equation (140) into equations (88)–(90) one obtains the following fixed-end moments and forces :

$$M_{fab} = \frac{1.}{\bar{b}^2} QR^2 [mm]\, \underline{\mathbf{A}}_4^{-1} \underline{\mathbf{FF}}_2 \tag{141}$$

$$M_{fba} = \frac{1.}{\bar{b}^2} QR^2 \left[mm \cdot e^{\lambda\alpha} \right] \underline{\mathbf{A}}_4^{-1} \underline{\mathbf{FF}}_2 \tag{142}$$

$$T_{fab} = \frac{1.}{\bar{b}^2} QR^2 [t\,t]\, \underline{\mathbf{A}}_4^{-1} \underline{\mathbf{FF}}_2 \tag{143}$$

$$T_{fba} = \frac{1.}{\bar{b}^2} QR^2 \left[t\,t \cdot e^{\lambda\alpha} \right] \underline{\mathbf{A}}_4^{-1} \underline{\mathbf{FF}}_2 \tag{144}$$

$$V_{fab} = \frac{1.}{\bar{b}^2} QR \left[v\,v \right] \underline{\mathbf{A}}_4^{-1} \underline{\mathbf{FF}}_2 \tag{145}$$

$$V_{fba} = \frac{1.}{\bar{b}^2} QR \left[v\,v \cdot e^{\lambda\alpha} \right] \underline{\mathbf{A}}_4^{-1} \underline{\mathbf{FF}}_2 \tag{146}$$

where [m m] , [mm $e^{\lambda\alpha}$] , [t t] , [t t $e^{\lambda\alpha}$], [v v] and [v v$e^{\lambda\alpha}$] are given in Appendix C.

The general solutions for the bending, twisting and warping moments and for the vertical shears can be obtained by combining equations (110) and (141)–(146) and they are

$$M_{ab} = M(0) + M_{fab} \tag{147}$$

$$M_{ba} = M(\alpha) + M_{fba} \tag{148}$$

$$T_{ab} = T(0) + T_{fab} \tag{149}$$

$$T_{ba} = T(\alpha) + T_{fba} \tag{150}$$

$$V_{ab} = V(0) + V_{fab} \tag{151}$$

$$V_{ba} = V(\alpha) + V_{fba} \tag{152}$$

or in the following form

$$\underline{\mathbf{F}}_4 = \left(\frac{EI}{R^3}\right)\underline{\mathbf{H}}_4 \cdot \underline{\mathbf{X}}_4 + \underline{\widetilde{\mathbf{F}}}_4 = \left(\frac{EI}{R^3}\right)\underline{\mathbf{H}}_4 \cdot \underline{\mathbf{A}}_4^{-1} \cdot \underline{\mathbf{D}}_4 + \underline{\widetilde{\mathbf{F}}}_4$$

$$= \underline{\mathbf{S}}_4 \cdot \underline{\mathbf{D}}_4 + \underline{\widetilde{\mathbf{F}}}_4 \tag{153}$$

where

$$\underline{\widetilde{\mathbf{F}}}_4 = \left[M_{fab} M_{fba} T_{fab} \; T_{fba} \; V_{fab} \; V_{fba} \right]^T \tag{154}$$

and $\underline{\mathbf{D}}_4$, $\underline{\mathbf{A}}_4$, $\underline{\mathbf{X}}_4$, $\underline{\mathbf{F}}_4$, $\underline{\mathbf{H}}_4$, $\underline{\mathbf{S}}_4$ have the same forms as those for free vibration case (c).

CHAPTER 4

NUMERICAL EXAMPLES

In the following examples, the effects of opening angles, warping and torsional inertia on the free and forced vibrations of beams having same height and different thickness will be investigated.

Example 1.

A three-span horizontally circular curved beam undergoing vertically free vibrations as shown in Fig. 5 will be analyzed for natural frequencies.

In this example the beam has height h =12 in., thickness t = 12 in. and radius of curvature R = 60 in. The properties of the beam materials are
$E = 29\ (10)^6$ psi and $G = 11.2\ (10)^6$ psi.

According to Oden [22] , the torsional constant J and the warping constant Γ

can be expressed as

$$J = 0.141 \cdot h \cdot t^3, \qquad \Gamma = \frac{h^3 t^3}{144}$$

For the beam supported as shown in Fig.5, it is assumed that no deflection or twist is allowed at the joints, thus the boundary conditions are

$$Y_a = Y_b = Y_c = Y_d = 0 \text{ and } B_a = B_b = B_c = B_d = 0$$

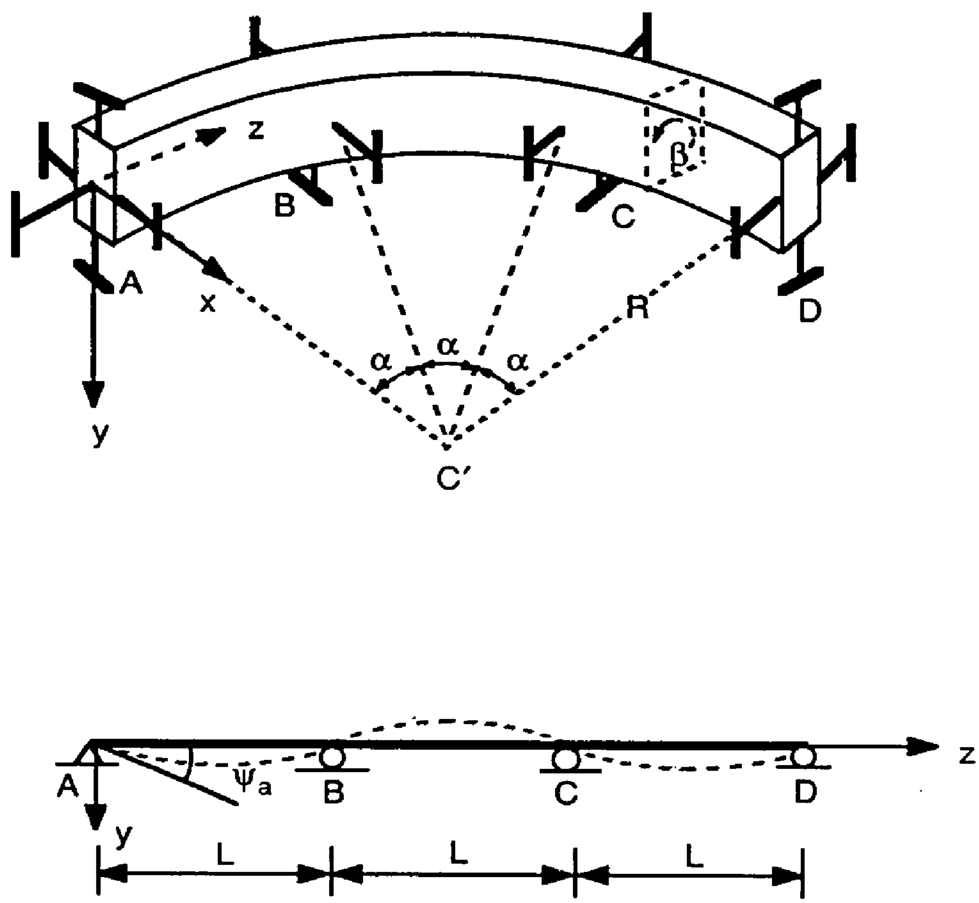

Figure 5. Three-span horizontally curved beam.

Consider first the effects of rotatory inertia r, shear deformation s, torsional inertia η and warping w, i.e., case (a) as described previously. The dynamic equilibrium equations for moments at A, B, C and D are

$$\left.\begin{aligned} &M_{ab} = 0 \\ &M_{ba} + M_{bc} = 0 \\ &M_{cb} + M_{cd} = 0 \\ &M_{dc} = 0 \end{aligned}\right\} \tag{155}$$

From expression (101) one obtains M_{ab} as

$$M_{ab} = \frac{EI}{R^2}\left[S_{11}R\Psi_a + S_{12}R\Psi_b + S_{17}R^2\tau_a + S_{18}R^2\tau_b\right] \tag{156}$$

According to the second assumption, the vibrations of the beam are considered small; and also the rate of change of the twisting angle β' is much smaller than the bending slope ψ; therefore β' can be neglected. Thus equation (87) reduces to $R^2\tau(\theta) = R\,\Psi(\theta)$.

Equations (156) now becomes

$$M_{ab} = \frac{EI}{R^2}\left[\left\{S_{11} + S_{17}\right\}_a R\Psi_a + \left\{S_{12} + S_{18}\right\}_a R\,\Psi_b\right] \tag{157}$$

Similarly, one can also obtain

$$M_{ba} = \frac{EI}{R^2}\left[\left\{S_{21}+S_{27}\right\}_a R\Psi_a + \left\{S_{22}+S_{28}\right\}_a R\Psi_b\right] \tag{158}$$

$$M_{bc} = \frac{EI}{R^2}\left[\left\{S_{11}+S_{17}\right\}_b R\Psi_b + \left\{S_{12}+S_{18}\right\}_b R\Psi_c\right] \tag{159}$$

$$M_{cb} = \frac{EI}{R^2}\left[\left\{S_{21}+S_{27}\right\}_b R\Psi_b + \left\{S_{22}+S_{28}\right\}_b R\Psi_c\right] \tag{160}$$

$$M_{cd} = \frac{EI}{R^2}\left[\left\{S_{11}+S_{17}\right\}_c R\Psi_c + \left\{S_{12}+S_{18}\right\}_c R\Psi_d\right] \tag{161}$$

$$M_{dc} = \frac{EI}{R^2}\left[\left\{S_{21}+S_{27}\right\}_c R\Psi_c + \left\{S_{22}+S_{28}\right\}_c R\Psi_d\right] \tag{162}$$

Substituting equations (157)–(162) into equations (155) yeilds the following frequency equation :

$$\begin{vmatrix} (S_{11}+S_{17})_a & (S_{12}+S_{18})_a & 0 & 0 \\ (S_{21}+S_{27})_a & (S_{22}+S_{28})_a+(S_{11}+S_{17})_b & (S_{12}+S_{18})_b & 0 \\ 0 & (S_{21}+S_{27})_b & (S_{22}+S_{28})_b+(S_{11}+S_{17})_c & (S_{12}+S_{18})_c \\ 0 & 0 & (S_{21}+S_{27})_c & (S_{22}+S_{28})_c \end{vmatrix} = 0$$

----------(163)

Since the beam for this example has three identical spans, expression (163) can be rewritten as

$$\begin{vmatrix} (S_{11}+S_{17}) & (S_{12}+S_{18}) & 0 & 0 \\ (S_{21}+S_{27}) & (S_{22}+S_{28})+(S_{11}+S_{17}) & (S_{12}+S_{18}) & 0 \\ 0 & (S_{21}+S_{27}) & (S_{22}+S_{28})+(S_{11}+S_{17}) & (S_{12}+S_{18}) \\ 0 & 0 & (S_{21}+S_{27}) & (S_{22}+S_{28}) \end{vmatrix} = 0 \tag{164}$$

A computer program has been written to find the natural frequencies and it is given in Appendix D. The program uses two I MSL library subroutines and they are

(i) ZPOLR--- for finding the roots of the characteristic equation, and

(ii) LEQ2C---compute the determinant values for complex matrices.

For case (c) with warping effect being neglected, the expression (164) can be written as

$$\begin{vmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{21} & S_{22}+S_{11} & S_{12} & 0 \\ 0 & S_{21} & S_{22}+S_{11} & S_{12} \\ 0 & 0 & S_{21} & S_{22} \end{vmatrix} = 0 \tag{165}$$

The computer results for the first 5 modes of the natural frequencies for cases (a), (b) and (c), respectively are given in Table 1 and are also been plotted In Figure 6 for comparison.

CASE / MODE / α	(a) $r \neq 0, s \neq 0, w \neq 0, \eta \neq 0$					(b) $r \neq 0, s \neq 0, w \neq 0, \eta = 0$					(c) $r \neq 0, s \neq 0, \eta \neq 0, w = 0$				
	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th
30	28.28	32.59	40.95	84.13	86.25	28.50	32.76	41.13	84.38	86.73	27.45	31.72	40.17	76.48	79.79
45	13.13	16.27	22.11	45.78	48.38	13.30	16.39	22.25	46.05	48.72	12.86	15.95	21.78	43.71	46.35
60	7.11	9.30	13.41	28.18	30.57	7.20	9.39	13.53	28.42	30.81	6.95	9.15	13.26	27.45	29.84
75	4.17	5.79	8.81	18.69	20.70	4.22	5.85	8.88	18.90	20.90	4.07	5.70	8.72	18.36	20.34
90	2.56	3.81	6.09	13.07	14.71	2.58	3.85	6.15	13.22	14.88	2.48	3.70	6.05	12.86	14.51

Table 1. Natural Frequencies for Example 1 (h = 12 in., t =12 in.)

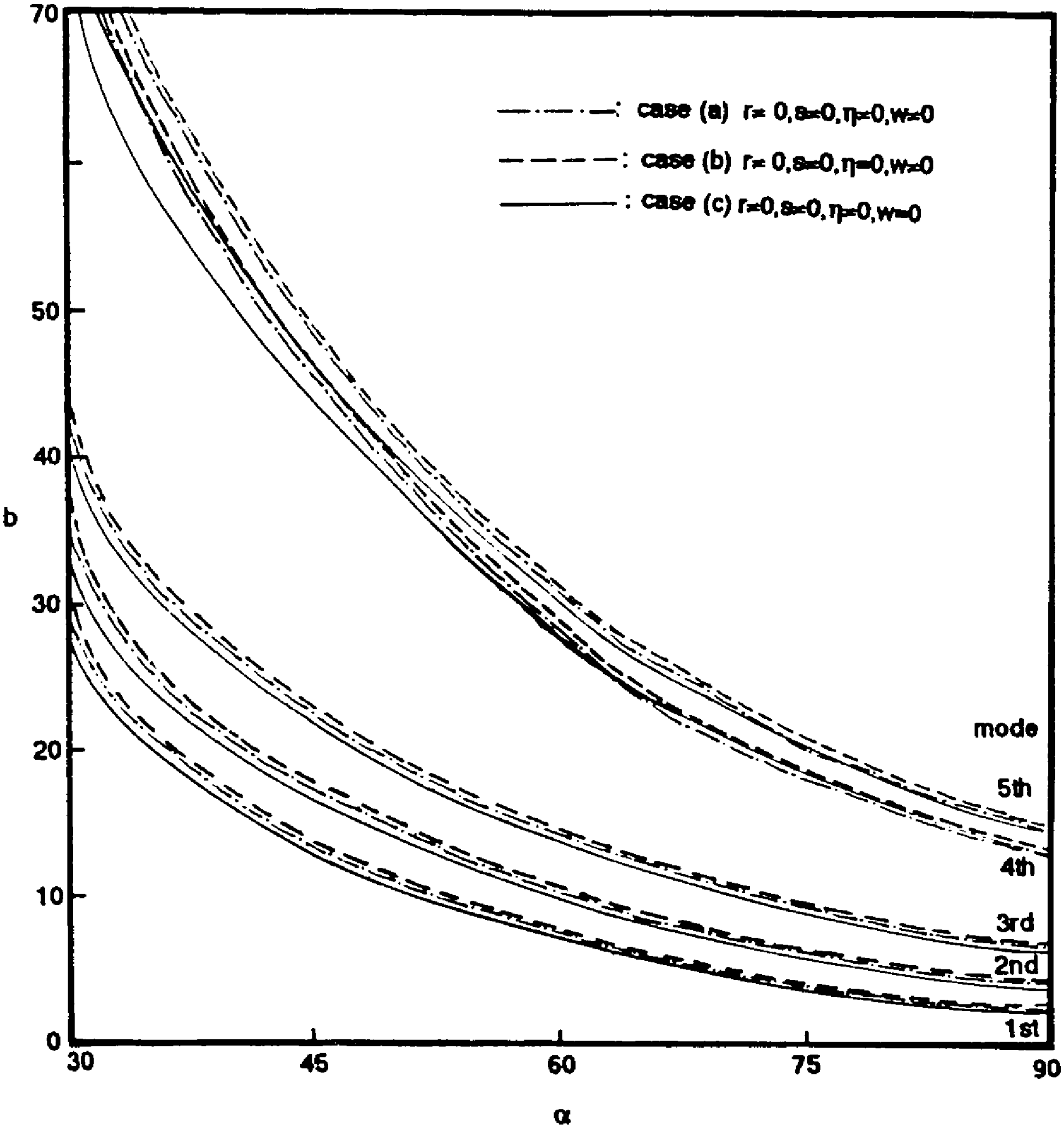

Figure 6. Variation of b with α for cases (a), (b) and (c) with dimensions h = 12 in, t = 12 in.

Example 2.

The same curved beam as example 1 will now be analyzed for natural frequencies with the following beam dimensions:

$h = 12$ in., $t = 6$ in., $J = 0.229 \cdot h \cdot t^3$ [22].

Applying the same procedure as used in the previous example, one obtains another three sets of the natural frequencies for cases (a), (b) and (c) as shown in the Table 2 and also Figure 7.

Example 3.

In this example, the dimensions of the beam discussed in the previous examples become

$h = 12$ in., $t = 4$ in., and $J = 0.263 \cdot h \cdot t^3$.

Again the three sets of the first 5 modes of the natural frequencies for cases (a), (b) and (c), respectively can be obtained and are shown in Table 3. The differences among these cases are found to be very small.

CASE / MODE / α	(a) $r \neq 0, s \neq 0, \eta \neq 0, w \neq 0$					(b) $r \neq 0, s \neq 0, w \neq 0, \eta = 0$					(c) $r \neq 0, s \neq 0, \eta \neq 0, w = 0$				
	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th
30	32.60	40.46	55.86	116.16	125.24	32.75	40.64	56.08	116.51	125.59	32.37	40.27	55.72	116.41	125.62
45	14.23	18.38	26.64	56.36	62.46	14.33	18.51	26.80	56.66	62.72	14.12	18.30	26.58	56.22	62.30
60	7.55	10.12	15.18	32.52	36.57	7.60	10.18	15.26	32.69	36.77	7.47	10.05	15.12	32.37	36.45
75	4.41	6.17	9.60	20.78	23.64	4.44	6.22	9.67	20.90	23.78	4.36	6.14	9.58	20.67	23.55
90	2.71	4.04	6.52	14.20	16.32	2.73	4.06	6.55	14.29	16.42	2.67	4.01	6.50	14.11	16.26

Table 2. Natural Frequencies for Example 2 (h =12 in., t = 6 in.)

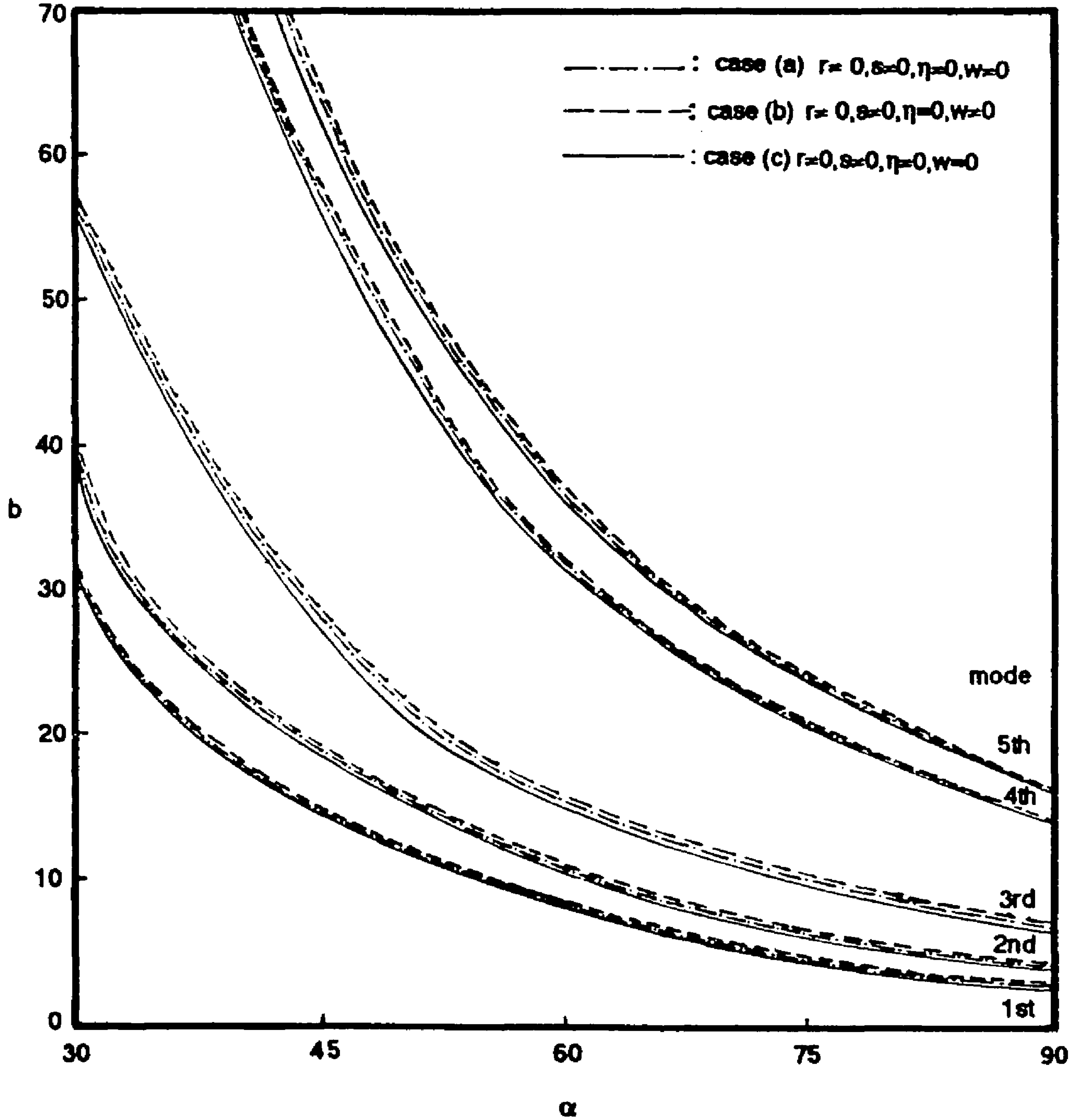

Figure7. Variation of b with α for cases (a), (b) and (c) with dimensions h = 12 in, t = 6 in.

CASE / MODE / α	(a) $r \neq 0, s \neq 0, \eta \neq 0, w \neq 0$					(b) $r \neq 0, s \neq 0, w \neq 0, \eta = 0$					(c) $r \neq 0, s \neq 0, \eta \neq 0, w = 0$				
	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th
30	33.69	42.76	60.96	128.16	141.41	33.83	42.93	61.21	128.77	142.23	33.50	42.60	60.90	129.27	143.25
45	14.43	18.91	27.84	59.38	66.73	14.56	19.02	28.01	59.77	67.24	14.38	18.84	27.82	59.40	66.90
60	7.65	10.30	15.60	33.62	38.20	7.68	10.35	15.67	33.77	38.39	7.58	10.25	15.55	33.50	38.11
75	4.41	6.25	9.79	21.26	24.35	4.49	6.30	9.83	21.37	24.48	4.42	6.23	9.76	21.17	24.29
90	2.75	4.09	6.61	14.46	16.69	2.76	4.10	6.64	14.52	16.78	2.72	4.06	6.60	14.38	16.64

Table 3. Natural Frequencies for Example 3 (h = 12 in., t = 4 in.)

Example 4

A three-span horizontally circular curved beam of a square cross-section, is subjected to a vertical dynamic distributed load as shown in Figure 8.

In this example the previous assumption that no deflections or twists are allowed at the joints will again be considered. The conditions of dynamic equilibrium at joints A, B, C and D may be written as

$$
\begin{aligned}
&M_{ab} = 0 \\
&M_{ba} + M_{bc} = 0 \\
&M_{cb} + \overline{M}_{cd} = 0 \\
&\overline{M}_{dc} = 0
\end{aligned}
\tag{166}
$$

From Figure 9 one can write

$$\overline{M}_{cd} = M_{cd} + M_{fcd} \tag{167}$$

$$\overline{M}_{dc} = M_{dc} - M_{fdc} \tag{168}$$

where M_{fcd} and M_{fdc} are the fixed-end moments for a uniformly distributed load acting on span CD and M_{cd} and M_{dc} are the joint moments.

Substituting equations (167) and (168) into equations (166) gives

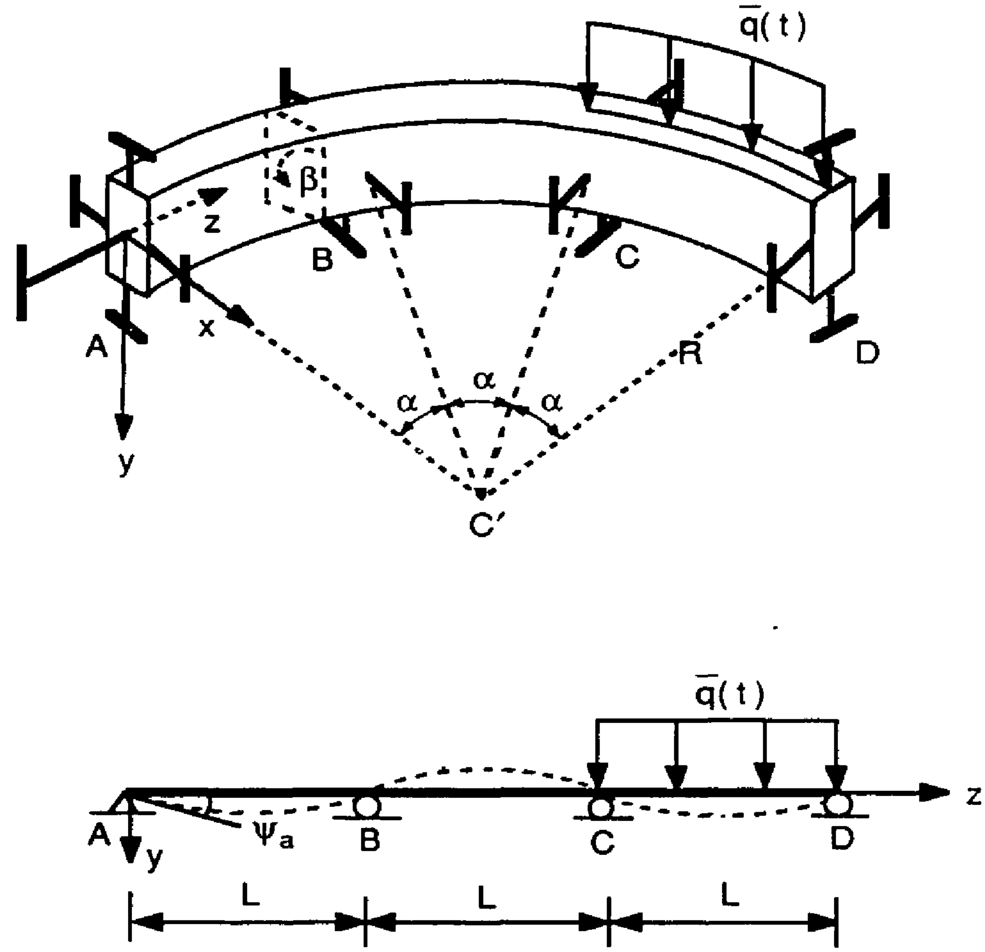

Figure 8. Three–span horizontally curved beam with a uniformly distributed load $\bar{q}$ (t) at span CD.

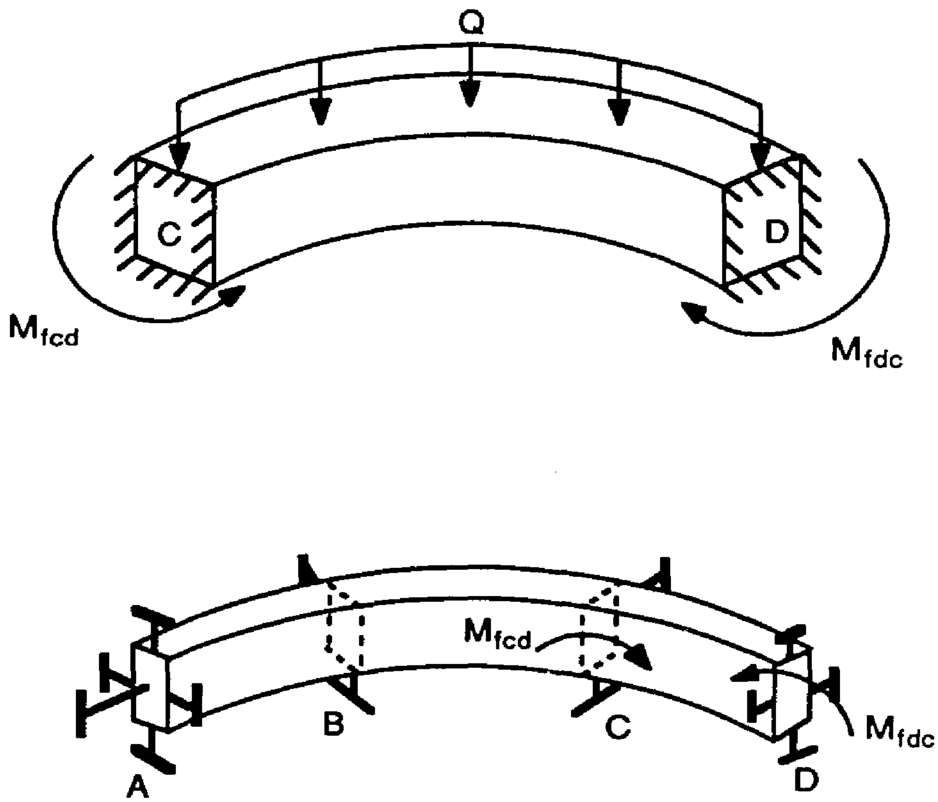

Figure 9. Combined moments of a curved beam sysytem with common factor $e^{i\bar{\Omega}t}$ omitted.

$$\left.\begin{aligned} &M_{ab} = 0 \\ &M_{ba} + M_{bc} = 0 \\ &M_{bc} + M_{cd} + M_{fcd} = 0 \\ &M_{dc} - M_{fcd} = 0 \end{aligned}\right\} \tag{169}$$

Consider case (a) ($r \neq 0, s \neq 0, \eta \neq 0, w \neq 0$) and

case (b) ($r \neq 0, s \neq 0, \eta = 0, w \neq 0$). Form expressions (95)-(102) one obtains

$$M_{ab} = \frac{EI}{R^2}\left[\left\{S_{11} + S_{17}\right\} R\Psi_a + \left\{S_{12} + S_{18}\right\} R\,\Psi_b\right] \tag{170}$$

$$\begin{aligned} M_{ba} + M_{bc} = \frac{EI}{R^2}\Big[&\left\{S_{21} + S_{27}\right\} R\Psi_a + \left(\left\{S_{22} + S_{28}\right\} + \left\{S_{11} + S_{17}\right\}\right) R\Psi_b \\ &+ \left\{S_{12} + S_{18}\right\} R\Psi_c\Big] \end{aligned} \tag{171}$$

$$\begin{aligned} &M_{cb} + M_{cd} = \frac{EI}{R^2}\Big[\left\{S_{21} + S_{27}\right\} R\Psi_b + \left(\left\{S_{22} + S_{28}\right\} + \left\{S_{11} + S_{17}\right\}\right) R\Psi_c \\ &+ \left\{S_{12} + S_{18}\right\} R\,\Psi_d\Big] + M_{fcd} \end{aligned} \tag{172}$$

$$M_{dc} = \frac{EI}{R^2}\left[\left\{S_{21} + S_{27}\right\} R\Psi_c + \left\{S_{22} + S_{28}\right\} R\,\Psi_d\right] - M_{fdc} \tag{173}$$

Substituting Expressions (170)–(173)into equations (169) one has the equations in the following matrix form :

$$\underline{SS}_1 \cdot \underline{DD}_1 = \underline{E}_{22} \tag{174}$$

in which the fixed-end force matrix $\underline{E}_{22}$, stiffness matrix $\underline{SS}_1$ and the displacement matrix $\underline{DD}_1$ are given as follows :

$$\underline{E}_{22} = \begin{bmatrix} 0 \\ 0 \\ -M_{fcd} \\ +M_{fdc} \end{bmatrix} \qquad \underline{DD}_1 = \begin{bmatrix} R\Psi_a \\ R\Psi_b \\ R\Psi_c \\ R\Psi_d \end{bmatrix}$$

$$\underline{SS}_1 = \begin{bmatrix} (S_{11}+S_{17}) & (S_{12}+S_{18}) & 0 & 0 \\ (S_{21}+S_{27}) & (S_{22}+S_{28})+(S_{11}+S_{17}) & (S_{12}+S_{18}) & 0 \\ 0 & (S_{21}+S_{27}) & (S_{22}+S_{28})+(S_{11}+S_{17}) & (S_{12}+S_{18}) \\ 0 & 0 & (S_{21}+S_{27}) & (S_{22}+S_{28}) \end{bmatrix}$$

and where M_{fcd} and M_{fdc} can be obtained from equations (121) and (122) as

$$M_{fcd} = \frac{1.}{\bar{b}^2} QR^2 \left[m \right] \underline{A}_3^{-1} \underline{FF} \tag{175}$$

$$M_{fdc} = \frac{1.}{\bar{b}^2} QR^2 \left[m \cdot e^{\lambda\alpha} \right] \Delta_3^{-1} \underline{FF} \tag{176}$$

Solving equations (174) for $\underline{DD}_1$ one has

$$\underline{DD}_1 = \underline{SS}_1^{-1} \cdot \underline{F}_{22} \tag{177}$$

A computer program has been written to find the unknown displacements ψ_a, ψ_b, ψ_c, ψ_d from equation (177) and the moments M_{cd}, M_{fcd} and M_{fdc} from equations (167), (175) and (176). In the process the same two IMSL subroutines ZPOLR and LEQ2C as for the free vibration cases have been used.

Using this program, the values of the moment coefficient f_{cd} ($= M_{cd}/QR^2$) were obtained for the first five modes of vibration for $\alpha = 30^\circ$, 60° and 90° with the frequency parameter b varying from 0 to 200. The results are shown in the Figures 10 to 11 with the effects of shear, rotatory inertia, warping and torsional inertia all being considered.

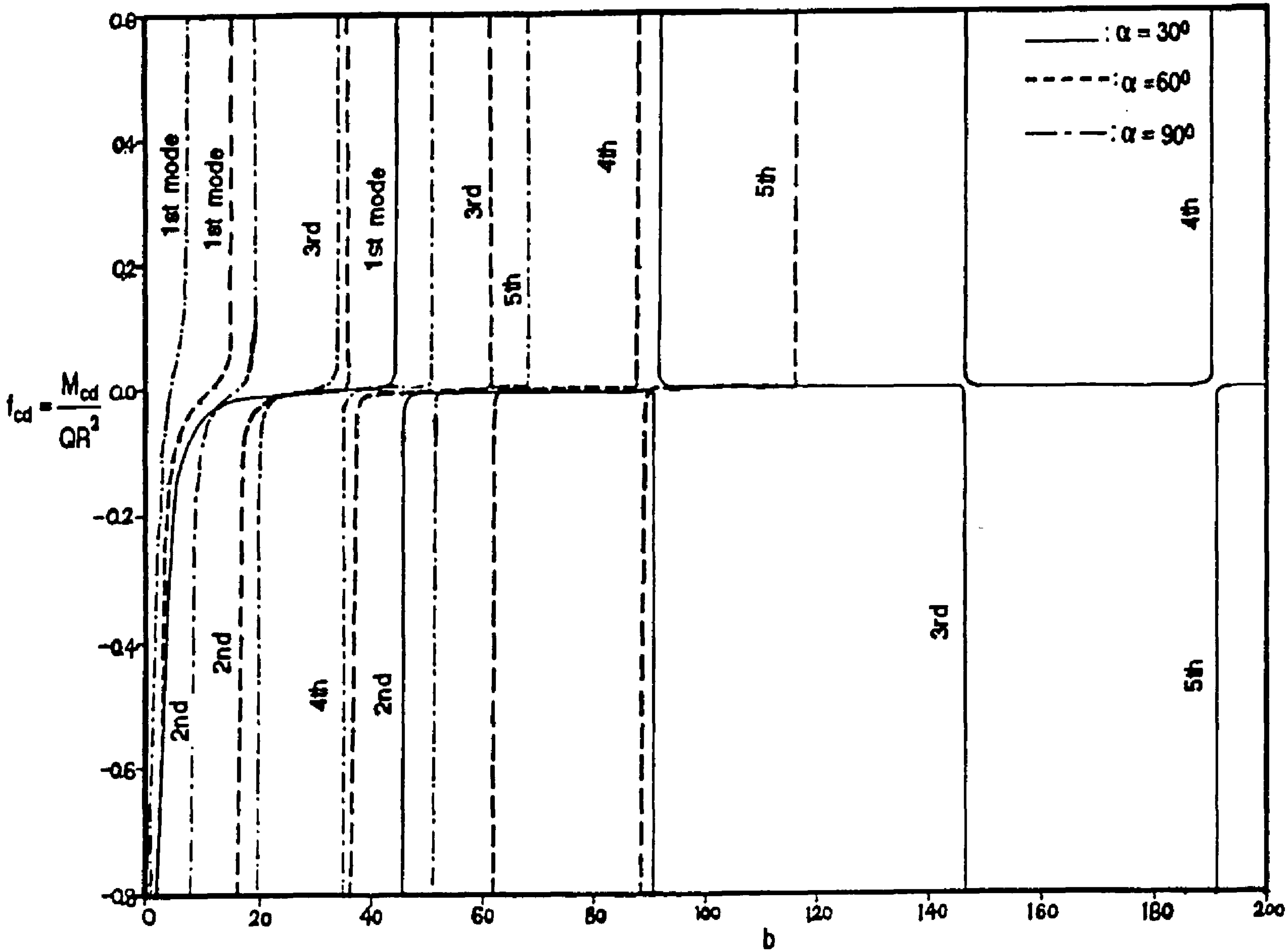

Figure 10. Case (b) for f_{cd} vs b with opening angle $\alpha = 30^0, 60^0, 90^0$

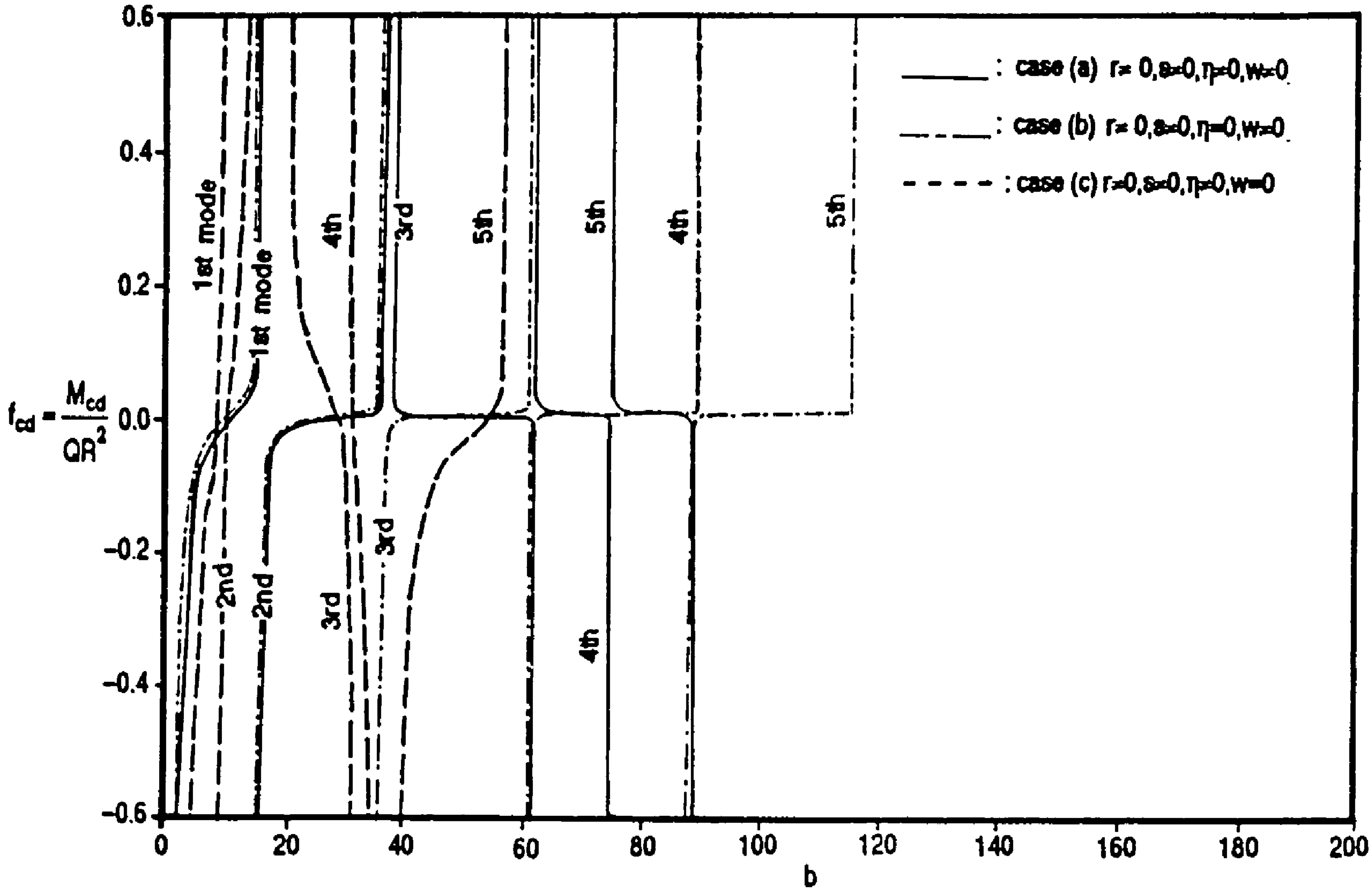

Figure 11. Variation of f_{cd} with b and opening angle $\alpha = 60^0$ for cases (a), (b) and (c).

CHAPTER 5

DISCUSSIONS AND CONCLUSIONS

5.1 Discussions

Four examples have been given to show the effects of rotatory inertia, shear deformation, warping and opening angles on natural frequencies and on moments of the curved beam. In each example, three cases are studied to show the influence of these various effects.

The first three examples are for free vibrations of curved beam with different cross-sections; a square one, a rectangular one and a thinner rectangular one. It is noted that as the beam become slender, the effects of rotatory inertia r, shear deformatiom s, torsional inertia η and warping w are all become small.

Table 1, 2 and 3 show that the natural frequencies of the beam increase as r and s decrease; and the effects of r and s on the natural frequencies increase with decreasing opening angle. From equation (18), it can be seen that the effects of warping and torsional inertia are in opposed direction. These opposite effects also can be observed from Figures 6 and 7 for the cases of free vibration and from Figures 10 to 11 for forced vibration case.

In these Figures, it is seen that the differences among cases (a), (b) and (c) are small at lower modes of vibration. However, for higher modes, the influence of trosional inertia on the natural frequencies of the beam become important. Also, for thick beams these curves reveal the significance of warping effect on natural frequencies of the beam.

The last example is for the case of forced vibrations of the beam. Again the same cases as for free vibrations are being investegated. From Figure 10, it is noticed that, for a beam of given section, the load frequency decreases as the opening angle increases. Figure 11 reveals that the significant effect of torsional inertia on the joint moment of a beam having the same section and opening angle.

5.2 Conclusions

From the previous discusions the following conclusions may be drawn :

1. For thick beams, the contributing of warping effect on the natural frequencies of the beam increases as the beam thickness increases.

2. At lower modes of vibration the warping effect on the beam is insignificant and therefore can be neglected.

3. For higher modes the effects of torsinal inertia and warping on the beam is become very important, and thus the both effects need to be considered.

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APPENDIX A

EXPANSION of OPERTOR Δ IN EXPRESSION (25)

$$\Delta = L_1\ (L_5 L_9 - L_6 L_8) - L_2\ (L_4 L_9 - L_6 L_7) + L_3\ (L_4 L_8 - L_5 L_7)$$

$$= (D^2 + b^2 s^2)(Rs^2)((-1./s^2) - \rho + b^2 r^2 + D^2 + wD^2)(1. - \rho D^2 - \eta b^2 r^2$$

$$+ wD^4) - (D^2 + b^2 s^2)(Rs^2)(-D - \rho D + wD^3)^2 - (-RD^2)(1. - \rho D^2 - \eta b^2 r^2$$

$$+ wD^4)$$

$$= \left(D^2 + b^2 s^2\right)\left(Rs^2\right)\left\{\left(-\frac{1.}{s^2} + \frac{\rho D^2}{s^2} - \frac{\eta b^2 r^2}{s^2} - \frac{wD^2}{s^2} - \rho + (\rho D)^2 + \rho\eta b^2 r^2 - \rho wD^4\right.\right.$$

$$+ b^2 r^2 - \rho b^2 r^2 D^2 - \eta b^4 r^4 + b^2 r^2 wD^4 + D^2 - \rho D^4 - \eta b^2 r^2 D^2 + wD^6 + wD^2 - \rho w D^4$$

$$\left.\left. - \eta w b^2 r^2 D^2 + w^2 D^6\right) - \left(D^2 + \rho^2 D^2 + w^2 D^6 + 2\rho D^2 - 2wD^4 - 2\rho wD^4\right)\right\}$$

$$+ RD^2\left(1. - \rho D^2 - \eta b^2 r^2 + wD^4\right)$$

$$= \left(Rs^2\right)\left\{ -\frac{D^2}{s^2} - b^2 + \frac{\rho D^4}{s^2} + \rho b^2 D^2 + \frac{\eta b^2 r^2 D^2}{s^2} + \eta b^4 r^2 - \frac{wD^6}{s^2} - wb^2 D^4 - \rho D^2\right.$$

$$- \rho b^2 s^2 + \rho\eta b^2 r^2 D^2 + \rho\eta b^4 r^2 s^2 + b^2 r^2 D^2 + b^4 r^2 s^2 - \rho b^2 r^2 D^4 - \rho b^4 r^2 s^2 D^2 - \eta b^4 r^2 D^2$$

$$-\eta b^6 r^4 s^2 + wb^2 s^2 D^6 + wb^4 r^2 s^2 D^4 - \rho D^6 - \rho b^2 s^2 D^4 - \eta b^2 r^2 D^4 - \eta b^4 r^2 s^2 D^2 + wD^8$$
$$+ wb^2 s^2 D^6 + wD^4 + wb^2 s^2 D^2 - \eta wb^2 r^2 D^4 - \eta wb^4 r^2 s^2 D^2 - 2\rho D^4 - 2\rho b^2 s^2 D^2$$
$$+ 2wD^6 + 2wb^2 s^2 D^4 \Big\} + RD^2\left(1. - \rho D^2 - \eta b^2 r^2 + wD^4\right)$$

Simplifying one obtains

$$\Delta = \{wD^8 + (wb^2 r^2 - \rho + wb^2 s^2 + 2w)D^6$$
$$+ (-wb^2 - \rho b^2 r^2 + wb^4 r^2 s^2 - \rho b^2 s^2 - \eta b^2 r^2 + w - \eta wb^2 r^2 - 2\rho + 2wb^2 s^2)$$
$$\cdot D^4 + (\rho b^2 - \rho + \rho\eta b^2 r^2 + b^2 r^2 - \rho b^4 r^2 s^2 - \eta b^4 r^2 - \eta b^4 r^2 s^2 + wb^2 s^2 - \eta wb^4$$
$$r^2 s^2 - 2\rho b^2 s^2) D^2 + (-b^2 + \eta b^4 r^2 - \rho b^2 s^2 + \rho\eta b^4 r^2 s^2 + b^4 r^2 s^2 - \eta b^6 r^2 s^2)\} \quad (27)$$

APPENDIX B

EXPRESSIONS FOR **[m], [t], [v], [q]** MATRICES

$$[m] = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 \end{bmatrix}$$

$$[m \cdot e^{\lambda\alpha}] = \begin{bmatrix} m_1 e^{\lambda\alpha_1} & m_2 e^{\lambda\alpha_2} & m_3 e^{\lambda\alpha_3} & m_4 e^{\lambda\alpha_4} & m_5 e^{\lambda\alpha_5} & m_6 e^{\lambda\alpha_6} & m_7 e^{\lambda\alpha_7} & m_8 e^{\lambda\alpha_8} \end{bmatrix}$$

$$[t] = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \end{bmatrix}$$

$$[t \cdot e^{\lambda\alpha}] = \begin{bmatrix} t_1 e^{\lambda\alpha_1} & t_2 e^{\lambda\alpha_2} & t_3 e^{\lambda\alpha_3} & t_4 e^{\lambda\alpha_4} & t_5 e^{\lambda\alpha_5} & t_6 e^{\lambda\alpha_6} & t_7 e^{\lambda\alpha_7} & t_8 e^{\lambda\alpha_8} \end{bmatrix}$$

$$[v] = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{bmatrix}$$

$$[v \cdot e^{\lambda\alpha}] = \begin{bmatrix} v_1 e^{\lambda\alpha_1} & v_2 e^{\lambda\alpha_2} & v_3 e^{\lambda\alpha_3} & v_4 e^{\lambda\alpha_4} & v_5 e^{\lambda\alpha_5} & v_6 e^{\lambda\alpha_6} & v_7 e^{\lambda\alpha_7} & v_8 e^{\lambda\alpha_8} \end{bmatrix}$$

$$[q] = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 & q_8 \end{bmatrix}$$

$$[q \cdot e^{\lambda\alpha}] = \begin{bmatrix} q_1 e^{\lambda\alpha_1} & q_2 e^{\lambda\alpha_2} & q_3 e^{\lambda\alpha_3} & q_4 e^{\lambda\alpha_4} & q_5 e^{\lambda\alpha_5} & q_6 e^{\lambda\alpha_6} & q_7 e^{\lambda\alpha_7} & q_8 e^{\lambda\alpha_8} \end{bmatrix}$$

APPENDIX C

EXPRESSIONS FOR [mm], [t t], [v v] MATRICES

$$[mm] = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \end{bmatrix}$$

$$[mm \cdot e^{\lambda\alpha}] = \begin{bmatrix} m_1 e^{\lambda_1 \alpha} & m_2 e^{\lambda_2 \alpha} & m_3 e^{\lambda_3 \alpha} & m_4 e^{\lambda_4 \alpha} & m_5 e^{\lambda_5 \alpha} & m_6 e^{\lambda_6 \alpha} \end{bmatrix}$$

$$[t\ t] = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{bmatrix}$$

$$[t\ t \cdot e^{\lambda\alpha}] = \begin{bmatrix} t_1 e^{\lambda_1 \alpha} & t_2 e^{\lambda_2 \alpha} & t_3 e^{\lambda_3 \alpha} & t_4 e^{\lambda_4 \alpha} & t_5 e^{\lambda_5 \alpha} & t_6 e^{\lambda_6 \alpha} \end{bmatrix}$$

$$[v\ v] = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}$$

$$[v\ v \cdot e^{\lambda\alpha}] = \begin{bmatrix} v_1 e^{\lambda_1 \alpha} & v_2 e^{\lambda_2 \alpha} & v_3 e^{\lambda_3 \alpha} & v_4 e^{\lambda_4 \alpha} & v_5 e^{\lambda_5 \alpha} & v_6 e^{\lambda_6 \alpha} \end{bmatrix}$$

APPENDIX D

```
C     ****************************************
C               PROGRAM  OPH3.FOR
C     ****************************************


C       THIS IS A COMPUTER PROGRAM FOR FINDING THE NATURAL FREQUENCIES
C       OF OUT-OF-PLANE VIBRATIONS OF THREE-SPAN CIRCULAR CURVED BEAM
C       CONSIDERING SHEAR DEFORMATION, ROTATORY INERTIAS AND WARPING
C       EFFECTS.

C     ************************************
C       DECLARE ALL VARIABLES.
C     ************************************

        IMPLICIT REAL*8 (A-H,O-Z)
        COMPLEX*16 ALAM(8),FN,UU,UD,UN,PN,VN,MN,CE,BM(8,8),WA(80),WA2(24)
       &,QN,TN,DET,A(8,8),DM(8,8),BC(4,4),DUMMY(8,8)
        REAL*8 G(9),WK(8),WK2(4)

C     ***************************************************************
C       WE ARE GOING TO USE LAGUERRE'S METHOD FOR FINDING QUADRATIC
C       FACTORS OF POLYNOMIALS.
C       SOLUTIONS OF POLYNOMIAL UP TO X**100 WITH REAL COEFFICIENTS.
C       NDEG DESIGNATES THE DEGREE OF THE POLYNOMIALS.
C     ***************************************************************

        OPEN(UNIT=1,FILE='OPV.DAT',STATUS='OLD')
        READ(1,*)EX,H,T,GR,CASE,ALPHA
        WRITE(2,11)EX,H,T,GR,CASE,ALPHA
     11 FORMAT(6X,'EXAMPLE=',F5.1,2X,'H=',F3.0,' INCHES,',' T=',F3.0,
       &' INCHES,',' CURVATURE=',F3.0,' INCHES,',//,6X,'CASE=',F3.0,
       &6X,' ALPHA=',F3.0,' DEGREES.',/)
        READ(1,*)IN,GG,C1
        WRITE(2,12)IN,GG,C1
     12 FORMAT(6X,'NUMBER OF ITERATION=',I5,5X,'INCREMENT INTERVAL=',
       &F7.2,//,6X,'COEFFICIENT OF TORSIONAL CONSTANT=',F6.3,//)
        DO 500 ID=1,IN
        FG=0.D0
        B=FG+(ID-1)*GG
        IF (ID.EQ.1)B=.01D0
        THETA=ALPHA

C     **************************************************
C       CALCULATE THE ROOTS OF CHARACTERISTIC EQUATION.
C     **************************************************

        CALL CHARAC(G,B,B2,R2,S2,THETA,ETA,ELO,W,H,T,C1,GR)
      9 NDEG=8
```

```
      CALL ZPOLR(G,NDEG,ALAM,IER)
      DO 604 NDEG=1,8
      WRITE(2,44) (NDEG,ALAM(NDEG))
  604 CONTINUE
  44  FORMAT(5X,'LAMBDA(',I2,') =',5X,2(F10.4,5X))

C    ****************************************
C      SETUP "A" AND "H" MATRICES.
C    ****************************************

  80  DO 100 J=1,8
      FN=(ALAM(J)**2+B2*S2)/ALAM(J)
      UU=(1.DO+ELO-W*ALAM(J)**2)*(ALAM(J)**2+B2*S2)
      UD=(1.DO-ELO*ALAM(J)**2-ETA*B2*R2+W*ALAM(J)**4)
      IF(UD.EQ.0.DO) GO TO 500
      UN=UU/UD
      PN=(UN*ALAM(J)+FN)
      CE=CDEXP(ALAM(J)*THETA)
      A(1,J)=-FN
      A(2,J)=-FN*CE
      A(3,J)=UN
      A(4,J)=UN*CE
      A(5,J)=(1.,0.)
      A(6,J)=CE
      A(7,J)=PN
      A(8,J)=PN*CE
      MN=UN-ALAM(J)*FN
      TN=(ELO*(ALAM(J)*UN+FN)-W*(ALAM(J)**3*UN+ALAM(J)**2*FN))
      VN=(ALAM(J)-FN)/S2
      QN=-W*(ALAM(J)**2*UN+ALAM(J)*FN)
      BM(1,J)=MN
      BM(2,J)=-MN*CE
      BM(3,J)=TN
      BM(4,J)=-TN
      BM(5,J)=VN
      BM(6,J)=VN*CE
      BM(7,J)=QN
      BM(8,J)=-QN*CE
 100    CONTINUE

C     ****************************************************
C       CALCULATE INVERSE MATRIX FOR "A".
C     ****************************************************

        DO 601 I=1,8
        DO 601 J=1,8
        DUMMY(I,J)=(0.,0.)
        IF(I.EQ.J) DUMMY(I,J)=(1.,0.)
```

```
  601   CONTINUE
        IA=8
        IB=8
        N1=8
        M1=8
        IJOB=0
        CALL LEQ2C  (A,N1,IA,DUMMY,M1,IB,IJOB,WA,WK,IER)
        DET=(1.,0.)
        DO 602 I=1,N1
        IPVT=WK(I)
        IF(IPVT.NE.I)DET=-DET
        INDX=I+(I-1)*N1
        DET=DET*WA(INDX)
  602   CONTINUE
        WRITE(2,101)DET
  101   FORMAT(5X,2D15.8)
        IF(CDABS(DET).LT..1D-8)  GOTO 500

C     ******************************************
C       FORMAT STIFFNESS MATRIX S  (=DM).
C     ******************************************

C        DO 103 KI=1,6
C        WRITE(2,102)  (A(KI,KJ),KJ=1,6)
C  102   FORMAT(//,3(5x,2(2D13.4),/))
C  103   CONTINUE
        CALL PRODCT(8,8,8,BM,DUMMY,DM)
        DO 600 I=1,4
        DO 600 J=1,4
        DUMMY(I,J)=(0.,0.)
        IF(I.EQ.J)DUMMY(I,J)=(1.,0.)
  600   BC(I,J)=(0.,0.)

C     ***************************************************
C       FIND NATURAL FREQUENCIES FOR THREE-SPAN BEAM.
C     ***************************************************

        BC(1,1)=DM(1,1)+DM(1,7)
        BC(1,2)=DM(1,2)+DM(1,8)
        BC(2,1)=DM(2,1)+DM(2,7)
        BC(2,2)=DM(2,2)+DM(2,8)+BC(1,1)
        BC(2,3)=BC(1,2)
        BC(3,2)=BC(2,1)
        BC(3,3)=BC(2,2)
        BC(3,4)=BC(1,2)
        BC(4,3)=BC(2,1)
        BC(4,4)=DM(2,2)+DM(2,8)
```

```
      IA=4
      IB=4
      N2=4
      M2=4
      IJOB=0
      CALL LEQ2C(BC,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DET=(1.,0.)
      DO 603 I=1,N2
      IPVT2=WK2(I)
      IF(IPVT2.NE.I)DET=-DET
      INDX2=I+(I-1)*N2
      DET=DET*WA2(INDX2)
 603   CONTINUE
      WRITE(2,250)DET
 250  FORMAT(5X,' DET. OF STIFFNESS MATRIX =',D15.8,'+',D15.8,'I',//)
 500  CONTINUE
1000  STOP
      END

     SUBROUTINE CHARAC(G,B,B2,R2,S2,THETA,ETA,ELO,W,H,T,CI,GR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 G(9)
CC    XI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO X---AXIS
CC    YI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO Y---AXIS
CC    POI=XI+YI
CC    XJ=TORSIONAL CONSTANT
CC    UNITS: INCHES , POUNDS, SECONDS
CC
      DO 10 I=1,9
 10   G(I)=0.D0
      U=0.3D0
      E=29.D+6
      GE=11.2D+6
      A=H*T
      CK=(10.D0+(1.D0+U))/(12.D0+11.D0*U)
      XI=H*T**3/12.D0
      YI=T*H**3/12.D0
      XJ=CI*H*T**3
      POI=XI+YI
      ETA=POI/XI
      ELO=(GE*XJ)/(E*XI)
      GAMMA=H**3*T**3/144.D0
      W=GAMMA/(XI*GR**2)
      R=DSQRT(XI/(A*GR**2))
      S=DSQRT(2.D0*(1.D0+U)/CK)*R
      B2=B**2
      B4=B2**2
      B6=B2**3
```

```
      S2=S**2
      R2=R**2
      R4=R2**2
      G(1)=1.D0
      G(3)=(2.D0+B2*R2+B2*S2)-ELO/W
      G(5)=(1.D0-B2+2.D0*B2*S2+B4*S2*R2-ETA*B2*R2)
     &-(2.D0*ELO+ELO*B2*R2+ELO*B2*S2+ETA*B2*R2)/W
      G(7)=(B2*S2-ETA*B4*S2*R2)+(ELO*ETA*B2*R2+B2*R2+ELO*B2-ELO
     &-2.D0*ELO*B2*S2-ELO*B4*S2*R2-ETA*B4*R4-ETA*B4*S2*R2)/W
      G(9)=(B4*R2*S2+ETA*B4*R2+ELO*ETA*B4*R2*S2-B2-ELO*B2*S2
     &-ETA*B6*R4*S2)/W
      THETA=THETA*3.1415926539/180.D0
      WRITE(2,300) R,B,THETA
300   FORMAT(2X,'R =',F8.5,5X,'B =',F12.5,5X,' THETA =',F8.5,/)
      RETURN
      END

      SUBROUTINE PRODCT(N1,M2,N2,A,B,EE)
      COMPLEX*16 D,A(N1,N2),B(N2,M2),EE(N1,M2)
      DO 270 I=1,N1
      DO 270 J=1,M2
      D=DCMPLX(0.,0.)
      DO 280 K=1,N2
      D=D+A(I,K)*B(K,J)
280   CONTINUE
      EE(I,J)=D
270   CONTINUE
      RETURN
      END
```

```
C   ****************************************************
C              PROGRAM  OPNT3.FOR
C   ****************************************************

C      THIS IS A COMPUTER PROGRAM FOR FINDING THE NATURAL FREQUENCIES
C      OF OUT-OF-PLANE VIBRATIONS OF THREE-SPAN CIRCULAR CURVED BEAM
C      CONSIDERING SHEAR DEFORMATION, ROTATORY INERTIA(NEGLECT TORSIONAL
C      INERTIA) AND WARPING EFFECTS.

C    **********************************
C      DECLARE ALL VARIABLES.
C    **********************************

       IMPLICIT REAL*8 (A-H,O-Z)
       COMPLEX*16 ALAM(8),FN,UU,UD,UN,PN,CE,CEE,DEE,VN,MN,BM(8,8),CE1(8)
      &,QN,TN,DET,A(8,8),DM(8,8),BC(4,4),DUMMY(8,8),WA(80),WA2(24),CE2
       REAL*8 G(9),WK(8),WK2(4)

C   ************************************************************
C      WE ARE GOING TO USE LAGUERRE'S METHOD FOR FINDING QUADRATIC
C      FACTORS OF POLYNOMIALS.
C      SOLUTIONS OF POLYNOMIAL UP TO X**100 WITH REAL COEFFICIENTS.
C      NDEG DESIGNATES THE DEGREE OF THE POLYNOMIALS.
C   ************************************************************

       OPEN(UNIT=1,FILE='OPV.DAT',STATUS='OLD')
       READ(1,*)EX,H,T,GR,CASE,ALPHA
       WRITE(2,11)EX,H,T,GR,CASE,ALPHA
    11 FORMAT(6X,'EXAMPLE',F5.1,2X,'h=',F3.0,' INCHES,',' t=',F3.0
      &,' INCHES,',' CURVATURE=',F3.0,' INCHES,',//,6X,'CASE=',F3.0
      &,6X,' ALPHA=',F3.0,' DEGREES.',/)
       READ(1,*)IN,GG,C1
       WRITE(2,12)IN,GG,C1
    12 FORMAT(6X,'NUMBER OF ITERATION=',I5,5X,'INCREMENT INTERVAL=',
      &F7.2,//,6X,'COEFFICIENT OF TORSIONAL CONSTANT=',F6.3,//)
        FG=0.D0
C       WRITE(2,607) FG
  607  FORMAT(5X,'INITIAL B = ',F7.3,/)
       DO 500 ID=1,IN
       B=FG+(ID-1)*GG
       IF (ID.EQ.1)B=.01D0
       THETA=ALPHA

C   ************************************************
C      CALCULATE THE ROOTS OF CHARACTERISTIC EQUATION.
C   ************************************************

       CALL CHARAC(G,B,B2,R2,S2,THETA,ETA,ELO,W,H,T,C1,GR)
```

```
  9  NDEG=8
     CALL ZPOLR(G,NDEG,ALAM,IER)
     DO 604 NDEG=1,8
C     WRITE(2,44) (NDEG,ALAM(NDEG))
 604 CONTINUE
 44  FORMAT(5X,'LAMBDA(',I2,') =',5X,2(F10.4,5X))

C   ***************************************
C    SETUP "A" AND "H" MATRICES.
C   ***************************************
 80  DO 100 J=1,8
     FN=(ALAM(J)**2+B2*S2)/ALAM(J)
     UU=(1.DO+ELO-W*ALAM(J)**2)*(ALAM(J)**2+B2*S2)
     UD=(1.DO-ELO*ALAM(J)**2+W*ALAM(J)**4)
     IF(UD.EQ.O.DO) GO TO 500
     UN=UU/UD
     PN=(UN*ALAM(J)+FN)
C      WRITE (2,*) ALAM(J)
     CEE=ALAM(J)*THETA
     CE=CDEXP(CEE)
     A(1,J)=-FN
     A(2,J)=-FN*CE
     A(3,J)=UN
     A(4,J)=UN*CE
     A(5,J)=DCMPLX(1.,0.)
     A(6,J)=CE
     A(7,J)=PN
     A(8,J)=PN*CE
     MN=(UN-ALAM(J)*FN)
     TN=(ELO-W*ALAM(J)**2)*(UN*ALAM(J)+FN)
     VN=(ALAM(J)-FN)/S2
     QN=-W*(ALAM(J)**2*UN+ALAM(J)*FN)
     BM(1,J)=MN
     BM(2,J)=-MN*CE
     BM(3,J)=TN
     BM(4,J)=-TN*CE
     BM(5,J)=VN
     BM(6,J)=VN*CE
     BM(7,J)=QN
     BM(8,J)=-QN*CE
 100   CONTINUE

C   *******************************************
C    CALCULATE INVERSE MATRIX FOR "A".
C   *******************************************

       DO 601 I=1,8
       DO 601 J=1,8
```

```
      DUMMY(I,J)=DCMPLX(0.,0.)
      IF(I.EQ.J) DUMMY(I,J)=DCMPLX(1.,0.)
  601 CONTINUE
      IA=8
      IB=8
      N1=8
      M1=8
      IJOB=0
      CALL LEQ2C (A,N1,IA,DUMMY,M1,IB,IJOB,WA,WK,IER)
      DET=DCMPLX(1.,0.)
      DO 602 I=1,N1
      IPVT=IDINT(WK(I))
      IF(IPVT.NE.I)DET=-DET
      INDX=I+(I-1)*N1
C     WRITE(2,*)WA(INDX)
C     IF(WA(INDX).EQ.0.D0) GOTO 500
      DET=DET*WA(INDX)
C     DET=DET*CE1(I)
  602 CONTINUE
      WRITE(2,101)DET
  101 FORMAT(5X,2D15.8)
      IF(CDABS(DET).LT..1D-8) GOTO 500
C    ****************************************
C       FORMAT STIFFNESS MATRIX S (=DM).
C    ****************************************
      CALL PRODCT(8,8,8,BM,DUMMY,DM)

C    ***************************************************
C       FIND NATURAL FREQUENCIES FOR THREE-SPAN BEAM.
C    ***************************************************

      DO 600 I=1,4
      DO 600 J=1,4
      DUMMY(I,J)=DCMPLX(0.,0.)
C     IF(I.EQ.J)DUMMY(I,J)=DCMPLX(1.,0.)
  600 BC(I,J)=DCMPLX(0.,0.)
      BC(1,1)=DM(1,1)+DM(1,7)
      BC(1,2)=DM(1,2)+DM(1,8)
      BC(2,1)=DM(2,1)+DM(2,7)
      BC(2,2)=DM(2,2)+DM(2,8)+BC(1,1)
      BC(2,3)=BC(1,2)
      BC(3,2)=BC(2,1)
      BC(3,3)=BC(2,2)
      BC(3,4)=BC(1,2)
      BC(4,3)=BC(2,1)
      BC(4,4)=DM(2,2)+DM(2,8)
```

```
      IA=4
      IB=4
      N2=4
      M2=4
      IJOB=1
      CALL LEQ2C(BC,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DET=DCMPLX(1.,0.)
      DO 603 I=1,N2
      IPVT2=WK2(I)
      IF(IPVT2.NE.I)DET=-DET
      INDX2=I+(I-1)*N2
C     WRITE(2,*)WA2(INDX2)
      DET=DET*WA2(INDX2)
  603  CONTINUE
      WRITE(2,250)DET
  250 FORMAT(5X,' DET. OF STIFFNESS MATRIX =',D15.8,'+',D15.8,'I',//)
  500 CONTINUE
 1000 STOP
      END

     SUBROUTINE CHARAC(G,B,B2,R2,S2,THETA,ETA,ELO,W,H,T,C1,GR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 G(9)
CC    XI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO X---AXIS
CC    YI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO Y---AXIS
CC    POI=XI+YI
CC    XJ=TORSIONAL CONSTANT
CC    UNITS: INCHES , POUNDS, SECONDS
CC
      DO 10 I=1,9
   10 G(I)=0.D0
      U=0.3D0
      E=29.D+6
      GE=11.2D+6
      A=H*T
      CK=(10.D0+(1.D0+U))/(12.D0+11.D0*U)
      XI=H*T**3/12.D0
      YI=T*H**3/12.D0
      XJ=C1*H*T**3
      POI=XI+YI
      ETA=POI/XI
      ELO=(GE*XJ)/(E*XI)
      GAMMA=H**3*T**3/144.D0
      W=GAMMA/(XI*GR**2)
      R=DSQRT(XI/(A*GR**2))
      S=DSQRT(2.D0*(1.D0+U)/CK)*R
      B2=B**2
      B4=B2**2
      B6=B2**3
```

```
      S2=S**2
      R2=R**2
      R4=R2**2
      G(1)=1.DO
      G(3)=(2.DO+B2*R2+B2*S2)-ELO/W
      G(5)=(1.DO-B2+2.DO*B2*S2+B4*S2*R2)-(2.DO*ELO+ELO*B2*R2
     1+ELO*B2*S2)/W
      G(7)=(B2*S2)+(B2*R2+ELO*B2-ELO
     1-2.DO*ELO*B2*S2-ELO*B4*S2*R2)/W
      G(9)=(B4*R2*S2-B2-ELO*B2*S2)/W
      THETA=THETA*3.1415926539/180.DO
      WRITE(2,300) R,B,THETA
300   FORMAT(2X,'R =',F8.5,5X,'B =',F12.5,5X,' THETA =',F8.5,/)
      RETURN
      END

      SUBROUTINE PRODCT(N1,M2,N2,A,B,EE)
      COMPLEX*16 D,A(N1,N2),B(N2,M2),EE(N1,M2)
      DO 270 I=1,N1
      DO 270 J=1,M2
      D=DCMPLX(0.,0.)
      DO 280 K=1,N2
      D=D+A(I,K)*B(K,J)
280   CONTINUE
      EE(I,J)=D
270   CONTINUE
      RETURN
      END
```

```
C     **************************************************
C                PROGRAM OPNW3.FOR
C     **************************************************

C       THIS IS A COMPUTER PROGRAM FOR FINDING THE NATURAL FREQUENCIES
C       OF OUT-OF-PLANE VIBRATIONS OF THREE-SPAN CIRCULAR CURVED BEAM
C       WITHOUT WARPING EFFECT.
C       (WITH SHEAR DEFORMATION AND ROTATORY INERTIA EFFECTS.)

C     ************************************
C       DECLARE ALL VARIABLES.
C     ************************************

        IMPLICIT REAL*8 (A-H,O-Z)
        COMPLEX*16 ALAM(6),FNN,FN2,UNN,VNN,MNN,CE,BM(6,6),WA(48)
       &,TNN,DET,A(6,6),DM(6,6),BC(4,4)  ,DUMMY(6,6),WA2(24)
        REAL*8 G(7),WK(6),WK2(4)

C     **************************************************************
C       WE ARE GOING TO USE LAGUERRE'S METHOD FOR FINDING QUADRATIC
C       FACTORS OF POLYNOMIALS.
C       SOLUTIONS OF POLYNOMIAL UP TO X**100 WITH REAL COEFFICIENTS.
C       NDEG  DESIGNATES THE DEGREE OF THE POLYNOMIAL.
C     **************************************************************

        OPEN(UNIT=1,FILE='OPV.DAT',STATUS='OLD')
        READ(1,*)EX,H,T,GR,CASE,ALPHA
        WRITE(2,11)EX,H,T,GR,CASE,ALPHA
     11 FORMAT(6X,'EXAMPLE=',F5.1,2X,'h=',F3.0,' INCHES,',' t=',F3.0
       &,' INCHES,',' CURVATURE=',F3.0,' INCHES,',//,6X,'CASE=',F3.0
       &,6X,' ALPHA=',F3.0,' DEGREES.'/)
        READ(1,*)IN,GG,C1
        WRITE(2,12)IN,GG,C1
     12 FORMAT(6X,'NUMBER OF ITERATION=',I5,5X,'INCREMENT INTERVAL='
       &,F7.2,//,6X,'COEFFICIENT OF TORSIONAL CONSTANT=',F6.3,//)
        FG=0.D0
C        WRITE(2,607) FG
  607   FORMAT(5X,'INITIAL B = ',F7.3,/)
        DO 500 ID=1,IN
        B=FG+(ID-1)*GG
        IF (ID.EQ.1)B=.01D0
        THETA=ALPHA

C     **************************************************************
C       CALCULATE THE ROOTS OF CHARACTERISTIC EQUATION.
C     **************************************************************

        CALL CHARAC(G,B,B2,B4,R2,S2,THETA,ELO,ETA,H,T,C1,GR)
```

```
    9  NDEG=6
       CALL ZPOLR(G,NDEG,ALAM,IER)
       DO 604 NDEG=1,6
C       WRITE(2,44) (NDEG,ALAM(NDEG))
  604  CONTINUE
  44   FORMAT(5X,'LAMBDA(',I2,') =',5X,2(F10.4,5X))

C    *********************************
C      SETUP "A" AND "H" MATRICES.
C    *********************************

  80   DO 100 J=1,6
       FN1=ELO*S2/((B2*R2*S2-ELO*S2-1.D0)*(1.D0-ETA*B2*R2))
       FN2=ALAM(J)**4+(2.D0+B2*R2+B2*S2+(ETA*B2*R2/ELO))*ALAM(J)**2
      &+(2.D0*B2*S2-B2-1.D0/(ELO*S2)+B4*S2*R2+(ETA*B4*R2*S2/ELO)
      &+(ETA*B2*R2/(ELO*S2)))
       FNN=ALAM(J)*FN1*FN2
       UN1=(1.D0+2.D0*ELO)/ELO
       UN2=(ELO/((1.D0+ELO)*(1.D0-ETA*B2*R2)))
       UNN=UN2*(ALAM(J)**4+(UN1+B2*R2+B2*S2)*ALAM(J)**2
      &+(UN1*B2*S2-B2+B4*R2*S2))
       CE=CDEXP(ALAM(J)*THETA)
       A(1,J)=FNN
       A(2,J)=FNN*CE
       A(3,J)=UNN
       A(4,J)=UNN*CE
       A(5,J)=DCMPLX(1.,0.)
       A(6,J)=CE
       MNN=UNN-ALAM(J)*FNN
       TNN=ELO*(FNN+ALAM(J)*UNN)
       VNN=(ALAM(J)-FNN)/S2
       BM(1,J)=MNN
       BM(2,J)=-MNN*CE
       BM(3,J)=TNN
       BM(4,J)=-TNN*CE
       BM(5,J)=VNN
       BM(6,J)=VNN*CE
  100    CONTINUE

C    ********************************************
C      CALCULATE INVERSE MATRIX FOR "A".
C    ********************************************

         DO 601 I=1,6
         DO 601 J=1,6
         DUMMY(I,J)=(0.,0.)
         IF(I.EQ.J) DUMMY(I,J)=(1.,0.)
  601    CONTINUE
```

```
      IA=6
      IB=6
      N1=6
      M1=6
      IJOB=0
      CALL LEQ2C(A,N1,IA,DUMMY,M1,IB,IJOB,WA,WK,IER)
      DET=(1.,0.)
      DO 602 I=1,N1
      IPVT=WK(I)
      IF(IPVT.NE.I) DET=-DET
      INDX=I+(I-1)*N1
      DET=DET*WA(INDX)
  602 CONTINUE
      WRITE(2,101)DET
  101 FORMAT(5X,2D15.8)
      IF(CDABS(DET).LT..1D-8) GOTO 500
C      DO 103 KI=1,6
C      WRITE(2,102) (A(KI,KJ),KJ=1,6)
C 102  FORMAT(//,3(5x,2(2D13.4),/))
C 103  CONTINUE

C     ******************************************
C       FORMAT STIFFNESS MATRIX S(=DM).
C     ******************************************

      CALL PRODCT(6,6,6,BM,DUMMY,DM)
      DO 600 I=1,4
      DO 600 J=1,4
      DUMMY(I,J)=(0.,0.)
      IF(I.EQ.J)DUMMY(I,J)=(1.,0.)

C     ****************************************************
C       FIND NATURAL FREQUENCIES FOR THREE-SPAN BEAM.
C     ****************************************************

  600 BC(I,J)=(0.,0.)
      BC(1,1)=DM(1,1)
      BC(1,2)=DM(1,2)
      BC(2,1)=DM(2,1)
      BC(2,2)=DM(2,2)+DM(1,1)
      BC(2,3)=DM(1,2)
      BC(3,2)=DM(2,1)
      BC(3,3)=DM(2,2)+DM(1,1)
      BC(3,4)=DM(1,2)
      BC(4,3)=DM(2,1)
      BC(4,4)=DM(2,2)
      IA=4
      IB=4
```

```
      N2=4
      M2=4
      IJOB=0
      CALL LEQ2C(BC,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DET=(1.,0.)
      DO 603 I=1,N2
      IPVT2=WK2(I)
      IF(IPVT2.NE.I) DET=-DET
      INDX2=I+(I-1)*N2
      DET=DET*WA2(INDX2)
  603 CONTINUE
      WRITE(2,250)DET
  250 FORMAT(5X,' DET. OF STIFFNESS MATRIX =',D15.8,'+',D15.8,'I',//)
  500 CONTINUE
 1000 STOP
      END

     SUBROUTINE CHARAC(G,B,B2,B4,R2,S2,THETA,ELO,ETA,H,T,C1,GR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 G(7)
CC    XI=MOMENT OF INERTIA WITH RESPECT TO X---AXIS
CC    YI=MOMENT OF INERTIA WITH RESPECT TO Y---AXIS
CC    POI=XI+YI
CC    XJ=TORSIONAL CONSTANT
CC    UNITS: INCHES , POUNDS, SECONDS

      DO 10 I=1,7
   10 G(I)=0.D0
      U=0.3D0
      E=29.D+6
      GE=11.2D+6
      A=H*T
      CK=(10.D0+(1.D0+U))/(12.D0+11.D0*U)
      XI=H*T**3/12.D0
      YI=T*H**3/12.D0
      XJ=DBLE(C1*H*T**3)
      POI=XI+YI
      ETA=POI/XI
      ELO=(GE*XJ)/(E*XI)
C     GAMMA=H**3*T**3/144.D0
C     W=GAMMA/(XI*GR**2)
      R=DSQRT(XI/(A*GR**2))
      S=DSQRT(2.D0*(1.D0+U)/CK)*R
      B2=B**2
      B4=B2**2
      B6=B2**3
      R2=R**2
      R4=R2**2
      S2=S**2
```

```
      G(1)=1.DO
      G(3)=2.DO+B2*R2+B2*S2+ETA*B2*R2/ELO
      G(5)=1.DO-B2-B2*R2/ELO+2.DO*B2*S2+B4*R2*S2
     &+ETA*((B4*R4+B4*S2*R2)/ELO-B2*R2)
      G(7)=B2/ELO+B2*S2-B4*R2*S2/ELO+ETA*((B6*R4*S2-B4*R2)/ELO
     &-B4*S2*R2)
      THETA=THETA*3.1415926539DO/180.DO
      WRITE(2,300) R,B,THETA
300   FORMAT(2X,'R=',F8.5,5X,'B=',F18.12,5X,'THETA =',F7.3)
      RETURN
      END


      SUBROUTINE PRODCT(N1,M2,N2,A,B,EE)
      COMPLEX*16 D,A(N1,N2),B(N2,M2),EE(N1,M2)
      DO 270 I=1,N1
      DO 270 J=1,M2
      D=DCMPLX(0.,0.)
      DO 280 K=1,N2
      D=D+A(I,K)*B(K,J)
280   CONTINUE
      EE(I,J)=D
270   CONTINUE
      RETURN
      END
```

```
C    ***************************************
C              PROGRAM  OPH3F.FOR
C    ***************************************

C      THIS IS A COMPUTER PROGRAM FOR FINDING THE JOINT MOMENTS OF
C      OUT-OF-PLANE VIBRATIONS OF THREE-SPAN CIRCULAR CURVED BEAM,
C      WITH A UNIFORMLY DISTRIBUTED LOAD q(t) ACTING ON THE RIGHTMOST
C      SPAN, CONSIDERING SHEAR DEFORMATION, ROTATORY INERTIAS AND
C      WARPING EFFECTS.

C    ************************
C      DECLARE ALL VARIABLES.
C    ************************

       IMPLICIT REAL*8  (A-H,O-Z)
       COMPLEX*16 ALAM(8),FN,UU,UD,UN,PN,CE,CEE,VN,MN,BM(8,8),SS1(4,4)
      &,QN,TN,DET,A(8,8),AA(8,8),DM(8,8),DM1(8,1),DM2(4,1),ST(4,4)
      &,ST1(1,4),DUMMY(8,8),WA(80),WA2(24),CE2,MFCD,MFDC,MCD,F22(4,1)
      &,FF(8,1)
       REAL*8 G(9),WK(8),WK2(4)
C
C    ******************************************************************
C      WE ARE GOING TO USE LAGUERRE'S METHOD FOR FINDING QUADRATIC
C      FACTORS OF POLYNOMIALS.
C      SOLUTIONS OF POLYNOMIAL UP TO X**100 WITH REAL COEFFICIENTS.
C      NDEG DESIGNATES THE DEGREE OF THE POLYNOMIALS.
C    ******************************************************************
C
       OPEN(UNIT=1,FILE='OPV.DAT',STATUS='OLD')
       READ(1,*) EX,H,T,GR,CASE,ALPHA
       WRITE(2,11)EX,H,T,GR,CASE,ALPHA
    11 FORMAT(6X,'EXAMPLE',F5.1,2X,'h=',F3.0,1X,'INCHES,',1X,'t=',F3.0,
      &1X,'INCHES,',1X,'CURVATURE=',F3.0,1X,'INCHES,',//,6X,'CASE='
      &,F3.0,6X,'ALPHA=',F3.0,'DEGREES.'/)
       READ(1,*) IN, GG,C1
       WRITE(2,12)IN,GG ,C1
    12 FORMAT(6X,'NUMBER OF ITERATION= ',I5,5X,'INCREMENT INTERVAL= '
      &,F7.2,//,6X,'COEFF. OF TORSIONAL CONSTANT=',F6.3,//)
       FG=0.D0
C       WRITE(2,607) FG
  607  FORMAT(5X,'INITIAL B = ',F7.3,/)
       DO 500 ID=1,IN
       B=FG+(ID-1)*GG
         IF(B.LE.(0.D0)) B=.01D0
       THETA=ALPHA

C      *************************************************
C      CALCULATE THE ROOTS OF CHARACTERISTIC EQUATION.
C      *************************************************
```

```
      CALL  CHARAC(G,B,B2,B4,R2,R4,S2,THETA,ETA,ELO,W,H,T,C1,GR)
    9 NDEG=8
      CALL  ZPOLR(G,NDEG,ALAM,IER)
      DO 604 NDEG=1,8
C      WRITE(2,44) (NDEG,ALAM(NDEG))
  604 CONTINUE
   44 FORMAT(5X,'LAMBDA(',I2,')  =',5X,2(F10.4,5X))

C     ***************************
C     SETUP "A" AND "H" MATRICES.
C     ***************************

   80 DO 100 J=1,8
      FN=(ALAM(J)**2+B2*S2)/ALAM(J)
      UU=(1.D0+ELO-W*ALAM(J)**2)*(ALAM(J)**2+B2*S2)
      UD=(1.D0-ELO*ALAM(J)**2-ETA*B2*R2+W*ALAM(J)**4)
      IF(UD.EQ.0.D0)  GO TO 500
      UN=UU/UD
      PN=(UN*ALAM(J)+FN)
C       WRITE  (2,*) ALAM(J)
      CEE=ALAM(J)*THETA
      CE=CDEXP(CEE)
      A(1,J)=-FN
      A(2,J)=-FN*CE
      A(3,J)=UN
      A(4,J)=UN*CE
      A(5,J)=DCMPLX(1.,0.)
      A(6,J)=CE
      A(7,J)=PN
      A(8,J)=PN*CE
      MN=(UN-ALAM(J)*FN)
      TN=(ELO-W*ALAM(J)**2)*(UN*ALAM(J)+FN)
      VN=(ALAM(J)-FN)/S2
      QN=-W*(ALAM(J)**2*UN+ALAM(J)*FN)
      BM(1,J)=MN
      BM(2,J)=-MN*CE
      BM(3,J)=TN
      BM(4,J)=-TN*CE
      BM(5,J)=VN
      BM(6,J)=VN*CE
      BM(7,J)=QN
      BM(8,J)=-QN*CE
  100   CONTINUE
        DO 601 I=1,8
        DO 601 J=1,8
        DUMMY(I,J)=DCMPLX(0.,0.)
        IF(I.EQ.J)  DUMMY(I,J)=DCMPLX(1.,0.)
  601   CONTINUE
```

```
      IA=8
      IB=8
      N1=8
      M1=8
      IJOB=0
C    ****************************************
C      CALCULATE INVERSE MATRIX FOR "A".
C    ****************************************

      CALL LEQ2C (A,N1,IA,DUMMY,M1,IB,IJOB,WA,WK,IER)
      DET=DCMPLX(1.,0.)
      DO 602 I=1,N1
      IPVT=IDINT(WK(I))
      IF(IPVT.NE.I)DET=-DET
      INDX=I+(I-1)*N1
C     WRITE(2,*)WA(INDX)
C     IF(WA(INDX).EQ.0.D0) GOTO 500
      DET=DET*WA(INDX)
C     DET=DET*CE1(I)
  602 CONTINUE
C      WRITE(2,101)DET
  101 FORMAT(5X,2D15.8)
      IF(CDABS(DET).LT..1D-8) GOTO 500

C    ******************************************
C      FORMAT STIFFNESS MATRIX S (= DM).
C    ******************************************

      CALL PRODCT(8,8,8,BM,DUMMY,DM)
      DO 621 II=1,8
      DO 621 JJ=1,8
      AA(II,JJ)=DUMMY(II,JJ)
  621 CONTINUE
      DO 600 I=1,4
      DO 600 J=1,4
      DUMMY(I,J)=DCMPLX(0.,0.)
      IF(I.EQ.J)DUMMY(I,J)=DCMPLX(1.,0.)
  600 ST(I,J)=DCMPLX(0.,0.)

C    **************************************************
C      FORMAT STIFFNESS MATRIX SS1(=ST).
C    **************************************************

      ST(1,1)=DM(1,1)+DM(1,7)
      ST(1,2)=DM(1,2)+DM(1,8)
      ST(2,1)=DM(2,1)+DM(2,7)
      ST(2,2)=DM(2,2)+DM(2,8)+ST(1,1)
```

```
      ST(2,3)=ST(1,2)
      ST(3,2)=ST(2,1)
      ST(3,3)=ST(2,2)
      ST(3,4)=ST(1,2)
      ST(4,3)=ST(2,1)
      ST(4,4)=DM(2,2)+DM(2,8)
      IA=4
      IB=4
      N2=4
      M2=4
      IJOB=1
      CALL LEQ2C(ST,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DET=DCMPLX(1.,0.)
      DO 603 I=1,N2
      IPVT2=WK2(I)
      IF(IPVT2.NE.I)DET=-DET
      INDX2=I+(I-1)*N2
C     WRITE(2,*)WA2(INDX2)
      DET=DET*WA2(INDX2)
  603  CONTINUE
      WRITE(2,250)DET
  250 FORMAT(5X,' DET. OF STIFFNESS MATRIX =',D15.8,'+',D15.8,'I')

C    *****************************************
C      SETUP INVERSE MATRIX FOR SS1.
C    *****************************************

      IJOB=0
      CALL LEQ2C (ST,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DO 703 I=1,N2
      DO 703 J=1,M2
      SS1(I,J)=DUMMY(I,J)
  703 CONTINUE
      DO 709 K=1,8
      FF(K,1)=(0.,0.)
  709 CONTINUE
      FF(3,1)=(1.,0.)
      FF(7,1)=(1.,0.)
      CALL PRODCT(8,1,8,AA,FF,DM1)

C    **********************************************
C      FIND FIXED MOMOENTS -MFCD/R AND MFDC/R.
C    **********************************************

      MFCD=(0.,0.)
      MFDC=(0.,0.)
      DO 704 I=1,8
      MFCD=MFCD+BM(1,I)*DM1(I,1)
      MFDC=MFDC+BM(2,I)*DM1(I,1)
  704 CONTINUE
```

```
      MFCD=(1/B2)*MFCD
      MFDC=(1/B2)*MFDC
      WRITE(2,705)MFCD,MFDC
  705 FORMAT(5X,' MFCD/QR**2 = ', F10.3,'+',F10.3,5X,' MFDC/QR**2 =',
     &F10.3,'+',F10.3)
      F22(1,1)=(0.,0.)
      F22(2,1)=(0.,0.)
      F22(3,1)=-MFCD
      F22(4,1)=MFDC

C    ***********************************************
C     FIND MOMENT MCD. (HERE DM2 REPRESENTS DD1)
C    ***********************************************

      CALL PRODCT (4,1,4,SS1,F22,DM2)
      ST1(1,1)=(0.,0.)
      ST1(1,2)=(0.,0.)
      ST1(1,3)=ST(1,1)
      ST1(1,4)=ST(1,2)
      MCD=(0.,0.)
      DO 706 I=1,4
      MCD=MCD + ST1(1,I)*DM2(I,1)
  706 CONTINUE
      WRITE(2,707)MCD
  707 FORMAT(5X,' fCD=(MCD/QR**2)=  ','(',F10.3,'+',F10.3,')',//)
  500 CONTINUE
 1000 STOP
      END

      SUBROUTINE CHARAC(G,B,B2,B4,R2,R4,S2,THETA,ETA,ELO,W,H,T,C1,GR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 G(9)
CC    XI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO X---AXIS
CC    YI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO Y---AXIS
CC    POI=XI+YI
CC    XJ=TORSIONAL CONSTANT
CC    UNITS: INCHES , POUNDS, SECONDS
CC
      DO 10 I=1,9
   10 G(I)=0.D0
      U=0.3D0
      E=29.D+6
      GE=11.2D+6
      A=H*T
      CK=(10.D0+(1.D0+U))/(12.D0+11.D0*U)
      XI=H*T**3/12.D0
      YI=T*H**3/12.D0
      XJ=C1*H*T**3
```

```
      POI=XI+YI
      ETA=POI/XI
      ELO=(GE*XJ)/(E*XI)
      GAMMA=H**3*T**3/144.DO
      W=GAMMA/(XI*GR**2)
      R=DSQRT(XI/(A*GR**2))
      S=DSQRT(2.DO*(1.DO+U)/CK)*R
      B2=B**2
      B4=B2**2
      B6=B2**3
      S2=S**2
      R2=R**2
      R4=R2**2
      G(1)=1.DO
      G(3)=(2.DO+B2*R2+B2*S2)-ELO/W
      G(5)=(1.DO-B2+2.DO*B2*S2+B4*S2*R2-ETA*B2*R2)
     &-(2.DO*ELO+ELO*B2*R2+ELO*B2*S2+ETA*B2*R2)/W
      G(7)=(B2*S2-ETA*B4*S2*R2)+(ELO*ETA*B2*R2+B2*R2+ELO*B2-ELO
     &-2.DO*ELO*B2*S2-ELO*B4*S2*R2-ETA*B4*R2*(R2+S2))/W
      G(9)=(B4*R2*S2-B2-ELO*B2*S2+ETA*B4*R2+ETA*ELO*B4*R2*S2
     &-ETA*B6*R4*S2)/W
      THETA=THETA*3.1415926539/180.DO
      WRITE(2,300) R,B,THETA
300   FORMAT(2X,'R =',F8.5,5X,'B =',F12.5,5X,' THETA =',F8.5)
      RETURN
      END
      SUBROUTINE PRODCT(N1,M2,N2,A,B,EE)
      COMPLEX*16 D,A(N1,N2),B(N2,M2),EE(N1,M2)
      DO 270 I=1,N1
      DO 270 J=1,M2
      D=DCMPLX(0.,0.)
      DO 280 K=1,N2
      D=D+A(I,K)*B(K,J)
280   CONTINUE
      EE(I,J)=D
270   CONTINUE
      RETURN
      END
```

```
C    ***************************************
C              PROGRAM  OPNT3F.FOR
C    ***************************************

C
C      THIS IS A COMPUTER PROGRAM FOR FINDING THE JOINT MOMENTS OF
C      OUT-OF-PLANE VIBRATIONS OF THREE-SPAN CIRCULAR CURVED BEAM,
C      WITH A UNIFORMLY DISTRIBUTED LOAD q(t) ACTING ON THE RIGHTMOST
C      SPAN, CONSIDERING SHEAR DEFORMATION, ROTATORY INERTIA(NEGLECT
C      TORSIONAL INERTIA) AND WARPING EFFECTS.

C    **************************************
C      DECLARE ALL VARIABLES.
C    **************************************

       IMPLICIT REAL*8 (A-H,O-Z)
       COMPLEX*16 ALAM(8),FN,UU,UD,UN,PN,CE,CEE,VN,MN,BM(8,8),SS1(4,4)
      &,QN,TN,DET,A(8,8),AA(8,8),DM(8,8),DM1(8,1),DM2(4,1),ST(4,4)
      &,ST1(1,4),DUMMY(8,8),WA(80),WA2(24),CE2,MFCD,MFDC,MCD,F22(4,1)
      &,FF(8,1)
       REAL*8 G(9),WK(8),WK2(4)

C    ****************************************************************
C      WE ARE GOING TO USE LAGUERRE'S METHOD FOR FINDING QUADRATIC
C      FACTORS OF POLYNOMIALS.
C      SOLUTIONS OF POLYNOMIAL UP TO X**100 WITH REAL COEFFICIENTS.
C      NDEG DESIGNATES THE DEGREE OF THE POLYNOMIALS.
C    ****************************************************************

       OPEN(UNIT=1,FILE='OPV.DAT',STATUS='OLD')
       READ(1,*)EX,H,T,GR,CASE,ALPHA
       WRITE(2,11)EX,H,T,GR,CASE,ALPHA
    11 FORMAT(6X,'EXAMPLE',F5.1,2X,'H=',F3.0,' INCHES,',' T=',F3.0,
      &' INCHES,',' CURVATURE=',F3.0,' INCHES,',//,6X,'CASE=',F3.1
      &,6X,'ALPHA=',F3.0,'DEGREES.',/)
       READ(1,*)IN,GG,C1
       WRITE(2,12)IN,GG,C1
    12 FORMAT(6X,'NUMBER OF ITERATION=',I5,5X,'INCREMENT OF INTERVAL=
      &',F7.2,//,6X,'COEFF. OF TORSIONAL CONSTANT=',F6.3,//)
       FG=0.D0
C       WRITE(2,607) FG
  607  FORMAT(5X,'INITIAL B = ',F7.3,/)
       DO 500 ID=1,IN
       B=FG+(ID-1)*GG
         IF(B.LE.(0.D0)) B=.01D0
       THETA=ALPHA

C    ****************************************************
C      CALCULATE THE ROOTS OF CHARACTERISTIC EQUATION.
C    ****************************************************
```

```
      CALL  CHARAC(G,B,B2,R2,S2,THETA,ETA,ELO,W,H,T,C1,GR)
    9 NDEG=8
      CALL  ZPOLR(G,NDEG,ALAM,IER)
      DO 604 NDEG=1,8
C      WRITE(2,44) (NDEG,ALAM(NDEG))
  604 CONTINUE
  44  FORMAT(5X,'LAMBDA(',I2,')  =',5X,2(F10.4,5X))

C    *********************************
C      SETUP "A" AND "H" MATRICES.
C    *********************************

  80  DO 100 J=1,8
      FN=(ALAM(J)**2+B2*S2)/ALAM(J)
      UU=(1.D0+ELO-W*ALAM(J)**2)*(ALAM(J)**2+B2*S2)
      UD=(1.D0-ELO*ALAM(J)**2+W*ALAM(J)**4)
      IF(UD.EQ.0.D0)  GO TO 500
      UN=UU/UD
      PN=(UN*ALAM(J)+FN)
C       WRITE  (2,*) ALAM(J)
      CEE=ALAM(J)*THETA
      CE=CDEXP(CEE)
      A(1,J)=-FN
      A(2,J)=-FN*CE
      A(3,J)=UN
      A(4,J)=UN*CE
      A(5,J)=DCMPLX(1.,0.)
      A(6,J)=CE
      A(7,J)=PN
      A(8,J)=PN*CE
      MN=(UN-ALAM(J)*FN)
      TN=(ELO-W*ALAM(J)**2)*(UN*ALAM(J)+FN)
      VN=(ALAM(J)-FN)/S2
      QN=-W*(ALAM(J)**2*UN+ALAM(J)*FN)
      BM(1,J)=MN
      BM(2,J)=-MN*CE
      BM(3,J)=TN
      BM(4,J)=-TN*CE
      BM(5,J)=VN
      BM(6,J)=VN*CE
      BM(7,J)=QN
      BM(8,J)=-QN*CE
 100    CONTINUE
        DO 601 I=1,8
        DO 601 J=1,8
        DUMMY(I,J)=DCMPLX(0.,0.)
        IF(I.EQ.J)  DUMMY(I,J)=DCMPLX(1.,0.)
 601    CONTINUE
```

```
C      ****************************************
C        CALCULATE INVERSE MATRIX FOR "A".
C      ****************************************

         IA=8
         IB=8
         N1=8
         M1=8
         IJOB=0
         CALL LEQ2C (A,N1,IA,DUMMY,M1,IB,IJOB,WA,WK,IER)
         DET=DCMPLX(1.,0.)
         DO 602 I=1,N1
         IPVT=IDINT(WK(I))
         IF(IPVT.NE.I)DET=-DET
         INDX=I+(I-1)*N1
C        WRITE(2,*)WA(INDX)
C        IF(WA(INDX).EQ.0.D0) GOTO 500
         DET=DET*WA(INDX)
C        DET=DET*CE1(I)
  602    CONTINUE
C         WRITE(2,101)DET
  101    FORMAT(5X,2D15.8)
         IF(CDABS(DET).LT..1D-8) GOTO 500

C     *****************************************
C        FORMAT STIFFNESS MATRIX S (=DM)
C     *****************************************

         CALL PRODCT(8,8,8,BM,DUMMY,DM)
         DO 621 II=1,8
         DO 621 JJ=1,8
         AA(II,JJ)=DUMMY(II,JJ)
  621    CONTINUE

C      *******************************************
C        FORMAT STIFFNESS MATRIX SS1(=ST).
C      *******************************************

         DO 600 I=1,4
         DO 600 J=1,4
         DUMMY(I,J)=DCMPLX(0.,0.)
         IF(I.EQ.J)DUMMY(I,J)=DCMPLX(1.,0.)
  600    ST(I,J)=DCMPLX(0.,0.)
         ST(1,1)=DM(1,1)+DM(1,7)
         ST(1,2)=DM(1,2)+DM(1,8)
         ST(2,1)=DM(2,1)+DM(2,7)
         ST(2,2)=DM(2,2)+DM(2,8)+ST(1,1)
```

```
      ST(2,3)=ST(1,2)
      ST(3,2)=ST(2,1)
      ST(3,3)=ST(2,2)
      ST(3,4)=ST(1,2)
      ST(4,3)=ST(2,1)
      ST(4,4)=DM(2,2)+DM(2,8)
      IA=4
      IB=4
      N2=4
      M2=4
      IJOB=1
      CALL LEQ2C(ST,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DET=DCMPLX(1.,0.)
      DO 603 I=1,N2
      IPVT2=WK2(I)
      IF(IPVT2.NE.I)DET=-DET
      INDX2=I+(I-1)*N2
C     WRITE(2,*)WA2(INDX2)
      DET=DET*WA2(INDX2)
 603   CONTINUE
C     WRITE(2,250)DET
 250  FORMAT(5X,' DET. OF STIFFNESS MATRIX =',D15.8,'+',D15.8,'I')

C    ********************************************
C       SETUP INVERSE MATRIX FOR SS1.
C    ********************************************

      IJOB=0
      CALL LEQ2C (ST,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DO 703 I=1,N2
      DO 703 J=1,M2
      SS1(I,J)=DUMMY(I,J)
 703  CONTINUE
      DO 709 K=1,8
      FF(K,1)=(0.,0.)
 709  CONTINUE
      FF(3,1)=(1.,0.)
      FF(7,1)=(1.,0.)
      CALL PRODCT(8,1,8,AA,FF,DM1)

C    ***************************************************
C       FIND FIXED MOMOENTS -MFCD/R AND MFDC/R.
C    ***************************************************

      MFCD=(0.,0.)
      MFDC=(0.,0.)
      DO 704 I=1,8
      MFCD=MFCD+BM(1,I)*DM1(I,1)
      MFDC=MFDC+BM(2,I)*DM1(I,1)
 704  CONTINUE
```

```
      MFCD=(1/B2)*MFCD
      MFDC=(1/B2)*MFDC
C     WRITE(2,705)MFCD,MFDC
  705 FORMAT(5X,'MFCD/QR**2 = ', F10.3,'+',F10.3,5X,'MFDC/QR**2 =',
     &F10.3,'+',F10.3,//)
      F22(1,1)=(0.,0.)
      F22(2,1)=(0.,0.)
      F22(3,1)=-MFCD
      F22(4,1)=MFDC

C    ****************************************
C       FIND MOMENT MCD.(HERE DM2 IS DD1)
C    ****************************************

      CALL PRODCT (4,1,4,SS1,F22,DM2)
      ST1(1,1)=(0.,0.)
      ST1(1,2)=(0.,0.)
      ST1(1,3)=ST(1,1)
      ST1(1,4)=ST(1,2)
      MCD=(0.,0.)
      DO 706 I=1,4
      MCD=MCD + ST1(1,I)*DM2(I,1)
  706 CONTINUE
      WRITE(2,707)MCD
  707 FORMAT(5X,' FCD=(MCD/QR**2)=  ','(',F10.3,'+',F10.3,')',//)
  500 CONTINUE
 1000 STOP
      END

      SUBROUTINE CHARAC(G,B,B2,R2,S2,THETA,ETA,ELO,W,H,T,C1,GR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 G(9)
CC    XI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO X---AXIS
CC    YI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO Y---AXIS
CC    POI=XI+YI
CC    XJ=TORSIONAL CONSTANT
CC    UNITS: INCHES , POUNDS, SECONDS
CC
      DO 10 I=1,9
   10 G(I)=0.D0
      U=0.3D0
      E=29.D+6
      GE=11.2D+6
      A=H*T
      CK=(10.D0+(1.D0+U))/(12.D0+11.D0*U)
      XI=H*T**3/12.D0
      YI=T*H**3/12.D0
      XJ=C1*H*T**3
```

```
      POI=XI+YI
      ETA=POI/XI
      ELO=(GE*XJ)/(E*XI)
      GAMMA=H**3*T**3/144.DO
      W=GAMMA/(XI*GR**2)
      R=DSQRT(XI/(A*GR**2))
      S=DSQRT(2.DO*(1.DO+U)/CK)*R
      B2=B**2
      B4=B2**2
      B6=B2**3
      S2=S**2
      R2=R**2
      R4=R2**2
      G(1)=1.DO
      G(3)=(2.DO+B2*R2+B2*S2)-ELO/W
      G(5)=(1.DO-B2+2.DO*B2*S2+B4*S2*R2)-(2.DO*ELO+ELO*B2*R2
     &+ELO*B2*S2)/W
      G(7)=(B2*S2)+(B2*R2+ELO*B2-ELO
     &-2.DO*ELO*B2*S2-ELO*B4*S2*R2)/W
      G(9)=(B4*R2*S2-B2-ELO*B2*S2)/W
      THETA=THETA*3.1415926539/180.DO
      WRITE(2,300) R,B,THETA
300   FORMAT(2X,'R =',F8.5,5X,'B =',F12.5,5X,' THETA =',F8.5)
      RETURN
      END

      SUBROUTINE PRODCT(N1,M2,N2,A,B,EE)
      COMPLEX*16 D,A(N1,N2),B(N2,M2),EE(N1,M2)
      DO 270 I=1,N1
      DO 270 J=1,M2
      D=DCMPLX(0.,0.)
      DO 280 K=1,N2
      D=D+A(I,K)*B(K,J)
280   CONTINUE
      EE(I,J)=D
270   CONTINUE
      RETURN
      END
```

```
C    ***************************************
C              PROGRAM  OPNW3F.FOR
C    ***************************************

C      THIS IS A COMPUTER PROGRAM FOR FINDING THE JOINT MOMENTS
C      OF OUT-OF-PLANE VIBRATIONS OF THREE-SPAN CIRCULAR CURVED BEAM,
C      WITH A UNIFORMLY DISTRIBUTED LOAD q(t) ACTING ON THE RIGHTMOST
C      SPAN, CONSIDERING SHEAR DEFORMATION, ROTATORY INERTIAS EFFECTS.

C    *****************************
C      DECLARE ALL VARIABVLES.
C    *****************************

       IMPLICIT REAL*8 (A-H,O-Z)
       COMPLEX*16 ALAM(6),FN,UU,UD,UN,PN,CE,CEE,VN,MN,BM(6,6),SS1(4,4)
     &,QN,TN,DET,A(6,6),AA(6,6),DM(6,6),DM1(6,1),DM2(4,1),ST(4,4)
     &,ST1(1,4),DUMMY(6,6),WA(48),WA2(24),CE2,MFCD,MFDC,MCD,F22(4,1)
     &,FF(6,1)
       REAL*8 G(7),WK(6),WK2(4)

C    *****************************************************************
C      WE ARE GOING TO USE LAGUERRE'S METHOD FOR FINDING QUADRATIC
C      FACTORS OF POLYNOMIALS.
C      SOLUTIONS OF POLYNOMIAL UP TO X**100 WITH REAL COEFFICIENTS.
C      NDEG DESIGNATES THE DEGREE OF THE POLYNOMIALS.
C    *****************************************************************

       OPEN(UNIT=1,FILE='OPV.DAT',STATUS='OLD')
       READ(1,*)EX,H,T,GR,CASE,ALPHA
       WRITE(2,11)EX,H,T,GR,CASE,ALPHA
    11 FORMAT(6X,'EXAMPLE',F5.1,2X,'H=',F3.0,1X,'INCHES,',1X,'T=',F3.0,
     &1X,'INCHES,',' CURVATURE=',F3.0,' INCHES,',//,6X,'CASE=',F3.0
     &,6X,'ALPHA=',F3.0,'DEGREES.',/)
       READ(1,*)IN,GG,C1
       WRITE(2,12)IN,GG,C1
    12 FORMAT(6X,'NUMBER OF ITERATION=',I5,5X,'INCREMENT OF INTERVAL'
     &,F7.2,//,6X,'COEFFICIENT OF TORSIONAL CONSTANT=',F6.3,//)
       FG=0.D0
C      WRITE(2,607) FG
  607  FORMAT(5X,'INITIAL B = ',F7.3,/)
       DO 500 ID=1,IN
       B=FG+(ID-1)*GG
         IF(B.LE.(0.D0)) B=.01D0
       THETA=ALPHA

C      ***********************************************
C      CALCULATE THE ROOTS OF CHARACTERISTIC EQUATION.
C      ***********************************************
```

```
      CALL  CHARAC(G,B,B2,B4,R2,R4,S2,THETA,ETA,ELO,H,T,C1,GR)
    9 NDEG=6
      CALL  ZPOLR(G,NDEG,ALAM,IER)
      DO 604 NDEG=1,6
C      WRITE(2,44) (NDEG,ALAM(NDEG))
  604 CONTINUE
  44  FORMAT(5X,'LAMBDA(',I2,') =',5X,2(F10.4,5X))

C    ****************************************
C       SETUP "A" AND "H" MATRICES.
C    ****************************************

  80  DO 100 J=1,6
      FN=(ALAM(J)**2+B2*S2)/ALAM(J)
      UU=(1.D0+ELO)*(ALAM(J)**2+B2*S2)
      UD=(1.D0-ELO*ALAM(J)**2-ETA*B2*R2)
      IF(UD.EQ.O.D0)  GO TO 500
      UN=UU/UD
C        WRITE  (2,*) ALAM(J)
      CEE=ALAM(J)*THETA
      CE=CDEXP(CEE)
      A(1,J)=-FN
      A(2,J)=-FN*CE
      A(3,J)=UN
      A(4,J)=UN*CE
      A(5,J)=DCMPLX(1.,0.)
      A(6,J)=CE
      MN=(UN-ALAM(J)*FN)
      TN=ELO*(UN*ALAM(J)+FN)
      VN=(ALAM(J)-FN)/S2
      BM(1,J)=MN
      BM(2,J)=-MN*CE
      BM(3,J)=TN
      BM(4,J)=-TN*CE
      BM(5,J)=VN
      BM(6,J)=VN*CE
 100    CONTINUE
        DO 601 I=1,6
        DO 601 J=1,6
        DUMMY(I,J)=DCMPLX(0.,0.)
        IF(I.EQ.J)  DUMMY(I,J)=DCMPLX(1.,0.)
  601   CONTINUE

C       ****************************************
C         CALCULATE  INVERSE  MATRIX  FOR  "A".
C       ****************************************
```

```
      IA=6
      IB=6
      N1=6
      M1=6
      IJOB=0
      CALL LEQ2C (A,N1,IA,DUMMY,M1,IB,IJOB,WA,WK,IER)
      DET=DCMPLX(1.,0.)
      DO 602 I=1,N1
      IPVT=IDINT(WK(I))
      IF(IPVT.NE.I)DET=-DET
      INDX=I+(I-1)*N1
C     WRITE(2,*)WA(INDX)
C     IF(WA(INDX).EQ.0.DO) GOTO 500
      DET=DET*WA(INDX)
C     DET=DET*CE1(I)
  602 CONTINUE
      WRITE(2,101)DET
  101 FORMAT(5X,2D15.8)
      IF(CDABS(DET).LT..1D-8) GOTO 500

C   ********************************************
C        FORMAT STIFFNESS MATRIX S (=DM).
C   ********************************************

      CALL PRODCT(6,6,6,BM,DUMMY,DM)
      DO 621 II=1,6
      DO 621 JJ=1,6
      AA(II,JJ)=DUMMY(II,JJ)
  621 CONTINUE

C   *******************************************
C        SETUP STIFFNESS MATRIX SS1(=ST).
C   *******************************************

      DO 600 I=1,4
      DO 600 J=1,4
      DUMMY(I,J)=DCMPLX(0.,0.)
      IF(I.EQ.J)DUMMY(I,J)=DCMPLX(1.,0.)
  600 ST(I,J)=DCMPLX(0.,0.)
      ST(1,1)=DM(1,1)
      ST(1,2)=DM(1,2)
      ST(2,1)=DM(2,1)
      ST(2,2)=DM(2,2)+DM(1,1)
      ST(2,3)=DM(1,2)
      ST(3,2)=DM(2,1)
      ST(3,3)=ST(2,2)
      ST(3,4)=DM(1,2)
      ST(4,3)=DM(2,1)
      ST(4,4)=DM(2,2)
```

```
      IA=4
      IB=4
      N2=4
      M2=4
      IJOB=1
      CALL LEQ2C(ST,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DET=DCMPLX(1.,0.)
      DO 603 I=1,N2
      IPVT2=WK2(I)
      IF(IPVT2.NE.I)DET=-DET
      INDX2=I+(I-1)*N2
C     WRITE(2,*)WA2(INDX2)
      DET=DET*WA2(INDX2)
 603   CONTINUE
      WRITE(2,250)DET
 250  FORMAT(5X,' DET. OF STIFFNESS MATRIX =',D15.8,'+',D15.8,'I')

C    ******************************************
C       SETUP INVERSE MATRIX FOR SS1.
C    ******************************************

      IJOB=0
      CALL LEQ2C (ST,N2,IA,DUMMY,M2,IB,IJOB,WA2,WK2,IER)
      DO 703 I=1,N2
      DO 703 J=1,M2
      SS1(I,J)=DUMMY(I,J)
 703  CONTINUE
      DO 709 K=1,6
      FF(K,1)=(0.,0.)
 709  CONTINUE
      FF(3,1)=(1.,0.)
      FF(6,1)=(1.,0.)
      CALL PRODCT(6,1,6,AA,FF,DM1)

C   ***************************************************************
C       FIND FIXED MOMOENTS -MFCD/R AND MFDC/R.
C   ***************************************************************

      MFCD=(0.,0.)
      MFDC=(0.,0.)
      DO 704 I=1,6
      MFCD=MFCD+BM(1,I)*DM1(I,1)
      MFDC=MFDC+BM(2,I)*DM1(I,1)
 704  CONTINUE
      MFCD=(1/B2)*MFCD
      MFDC=(1/B2)*MFDC
```

```
      WRITE(2,705)MFCD,MFDC
  705 FORMAT(5X,' MFCD/QR**2 = ', F10.3,'+',F10.3,5X,'MFDC/QR**2 =',
     &F10.3,'+',F10.3)
      F22(1,1)=(0.,0.)
      F22(2,1)=(0.,0.)
      F22(3,1)=MFCD
      F22(4,1)=-MFDC
C    ******************************************
C        FIND MOMENT MCD.(DM2 STANDS FOR DD1)
C    ******************************************
      CALL PRODCT (4,1,4,SS1,F22,DM2)
      ST1(1,1)=(0.,0.)
      ST1(1,2)=(0.,0.)
      ST1(1,3)=ST(1,1)
      ST1(1,4)=ST(1,2)
      MCD=(0.,0.)
      DO 706 I=1,4
      MCD=MCD + ST1(1,I)*DM2(I,1)
  706 CONTINUE
      WRITE(2,707)MCD
  707 FORMAT(5X,' fCD=(MCD/QR**2)=  ','(',F10.3,'+',F10.3,')',//)
  500 CONTINUE
 1000 STOP
      END

     SUBROUTINE CHARAC(G,B,B2,B4,R2,R4,S2,THETA,ETA,ELO,H,T,C1,GR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 G(7)
CC    XI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO X---AXIS
CC    YI=MOMENT OF INERTIA OF BEAM WITH RESPECT TO Y---AXIS
CC    POI=XI+YI
CC    XJ=TORSIONAL CONSTANT
CC    UNITS: INCHES , POUNDS, SECONDS
CC
      DO 10 I=1,7
   10 G(I)=0.D0
      U=0.3D0
      E=29.D+6
      GE=11.2D+6
      A=H*T
      CK=(10.D0+(1.D0+U))/(12.D0+11.D0*U)
      XI=H*T**3/12.D0
      YI=T*H**3/12.D0
      XJ=C1*H*T**3
      POI=XI+YI
      ETA=POI/XI
      ELO=(GE*XJ)/(E*XI)
```

```
      GAMMA=H**3*T**3/144.DO
      W=GAMMA/(XI*GR**2)
      R=DSQRT(XI/(A*GR**2))
      S=DSQRT(2.DO*(1.DO+U)/CK)*R
      B2=B**2
      B4=B2**2
      B6=B2**3
      S2=S**2
      R2=R**2
      R4=R2**2
      G(1)=1.DO
      G(3)=(2.DO+B2*R2+B2*S2+ETA*B2*R2/ELO)
      G(5)=-ETA*B2*R2-B2*R2/ELO-B2+1.DO+2.DO*B2*S2+B4*S2*R2
     &+ETA*B4*R2*(R2+S2)/ELO
      G(7)=(-B4*R2*S2+B2+ETA*(B6*R4*S2-B4*R2))/ELO+B2*S2
     &-B4*R2*S2*ETA
      THETA=THETA*3.1415926539/180.DO
      WRITE(2,300) R,B,THETA
300   FORMAT(2X,'R =',F8.5,5X,'B =',F12.5,5X,' THETA =',F8.5)
      RETURN
      END

      SUBROUTINE PRODCT(N1,M2,N2,A,B,EE)
      COMPLEX*16 D,A(N1,N2),B(N2,M2),EE(N1,M2)
      DO 270 I=1,N1
      DO 270 J=1,M2
      D=DCMPLX(O.,O.)
      DO 280 K=1,N2
      D=D+A(I,K)*B(K,J)
280   CONTINUE
      EE(I,J)=D
270   CONTINUE
      RETURN
      END
```