Chapter 9
Twist Deformations of Quantum Integrable Spin Chains

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Twist deformations of spacetime lead to deformed field theories with twisted symmetries. Twisted symmetries are quantum group symmetries. Most integrable spin systems have dynamical symmetries related to appropriate quantum groups. We discuss the changes of the properties of these systems under twist transformations of quantum groups. A main example is the isotropic Heisenberg spin chain and the jordanian twist of the universal enveloping algebra of $sl(2)$. It is shown that the spectrum of the XXX spin chain is preserved under the twist deformation while the structure of the eigenstates depends on the choice of boundary conditions. Another example is provided by abelian twists, these give physical deformations of closed spin chains corresponding to higher rank Lie algebras, e.g., $gl(n)$. The energy spectrum of these integrable models is changed and correspondingly their eigenvectors.

9.1 Introduction

One of the cornerstone of the quantum inverse scattering method was the isotropic Heisenberg spin chain [1] exactly solved by H. Bethe [2]. The development of the quantum inverse scattering method (QISM) [3–7], as an approach to the construction and solution of quantum integrable systems, has led to the foundations of the theory of quantum groups [8–11]. Both in QISM and in quantum groups a fundamental, defining object is the R-matrix. V. Drinfel'd introduced an important transformation of quantum groups: a twist of coproduct map. The R-matrix is changed under Drinfel'd twist transformations. We would like to discuss the corresponding changes in integrable models taking as examples the isotropic XXX (9.1) and the anisotropic XXZ spin chains (9.20). These systems are more elementary than the field theories on noncommutative spaces discussed in the previous chapters. The aim is to see what kinds of modifications on these physical systems are produced by twisting their underlying symmetry structures.

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To explain magnetic properties of solids in quantum theory a model of interacting half-integer spins was proposed by W. Heisenberg in 1928 [1]. The hamiltonian of the isotropic model (XXX spin chain) is given in terms of Pauli sigma matrices $\sigma_k^\alpha, \alpha = x,y,z$; at each site $k = 1,2,\ldots,N$ of a one-dimensional chain

$$H_{XXX} = \sum_{m=1}^{N} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \left(\sigma_m^z \sigma_{m+1}^z - 1\right)\right) - \frac{h}{2} \sum_{m=1}^{N} \sigma_m^z. \tag{9.1}$$

The following periodic ($\kappa = 1$) or quasi-periodic ($\kappa \neq 1$) boundary conditions are imposed

$$\sigma_{N+1}^z = \sigma_1^z, \qquad \sigma_{N+1}^\pm = \kappa^{\pm 1} \sigma_1^\pm, \qquad \sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y).$$

(Often the quasi-periodic boundary conditions are referred to as the twisted ones, but the word "twist" is reserved in this book for the theory of quantum groups.) The hamiltonian in (9.1) is an operator in the Hilbert space of spin states

$$\mathscr{H} = \bigotimes_{m=1}^{N} \mathbb{C}_m^2,$$

which is the tensor product of the two-dimensional Hilbert spaces associated with each site of the chain $m = 1,2,\ldots,N$. The explicit form of these sigma matrices $\sigma^\alpha, \alpha = x,y,z,$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9.2}$$

enables one to write the hamiltonian density for zero magnetic field $h = 0$ as a permutation operator $\mathscr{P}_{m\,m+1}$ of neighboring spaces $\mathbb{C}_m^2 \otimes \mathbb{C}_{m+1}^2$: $\mathscr{P}(v \otimes w) = w \otimes v$, where $v, w \in \mathbb{C}^2$. Indeed

$$\sum_\alpha \sigma_m^\alpha \sigma_{m+1}^\alpha = 2\mathscr{P}_{m\,m+1} - I_{m\,m+1}\,; \tag{9.3}$$

here $I_{m\,m+1}$ is the identity matrix. Taking the basis vectors of $\mathbb{C}^2$ as $e^{(+)} = \binom{1}{0}, e^{(-)} = \binom{0}{1}$, so that $\sigma^\pm e^{(\pm)} = 0$, and the basis vectors of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$e^{(+)} \otimes e^{(+)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^{(+)} \otimes e^{(-)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$e^{(-)} \otimes e^{(+)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{(-)} \otimes e^{(-)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{9.4}$$

the permutation (flip) matrix is

$$\mathscr{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{9.5}$$

In terms of the permutation operators $\mathscr{P}_{mm+1}$ the hamiltonian in (9.1), that from now on we consider with zero magnetic field $h = 0$, reads

$$H_{XXX} = 2 \sum_{m=1}^{N} \left(\mathscr{P}_{mm+1} - I_{mm+1} \right) . \tag{9.6}$$

H_{XXX} is an element of the group algebra $\mathbb{C}[\mathscr{S}_N]$ of the symmetric group $\mathscr{S}_N$ (the group of permutations of N objects). See (7.52) for the definition of group algebra $\mathbb{C}[\mathscr{S}_N]$. One can rewrite this hamiltonian using raising σ^+ and lowering σ^- matrices,

$$H_{XXX} = 2 \sum_{m=1}^{N} \left(\sigma_m^+ \sigma_{m+1}^- + \sigma_m^- \sigma_{m+1}^+ + \frac{1}{2} (\sigma_m^z \sigma_{m+1}^z - 1) \right) . \tag{9.7}$$

Then it is easy to see that the tensor product state

$$\Omega = \bigotimes_{k=1}^{N} e_k^{(+)} = \bigotimes_{k=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_k \tag{9.8}$$

is an eigenvector of H_{XXX} with zero eigenvalue Ω,

$$H_{XXX}\, \Omega = 0.$$

This state Ω corresponds to all spins up, and it is called the ferromagnetic state.

The complete spectrum of the energy operator H_{XXX} and its eigenvectors were found by H. Bethe in 1931 [2]. Due to the obvious rotational invariance H_{XXX} commutes with the generators of rotations (global spin):

$$[H_{XXX}, S^\alpha] = 0, \quad S^\alpha = \frac{1}{2} \sum_{k=1}^{N} \sigma_k^\alpha, \quad [S^\alpha, S^\beta] = i\varepsilon^{\alpha\beta\gamma} S^\gamma. \tag{9.9}$$

Hence, the Hilbert space of states $\mathscr{H} = \bigotimes_1^N \mathbb{C}^2$ can be decomposed into invariant subspaces with fixed value of the third component S^z:

$$\mathscr{H} = \bigotimes_{1}^{N} \mathbb{C}^2 = \bigoplus_{M=0}^{N} \mathscr{H}_{\frac{1}{2}N-M} . \tag{9.10}$$

Consider the shift operator U

$$U = \mathscr{P}_{1N} \dots \mathscr{P}_{13} \mathscr{P}_{12} =: \prod_{k=1}^{N-1} \mathscr{P}_{1k+1}, \qquad U \sigma_k^\alpha = \sigma_{k+1}^\alpha U \,. \tag{9.11}$$

It commutes with the hamiltonian

$$[H_{XXX}, U] = 0 \,. \tag{9.12}$$

Then it is easy to see that the one-magnon state

$$\Psi(z) = \sum_{k=1}^{N} z^k \sigma_k^- \Omega \tag{9.13}$$

is a common eigenvector of H_{XXX} and U,

$$U\Psi(z) = z^{-1}\Psi(z), \quad H_{XXX}\Psi(z) = 2(z + z^{-1} - 2)\Psi(z), \tag{9.14}$$

provided that the quasimomentum z satisfies the quantization condition

$$z^N = 1, \quad \log z = 2\pi i k/N, \quad k = 1, 2, \dots, N-1. \tag{9.15}$$

The module $|z|$ is equal to 1. Hence the magnon energy is negative, and to find the ground state with the lowest energy one needs to analyze states with many magnons.

Bethe's proposal was to search for eigenvectors of H_{XXX} in the form of the so-called (coordinate) Bethe ansatz: a linear combination of products of one-magnon states

$$\Psi(z_1, \dots, z_M) = \sum_{1 \le n_1 < n_2 < \dots < n_M \le N} \sum_{\pi \in \mathscr{S}_M} A\left(\pi, \{z_j\}_1^M\right) z_{\pi(1)}^{n_1} z_{\pi(2)}^{n_1} \dots z_{\pi(M)}^{n_M} \prod_{j=1}^{M} \sigma_{n_j}^- \Omega \,. \tag{9.16}$$

Here z_j are the quasimomenta of the M magnons, $\mathscr{S}_M$ is the symmetric group with $M!$ elements $\{\pi\}$, and $A(\pi, \{z_j\}_1^M)$ are the amplitudes depending on $\{z_j\}_1^M$ and π.

The description of a thermodynamic limit $N \to \infty$ corresponding to an infinite antiferromagnetic chain was given by L. Hulthen in 1938 [12].

The requirement is that the M magnon vector (9.16) is an eigenvector of H_{XXX}, and the spin chain periodicity condition results in the explicit form of the coefficients $A(\pi; \{z_j\}_1^M)$ and the quantization conditions of quasimomenta $\{z_j\}_1^M$ (the so-called Bethe equations):

$$z_j^N = \prod_{k \neq j}^{M} \frac{z_j z_k + 1 - 2z_j}{2z_k - z_j z_k - 1}, \quad j = 1, 2, \dots, M. \tag{9.17}$$

The corresponding energy is

$$E_M = \sum_{j=1}^{M} 2(z_j + z_j{}^{-1} - 2). \tag{9.18}$$

The factors on the RHS of (9.17) are scalar two-magnon scattering matrices $S(z_j,\ z_k) = S(z_k,\ z_j)^{-1}$. A detailed deduction of these relations can be found in monographs (e.g. [13–15]). We will obtain them using the QISM in the next section.

There is also a different parameterization λ of quasimomenta

$$z(\lambda) = \frac{\lambda + \eta/2}{\lambda - \eta/2},$$

which is more convenient for the QISM formalism, where λ is known also as a spectral parameter. Although by a scaling $\lambda \to \eta\lambda$ the parameter η can be omitted it is useful to preserve it for the future discussions, e.g., of the quasiclassical limit $\eta \to 0$. Usually, one puts $\eta = i$ to get real-valued λ for $|z| = 1$. The one-magnon energy in terms of λ and $\eta = i$ is $E(\lambda) = -4/(4\lambda^2 + 1)$, and the Bethe equations (9.17) in terms of λ reads as follows:

$$\left(\frac{\lambda_j + \eta/2}{\lambda_j - \eta/2}\right)^N = \prod_{k \neq j}^{M} \frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta}. \tag{9.19}$$

It is instructive to mention two obvious algebras related to the isotropic Heisenberg spin chain: the rotational symmetry Lie algebra $sl(2)$ of H_{XXX} (9.6), (9.9), and the group algebra $\mathbb{C}[\mathscr{S}_N]$ of the symmetric group $\mathscr{S}_N$. We already remarked that the expression of the hamiltonian density in terms of permutation operators (9.6) shows that $H_{XXX} \in \mathbb{C}[\mathscr{S}_N]$. There is also a much bigger dynamical symmetry algebra, the so-called Yangian $\mathscr{Y}(sl(2))$ [4] which includes all the observables of the model (see Sect. 9.2).

Similar solution using the coordinate Bethe ansatz was constructed by R. Orbach in 1958 [16] for the anisotropic Heisenberg spin chain

$$H_{XXZ} = \sum_{k=1}^{N} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta(\sigma_k^z \sigma_{k+1}^z - 1)\right), \tag{9.20}$$

where $\Delta \in (-\infty, \infty)$ is an anisotropy parameter. The only obvious symmetry of this spin chain is the $U(1)$ group with the Lie algebra generator S^z (9.9). The space of states is also decomposed according to the eigenvalues of S^z

$$\mathscr{H} = \bigoplus_{M=0}^{N} \mathscr{H}_{\frac{1}{2}N-M}. \tag{9.21}$$

However, under a minor modification of the XXZ model hamiltonian (concerning an appropriate boundary condition instead of the periodicity one, cf. Sect. 9.2) the symmetry algebra is "similar" to the $sl(2)$ one; it is the quantum algebra $\mathscr{U}_q(sl(2))$ with three generators [8] (see also Sect. 7.4). As a second algebra of this XXZ model one has the Hecke algebra $\mathscr{H}_N(q)$ instead of $\mathbb{C}[\mathscr{S}_N]$. Finally a dynamical symmetry algebra for this model is the quantum affine algebra $\mathscr{U}_q(\widehat{sl}(2))$ [17].

In the next section we solve the XXX and XXZ models by a pure algebraic approach using the quantum inverse scattering method (QISM). For this reason now we write down only the spectrum of H_{XXZ} and we consider the corresponding Bethe equations for the quasimomenta with a different parameterization $\{\mu_j\}_1^M$,

$$H_{XXZ}\Psi(\{\mu_j\}_1^M) = E_M(\{\mu_j\}_1^M)\Psi(\{\mu_j\}_1^M), \tag{9.22}$$

$$E_M(\{\mu_j\}_1^M) = \sum_{j=1}^{M} \frac{\Delta^2-1}{\Delta-\cos 2\mu_j} = \sum_{j=1}^{M} \frac{(\cosh\eta)^2-1}{\cosh\eta-\cos 2\mu_j}, \tag{9.23}$$

$$\left(\frac{\sinh(\mu_j+\frac{1}{2}\eta)}{\sinh(\mu_j-\frac{1}{2}\eta)}\right)^N = \prod_{k\neq j}^{M} \frac{\sinh(\mu_j-\mu_k+\eta)}{\sinh(\mu_j-\mu_k-\eta)}. \tag{9.24}$$

We are using the standard parameterization of the anisotropy parameter Δ,

$$\Delta = \frac{1}{2}(q+q^{-1}) = \cosh(\eta), \quad q=\exp(\eta). \tag{9.25}$$

We finish this introduction by recalling that integrable quantum spin chains are closely related to exactly solved models of statistical mechanics on square lattice (à la two-dimensional Ising model) [15]. The trace of the transfer matrix $t(\lambda)$, which is the generating function of the integrals of motion of the spin system, leads to the partition function Z of the corresponding lattice statistical model. The entries of the R-matrix, a fundamental object of the QISM, are the Boltzmann weights of the local configurations [13–15].

9.2 Algebraic Bethe ansatz (QISM)

In this section we review the QISM formalism. We obtain the eigenvectors (9.22), the eigenvalues (9.23), and the quantization conditions (9.24) of the XXZ model, and the corresponding ones of the XXX model, via an algebraic approach (algebraic Bethe ansatz). This algebraic method is analogous to the treatment à la Dirac of the quantum harmonic oscillator with creation and annihilation operators. We construct a particular transformation converting the variables σ_k^α into a new set of operators. More precisely the aim is to transform the original spin $1/2$ operators σ_k^α (that are local operators because they act only on the kth site) to a set of new nonlocal operators in $\mathscr{H}$ with peculiar algebraic properties independent from the number of sites N. We denote these nonlocal operators by $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$. The hamiltonian H_{XXX} is expressed in terms of these operators, and by acting on the vacuum state Ω with the creation operators $B(\lambda_j)$ we also construct its eigenstates. A similar scheme holds also for the XXZ model.

Next we briefly discuss the underlying dynamical symmetry algebras. These are the Yangian $\mathscr{Y}(sl(2))$ for the XXX model and the quantum affine algebra $\mathscr{U}_q(\widehat{sl(2)})$ for the XXZ model. Deformations of the XXX and XXZ models obtained by twisting of these dynamical symmetries are then discussed in Sect. 9.3.

9.2.1 QISM for the XXX model

The main object of the transformation from the local operators $\sigma_k \in \mathrm{End}(\mathscr{H})$ to the nonlocal ones $A(\lambda)$, $B(\lambda)$, $C(\lambda), D(\lambda) \in \mathrm{End}(\mathscr{H})$ is an auxiliary operator: the L-matrix. It is a 2×2 matrix on an auxiliary space. The matrix entries depend on the local observables σ_k^α at a given site k and on the spectral parameter λ. In the case of the XXX model the L-matrix is

$$L_{ak}(\lambda) = \lambda I + \frac{1}{2}\eta \sum_\alpha \sigma^\alpha \otimes \sigma_k^\alpha = \begin{pmatrix} \lambda+\eta/2 & 0 & 0 & 0 \\ 0 & \lambda-\eta/2 & \eta & 0 \\ 0 & \eta & \lambda-\eta/2 & 0 \\ 0 & 0 & 0 & \lambda+\eta/2 \end{pmatrix}. \tag{9.26}$$

The indices a and k refer to the auxiliary space the matrices σ^α act and to the quantum space $\mathbb{C}_k^2$ (a factor in the definition of $\mathscr{H}$). On the other factors of $\mathscr{H}$ the L_{ak}-matrix acts as the identity. The L-operator in (9.26) is written as a 4×4 matrix in $\mathbb{C}_a^2 \otimes \mathbb{C}_k^2$ and one can recognize the local operators σ_k^α as 2×2 blocks of the 4×4 matrix.

Using the L-operator (9.26) a new set of variables (operators in the Hilbert space $\mathscr{H}$ (9.10) depending on the parameter λ) is introduced by an ordered product of $L_{ak}(\lambda)$ as 2×2 matrices on the auxiliary space $\mathbb{C}_a^2$,

$$T(\lambda) := L_{aN}(\lambda) L_{aN-1}(\lambda) \ldots L_{a1}(\lambda).$$

This new operator $T(\lambda)$ is the QISM monodromy matrix [3–6]. It is a 2×2 matrix in the auxiliary space $V_a \simeq \mathbb{C}^2$. Its entries

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{9.27}$$

are operators in $\mathscr{H}$. They are the new nonlocal variables. The commutation relations of these new operators $A(\lambda),\ldots,D(\lambda) \in \mathrm{End}(\mathscr{H})$ can be obtained from the local relation for the L-operator at one site (see for example [6]):

$$R_{12}(\lambda-\mu) L_{1k}(\lambda) L_{2k}(\mu) = L_{2k}(\mu) L_{1k}(\lambda) R_{12}(\lambda-\mu), \tag{9.28}$$

where the R-matrix is found from the previous equation to be

$$R(\lambda) = \lambda I + \eta \mathscr{P}. \tag{9.29}$$

The R-matrix in (9.29) acts on the tensor product of two auxiliary spaces $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2$. Equation (9.28) involves operators on $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \mathbb{C}_k^2$, where $\mathbb{C}_k^2$ is the space of spin quantum states at site k. The operators R and L_{ak} are understood to act in $\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \mathbb{C}_k^2$ via the embeddings

$$R_{12}(\lambda - \mu) = R(\lambda - \mu) \otimes 1 \,, \qquad L_{2k}(\mu) = 1 \otimes L_{ak}(\mu) \,,$$

and similarly $L_{1k}(\lambda)$ acts as $L_{ak}(\lambda)$ on $\mathbb{C}_1^2 \otimes \mathbb{C}_k^2$ and as the identity on the remaining factor $\mathbb{C}_2^2, L_{1k}(\lambda) = \mathscr{P}_{12} L_{2k}(\lambda) \mathscr{P}_{12}$.

Taking into account (9.5) one can see that the R-matrix (9.29) coincides with the L-matrix (9.26) up to a shift of the spectral parameter

$$R(\lambda) = L(\lambda + \frac{\eta}{2}). \tag{9.30}$$

Then by trivially shifting the spectral parameters λ and μ in (9.28) we obtain the Yang–Baxter equation (YBE) [3]

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu) \,. \tag{9.31}$$

This matrix equation is written in the auxiliary space $\mathrm{End}(\mathbb{C}_1^2 \otimes \mathbb{C}_2^2 \otimes \mathbb{C}_3^2)$, and $R_{12} := R \otimes I,\ R_{23} := I \otimes R,\ R_{13} := \mathscr{P}_{12} R_{23} \mathscr{P}_{12}$. The solution (9.29) is called the Yang R-matrix and there is an obvious extension of it to higher dimensional spaces $\mathbb{C}^n \otimes \mathbb{C}^n$ as the $n^2 \times n^2$ matrix $R(\lambda) = \lambda I + \eta \mathscr{P}$ which also satisfies the YBE (9.31).

The commutation relation for the L-matrix (9.28) induces the commutation relations for the monodromy matrix $T(\lambda)$. These latter have the same form [3–7]

$$R_{12}(\lambda - \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda - \mu), \tag{9.32}$$

where a convenient notation for tensor products is used $T_1(\lambda) := T(\lambda) \otimes I$, $T_2(\mu) = I \otimes T(\mu)$ [3, 4], see also (7.19) and (7.20). One can extract 16 commutation relations for the entries of $T(\lambda)$ (see e.g. [13]). We will use only few of them (the entries 13, 34, and 14) to algebraically construct the eigenvectors of the hamiltonian H_{XXX}:

$$A(\lambda) B(\mu) = f(\lambda - \mu) B(\mu) A(\lambda) + g(\lambda - \mu) B(\lambda) A(\mu), \tag{9.33}$$
$$D(\lambda) B(\mu) = f(\mu - \lambda) B(\mu) D(\lambda) + g(\mu - \lambda) B(\lambda) D(\mu), \tag{9.34}$$
$$B(\lambda) B(\mu) = B(\mu) B(\lambda), \tag{9.35}$$

where $f(\lambda - \mu) = (\lambda - \mu - \eta)/(\lambda - \mu)$, $g(\lambda - \mu) = \eta/(\lambda - \mu)$. Multiplying the RTT-relation (9.32) by $R_{12}^{-1}(\lambda - \mu)$ and taking the trace over two auxiliary spaces one gets the commutativity property of the transfer matrix $t(\lambda)$,

$$t(\lambda) := \mathrm{tr}\, T(\lambda) = A(\lambda) + D(\lambda), \quad t(\lambda) t(\mu) = t(\mu) t(\lambda). \tag{9.36}$$

The transfer matrix $t(\lambda)$ is a generating function of integrals of motion. Due to the regularity property of the Yang R-matrix

$$R(\lambda;\eta)\big|_{\lambda=0} = \eta\mathscr{P} \tag{9.37}$$

(that in terms of the L-matrix reads $L(\lambda)\big|_{\lambda=\eta/2} = \eta\mathscr{P}$, where $\mathscr{P}$ is the permutation operator (cf. (9.5))) we have that $t(\lambda)\big|_{\lambda=\eta/2}$ is proportional to the shift operator U (9.12). Using the obvious property $\frac{d}{d\lambda}L(\lambda) = I$ it can then further be shown that the logarithmic derivative of $t(\lambda)$ at the point $\lambda = \eta/2$ yields the hamiltonian,

$$H_{XXX} \simeq \frac{d}{d\lambda}\log t(\lambda)\big|_{\lambda=\eta/2}, \tag{9.38}$$

where $\simeq$ stands for equality up to a proportionality factor and a constant additive term (proportional to N). The transfer matrix $t(\lambda)$ is the generating function of the mutually commuting integrals of motions $\mathscr{I}_n = \frac{d^n}{d\lambda^n}\log t(\lambda)\big|_{\lambda=\eta/2}$. These integrals are local densities (a natural and desirable physical property) in the sense that $\mathscr{I}_n$ is a sum of operators each of which acts nontrivially at no more than $n+1$ neighboring sites of the lattice.

We have seen that the hamiltonian can be written in terms of the $A(\lambda)$ and $D(\lambda)$ operators. On the other hand the operators $B(\lambda)$, for different values of λ, generate the eigenvectors of the hamiltonian. They act on the vacuum state (the highest weight vector) Ω defined in (9.8):

$$\Omega = \bigotimes_1^N e_m^{(+)}, \quad \sigma_m^z e_m^{(\pm)} = \pm e_m^{(\pm)}, \quad \sigma_m^+ e_m^{(+)} = 0, \quad \sigma_m^- e_m^{(+)} = e_m^{(-)},$$

as creation operators for magnons. In order to show that they are creation operators we first observe that

$$C(\lambda)\Omega = 0, \quad A(\lambda)\Omega = a_N(\lambda)\Omega, \quad D(\lambda)\Omega = d_N(\lambda)\Omega,$$

where $a_N(\lambda) = (\lambda + \frac{\eta}{2})^N$, $d_N(\lambda) = (\lambda - \frac{\eta}{2})^N$. This follows from the upper triangular form of the L-matrix on Ω and by recalling the expression of the monodromy matrix $T(\lambda)$ in terms of the $L(\lambda)$ matrices. Next, from the quadratic relation (9.33) for $A(\lambda)$ and $B(\mu)$, we have

$$\begin{aligned} A(\lambda)\prod_{j=1}^M B(\mu_j) = \prod_{j=1}^M f(\lambda-\mu_j)B(\mu_j)A(\lambda) \\ + \sum_{k=1}^M g(\lambda-\mu_k)B(\lambda)\prod_{j\neq k}^M f(\mu_k-\mu_j)B(\mu_j)A(\mu_k), \end{aligned} \tag{9.39}$$

and a similar relation for $D(\lambda)$ and the product of $B(\mu_j)$. The sum of these relations acting on the vacuum Ω gives the eigenvector (9.16) of the transfer matrix $t(\lambda)$

$$\Psi(\{\mu_j\}_1^M) = \prod_{j=1}^{M} B(\mu_j)\Omega,$$

$$t(\lambda)\prod_{j=1}^{M} B(\mu_j)\Omega = \Lambda(\lambda|\{\mu_k\}_1^M)\prod_{j=1}^{M} B(\mu_j)\Omega, \tag{9.40}$$

under the condition that the parameters μ_k satisfy the Bethe equations $(k = 1,\dots,M)$

$$\frac{a_N(\mu_k)}{d_N(\mu_k)} = \prod_{j\neq k}^{M} \frac{f(\mu_j-\mu_k)}{f(\mu_k-\mu_j)}. \tag{9.41}$$

This condition yields the vanishing of "unwanted terms" containing the operator $B(\lambda)$ and the operators $A(\mu_k), D(\mu_j)$ that as a result of the commutation relations (9.39) have arguments different from μ_j and λ, respectively.

The eigenvalue of the transfer matrix $t(\lambda)$ is

$$\Lambda(\lambda|\{\mu_k\}_1^M) = a_N(\lambda)\prod_{j=1}^{M} f(\lambda-\mu_j) + d_N(\lambda)\prod_{j=1}^{M} f(\mu_j-\lambda).$$

This construction of the eigenvectors of quantum integrable models was coined as the algebraic Bethe ansatz [3].

We conclude by observing that the eigenstates $\Psi = \prod_{j=1}^{M} B(\mu_j)\Omega$, $M \leq [N/2]$ (where $[N/2]$ stands for integer part of $N/2$) are highest weight vectors for the global symmetry algebra $sl(2)$ with generators $S^\alpha = \frac{1}{2}\sum_{n=1}^{N} \sigma_n^\alpha$ (cf. (9.9)),

$$S^+\Psi(\mu_1,\dots,\mu_M) = 0\,, \quad S^z\Psi(\mu_1,\dots,\mu_M) = \left(\frac{N}{2} - M\right)\Psi(\mu_1,\dots,\mu_M)\,. \tag{9.42}$$

The proof is purely algebraic and follows from the RTT-relation and the asymptotic behaviors of the monodromy matrix and of the R-matrix [18],

$$T(\lambda) = \lambda^N I + \eta\lambda^{N-1}\sum_\alpha \sigma_a^\alpha \otimes S^\alpha + O(\lambda^{N-2}), \tag{9.43}$$

$$R_{12}(\lambda-\mu) \simeq I + \frac{\eta}{2\lambda}\left(\sum_\alpha \sigma_1^\alpha \otimes \sigma_2^\alpha + I\right) + O\left(\frac{1}{\lambda^2}\right). \tag{9.44}$$

Indeed, substituting these two asymptotics into the RTT-relation one gets

$$\left[\left(\left(1+\frac{\eta}{2\lambda}\right)I + \frac{\eta}{\lambda}\sum_\alpha \sigma_1^\alpha \otimes \left(\frac{1}{2}\sigma_2^\alpha \otimes 1 + 1 \otimes S^\alpha\right)\right), T_2(\mu)\right] = 0,$$

or

$$\frac{1}{2}\left[\sigma^\alpha, T(\mu)\right] = \left[T(\mu), S^\alpha\right], \quad \frac{1}{2}\left[\sigma^\alpha, T(\mu)\right]_{xy} = \left[T(\mu)_{xy}, S^\alpha\right].$$

The LHS is the commutator of 2×2 matrices while the RHS is the 2×2 matrix of the commutators of the entries of $T(\mu)$ with the global spin generators, e.g.,

$$[S^z, B(\mu)] = -B(\mu), \qquad \left[S^+, B(\mu)\right] = \frac{1}{2}\left(A(\mu) - D(\mu)\right).$$

These relations and (9.33) and (9.34) permit to prove the property (9.42) provided the Bethe equations (9.41) are valid.

9.2.1.1 The Yangian $\mathscr{Y}(sl(2))$

Consider the entries of the 2×2 monodromy matrix $T(\lambda)$ as abstract operators obeying the RTT-relation, divide them by λ^{-N}, and let N be arbitrarily big. We denote by $\mathscr{T}(\lambda)$ this series of 2×2 matrices, with coefficients $t_{ij}^{(n)}$ as abstract generators

$$\mathscr{T}(\lambda)_{ij} = \sum_{n=0}^{\infty} t_{ij}^{(n)} \frac{1}{\lambda^n}, \quad t_{ij}^{(0)} = \delta_{ij}. \tag{9.45}$$

The RTT-relation (9.32) for $\mathscr{T}(\lambda)$ defines an infinite-dimensional Hopf algebra, the Yangian $\mathscr{Y}(gl(2))$. One can define a q-determinant of the matrix $\mathscr{T}(\lambda)$, it is central in $\mathscr{Y}(gl(2))$ and setting it to 1 gives the Yangian $\mathscr{Y}(sl(2))$. The Yangian's coproduct $\Delta : \mathscr{Y}(sl(2)) \to \mathscr{Y}(sl(2)) \otimes \mathscr{Y}(sl(2))$ on the generators $t_{ij}^{(n)}$ can be written in a compact matrix form [19, 20]

$$\Delta(\mathscr{T}(\lambda)_{ij}) = \sum_k \mathscr{T}(\lambda)_{ik} \otimes \mathscr{T}(\lambda)_{kj}. \tag{9.46}$$

According to (9.43) the first nontrivial term $t_{ij}^{(1)}/\lambda$ yields generators of the Lie algebra $sl(2)$ and their coproduct is primitive

$$\Delta(S^\alpha) = S^\alpha \otimes 1 + 1 \otimes S^\alpha.$$

Hence, the universal enveloping algebra $\mathscr{U}(sl(2))$ is a Hopf subalgebra of the Yangian $\mathscr{U}(sl(2)) \subset \mathscr{Y}(sl(2))$. This embedding permits to use twist elements found in $\mathscr{U}(sl(2))^{\otimes 2}$ to perform twisting also of the Yangian (see below and [21]). The Yangian $\mathscr{Y}(g)$ of a Lie algebra g is a deformation of the Lie algebra of polynomial maps $\mathbb{C} \to g$ (or the current algebra $g[t]$), it can also be considered as a "degenerate" version of the quantum affine algebra $\mathscr{U}_q(\hat{g})$, this is a deformation of the central extension $\widehat{L(g)}$ of the loop algebra $L(g)$ (the current algebra $g[t, t^{-1}]$) [9, 20].

9.2.1.2 Higher spins and generalizations

One can take as L-operator the expression similar to (9.26) with an arbitrary representation s_k^α of spin s ($s = 1, 3/2, \ldots$) instead of σ_k^α [7]

$$L_{ak}(\lambda) = \lambda I + \frac{1}{2}\eta \sum_{\alpha} \sigma_a^{\alpha} \otimes s_k^{\alpha} . \tag{9.47}$$

The main QISM relation (9.28) will be still valid with the same 4×4 R-matrix (9.29). This gives us a generalization of the spin 1/2 XXX model to higher spins, i.e., the isotropic spin s model XXX_s [7].

More generally we can consider a solution $R(\lambda;\eta)$ of the YBE (9.31) which has the regularity property $R(\lambda;\eta)\big|_{\lambda=\lambda_0} = \eta\mathscr{P}$ for some value $\lambda = \lambda_0$ (cf. (9.37)) and construct a corresponding quantum integrable system. As before we define the monodromy matrix $T(\lambda)$ as an ordered product of R-matrices (that are related to L-matrices via a formula similar to (9.30)), then the first logarithmic derivative of $t(\lambda)$ gives the hamiltonian H of a spin model

$$H \simeq \frac{d}{d\lambda} \log t(\lambda)\big|_{R(\lambda)=R(\lambda_0)} , \tag{9.48}$$

where, similarly to (9.6),

$$H \simeq \sum_{n=1}^{N} \check{R}_{nn+1} \tag{9.49}$$

and

$$\check{R}_{nn+1} = \mathscr{P}_{nn+1} \frac{d}{d\lambda} R(\lambda)_{nn+1}\big|_{\lambda=\lambda_0} . \tag{9.50}$$

Higher logarithmic derivatives of $t(\lambda)$ give mutually commuting integral of motions.

For the XXX and XXZ models, with chains carrying an arbitrary representation of spin s ($s = 1, 3/2, \ldots$), the constant $\check{R}_{nn+1}$ matrix (9.50) (that is proportional to the permutation matrix P_{nn+1} in the XXX model) satisfies the YBE in the braid group form

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} . \tag{9.51}$$

9.2.2 *Anisotropic* XXZ *spin chain*

The QISM approach to the XXZ model is almost identical to the one discussed for the XXX model. This is so because the corresponding L-matrices and the R-matrices have the same structure and satisfy the same fundamental relations (9.28) and (9.31). We explicitly have

$$L_{XXZ}(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta\sigma_k^{\alpha}/2) & \sinh(\eta)\,\sigma_k^{-} \\ \sinh(\eta)\,\sigma_k^{+} & \sinh(\lambda - \eta\sigma_k^{\alpha}/2) \end{pmatrix} , \tag{9.52}$$

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \tag{9.53}$$

where the entries of the R-matrix are

$$a(\lambda) = \sinh(\lambda + \eta), \quad b(\lambda) = \sinh(\lambda), \quad c(\lambda) = \sinh(\eta).$$

The hamiltonian of the XXZ model is given in (9.20). As in the XXX model the ferromagnetic state Ω is the highest eigenstate of H_{XXZ}, and the L-operator (9.52) and the monodromy matrix $T(\lambda)$ on Ω have an upper triangular structure. Hence the eigenstates, the Bethe equations (9.24), and the energy spectrum are produced by the same algebraic procedure (algebraic Bethe ansatz) that consists of creating magnon states by applying to Ω products of the mutually commuting operators $B(\mu_j)$.

From the quantum group point of view it is more convenient to consider a non-symmetric R-matrix instead of (9.53),

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c_+(\lambda) & 0 \\ 0 & c_-(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \qquad c_\pm(\lambda) = \exp(\pm\lambda)\sinh\eta. \tag{9.54}$$

It is useful to prove directly that due to the commutativity of the R-matrix (9.53) with the primitive coproduct of the Cartan generator $h = \sigma^z$, $[R(\lambda),\ h\otimes 1 + 1\otimes h] = 0$, the transformed R-matrix

$$\exp(x\lambda\ h_1) R_{12} \exp(-x\lambda\ h_1)$$

(where $h_1 = h\otimes 1$, $h_2 = 1\otimes h$) satisfies the YBE (9.31). To obtain (9.54) set $x = \frac{1}{2}$.

The R-matrices (9.53) and (9.54) give the same XXZ model with periodic boundary conditions, but, as we now explain, it is the R-matrix (9.54) that is relevant for the XXZ model with open boundary conditions and that is directly related to the quantum algebras $\mathscr{U}_q(sl(2)) \subset \mathscr{U}_q(\widehat{sl(2)})$. This relation is given via a linear combination of constant R-matrices

$$R(\lambda;q) = \exp(\lambda)R^{(+)}(q) - \exp(-\lambda)R^{(-)}(q), \quad q = \exp\eta,$$

where $R^{(-)}(q) = (R^{(+)}_{21}(q))^{-1} := \mathscr{P}(R^{(+)}_{12})^{-1}\mathscr{P}$. The constant R-matrix

$$R^{(+)}(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \omega & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \omega = q - \frac{1}{q} = 2\sinh\eta \tag{9.55}$$

enters the RLL relations defining $\mathscr{U}_q(sl(2))$, they are given in (7.63) and (7.64). These relations can be used to prove the RLL relations (9.28) for the L- and R-matrices with spectral parameter (9.52) and (9.53).

By multiplying the $R^{(+)}$-matrix by the permutation $\mathscr{P}$ one gets the matrices

$$\check{R}_k(q) \equiv \check{R}_{k\,k+1}(q) := \mathscr{P} R^{(+)}_{k\,k+1}(q)\,, \quad k = 1, 2, \ldots, N-1\,. \tag{9.56}$$

They satisfy the braid group relation (9.51) and additionally the quadratic relation [20]

$$\check{R}_k(q)^2 = \left(q - \frac{1}{q}\right) \check{R}_k(q) + I. \tag{9.57}$$

The $N-1$ elements $\check{R}_k(q)$ that satisfy (9.51) and (9.57) are the generators of the Hecke algebra $\mathscr{H}_N(q)$.

According to the theory of quantum groups the Hecke algebra $\mathscr{H}_N(q)$ with generators $\check{R}_k(q)$ (9.56) is the centralizer of the diagonal action of $\mathscr{U}_q(sl(2))$ in the space $\otimes_1^N \mathbb{C}^2$,

$$[\check{R}_k(q), \Delta^N(X)] = 0, \quad X \in \mathscr{U}_q(sl(2))\,,$$

where $\Delta^N(X)$ is understood in the representation space $\otimes_1^N \mathbb{C}^2$, and the diagonal action is given by the N-fold coproduct map[1] $\Delta^N : \mathscr{U}_q(sl(2)) \to \mathscr{U}_q(sl(2))^{\otimes N}$,

$$\Delta^N := (\Delta \otimes id \otimes id \otimes \ldots id)(\Delta \otimes id \otimes \ldots id) \ldots (\Delta \otimes id)\Delta\,. \tag{9.58}$$

Let us now consider the hamiltonian of the XXZ model with open boundary conditions

$$H_{XXZ} = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \cosh\eta\, (\sigma_k^z \sigma_{k+1}^z - 1)\right) + \sinh\eta\, (\sigma_1^z - \sigma_N^z). \tag{9.59}$$

This open spin chain hamiltonian is explicitly $\mathscr{U}_q(sl(2))$ invariant because its density is a cross Casimir of $\mathscr{U}_q(sl(2))$,

$$c_2^{\otimes}(q) = 2(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+) + \cosh\eta\, \sigma_k^z \sigma_{k+1}^z + \sinh\eta\, (\sigma_k^z - \sigma_{k+1}^z).$$

This expression, in accordance with (9.49), essentially coincides with the Hecke algebra generator $\check{R}_k(q)$ (9.56).

Finally we comment on the difference between open and closed (periodic) boundary conditions for the XXX and XXZ models. In the XXX model the difference between open and closed boundary conditions is given by the element $\check{R}_{N1}^{XXX}(q) = P_{N1} = P_{1N}$, that belongs to the symmetry group $\mathscr{S}_N$, so that also H_{XXX} with periodic boundary conditions is an element of the group algebra $\mathbb{C}[\mathscr{S}_N]$, and we

[1] The map Δ^N is the composition of $N-1$ coproduct maps. For example, for $N = 3$ we have $\Delta^3 = (\Delta \otimes id)\Delta$. Coassociativity of Δ (cf. (9.79)) then implies that $\Delta^3 = (id \otimes \Delta)\Delta$; similarly Δ^N is independent from the position of Δ in the tensor products (9.58).

have $\mathscr{Y}(sl(2))$ dynamical symmetry. The situation is different in the XXZ model. Indeed in this case the hamiltonian with periodic boundary condition has together with $\check{R}_{kk+1}(q)$ the summand $\check{R}_{N1}(q)$. This latter addend does not belong to the Hecke algebra. This explains why the open spin chain hamiltonian (9.59) is $\mathscr{U}_q(sl(2))$ invariant while the closed spin chain hamiltonian with periodic boundary condition (9.20) is not.

9.3 Twists and QISM

In this section we consider what kind of changes can be induced in integrable spin chains using twist transformations of the related quantum groups.

We see that twists naturally arise when considering scaling limits, for example, the XXX and XXZ models can be related by two inequivalent elementary scaling transformations, and we propose a treatment of the relations obtained via the second scaling limit in terms of a corresponding twist. This leads to the example of the so-called jordanian twist.

In Sect. 9.3.2 on the other hand we consider an abelian twist and study the changes in the hamiltonian of the XXZ model with periodic boundary conditions under this twist transformation.

Section 9.3.3 first details the relation between quantum groups and integrable systems. We then see how, in the case of open spin chains, the twisting of a quantum group leads to the corresponding twisting of the integrable system. Contrary to the case of closed spin chains considered in Sects. 9.3.1 and 9.3.2, the original open spin chain hamiltonian H and the twisted ones $H^{(t)}$ can be easily compared, they are related by a similarity transformation.

In Sect. 9.3.4 we consider another example of twist (coboundary twist) this is in general a trivial twist. Under scaling transformations these coboundary twist can however become nontrivial, this is yet another way to obtain (extended) jordanian twists and their related integrable systems.

Scaling limit $XXZ \to XXX$. It is easy to get the isotropic XXX spin chain from the anisotropic XXZ one via a scaling limit $\varepsilon \to 0$ of parameters

$$\lambda \to \varepsilon\lambda,\, \eta \to \varepsilon\eta,\, q \to 1+\varepsilon\eta,\, \sinh(\lambda-\eta) \to \varepsilon(\lambda-\eta),\, \cosh\eta \to 1+\frac{1}{2}\varepsilon^2\eta^2. \tag{9.60}$$

The hamiltonians (9.20), eigenvectors, and Bethe equations (9.24) are clearly connected in this limit $\varepsilon \to 0$, as well as R-matrices (9.53) and (9.29),

$$R_{XXZ}(\varepsilon\lambda;\varepsilon\eta) \to \varepsilon(\lambda I+\eta\mathscr{P}) = \varepsilon R_{XXX}(\lambda;\eta), \tag{9.61}$$

and the algebraic Bethe ansatz.

Scaling limit $XXZ \to XXX_\xi$. A nontrivial scaling limit (contraction) of the XXZ model is obtained by applying additionally a similarity transformation with the matrix $M(\xi) = \exp(\xi\sigma^+) \in Mat(\mathbb{C}^2)$

$$M(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \tag{9.62}$$

to the main objects of the QISM. The YBE is obviously invariant with respect to the factorized similarity transformations of its solution $R \to \mathrm{Ad}M^{\otimes 2}R$ [7]. Then the scaling limit (9.60) with a singular behavior of the parameter ξ with respect to ε: $\xi \to \xi/\varepsilon$ yields a deformed XXX spin chain. One obtains the closed spin chain hamiltonian

$$\mathrm{Ad}M(\xi/\varepsilon)^{\otimes N} H_{XXZ} \to H_{XXX_\xi} := H_{XXX} + \sum_{n=1}^{N} \left(\xi^2 \sigma_n^+ \sigma_{n+1}^+ + \xi(\sigma_n^+ - \sigma_{n+1}^+)\right). \tag{9.63}$$

It is a hamiltonian of a deformed XXX model with ξ as the deformation parameter [21]. The similarity transformation does not change the spectrum of H_{XXZ}. Thus in this limit one produces the standard spectrum of the XXX model, although the hamiltonian is now non-hermitian and it depends on ξ. This change of hermiticity comes from the triangularity of the matrix M. In the scaling limit we get additional degeneracy of the spectrum, and some jordanian cells appear. Here is a two-dimensional example of this phenomenon $(\xi \to \xi/(x_2 - x_1), x_2 \to x_1)$

$$AdM(\xi)\cdot\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} x_1 & \xi(x_2 - x_1) \\ 0 & x_2 \end{pmatrix} \underset{(x_2 \to x_1)}{\longrightarrow} \begin{pmatrix} x_1 & \xi \\ 0 & x_1 \end{pmatrix}.$$

The eigenvector $\binom{1}{0}$ survives, while the second eigenvector becomes an adjoint eigenvector.

After this transformation the limiting R-matrix and L-operators, similarly to the hamiltonian (9.63), have also extra terms

$$\mathrm{Ad}M(\xi/\varepsilon)^{\otimes 2} R_{XXZ}(\varepsilon\lambda; \varepsilon\eta) \to \lambda R(\xi) + \eta\mathscr{P}, \tag{9.64}$$

$$(\mathrm{Ad}M_a(\xi) \otimes \mathrm{Ad}M_k(\xi))\, L_{ak}^{(XXZ)}(\lambda) \to L_{ak}^{(XXX)}(\lambda;\xi), \tag{9.65}$$

where

$$\begin{aligned} R(\xi) &= I + \xi\left(\sigma^+ \otimes \sigma^z - \sigma^z \otimes \sigma^+ + \xi\sigma^+ \otimes \sigma^+\right) \\ &= \exp(\xi\sigma^+ \otimes \sigma^z)\exp(-\xi\sigma^z \otimes \sigma^+) \end{aligned} \tag{9.66}$$

and

$$\begin{aligned} L_{ak}^{(XXX)}(\lambda;\xi) &= \lambda I + \frac{\eta}{2}\sum_\alpha (\sigma_a^\alpha \otimes \sigma_k^\alpha) \\ &\quad + \xi(\lambda - \eta/2)\left(\sigma_a^+ \otimes \sigma_k^z - \sigma_a^z \otimes \sigma_k^+ + \xi\sigma_a^+ \otimes \sigma_k^+\right). \end{aligned} \tag{9.67}$$

9.3.1 Jordanian twist

The constant $R(\xi)$-matrix satisfies the YBE without spectral parameter. It is a triangular R-matrix: $R_{12}(\xi)R_{21}(\xi) = 1$. It is an image of a universal R-matrix $\mathscr{R} = \mathscr{F}_{21}\mathscr{F}_{12}^{-1}$ obtained by means of a jordanian twist of the universal enveloping algebra $\mathscr{U}(sl(2))$:

$$\mathscr{F}^{(j)} = \exp\left(\frac{1}{2}h \otimes \ln(1+2\xi X^+)\right), \quad w := \ln(1+2\xi X^+), \tag{9.68}$$

where h, $X^\pm$ are the generators of the Lie algebra $sl(2)$: $[h, X^\pm] = \pm 2X^\pm$, $[X^+, X^-] = h$. Let us write down for completeness the twisted coproduct maps for the generators:

$$\begin{aligned}
\Delta_t(h) &:= \mathscr{F}^{(j)}\Delta(h)(\mathscr{F}^{(j)})^{-1} = h \otimes e^{-w} + 1 \otimes h,\\
\Delta_t(X^+) &= X^+ \otimes 1 + 1 \otimes X^+ + 2\xi X^+ \otimes X^+ = X^+ \otimes e^w + 1 \otimes X^+,\\
\Delta_t(w) &= w \otimes 1 + 1 \otimes w,\\
\Delta_t(X^-) &= X^- \otimes e^{-w} + 1 \otimes X^- + \xi h \otimes h e^{-w} + \xi \left(h - \frac{1}{2}h^2\right) \otimes (e^{-w} - e^{-2w}).
\end{aligned}$$

Introducing the new combination $\tilde{X}^- = X^- - \frac{1}{2}\xi h^2$ one obtains a quasiprimitive coproduct also for $\tilde{X}^-$

$$\Delta_t(\tilde{X}^-) = \tilde{X}^- \otimes e^{-w} + 1 \otimes \tilde{X}^-.$$

In the spin 1/2 representation we have $F^{(j)} = \exp(\xi\sigma^z \otimes \sigma^+)$ and $R_{12}(\xi) = \exp(\xi\sigma^+ \otimes \sigma^z)\exp(-\xi\sigma^z \otimes \sigma^+)$.

The scaling limit procedure $XXZ \to XXX_\xi$ does not lead to fully solve the XXX_ξ model because in this limit many eigenstates of the XXZ model become singular (e.g., $\Omega_- = \otimes_k e_k^{(-)}$). New ones have therefore to be found. The study of this closed spin chain quantum integrable system via its R-matrix (9.64) is nontrivial because the form of (9.64) is more complicated than that of (9.61). In particular the commutation relations among the operators $A(\lambda), \ldots, D(\lambda)$ are more involved. Although the monodromy matrix $T(\lambda)$ still has an upper triangular structure when acting on the ferromagnetic state $\Omega = \otimes_1^N e_k^{(+)}$ (9.8), and therefore the operator $B(\lambda)$ is still a creation operator, the algebraic Bethe ansatz is quite elaborated.

Deformations of integrable spin systems related to higher rank Lie algebras, e.g., $gl(n)$ or Lie superalgebras $gl(m|n)$, can be similarly obtained using extended jordanian twists [22, 23]. In particular, a generalization of the isotropic XXX model to the case of $gl(n)$ is given by the hamiltonian

$$H_{gl(n)} = \sum_{m=1}^{N} \mathscr{P}_{mm+1} = \sum_{m=1}^{N}\sum_{i,j=1}^{n} e_{ij}^{(m)} \otimes e_{ji}^{(m+1)}, \tag{9.69}$$

where $\mathscr{P}_{m\,m+1}$ is the permutation operator of $\mathbb{C}^n_m \otimes \mathbb{C}^n_{m+1}$ while $e^{(m)}_{ij}$ are the basic matrices on $\mathbb{C}^n_m$ (with matrix entries $(e^{(m)}_{ij})_{kl} = \delta_{ik}\delta_{jl}$). An extended jordanian twist, e.g., for $n=3$, is [22]

$$\begin{aligned}\mathscr{F}^{(j_{ext})} &= \exp(2\xi E_{12}\otimes E_{23}\exp(-w_{13}))\exp\left(\frac{1}{2}h\otimes\ln(1+2\xi E_{13})\right), \quad (9.70)\\ w_{13} &= \ln(1+2\xi E_{13})),\end{aligned}$$

where h, E_{ij} are the generators of $sl(3)$, $[h,E_{ij}] = (\delta_{1i}+\delta_{3j})E_{ij}$, $[E_{13},E_{31}] = h.$

9.3.2 Abelian twist

One can add more parameters to the R-matrix of the XXZ model (9.54) using an abelian twist related to the quantum algebra $\mathscr{U}_q(sl(2)) \subset \mathscr{U}_q(\widehat{sl(2)})$. The generator h of $\mathscr{U}_q(sl(2))$ still has the primitive coproduct:

$$\Delta(h) = h\otimes 1 + 1\otimes h := h_1 + h_2 \in \mathscr{U}_q(sl(2))^{\otimes 2}.$$

Extending this quantum algebra by a central element κ which has also the primitive coproduct $\Delta(\kappa) = \kappa_1 + \kappa_2$, a twist with the carrier space in abelian Lie subalgebra $\mathbb{C}[\kappa, h] \subset \mathscr{U}_q(gl(2))$ can be constructed (i.e., an abelian twist)

$$\mathscr{F}^{(ab)} = \exp(\theta(\kappa\otimes h - h\otimes\kappa)). \quad (9.71)$$

The transformation of the universal R-matrix is

$$\mathscr{R}^{(t)} = \mathscr{F}_{21}\mathscr{R}\mathscr{F}_{12}^{-1} = \mathscr{F}_{12}^{-1}\mathscr{R}\mathscr{F}_{12}^{-1}, \quad (9.72)$$

the last equality is due to the property $\mathscr{F}_{21} = \mathscr{F}_{1\,2}^{-1}$ valid for the twist (9.71). Spin $\frac{1}{2}$ representations with fixed central elements $\kappa = \frac{1}{4}$ for both representation spaces $\mathbb{C}^2\otimes\mathbb{C}^2$ yield

$$R^{(t)}_{12}(\lambda) = \exp\left(\frac{\theta}{4}(\sigma^z_1 - \sigma^z_2)\right)R_{12}(\lambda)\exp\left(\frac{\theta}{4}(\sigma^z_1 - \sigma^z_2)\right). \quad (9.73)$$

One explicitly obtains

$$R^{(t)}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0\\ 0 & b_+(\lambda) & c_+(\lambda) & 0\\ 0 & c_-(\lambda) & b_-(\lambda) & 0\\ 0 & 0 & 0 & a(\lambda)\end{pmatrix}, \qquad \begin{aligned} b_\pm(\lambda) &= p^{\pm 1}\sinh(\lambda), p = \exp(\theta)\,,\\ c_\pm(\lambda) &= \exp(\pm\lambda)\sinh\eta\,.\end{aligned} \quad (9.74)$$

The matrix structure of the $R^{(t)}$-matrix is the same as that of (9.54), just the diagonal elements are different. Similarly the L-operator has the same matrix structure. Hence the algebraic Bethe ansatz is also the same. However, since different functions enter relations (9.33) and (9.34), the result is a change of the Bethe equations (9.24) that now read

$$\left(\frac{1}{p}\frac{\sinh(\mu_j+\frac{1}{2}\eta)}{\sinh(\mu_j-\frac{1}{2}\eta)}\right)^N=\prod_{k\neq j}^{M}\frac{\sinh(\mu_j-\mu_k+\eta)}{\sinh(\mu_j-\mu_k-\eta)}. \tag{9.75}$$

Also the hamiltonian depends on the twist parameter $p=\exp(\theta)$,

$$H_{XXZp}=2\sum_{k=1}^{N}\left(\left(p\,\sigma_k^+\sigma_{k+1}^-+\frac{1}{p}\,\sigma_k^-\sigma_{k+1}^+\right)+\frac{1}{2}\cosh\eta\,(\sigma_k^z\sigma_{k+1}^z-1)\right)\ . \tag{9.76}$$

This is the hamiltonian of the closed XXZ_p spin chain, and it is hermitian for $|p|=1$. References on this model studied as spin chain and as $2d$ classical statistical system (6 vertex model) can be found in [24].

The method of constructing new quantum integrable systems via an abelian twist is quite general. There are quantum integrable spin chains corresponding to higher rank ($r>1$) Lie algebras, e.g., $gl(n)$, or Lie superalgebras, e.g., $gl(m|n)$. Then one has an r-dimensional abelian Lie subalgebra, with generators $\{h_j\}_1^r$, and one can construct an abelian twist with more parameters to deform the spin model [25]

$$\mathscr{F}^{(ab)}=\exp\left(\sum\theta^{ij}h_i\otimes h_j\right). \tag{9.77}$$

This twist element is similar to the one used to construct the θ-deformed Poincaré algebra (see [26] and previous chapters).

9.3.3 Generalities on twist transformations

The algebraic structure underlying the main operators entering the QISM: the R-matrix, the L-operator, and the monodromy matrix $T(\lambda)$, is that of a quantum group. In quantum groups a key role is played by the universal R-matrix $\mathscr{R}$ and by the coproduct map Δ. By representing the universal $\mathscr{R}$-matrix and by using the coproduct map Δ one obtains the R-, L-, and T- operators. By twisting the quantum group coproduct map Δ one obtains a new (twisted) quantum group and can consider the corresponding changes of the R-matrix, the L-operator, and the monodromy matrix $T(\lambda)$ that in turn define a new integrable system.

Given a quasitriangular Hopf algebra $\mathscr{U}(m,\Delta,S,\mathscr{R})$ with product m, coproduct Δ, and antipode S, and a twist $\mathscr{F}\in\mathscr{U}\otimes\mathscr{U}$, the corresponding twisted quasitriangular Hopf algebra has a transformed coproduct map Δ_t, for all $a\in\mathscr{U}$,

$$\Delta_t(a)=\mathscr{F}\Delta(a)\mathscr{F}^{-1} \tag{9.78}$$

(cf. Chap. 8.2.1). Coassociativity of this deformed coproduct, i.e.,

$$(\Delta_t\otimes id)\Delta_t=(id\otimes\Delta_t)\Delta_t \tag{9.79}$$

is implied by the Drinfel'd twist equation

$$\mathscr{F}_{12}(\Delta\otimes id)\mathscr{F}=\mathscr{F}_{23}(id\otimes\Delta)\mathscr{F}\,. \tag{9.80}$$

The corresponding twist-transformed universal R-matrix is

$$\mathscr{R}^{(t)}=\mathscr{F}_{21}\mathscr{R}\mathscr{F}^{-1}. \tag{9.81}$$

Defining Δ^{op} (and similarly Δ_t^{op}) by $\Delta_{12}^{op}(a)=\Delta_{21}(a)$ for all $a\in\mathscr{U}$, we have, again for all $a\in\mathscr{U}$, $\Delta_t^{op}(a)=\mathscr{F}_{21}\Delta^{op}(a)\mathscr{F}_{21}^{-1}$, and

$$\mathscr{F}_{21}\mathscr{R}\Delta(a)\mathscr{F}^{-1}=\mathscr{F}_{21}\Delta^{op}(a)\mathscr{R}\mathscr{F}^{-1}.$$

These two last relations imply the intertwining relation (for all $a\in\mathscr{U}$)

$$\mathscr{R}^{(t)}\Delta_t(a)=\Delta_t^{op}(a)\mathscr{R}^{(t)}.$$

In order to obtain the R-, L-, and T- operators from the universal R-matrix and the coproduct Δ we consider the universal L-matrix. It is an image of the universal R-matrix in a representation ρ corresponding to an auxiliary space V_a

$$\mathscr{L}=(\rho\otimes id)\mathscr{R},\quad \text{or}\quad \mathscr{L}^{(t)}=(\rho\otimes id)\mathscr{F}_{21}\mathscr{R}\mathscr{F}^{-1}.$$

The L-matrix of the previous sections is then obtained by representing $\mathscr{L}$ on the vector space V_k. The monodromy matrix T of a chain with N sites

$$T_N=L_{aN}L_{aN-1}\ldots L_{a1}$$

can be obtained by the action of the N-fold coproduct $\Delta^N:\mathscr{U}\to\mathscr{U}^{\otimes N}$ as defined in (9.58). In fact taking into account the factorization property of the universal R-matrix [9],

$$(id\otimes\Delta)\mathscr{R}=\mathscr{R}_{13}\mathscr{R}_{12},$$

we have

$$(id\otimes\Delta^3)\mathscr{R}:=(id\otimes\Delta\otimes id)(id\otimes\Delta)\mathscr{R}=\mathscr{R}_{14}\mathscr{R}_{13}\mathscr{R}_{12}\,,$$

hence,

$$\mathscr{T}_N=(\rho\otimes id)\Delta^N\mathscr{R}=(\rho\otimes id)\mathscr{R}_{aN}\mathscr{R}_{aN-1}\cdots\mathscr{R}_{a1}\in\mathrm{End}(V_a)\otimes\mathscr{U}^{\otimes N}\,, \tag{9.82}$$

$$T_N=(id\otimes\rho^{\otimes N})\mathscr{T}_N\,. \tag{9.83}$$

Now we consider twist transformations of the monodromy matrix. From (9.82) we see that it is obtained by twisting the universal $\mathscr{R}$-matrix and the N-fold coproduct Δ^N. From the definition of twisted coproduct we have the relation for 3-fold coproducts, for all $a\in\mathscr{U}$,

$$\Delta_t^3(a) = (\Delta_t \otimes id)\Delta_t(a) = \mathscr{F}_{12}\mathscr{F}_{(12)3}\,\Delta^3(a)\,\mathscr{F}_{(12)3}^{-1}\mathscr{F}_{12}^{-1},$$

here $\mathscr{F}_{(12)3} = (\Delta \otimes id)\mathscr{F}$. The N-fold coproduct $\Delta^N : \mathscr{U} \to \mathscr{U}^{\otimes N}$ is also similarly transformed by the twist

$$\Delta_t^N = \mathscr{F}^{(N)}\Delta^N(\mathscr{F}^{(N)})^{-1}, \tag{9.84}$$

where

$$\mathscr{F}^{(N)} := \mathscr{F}_{12}\mathscr{F}_{(12)3}\cdots\mathscr{F}_{(12\ldots N-1)N},$$

and $\mathscr{F}_{(123)4} = (\Delta^3 \otimes id)\mathscr{F}$, and similarly for all other factors up to $\mathscr{F}_{(12\ldots N-1)N} = (\Delta^N \otimes id)\mathscr{F}$.[2]

It is instructive to prove that the relation $\check{\mathscr{R}}^{(t)} = \mathscr{F}\check{\mathscr{R}}\mathscr{F}^{-1}$, defining the twist-transformed $\check{R}$-matrix (cf. (9.81) and (9.50)), in $\mathscr{U}^{\otimes N}$ reads

$$\check{\mathscr{R}}^{(t)}_{nn+1} = \mathscr{F}_{nn+1}\check{\mathscr{R}}_{nn+1}(\mathscr{F}_{nn+1})^{-1} = \mathscr{F}^{(N)}\check{\mathscr{R}}_{nn+1}(\mathscr{F}^{(N)})^{-1}\ . \tag{9.85}$$

The last equality shows that the operator $\mathscr{F}_{nn+1}$ that defines the similarity transformation $\check{\mathscr{R}}_{nn+1} \to \mathscr{F}_{nn+1}\check{\mathscr{R}}_{nn+1}(\mathscr{F}_{nn+1})^{-1}$, and that is local because it depends on the sites n and $n+1$, can be replaced by the operator $\mathscr{F}^{(N)}$ that is global because it is independent from the positions n and $n+1$.

Equality (9.85) allows to compare the hamiltonian $H^{(t)}$ of an open spin chain described by a twisted quantum group to the untwisted one H. Recalling (9.48) we see that

$$H^{(t)} = \sum_{n=1}^{N-1}\check{R}^{(t)}_{nn+1} = F^{(N)}\left(\sum_{n=1}^{N-1}\check{R}_{nn+1}\right)(F^{(N)})^{-1} = F^{(N)}\,H\,(F^{(N)})^{-1}\ , \tag{9.86}$$

where $H^{(t)}, F_{nn+1}$, and $\check{R}_{nn+1}$ are written in a representation. Contrary to the closed spin chains of Sects. 9.3.1 and 9.3.2, we see that the open spin chain hamiltonian $H^{(t)}$ has the same spectrum as H and that its eigenvectors are transformed via $F^{(N)}$.

9.3.4 Coboundary twists and the jordanian deformation

Coboundary twists are twists constructed with any invertible element u of a Hopf algebra $\mathscr{U}$:

$$\mathscr{F}^{(cob)} = (u\otimes u)\Delta(u^{-1}).$$

The Hopf algebra constructed with a coboundary twist has the coproduct $\widetilde{\Delta} = \mathscr{F}^{(cob)}\Delta(\mathscr{F}^{(cob)})^{-1}$ and is isomorphic (as a Hopf algebra) to the original one. They are in fact related by the similarity transformation $\varphi_u : \mathscr{U} \to \mathscr{U}, \quad a \to uau^{-1}$,

[2] Due to the Drinfel'd twist equation (9.80), the N-fold twist $\mathscr{F}^{(N)}$ admits similar and equivalent factorizations with a different order of the $N-2$ coproduct maps acting on different factors of $\mathscr{F}$.

$$\widetilde{\Delta} \circ \varphi_u = (\varphi_u \otimes \varphi_u) \circ \Delta \ .$$

The universal R-matrix of $\mathscr{U}$ (if $\mathscr{U}$ is quasitriangular) is transformed with this twist just by the similarity transformation

$$\mathscr{R} \to \mathrm{Ad}(u \otimes u)\mathscr{R}.$$

We now exploit the very definition of coboundary twist and concoct a coboundary twist of the Hopf algebra $\mathscr{U}_q(sl(2))$ given by an element $u(q,t) \in \mathscr{U}_q(sl(2))$ (where t is a parameter that we later relate to ξ), such that

$$\mathscr{F}^{(cob)}(q,t) = (u \otimes u)\Delta(u^{-1}) \in \mathscr{U}_q(sl(2)) \otimes \mathscr{U}_q(sl(2))$$

is nonsingular in the limit $q \to 1$, while the corresponding element $u(q,t)$ is singular. This coboundary twist in the $q \to 1$ limit is no more a coboundary and leads to the jordanian twist $\mathscr{F}^{(j)}$. Hence, instead of performing a singular contraction of the XXZ model, one can apply the appropriate twist transformation to the whole QISM machinery of the XXZ model and then consider the limit $q \to 1$. An element $u(q,t)$ with these properties is [27]

$$u(q,t) = \exp_{q^2}\left(\frac{t}{1-q^2}X^+\right), \tag{9.87}$$

where

$$\exp_q(x) := \sum_{n=1}^{\infty} \frac{x^n}{(n)_q!} = \exp\left(\sum_{n=1}^{\infty} \frac{(1-q)^{n-1}x^n}{n\,(n)_q}\right), \quad (\exp_q(x))^{-1} = \exp_{q^{-1}}(-x), \tag{9.88}$$

and $(n)_q := (1-q^n)/(1-q)$, $(n)_q! := (1)_q(2)_q\cdots(n)_q$.

Since the generator X^+ of the quantum algebra $\mathscr{U}_q(sl(2))$ has the following coproduct

$$\Delta(X^+) = X^+ \otimes 1 + K^{-2} \otimes X^+,$$

then the coboundary twist element is

$$\begin{aligned} \mathscr{F}^{(cob)}(q) &= (u(q) \otimes u(q))\Delta(u(q)^{-1}) \\ &= \exp_{q^2}\left(\frac{t}{1-q^2}X^+\right) \otimes \exp_{q^2}\left(\frac{t}{1-q^2}X^+\right) \\ &\qquad \exp_{q^{-2}}\left(-\frac{t}{1-q^2}(X^+ \otimes 1 + K^{-2} \otimes X^+)\right). \end{aligned} \tag{9.89}$$

We now use a functional equation for the q-exponential of a sum of noncommuting arguments. Provided that $yx = qxy$ we have

$$\exp_q(x+y) = \exp_q(x)\exp_q(y).$$

Recalling the $\mathscr{U}_q(sl(2))$ commutation relations $K^{-2}X^+ = q^{-2}X^+K^{-2}$ (cf. (7.39) and (7.40)) we can then factorize the third q-exponential in (9.89). Then the expression for $\mathscr{F}^{(cob)}(q)$ simplifies to

$$\mathscr{F}^{(cob)}(q) = \exp_{q^2}\left(\frac{t}{1-q^2}1\otimes X^+\right)\exp_{q^{-2}}\left(-\frac{t}{1-q^2}(K^{-2}\otimes X^+)\right).$$

Using the representation of the q-exponential as standard exponential of the q-dilogarithm (9.88), the realization $K^2 = q^h$, and commutativity of the elements $1\otimes X^+$, $K^{-2}\otimes X^+$, one can show that there are no singular terms in $\mathscr{F}^{(cob)}(q)$ in the limit $q \longrightarrow 1$. The explicit expression is

$$\lim_{q\to 1}\mathscr{F}^{(cob)}(q) = \exp\left(\sum_{n=1}^{\infty} -\frac{1}{2}h\otimes\frac{(tX^+)^n}{n}\right) = \exp\left(\frac{1}{2}h\otimes\ln(1-tX^+)\right),$$

which gives for $t=-2\xi$ the jordanian twist $\mathscr{F}^{(j)}$ (9.68).

9.4 Conclusions

By transforming a given quantum group with a twist we obtain a new quantum group with universal R-matrix changed according to $\mathscr{F}_{21}\mathscr{R}\mathscr{F}^{-1}$. As a result there is a corresponding change in the integrable model associated with the initial quantum group and its representations. It was demonstrated that depending on the properties of the twist the energy spectrum for closed spin chains can be preserved (XXX_ξ model (9.63)) or changed (asymmetric XXZ_p model (9.76)). In both these cases the structure of the eigenstates is also twist dependent. On the other hand, for an open spin chain the twisting procedure simply generates a similarity transformation of the hamiltonian and its eigenstates.

Finally all the new quantum integrable systems obtained by twisting a given quantum integrable system share the same amount of symmetry as the initial one because the amount of symmetry in a group or in a twisted deformation of the group is the same.

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