

Twist operators in higher dimensions

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ABSTRACT: We study twist operators in higher dimensional CFT's. In particular, we express their conformal dimension in terms of the energy density for the CFT in a particular thermal ensemble. We construct an expansion of the conformal dimension in power series around $n = 1$, with n being replica parameter. We show that the coefficients in this expansion are determined by higher point correlations of the energy-momentum tensor. In particular, the first and second terms, i.e. the first and second derivatives of the scaling dimension, have a simple universal form. We test these results using holography and free field theory computations, finding agreement in both cases. We also consider the 'operator product expansion' of spherical twist operators and finally, we examine the behaviour of correlators of twist operators with other operators in the limit $n \rightarrow 1$.

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1 Introduction

Recently, entanglement entropy and related theoretical tools have received considerable attention in areas ranging from condensed matter physics, e.g., [1–5] to quantum gravity, e.g., [6–12]. In particular, holographic entanglement entropy [13–16] is now playing an important role in developing our understanding of gauge/gravity duality. This concept has evolved to become a universal tool that intertwines the non-perturbative structure of the boundary field theory and the quantum nature of the bulk spacetime.

A challenge to gaining a better understanding of entanglement entropy in quantum field theory (QFT) remains simply a deficiency of computational tools. One of the most commonly used techniques for evaluating entanglement entropy is the ‘replica trick’ [4, 5]. In this approach, given the reduced density matrix ρ_A describing the QFT restricted to a certain spatial region A , one first evaluates the Rényi entropies

$$S_n = \frac{1}{1-n} \log \text{Tr} \rho_A^n = \frac{1}{n-1} (n \log Z_1 - \log Z_n) , \quad (1.1)$$

where Z_n is the partition function on an n -fold covering geometry, with cuts introduced on region A . The entanglement entropy is then determined as the limit: $S_{\text{EE}} = \lim_{n \rightarrow 1} S_n$. In this construction, the entangling surface Σ , which encloses the region A , becomes the branch-point of the cut which separates different copies in the n -fold cover. It is convenient to think of these boundary conditions as produced by the insertion of a $(d-2)$ -dimensional surface operator, i.e., the twist operator σ_n , at Σ which interlaces n copies of the QFT on a single copy of the background geometry [4, 5]. These twist operators will be in the focus of our current study.

In two (spacetime) dimensions, twist operators are local operators [4, 5] and for a two-dimensional conformal field theory (CFT), they are in fact conformal primaries whose scaling dimension is given by

$$h_n = \frac{c}{12} \left(n - \frac{1}{n} \right). \tag{1.2}$$

However, in general dimensions, the replica-trick construction provides only a formal definition of twist operators for higher dimensions and so in practice beyond $d = 2$, the properties of these operators is not well understood.¹ In [17], a holographic study suggested a new approach to evaluate a generalized notion of the conformal dimension h_n of the twist operators in higher dimensional CFT's, which we will review below. Further, this work [17] revealed in a wide variety of holographic theories, this conformal weight satisfied a simple and intriguing relation:

$$\partial_n h_n|_{n=1} = 2\pi^{\frac{d}{2}+1} \frac{\Gamma(d/2)}{\Gamma(d+2)} C_T, \tag{1.3}$$

where C_T is the central charge appearing in the two-point function of the stress tensor — see eq. (2.29) below. One of our results here is to explain that this expression is, in fact, universal applying for twist operators in any CFT. Further, we will also show that $\partial_n^2 h_n|_{n=1}$ has a similar universal form involving the CFT parameters which determine the three-point function of the stress tensor.

The remainder of the paper is organized as follows: in section 2, we review the construction of [17, 18] which allows us to evaluate the scaling dimension of twist operators in higher dimensional CFT's in terms of the energy density of a thermal ensemble on a certain hyperbolic geometry. Next, we use the resulting expression to make an expansion of the conformal dimension in power series around $n = 1$, and show that the k -th order coefficient is determined by the $(k+1)$ - and k -point correlation functions of the stress tensor. As noted above, this yields simple universal expressions for the first and second terms, i.e., the first and second derivatives of the scaling dimension. In section 2.3, we extend these results to directly expand the Rényi entropy about $n = 1$, as was considered recently in [19]. We find our results are in complete agreement with the latter reference. In section 3, we compare our results for $\partial_n h_n|_{n=1}$ and $\partial_n^2 h_n|_{n=1}$ with explicit computations in various holographic models and in free field theories. Section 4 reviews the generalized OPE expansion for twist operators and we extract certain coefficients in the OPE expansion of a spherical twist operator.

¹However, see [20] for an earlier discussion of twist operators in higher dimensional CFT's, as well as [21] for related comments.

In section 5, we propose that the twist operator can be effectively represented by a construction involving the modular Hamiltonian. This construction allows us to consider an expansion for small $(n-1)$ of the twist operators themselves. Finally, we close with a brief discussion of our results and future directions in section 6. The evaluation of a useful integral used in section 2 is presented in appendix A. Further the details of heat kernel computations of the conformal dimensions of twist operators in free theories are presented in appendix B.

2 Twist operators for higher dimensional CFT's

As discussed above, just as in two dimensions, twist operators are naturally defined in higher dimensions through the replica trick. In d dimensions then, a twist operator σ_n is a $(d-2)$ -dimensional *surface* operator which introduces a branch cut at the entangling surface in the path integral over the n -fold replicated theory. In this section, we examine this formal definition more carefully to produce certain explicit results for these surface operators. However, we will restrict our attention to twist operators in higher dimensional CFT's. More specifically, we consider a CFT in its vacuum state in d -dimensional flat space and we choose the entangling surface to be a sphere of radius R (on a constant time slice). In this case, it was shown that the entanglement entropy can be evaluated as the thermal entropy of the CFT on a hyperbolic cylinder $R \times H^{d-1}$, where the temperature and curvature are fixed by the radius of the original entangling surface [18]. Further, this approach is easily extended to a calculation of the Rényi entropies by varying the temperature in this thermal ensemble [17]. As will be described in the following, we use this construction here to produce a better understanding of the corresponding twist operators.

2.1 Spherical twist operators

A key point in the analysis of [17, 18] is to take advantage of the fact that the underlying theory is a conformal field theory and to find a conformal transformation mapping the theory between flat space and the hyperbolic cylinder. Hence, we begin with a review of this transformation for the corresponding Euclidean signature geometries: following [17], the metric on flat (Euclidean) space may be written in terms of a complex coordinate $\omega = r + it_E$:

$$ds_{R^d}^2 = dt_E^2 + dr^2 + r^2 d\Omega_{d-2}^2 = d\omega d\bar{\omega} + \left(\frac{\omega + \bar{\omega}}{2}\right)^2 d\Omega_{d-2}^2, \tag{2.1}$$

where $d\Omega_{d-2}^2$ denotes a standard round metric on a unit $(d-2)$ -sphere. The spherical entangling surface will be located at $(t_E, r) = (0, R)$, or in terms of the complex coordinate, at $\omega = R$. Now to construct the desired conformal transformation, we introduce a second complex coordinate $\sigma = u + i\frac{t_E}{R}$ and then make the coordinate transformation

$$e^{-\sigma} = \frac{R - \omega}{R + \omega}. \tag{2.2}$$

The metric (2.1) then becomes

$$ds_{R^d}^2 = \Omega^{-2} R^2 \left[d\sigma d\bar{\sigma} + \sinh^2 \left(\frac{\sigma + \bar{\sigma}}{2} \right) d\Omega_{d-2}^2 \right], \tag{2.3}$$

where

$$\Omega = \frac{2R^2}{|R^2 - \omega^2|} = |1 + \cosh \sigma|. \quad (2.4)$$

Removing the Ω^{-2} prefactor by a simple Weyl rescaling, the resulting metric reduces to

$$ds_{H^{d-1} \times S^1}^2 = \Omega^2 ds_{R^d}^2 = d\tau_E^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2). \quad (2.5)$$

This conformally transformed geometry corresponds to $S^1 \times H^{d-1}$, where u is the (dimensionless) radial coordinate on the hyperboloid H^{d-1} and τ_E is the Euclidean time coordinate on S^1 . As is clear from eq. (2.5), the curvature radius of H^{d-1} is R , the radius of the original spherical entangling surface.

To confirm that the τ_E direction is indeed periodic, we can examine the transformation (2.2) in the vicinity of the entangling surface. That is, if we choose $\omega = R - \delta r - i\delta t_E$ with $\delta r, \delta t_E \ll R$, then to leading order, eq. (2.2) becomes

$$e^{-u - i\frac{\tau_E}{R}} \simeq \frac{\delta r + i\delta t_E}{2R}. \quad (2.6)$$

This expression is the usual exponential mapping between the plane R^2 and the cylinder $R \times S^1$. Hence it makes clear that the τ_E coordinate lives on a circle and further that we should identify the period as $\Delta\tau_E = 2\pi R$ to ensure that the geometry is smooth at the entangling surface, i.e., at the origin of the $(\delta r, \delta t_E)$ -plane. We also note that from eq. (2.6), it is apparent that the entangling surface at $\omega = R$ has been pushed out to $u \rightarrow \infty$, the asymptotic boundary of the hyperbolic geometry.

Given the periodicity of the Euclidean time coordinate τ_E , it is evident that the CFT on the hyperbolic geometry is at finite temperature with

$$T_0 = \frac{1}{2\pi R}. \quad (2.7)$$

Hence under the conformal mapping, the reduced density matrix describing the CFT on the interior of the spherical entangling surface is transformed to a thermal density matrix,

$$\rho_A = U^{-1} \rho_{\text{thermal}}(T_0) U = U^{-1} \frac{e^{-H/T_0}}{Z(T_0)} U. \quad (2.8)$$

Here U denotes the unitary transformation implementing the conformal transformation. Since the entropy is insensitive to such a unitary transformation, the desired entanglement entropy just equals the thermal entropy in the transformed space [18].

Now to evaluate the Rényi entropy S_n as in eq. (1.1), we must consider the n 'th power of the density matrix

$$\rho_A^n = U^{-1} \frac{e^{-n H/T_0}}{Z(T_0)^n} U. \quad (2.9)$$

That is, we must consider the thermal ensemble with the temperature $T = T_0/n$ on the same hyperbolic geometry. Hence the period $\Delta\tau_E$ is extended to $2\pi n R$. Now applying the same conformal mapping (2.2) in this case, will again yield the flat space metric (2.1). However, examining the geometry near the entangling surface with eq. (2.6), we see that the

origin is now circled n times as τ_E runs over its full period. Therefore, as might have been anticipated, the transformation (2.2) actually maps the thermal background $S^1 \times H^{d-1}$ to an n -fold cover of R^d with an orbifold singularity located precisely at the entangling surface, i.e., the $(d-2)$ -dimensional sphere given by $r = R$ (and $t_E = 0$). Hence the path integral on this new geometry would yield precisely the partition function Z_n of an n -fold replicated theory with a spherical twist operator inserted at $r = R$.

While this twist operator is the focus of our study, let us add that since we are studying a CFT, we may equate $Z_n = Z(T_0/n)$ because the two path integrals are simply related by a conformal transformation.² Hence the Rényi entropy (1.1) may be re-expressed in terms of these thermal partition functions,

$$S_n = \frac{1}{n-1} (n \log Z(T_0) - \log Z(T_0/n)) . \tag{2.10}$$

Then using the standard thermodynamic identity $S_{\text{therm}}(T) = \partial_T [T \log Z(T)]$, we may express the Rényi entropy in terms of the thermal entropy [17],

$$S_n = \frac{n}{n-1} \frac{1}{T_0} \int_{T_0/n}^{T_0} S_{\text{therm}}(T) dT . \tag{2.11}$$

2.2 Conformal dimension

As noted in the introduction, in a higher dimensional CFT, the twist operators may be assigned a generalized notion of conformal dimension [22]. As in $d = 2$, the latter is defined by the leading singularity in the correlator $\langle T_{\mu\nu} \sigma_n \rangle$. We review the structure of this singularity here following the discussion in [17]: first in flat (Euclidean) space, we make an insertion of the stress tensor in the vicinity of a *planar* twist operator σ_n . We align the Cartesian coordinates x^μ on R^d with the twist operator, so that this surface operator is positioned at $x^1 = 0 = x^2$ while it extends throughout the remaining coordinates with $\mu = a \in \{3, \dots, d\}$. With the stress tensor inserted at $x^\mu = \{y^i, x^a\}$, the perpendicular distance to the twist operator is defined as $y = \sqrt{(y^1)^2 + (y^2)^2}$. Now symmetry dictates the form of the corresponding correlator up to a single constant, i.e., the conformal dimension. Specifically, the basic geometric structures appearing in the correlator are determined by the residual translational and rotational symmetries, which remain in the presence of the twist operator. Then the relative normalization of various contributions is fixed by the tracelessness and conservation of the stress tensor, i.e., by imposing $\langle T^\mu{}_\mu \sigma_n \rangle = 0 = \nabla^\mu \langle T_{\mu\nu} \sigma_n \rangle$. Subject to all of these constraints, the correlator is restricted to take the following form³

$$\begin{aligned} \langle T_{ab} \sigma_n \rangle &= -\frac{h_n}{2\pi} \frac{\delta_{ab}}{y^d}, & \langle T_{ai} \sigma_n \rangle &= 0, \\ \langle T_{ij} \sigma_n \rangle &= \frac{h_n}{2\pi} \frac{(d-1)\delta_{ij} - dn_i n_j}{y^d}, \end{aligned} \tag{2.12}$$

where a, b (i, j) denote tangential (normal) directions to the twist operator and $n^i = y^i/y$ is the unit vector directed orthogonally from the twist operator to the $T_{\mu\nu}$ insertion. Thus

²Of course, with the choice $n = 1$, we have the original one-to-one mapping from $S^1 \times H^{d-1}$ to R^d .

³These expressions are implicitly normalized by dividing by $\langle \sigma_n \rangle$ but we left this normalization implicit to avoid the clutter that would otherwise be created.

the correlator is completely fixed up to the single constant h_n . The latter is commonly referred to as the conformal dimension of σ_n , since its appearance above is analogous to that of the scaling dimension of a local primary operator. In particular then, if one reduces these expressions to $d = 2$ (in which case the twist operators are local primaries), one finds that the present definition for h_n matches with the standard definition, as given in [4, 5], and h_n is precisely the total scaling dimension given in eq. (1.2). Further, note that we are assuming that $T_{\mu\nu}$ corresponds to the total stress tensor for the entire n -fold replicated CFT, i.e., $T_{\mu\nu}$ is inserted on all n sheets of the universal cover.

Given the basic definition of h_n , we can now use the conformal mapping described in the previous section to gain further insights about this parameter [17]. On one side of this mapping, we have the CFT in a thermal ensemble on the hyperbolic cylinder or rather the Euclidean CFT lives on the background $S^1 \times H^{d-1}$. On general grounds, the expectation value of the stress tensor will then take the form

$$\langle T^\mu{}_\nu \rangle = \text{diag}(-\mathcal{E}(T), p(T), \dots, p(T)), \tag{2.13}$$

where the energy density $\mathcal{E}(T)$ and the pressure $p(T)$ are constant throughout the hyperbolic background.⁴ Further the trace of this expression must vanish in a CFT⁵ and hence,

$$p(T) = \mathcal{E}(T)/(d - 1). \tag{2.14}$$

Next we can relate the thermal energy density to the correlator (2.12) by applying the conformal map from $S^1 \times H^{d-1}$ to the n -fold cover of R^d described above. In particular, recall that the n -fold cover is produced when the temperature is tuned to $T = T_0/n$. Now under this conformal mapping, the stress tensor becomes

$$\langle T_{\alpha\beta} \sigma_n \rangle = \Omega^{d-2} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} \left(\langle T_{\mu\nu}(T_0/n) \rangle - \mathcal{A}_{\mu\nu} \right), \tag{2.15}$$

where α, β and μ, ν denote indices on the flat geometry and $S^1 \times H^{d-1}$, respectively. Since the conformal mapping generates an orbifold singularity in the (otherwise) flat covering space, the expectation on the left-hand side of eq. (2.15) has been interpreted as the expectation value of the stress tensor in the presence of the corresponding twist operator σ_n (on the sphere at $r = R$ and $t_E = 0$). Further, the stress tensor is not a primary operator and so an anomalous contribution $\mathcal{A}_{\mu\nu}$ also appears on the right-hand side. This contribution is the higher dimensional analog of the usual Schwarzian term appearing in two dimensions [23]. We observe that this anomalous term depends entirely on the details of the transformation (2.2), but it is independent of the temperature in the hyperbolic background, i.e., the period of the S^1 . Therefore this term can be fixed by noting that eq. (2.2) produces a one-to-one mapping from $S^1 \times H^{d-1}$ to R^d with $n = 1$. Hence since there is no orbifold singularity in this case, the left-hand side becomes $\langle T_{\alpha\beta} \rangle$, i.e., the

⁴The signs are chosen here in eq. (2.13) for a Euclidean signature, i.e., $\mathcal{E}(T) = -\langle T_{\tau_E \tau_E} \rangle$.

⁵In principle, the trace anomaly could lead to a nonvanishing trace in even dimensions. However, one can readily verify that in fact the trace anomaly vanishes for the background geometry $S^1 \times H^{d-1}$. In particular, the Euler density vanishes because the background is the direct product of two lower dimensional geometries. Further this background is conformally flat and so any conformal invariants also vanish.

vacuum expectation value of the stress energy in flat space, and so it simply vanishes. Since the left-hand side vanishes with $n = 1$, we conclude that $\mathcal{A}_{\mu\nu} = T_{\mu\nu}(T_0)$. Hence eq. (2.15) becomes

$$\langle T_{\alpha\beta} \sigma_n \rangle = \Omega^{d-2} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} \left(\langle T_{\mu\nu}(T_0/n) \rangle - \langle T_{\mu\nu}(T_0) \rangle \right). \quad (2.16)$$

Recall that the conformal factor Ω is given by eq. (2.4).

Note that the conformal mapping above generates a *spherical* twist operator while conformal dimension was defined in eq. (2.12) by the correlator of the stress tensor with a *planar* twist operator. However, h_n can be identified here by bringing the insertion of the stress tensor very close to the spherical twist operator, in which case the leading singularity in eq. (2.16) will emerge with the same form as in eq. (2.12). To evaluate this singularity, we begin by examining eq. (2.2) which yields

$$\begin{aligned} \frac{\partial u}{\partial t_E} &= -\frac{1}{R} \frac{\partial \tau_E}{\partial r} = \frac{iR(\omega^2 - \bar{\omega}^2)}{(R^2 - \omega^2)(R^2 - \bar{\omega}^2)}, \\ \frac{\partial u}{\partial r} &= \frac{1}{R} \frac{\partial \tau_E}{\partial t_E} = \frac{2R^3 - R(\omega^2 + \bar{\omega}^2)}{(R^2 - \omega^2)(R^2 - \bar{\omega}^2)}. \end{aligned} \quad (2.17)$$

Of course, the first equality in each of the above expressions corresponds to the standard Cauchy-Riemann conditions. Next, to simplify the analysis, we insert $T_{\alpha\beta}$ at $t_E = 0$ and $r = R - y$ with $y \ll R$ (as well as some fixed angles). With this choice, eq. (2.17) simplifies to $\frac{\partial u}{\partial t_E} = 0$ and $\frac{\partial \tau_E}{\partial t_E} \simeq R/y$ and further, eq. (2.4) gives $\Omega \simeq R/y$. Given these expressions and setting $\alpha = t_E = \beta$, eq. (2.16) yields

$$\begin{aligned} \langle T_{t_E t_E} \sigma_n \rangle &= \Omega^{d-2} \left(\frac{\partial \tau_E}{\partial t_E} \right)^2 \left(T_{\tau_E \tau_E}(T_0/n) - T_{\tau_E \tau_E}(T_0) \right) \\ &= -\left(\frac{R}{y} \right)^d \left(\mathcal{E}(T_0/n) - \mathcal{E}(T_0) \right) + \dots \end{aligned} \quad (2.18)$$

This result should be compared to the $i = j = t_E$ component in eq. (2.12), i.e.,

$$\langle T_{t_E t_E} \sigma_n \rangle = \frac{d-1}{2\pi} \frac{h_n}{y^d}. \quad (2.19)$$

However, first, we recall that the expectation value in eq. (2.19) involves the total stress tensor for the entire n -fold replicated CFT, while in eq. (2.18), we have an insertion of $T_{t_E t_E}$ on a single sheet of the universal cover. Hence the latter must be multiplied by an extra factor of n before comparing the two expressions. The final result for the scaling dimension is

$$h_n = \frac{2\pi n}{d-1} R^d \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n) \right). \quad (2.20)$$

We now turn to the intriguing result (1.3) which was found in [17] to apply for a variety of holographic models:

$$\partial_n h_n|_{n=1} = 2\pi^{\frac{d}{2}+1} \frac{\Gamma(d/2)}{\Gamma(d+2)} C_T. \quad (2.21)$$

Here C_T is the central charge defined by the two-point function of the stress tensor.⁶ In fact, we will now show below that eq. (2.21) is a universal result that applies for the scaling dimension of twist operators in any CFT.

Our proof that eq. (2.21) is universal begins with eq. (2.20), which again applies for any CFT. The energy densities in the latter equation are evaluated in a thermal ensemble on the hyperbolic cylinder and so can be written as

$$\mathcal{E}(T) = -\langle T_{\tau_E \tau_E} \rangle = -\text{Tr} [\rho_{\text{thermal}}(T) T_{\tau_E \tau_E}] = -\frac{1}{Z(T)} \text{Tr} \left[e^{-H/T} T_{\tau_E \tau_E} \right]. \quad (2.22)$$

Combining eqs. (2.20) and (2.22), the expression on the left-hand side of eq. (2.21) becomes

$$\begin{aligned} \partial_n h_n|_{n=1} &= -\frac{2\pi R^d}{(d-1)T_0} \langle H T_{\tau_E \tau_E}(x_0) \rangle_c \\ &= \frac{2\pi R^d}{(d-1)T_0} \int_{H^{d-1}} d^{d-1} \tilde{x} \sqrt{\gamma} \langle T_{\tau_E \tau_E}(\tilde{x}) T_{\tau_E \tau_E}(x_0) \rangle_c \end{aligned} \quad (2.23)$$

where x_0 is an arbitrary point, γ stands for the determinant of the induced metric on the Cauchy surface on which H is evaluated and the subscript ‘ c ’ denotes the connected part of the thermal correlator, i.e.,

$$\langle T_{\tau_E \tau_E}(\tilde{x}) T_{\tau_E \tau_E}(x_0) \rangle_c = \langle T_{\tau_E \tau_E}(\tilde{x}) T_{\tau_E \tau_E}(x_0) \rangle - \langle T_{\tau_E \tau_E}(\tilde{x}) \rangle \langle T_{\tau_E \tau_E}(x_0) \rangle. \quad (2.24)$$

Further with n set to 1 on the left-hand side of eq. (2.23), we must evaluate the corresponding thermal expectation values at $T = T_0$.

Now, given the expression in eq. (2.23) where $\partial_n h_n|_{n=1}$ is expressed in terms of a two-point function of the stress tensor, it is natural to expect that the final result should be proportional to C_T . However, we may further evaluate the precise constant of proportionality in order to establish the universality of the result in eq. (2.21).

In eq. (2.23), we introduced a specific location x_0^α for the insertion of the stress tensor. Of course, the choice of this location is arbitrary since the thermal bath on the hyperbolic geometry is homogeneous. Similarly, in the final expression, the Hamiltonian can be evaluated with an integration over any Cauchy surface because the stress tensor is conserved. To simplify the following analysis, we fix the location x_0^α to be at $u = 0$ and $\tau_E = 0$ and also evaluate the Hamiltonian by integrating over the surface $\tau_E = 0$. With this choice, the correlator in eq. (2.23) is spherically symmetric and so using eq. (2.5), we may write

$$\partial_n h_n|_{n=1} = \frac{2\pi R^{2d-1}}{(d-1)T_0} \Omega_{d-2} \int du \sinh^{d-2} u \langle T_{\tau_E \tau_E}(\tau_E = 0, u) T_{\tau_E \tau_E}(\tau_E = 0, u = 0) \rangle_c \quad (2.25)$$

where Ω_{d-2} denotes the volume of a unit $(d-2)$ -sphere.

Now using the conformal transformation described in the previous two sections, we can map the two-point correlator of the stress tensor on the Euclidean background $S^1 \times H^{d-1}$ with $\beta = 1/T_0$ to the two-point correlator in the CFT vacuum on R^d . That is, we can relate

⁶Note that our normalization for C_T above corresponds to that introduced in [24, 25] but not the same as in [17]. Hence the numerical factor in eq. (2.21) is different than given in the original reference [17].

the thermal correlator in eq. (2.23) to the usual two-point correlator in flat space which defines C_T — see below. In particular, with special choice of insertion points described above, eq. (2.16) yields⁷

$$\langle T_{\tau_E \tau_E}(0, u) T_{\tau_E \tau_E}(0, 0) \rangle_c = \left[\frac{1}{\Omega^{d-2}} \left(\frac{\partial t_E}{\partial \tau_E} \right)^2 \right]_{\tau_E=0, u} \left[\frac{1}{\Omega^{d-2}} \left(\frac{\partial t_E}{\partial \tau_E} \right)^2 \right]_{\tau_E=0, u=0} \quad (2.26)$$

$$\times \langle T_{t_E t_E}(t_E = 0, r) T_{t_E t_E}(t_E = 0, r = 0) \rangle_c \Big|_{r=R \tanh u/2},$$

where eq. (2.2) was used to determine $(\tau_E = 0, u) \rightarrow (t_E = 0, r = R \tanh u/2)$. Implicitly, we have also used eq. (2.17) to show $\partial r / \partial \tau_E|_{\tau_E=0} = 0$. Further, this equation also yields

$$\frac{\partial t_E}{\partial \tau_E} \Big|_{\tau_E=0} = \frac{R^2 - r^2}{2R^2} = \frac{1}{\Omega} \Big|_{t_E=0, r \leq R}. \quad (2.27)$$

Note that we have the following additional simplifications for $t_E = 0$: $\sinh u = \Omega r/R$ and $\partial u / \partial r = \Omega/R$, as well as $\Omega|_{t_E=0, r=0} = 2$. Combining these results, eq. (2.25) becomes

$$\partial_n h_n|_{n=1} = \frac{R^{d-1}}{(d-1)2^d T_0} \Omega_{d-2} 2\pi \int_0^R dr r^{d-2} \frac{R^2 - r^2}{2R} \langle T_{t_E t_E}(t_E = 0, r) T_{t_E t_E}(t_E = 0, r = 0) \rangle_c. \quad (2.28)$$

Since this correlator is now evaluated in R^d , we may drop the subscript c since the corresponding one-point expectation values vanish — in particular, $\langle T_{t_E t_E} \rangle = 0$. Further, we observe that the standard Hamiltonian on the hyperbolic space appearing in eq. (2.23) has been transformed in the above expression to the corresponding entanglement Hamiltonian for a spherical region in flat space found in [18]⁸

Now recall that the two-point function of the stress tensor for a general CFT in R^d takes the form [24, 25]

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(0) \rangle = \frac{C_T}{x^{2d}} \mathcal{I}_{\mu\nu, \alpha\beta}(x), \quad (2.29)$$

where C_T is a constant and the tensor structure is given by

$$\mathcal{I}_{\mu\nu, \alpha\beta} = \frac{1}{2}(I_{\mu\alpha} I_{\nu\beta} + I_{\mu\beta} I_{\nu\alpha}) - \frac{1}{d} \delta_{\mu\nu} \delta_{\alpha\beta} \quad \text{with} \quad I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}. \quad (2.30)$$

Hence the desired correlator becomes

$$\langle T_{t_E t_E}(0, r) T_{t_E t_E}(0, 0) \rangle = \frac{d-1}{d} \frac{C_T}{r^{2d}}. \quad (2.31)$$

Now substituting this expression, as well as $\Omega_{d-2} = 2\pi^{(d-1)/2} / \Gamma((d-1)/2)$ and $T_0 = 1/(2\pi R)$, into eq. (2.28), we find the final result

$$\begin{aligned} \partial_n h_n|_{n=1} &= \frac{\pi^2 R^{d+1}}{d 2^{d-2}} \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_0^R dr r^{d-2} \frac{R^2 - r^2}{2R^2} \frac{C_T}{r^{2d}} \\ &= \frac{\pi^{(d+3)/2}}{2^{d-3} d (d^2 - 1) \Gamma((d-1)/2)} C_T. \end{aligned} \quad (2.32)$$

⁷Note that the anomalous terms do not contribute here in the transformation of the connected correlator.

⁸More precisely, the conformal mapping takes $H/T_0 \rightarrow H_m$ [26] — see eq. (5.6) below.

Of course, the integral in the first line contains a divergence at $r = 0$ where the insertion points of the two energy-momentum tensors collide. However, recall that the final result must be independent of the precise choice of the insertion point x_0^α . Hence to regulate the singularity, we could instead evaluate eq. (2.23) with x_0^α shifted slightly away from $\tau_E = 0$. A simpler approach is to simply evaluate the integral above using dimensional regularization, which yields

$$\int_0^R \frac{dr}{r^{d+2}} \frac{R^2 - r^2}{2R^2} = \frac{1}{d^2 - 1} \frac{1}{R^{d+1}}. \quad (2.33)$$

We have explicitly verified that both approaches lead to the same result for eq. (2.32). Finally it is straightforward to show that the coefficient appearing in eq. (2.32) precisely matches the coefficient appearing in eq. (2.21). Hence we have established that the latter is a universal result that applies for twist operators in any CFT.

The previous analysis is easily extended to higher derivatives of the conformal weight. In particular, eq. (2.23) generalizes to

$$\begin{aligned} h_{n,1} &\equiv \partial_n h_n|_{n=1} = -\frac{2\pi R^d}{(d-1)T_0} \langle H T_{\tau_E \tau_E}(x_0) \rangle_c, \\ h_{n,2} &\equiv \partial_n^2 h_n|_{n=1} = \frac{2\pi R^d}{(d-1)T_0^2} \left(\langle H H T_{\tau_E \tau_E}(x_0) \rangle_c - 2T_0 \langle H T_{\tau_E \tau_E}(x_0) \rangle_c \right), \\ h_{n,k} &\equiv \partial_n^k h_n|_{n=1} = (-1)^k \frac{2\pi R^d}{(d-1)T_0^k} \left(\langle \underbrace{H \cdots H}_k T_{\tau_E \tau_E}(x_0) \rangle_c - k T_0 \langle \underbrace{H \cdots H}_{k-1} T_{\tau_E \tau_E}(x_0) \rangle_c \right), \\ &k \geq 2, \end{aligned} \quad (2.34)$$

where as before, the expectation values on the right-hand side are the connected parts of the thermal expectation values on the hyperbolic space evaluated at the temperature T_0 . We remind the readers that these expressions are independent of the arbitrary point of insertion x_0 of the stress tensor. With these expressions, we can construct a Taylor series for the conformal dimension around $n = 1$, i.e.,

$$h_n = \sum_{k=1}^{\infty} \frac{1}{k!} h_{n,k} (n-1)^k \quad (2.35)$$

Note that the series above begins with $k = 1$ because, as is evident from eq. (2.20), $\lim_{n \rightarrow 1} h_n = 0$. Therefore eq. (2.34) shows that this expansion is determined by the correlation functions of the energy-momentum tensor. Our previous analysis provided a universal expression fixing $h_{n,1}$ in terms the central charge C_T . For general k , $h_{n,k}$ will be determined by the $(k+1)$ - and k -point correlators of the stress tensor and so these expressions will depend on the details of the underlying CFT, e.g., on the full spectrum of primary operators.⁹ However, to evaluate the second derivative above, we only need the

⁹Of course, $d = 2$ is an exception to this general rule. In this case, the relevant expressions are all completely determined by the central charge c . In fact, given the full expression in eq. (1.2), one finds $h_{n,k} = \frac{c}{12} \left((-1)^{k+1} \Gamma(k+1) + \delta_{k,1} \right)$. It is straightforward to confirm that this result agrees with eq. (2.21) for $k = 1$ after identifying $c = 2\pi^2 C_T$. Further for $k = 2$, agreement is found with eq. (2.45) after substituting in $d = 2$ and using eq. (2.46).

three- and two-point functions, which are both completely fixed by conformal invariance. Hence $h_{n,2}$ also has a universal form which depends on the (three) parameters appearing in the three-point correlator of the stress tensor [24, 25]. Therefore we turn to deriving this universal expression next.

Following the previous discussion, the correlator implicitly appearing in $h_{n,2}$ is

$$\langle T_{\tau_E \tau_E}(\tau_{E1}, \vec{u}_1) T_{\tau_E \tau_E}(\tau_{E2}, \vec{u}_2) T_{\tau_E \tau_E}(\tau_{E3}, \vec{u}_3) \rangle_c, \quad (2.36)$$

where \vec{u}_i denotes both the radius and the angles at which each of these insertions is positioned on H^{d-1} . However, as before, the results will be independent of the precise choice made for the time slices for the first two stress tensors, which appear in the Hamiltonians in eq. (2.34), and for the position of the third stress tensor. Hence, it will be convenient to set $\tau_{E1} = 0 = \tau_{E2}$, with which these two insertions will be mapped to the slice $t_E = 0$ and $r = R \tanh u/2 \leq R$ in flat space, i.e., within the entangling surface. However, we will choose $\tau_{E3} = \pi R$, which also maps the third insertion to $t_E = 0$ but with $r = R \coth u/2 \geq R$, i.e., outside of the entangling surface. Further, we will take $u_3 \ll 1$ below which will correspond to a limit where $r_3 \gg R$. Now in analogy to eq. (2.26), we map eq. (2.36) to the corresponding flat space correlator with

$$\begin{aligned} \langle T_{\tau_E \tau_E}(0, \vec{u}_1) T_{\tau_E \tau_E}(0, \vec{u}_2) T_{\tau_E \tau_E}(\pi R, \vec{u}_3) \rangle_c &= \prod_{i=1}^2 \left[\frac{1}{\Omega^{d-2}} \left(\frac{\partial t_E}{\partial \tau_E} \right)^2 \right]_{\tau_i=0, \vec{u}_i} \\ &\times \left[\frac{1}{\Omega^{d-2}} \left(\frac{\partial t_E}{\partial \tau_E} \right)^2 \right]_{\tau_3=\pi R, \vec{u}_3} \\ &\times \langle T_{t_E t_E}(t_E = 0, \vec{r}_1) T_{t_E t_E}(t_E = 0, \vec{r}_2) T_{t_E t_E}(t_E = 0, \vec{r}_3) \rangle \Big|_{r_{1,2}=R \tanh u_{1,2}/2; r_3=R \coth u_3/2}. \end{aligned} \quad (2.37)$$

Now the first two factors can be simplified using eq. (2.27) and similarly, eq. (2.17) yields

$$\frac{\partial t_E}{\partial \tau_E} \Big|_{\tau_E=\pi R} = \frac{R^2 - r^2}{2R^2} = -\frac{1}{\Omega} \Big|_{t_E=0, r \geq R}. \quad (2.38)$$

Combining these results, the three-point contribution to $h_{n,2}$ becomes

$$\begin{aligned} \langle H H T_{\tau_E \tau_E}(\tau_{E3} = \pi R, u_3) \rangle_c &= \left(\frac{r_3^2 - R^2}{2R^2} \right)^d \prod_{i=1}^2 \left[\int d\Omega_i \int_0^R dr_i r_i^{d-2} \frac{R^2 - r_i^2}{2R^2} \right] \\ &\times \langle T_{t_E t_E}(t_E = 0, \vec{r}_1) T_{t_E t_E}(t_E = 0, \vec{r}_2) T_{t_E t_E}(t_E = 0, \vec{r}_3) \rangle \end{aligned} \quad (2.39)$$

where the three-point function on the right-hand side is evaluated in R^d (with $r_3 > R$). As in eq. (2.28), we also observe that the standard Hamiltonians on the left-hand side have become entanglement Hamiltonians for the spherical region on the right-hand side.

Now we may employ the results of [24, 25] which give the three-point function of the stress tensor in R^d . Recall that this correlator has a universal form that is completely fixed by conformal invariance, tracelessness of the stress tensor and energy conservation, up to three constant parameters which characterize the underlying CFT. The resulting expression is quite complicated in general and so we simplify our calculation by using the

remaining freedom in choosing \vec{r}_3 , the position of the third stress tensor. In particular, if we choose $r_3 \gg R \geq r_{1,2}$ (or on the hyperbolic space, $u_3 \ll 1$), the three-point correlator becomes

$$\langle T_{t_{\text{E}}t_{\text{E}}}(0, \vec{r}_1) T_{t_{\text{E}}t_{\text{E}}}(0, \vec{r}_2) T_{t_{\text{E}}t_{\text{E}}}(0, \vec{r}_3) \rangle \simeq \frac{K}{|\vec{r}_1 - \vec{r}_2|^d r_3^{2d}}, \quad (2.40)$$

where

$$K = \frac{8(d-1)(d-2)\hat{a} - 2d\hat{b} - (5d-4)\hat{c}}{d^2}, \quad (2.41)$$

with \hat{a} , \hat{b} and \hat{c} being the parameters characterizing the CFT.¹⁰ Substituting this expression into eq. (2.37) yields

$$\begin{aligned} \langle H H T_{\tau_{\text{E}}\tau_{\text{E}}}(\tau_{\text{E}3} = \pi R, u_3 = 0) \rangle_c &= \left(\frac{1}{2R^2} \right)^d \prod_{i=1}^2 \left[\int d\Omega_i \int_0^R dr_i r_i^{d-2} \frac{R^2 - r_i^2}{2R^2} \right] \frac{K}{|\vec{r}_1 - \vec{r}_2|^d} \\ &= \frac{K}{(2R)^{d+2}} I \end{aligned} \quad (2.42)$$

where we have written the remaining integral as

$$I = \prod_{i=1}^2 \left[\int d\Omega_i \int_0^1 dx_i x_i^{d-2} (1 - x_i^2) \right] \frac{1}{|\vec{x}_1 - \vec{x}_2|^d}. \quad (2.43)$$

with $x_i = r_i/R$. We evaluate this integral in appendix A and with this result, eq. (2.42) becomes

$$\langle H H T_{\tau_{\text{E}}\tau_{\text{E}}}(\tau_{\text{E}3} = \pi R, u_3 = 0) \rangle_c = -\frac{2\pi^{d-2}}{d(d+2)\Gamma(d-1)} \frac{K}{R^{d+2}}. \quad (2.44)$$

Combining this correlator with eqs. (2.32) and (2.34), we finally obtain

$$\begin{aligned} \partial_n^2 h_n|_{n=1} &= -\frac{16\pi^{d+1}}{d^2(d+2)\Gamma(d+1)} (8(d-1)(d-2)\hat{a} - 2d\hat{b} - (5d-4)\hat{c}) + 2h_{n,1} \\ &= -\frac{16\pi^{d+1}}{d^2\Gamma(d+3)} (2(d-2)(3d^2 - 3d - 4)\hat{a} - 2d(d-1)\hat{b} - (3d-4)(d+1)\hat{c}) \end{aligned} \quad (2.45)$$

where we have simplified the final expression using [24, 25]

$$C_T = \frac{4\Omega_{d-1}}{d(d+2)} \left((d+3)(d-2)\hat{a} - 2\hat{b} - (d+1)\hat{c} \right). \quad (2.46)$$

2.3 Comparison with [19]

Recently, [19] considered an expansion of the Rényi entropy S_n in the vicinity of $n = 1$ and found that the first derivative had a universal form similar to eq. (2.21). Hence it is

¹⁰Here we are adopting the parametrization of the three-point function of the stress tensor in [24], in terms of $\hat{a}, \hat{b}, \hat{c}$. We note that a slightly different parametrization is introduced in [25], which is also widely used, e.g., [27, 28]. The parameters there are often denoted $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and the relation between these two sets of parameters is given by: $\mathcal{A} = 8\hat{a}$, $\mathcal{B} = 8(2\hat{a} + \hat{b})$, $\mathcal{C} = 2\hat{c}$.

instructive to compare our results with the expansion in [19]. As argued in [17], Rényi entropy for a spherical region in flat space can be written as

$$S_n = \frac{n}{1-n} \frac{1}{T_0} [F(T_0) - F(T_0/n)] = \frac{n}{1-n} \beta_0 F(\beta_0) - \frac{1}{1-n} \beta_n F(\beta_n), \quad (2.47)$$

where $F(\beta)$ is the free energy of the CFT at temperature $T = 1/\beta$ on the hyperbolic background $R \times H^{d-1}$. Further $\beta_0 \equiv 1/T_0 = 2\pi R$ and $\beta_n \equiv n/T_0 = 2\pi R n$ — compare to eq. (2.10). In particular, expanding this result in the vicinity of $n = 1$ or equivalently around β_0 , yields

$$S_n = \frac{n}{1-n} \beta_0 F(\beta_0) - \frac{1}{1-n} \left(\beta_0 F(\beta_0) + (\beta_n - \beta_0) \left. \frac{\partial}{\partial \beta_n} (\beta_n F(\beta_n)) \right|_{\beta_n = \beta_0} \right. \\ \left. + \frac{1}{2} (\beta_n - \beta_0)^2 \left. \frac{\partial^2}{\partial \beta_n^2} (\beta_n F(\beta_n)) \right|_{\beta_n = \beta_0} + \frac{1}{6} (\beta_n - \beta_0)^3 \left. \frac{\partial^3}{\partial \beta_n^3} (\beta_n F(\beta_n)) \right|_{\beta_n = \beta_0} + \dots \right) \quad (2.48)$$

Now this expression may be simplified using $\beta_n - \beta_0 = (n - 1) \beta_0$ and

$$\left. \frac{\partial \beta_n F(\beta_n)}{\partial \beta_n} \right|_{\beta_n = \beta_0} = R^{d-1} V_\Sigma \mathcal{E}(\beta_0), \quad (2.49)$$

where $R^{d-1} V_\Sigma$ denotes the (regulated) volume of the hyperbolic space¹¹ H^{d-1} and, as before, \mathcal{E} is the energy density. Thus eq. (2.48) becomes the expansion presented in [19]:

$$S_n = S(\beta_0) + 2\pi R^d V_\Sigma \left[\frac{1}{2} (n-1) \beta_0 \frac{\partial \mathcal{E}(\beta_0)}{\partial \beta_0} + \frac{1}{6} (n-1)^2 \beta_0^2 \frac{\partial^2 \mathcal{E}(\beta_0)}{\partial \beta_0^2} + \dots \right], \quad (2.50)$$

where $S(\beta_0)$ is the thermodynamic entropy on $R \times H^{d-1}$ at temperature $T = T_0$, which equals the entanglement entropy across the sphere of radius R in flat space [18]. Now if we examine eq. (2.20), we see that h_n can be written in terms of a similar expansion with derivatives of the energy density

$$h_n = -\frac{2\pi R^d}{d-1} \left((n-1) \beta_0 \frac{\partial \mathcal{E}(\beta_0)}{\partial \beta_0} + \frac{(n-1)^2}{2} \left(\beta_0^2 \frac{\partial^2 \mathcal{E}(\beta_0)}{\partial \beta_0^2} + 2\beta_0 \frac{\partial \mathcal{E}(\beta_0)}{\partial \beta_0} \right) + \dots \right). \quad (2.51)$$

Hence comparing the leading coefficient in the two expansions, we find

$$\partial_n S_n|_{n=1} = -\frac{d-1}{2} V_\Sigma \partial_n h_n|_{n=1}. \quad (2.52)$$

Similarly, comparing the expansions at higher orders yields

$$\partial_n^2 S_n|_{n=1} = -\frac{d-1}{3} V_\Sigma (h_{n,2} - 2h_{n,1}), \quad (2.53)$$

$$\partial_n^k S_n|_{n=1} = -\frac{d-1}{k+1} V_\Sigma (h_{n,k} - k h_{n,k-1}),$$

¹¹Here, we are using the notation introduced in [17].

where we are using the notation introduced in eq. (2.34) here, i.e., $h_{n,k} = \partial_n^k h_n|_{n=1}$. In particular, substituting eq. (2.21) into eq. (2.52), we recover the result derived in [19]

$$\partial_n S_n|_{n=1} = -\pi^{\frac{d}{2}+1} \frac{(d-1)\Gamma(d/2)}{\Gamma(d+2)} V_\Sigma C_T. \tag{2.54}$$

Further, using eqs. (2.45) and (2.46), we could express the second derivative $\partial_n^2 S_n|_{n=1}$ in terms of the CFT parameters \hat{a} , \hat{b} and \hat{c} .¹² However, as with the expansion of h_n , the higher derivatives of the Rényi entropy would not have a simple universal form.

3 Explicit examples

In this section, we explicitly evaluate the conformal weight h_n in several theories using eq. (2.20). Given the expression for h_n , we can calculate the derivatives $h_{n,1}$ and $h_{n,2}$ and then compare the results to eqs. (2.21) and (2.45). In this way, it is first observed for a variety of holographic models [17] that the first derivative $h_{n,1}$ had a simple universal form and the latter then motivated the general proof for a generic CFT, which we presented in the previous section. Since this proof extends to second derivatives, this new expression in eq. (2.45) can also be compared with the results for $h_{n,2}$ obtained from holography in sections 3.1 and 3.2. Free fields, e.g., a massless fermion or a conformally coupled scalar, are another case where the relevant computations can be explicitly performed. We present the results of our comparison for these free theories in section 3.3, while the details of the heat kernel calculations for the free fields appear in appendix B.

3.1 Holographic Lovelock gravity

Let us begin with a holographic framework where the bulk is described by Lovelock gravity with up to six-derivatives in $d+1$ dimensions [30, 31].¹³ The inclusion of the six-derivative terms here extends the results for $h_{n,1}$ in Gauss-Bonnet gravity already described in [17]. The gravitational action is given by

$$I_{\text{Lovelock}} = \frac{1}{2\ell_{\text{P}}^{d-1}} \int d^{d+1}x \sqrt{-g} \left[\frac{d(d-1)}{L^2} + R + \frac{L^2 \lambda_{\text{GB}}}{(d-2)(d-3)} \mathcal{L}_4 + \frac{L^4 \mu}{(d-2)(d-3)(d-4)(d-5)} \mathcal{L}_6 \right], \tag{3.1}$$

where the two higher curvature interactions take the form

$$\mathcal{L}_4 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2, \tag{3.2}$$

¹²The higher order terms in the expansion (2.50) were recently studied in [29] and our results are in complete agreement between their general expression:

$$\partial_n^2 S_n|_{n=1} = \frac{16\pi^{d+1}}{3d^3(d+2)\Gamma(d-1)} (8(d-1)(d-2)\hat{a} - 2d\hat{b} - (5d-4)\hat{c}). \tag{2.55}$$

¹³For holographic studies of Lovelock gravity, see for example [27, 32–35].

$$\begin{aligned} \mathcal{L}_6 = & 4 R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\tau\chi} R_{\tau\chi}{}^{\mu\nu} - 8 R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} R_{\rho}{}^{\tau}{}_{\sigma}{}^{\chi} R_{\tau}{}^{\mu}{}_{\chi}{}^{\nu} - 24 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}{}_{\tau} R^{\sigma\tau} + 3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R \\ & + 24 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 16 R_{\mu}{}^{\nu} R_{\nu}{}^{\rho} R_{\rho}{}^{\mu} - 12 R_{\mu}{}^{\nu} R_{\nu}{}^{\mu} R + R^3. \end{aligned} \quad (3.3)$$

The scale L appearing in the action (3.1) is related to the radius of curvature \tilde{L} of the corresponding AdS vacuum by

$$\tilde{L}^2 = L^2/f_{\infty}, \quad \text{where } 1 - f_{\infty} + \lambda_{\text{GB}} f_{\infty}^2 + \mu f_{\infty}^3 = 0. \quad (3.4)$$

Now following [17], the thermodynamic properties of the dual CFT in the background $R \times H^{d-1}$ are determined by studying hyperbolic AdS black holes. In particular, the temperature of such an AdS black hole can be written as

$$T = \frac{1}{2\pi R x} \left(1 + \frac{d}{2f_{\infty}} \frac{x^6 - f_{\infty} x^4 + \lambda_{\text{GB}} f_{\infty}^2 x^2 + \mu f_{\infty}^3}{x^4 - 2\lambda_{\text{GB}} f_{\infty} x^2 - 3\mu f_{\infty}^2} \right) \quad (3.5)$$

where $x = r_{\text{H}}/\tilde{L}$ with r_{H} being the horizon radius. However, recall that we are particularly interested in temperatures

$$T = T_0/n = \frac{1}{2\pi R n}, \quad (3.6)$$

and with this choice, eq. (3.5) determines $x = x_n$ for a given n .

Now the energy density can be determined from the black hole solutions and then the scaling dimension h_n of the twist operators can be obtained using eq. (2.20). However, the calculations are simpler if the latter is re-expressed the latter in terms of the entropy [17]

$$h_n = \frac{2\pi R n}{(d-1)V_{\Sigma}} \int_{x_n}^1 dx T(x) \frac{dS(x)}{dx}. \quad (3.7)$$

The horizon entropy can be evaluated with Wald's formula [36–38] and for completeness, we give the final expression for the hyperbolic AdS black holes in Lovelock gravity

$$S(x) = 2\pi \left(\frac{\tilde{L}}{\ell_{\text{P}}} \right)^{d-1} V_{\Sigma} x^{d-1} \left(1 - 2 \frac{d-1}{d-3} \frac{\lambda_{\text{GB}} f_{\infty}}{x^2} - 3 \frac{d-1}{d-5} \frac{\mu f_{\infty}^2}{x^4} \right). \quad (3.8)$$

eq. (3.7) then yields

$$h_n = \pi \left(\frac{\tilde{L}}{\ell_{\text{P}}} \right)^{d-1} n x_n^{d-6} (x_n^2 - 1) (\mu f_{\infty}^2 + x_n^4 (\lambda_{\text{GB}} f_{\infty} + \mu f_{\infty}^2 - 1) + x_n^2 (\lambda_{\text{GB}} f_{\infty} + \mu f_{\infty}^2)). \quad (3.9)$$

Since we will be taking the limit $n \rightarrow 1$ at the end, we need only to obtain a perturbative solution for x_n around $n = 1$. This can be readily solved, giving

$$\begin{aligned} x_n &= 1 + \alpha_1(n-1) + \alpha_2(n-1)^2 + \dots \quad \text{with} \quad (3.10) \\ \alpha_1 &= -\frac{1}{d-1}, \quad \alpha_2 = -\frac{d(3 - 10\lambda_{\text{GB}} f_{\infty} - 21\mu f_{\infty}^2 - 2d(1 - 2\lambda_{\text{GB}} f_{\infty} - 3\mu f_{\infty}^2))}{2(d-1)^3(1 - 2\lambda_{\text{GB}} f_{\infty} - 3\mu f_{\infty}^2)}, \end{aligned}$$

which in turn, yields

$$h_{n,1} = \frac{2\pi}{d-1} \left(\frac{\tilde{L}}{\ell_P} \right)^{d-1} (1 - 2\lambda_{\text{GB}} f_\infty - 3\mu f_\infty^2), \quad (3.11)$$

$$h_{n,2} = -\frac{2\pi}{(d-1)^3} \left(\frac{\tilde{L}}{\ell_P} \right)^{d-1} \left(1 - 6\lambda_{\text{GB}} f_\infty - 15\mu f_\infty^2 - 4d(1 - 4\lambda_{\text{GB}} f_\infty - 9\mu f_\infty^2) + 2d^2(1 - 2\lambda_{\text{GB}} f_\infty - 3\mu f_\infty^2) \right).$$

Finally we need rewrite the above expressions in terms of parameters from the boundary CFT. Three such CFT parameters which are readily determined in terms of the free parameters in the gravitational theory, i.e., λ_{GB} , μ and \tilde{L}/ℓ_P , are [27]

$$C_T = \frac{(d+1)\Gamma[d+1]}{(d-1)\pi^{d/2}\Gamma[d/2]} \left(\frac{\tilde{L}}{\ell_P} \right)^{d-1} (1 - 2\lambda f_\infty - 3\mu f_\infty^2), \quad (3.12)$$

$$t_2 = \frac{4d(d-1)(\lambda f_\infty + 3\mu f_\infty^2)}{(d-2)(d-3)(1 - 2\lambda f_\infty - 3\mu f_\infty^2)}, \quad t_4 = 0. \quad (3.13)$$

where t_2 and t_4 are constants appearing in certain thought experiments involving the measurement of energy fluxes [39] — see also [27, 28]. Now for any CFT, these coefficients C_T , t_2 and t_4 are related to the parameters \hat{a} , \hat{b} , \hat{c} , described in the previous section, with [27, 39]

$$\begin{aligned} C_T &= 4\Omega_{d-1} \frac{(d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c}}{d(d+2)} \\ t_2 &= \frac{2(d+1)((d-1)(d^2+8d+4)\hat{a} + 3d^2\hat{b} - d(2d+1)\hat{c})}{d((d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c})} \\ t_4 &= -\frac{(d+1)(d+2)(3(2d+1)(d-1)\hat{a} + 2d^2\hat{b} - d(d+1)\hat{c})}{d((d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c})}. \end{aligned} \quad (3.14)$$

Since t_4 vanishes in Lovelock gravity, there is a constraint which allows us to eliminate one of the parameters \hat{a} , \hat{b} and \hat{c} in favour of the other two. In particular, we write

$$\hat{c} = \frac{3(2d+1)(d-1)\hat{a} + 2d^2\hat{b}}{d(d+1)}. \quad (3.15)$$

Taking eqs. (3.14) and (3.15) into account, eq. (3.11) becomes

$$h_{n,1} = \frac{16\pi^{d+1}}{d\Gamma(d+3)} \left((d^2 - 6d + 3)\hat{a} - 2d\hat{b} \right), \quad (3.16)$$

$$h_{n,2} = -\frac{16\pi^{d+1}}{d^3\Gamma(d+3)} \left((6d^4 - 36d^3 + 37d^2 + 13d - 12)\hat{a} - 2d^2(4d - 5)\hat{b} \right), \quad (3.17)$$

which is in perfect agreement with our CFT results, i.e., with eqs. (2.21) and (2.45) when we substitute in eq. (3.15).

3.2 Holographic quasi-topological gravity

To explore more general holographic CFT's, i.e., with $t_4 \neq 0$, we also consider quasi-topological gravity [28, 40] with four boundary dimensions, as in [17]. For completeness, the action is given by

$$I_{\text{quasi-top}} = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[\frac{12}{L^2} + R + \frac{1}{2} L^2 \lambda_{\text{GB}} \mathcal{L}_4 + \frac{7}{8} L^4 \mu_{\text{QT}} \mathcal{Z}'_5 \right] \quad (3.18)$$

where

$$\begin{aligned} \mathcal{Z}'_5 = & R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} + \frac{1}{14} (21 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R - 120 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}{}_{\alpha} R^{\sigma\alpha} \\ & + 144 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 128 R_{\mu}{}^{\nu} R_{\nu}{}^{\rho} R_{\rho}{}^{\mu} - 108 R_{\mu}{}^{\nu} R_{\nu}{}^{\mu} R + 11 R^3). \end{aligned} \quad (3.19)$$

The expressions for the temperature and entropy of a hyperbolic AdS black hole are precisely as in eqs. (3.5) and (3.8), respectively, upon making the replacement $\mu \rightarrow \mu_{\text{QT}}$ and taking $d = 4$. Since the formula for the temperature stays the same, the perturbative solution for x_n in eq. (3.10) also remains unchanged. Similarly, the expression for C_T in eq. (3.12) is the same in the present case of quasi-topological gravity. The new ingredients are the values for the parameters t_2 and t_4 , for which we have [28]

$$\begin{aligned} t_2 &= 24 \frac{\lambda_{\text{GB}} f_{\infty} - 87 \mu_{\text{QT}} f_{\infty}^2}{1 - 2\lambda_{\text{GB}} f_{\infty} - 3\mu_{\text{QT}} f_{\infty}^2}, \\ t_4 &= 3780 \frac{\mu_{\text{QT}} f_{\infty}^2}{1 - 2\lambda_{\text{GB}} f_{\infty} - 3\mu_{\text{QT}} f_{\infty}^2}. \end{aligned} \quad (3.20)$$

Again collecting these results and applying eq. (3.14), we arrive at

$$h_{n,1} = \frac{\pi^5}{180} (14\hat{a} - 2\hat{b} - 5\hat{c}), \quad (3.21)$$

$$h_{n,2} = -\frac{\pi^5}{90} (16\hat{a} - 3\hat{b} - 5\hat{c}). \quad (3.22)$$

which are again in perfect agreement with eqs. (2.21) and (2.45) with $d = 4$.

3.3 Free fields

In this subsection, we consider making a comparison to eqs. (2.21) and (2.45) in the special cases of free massless fermions and free conformally coupled scalar fields. As expected, a match is found for $h_{n,1}$ and $h_{n,2}$ calculated with heat kernel techniques for these free fields. We present here only our final findings, whereas the details of the computations are relegated to appendix B.

As shown in [24], the free fields under consideration satisfy

$$C_T = \frac{d}{\Omega_{d-1}^2} \left(\frac{n_s}{d-1} + \frac{n_f}{2} \right), \quad (3.23)$$

where n_s and n_f denote the number of (massless) degrees of freedom contributed by the scalar and fermion fields, respectively. In particular for a real scalar field, $n_s = 1$, whereas

for a Dirac fermion,¹⁴ $n_f = 2^{\lfloor \frac{d}{2} \rfloor}$. Similarly, the parameters appearing in the three-point correlator of the stress tensor take the form [24]

$$\begin{aligned} \hat{a} &= \frac{d^3}{8\Omega_{d-1}^3(d-1)^3} n_s, & \hat{b} &= -\frac{d^2}{8\Omega_{d-1}^3} \left(\frac{d^2}{(d-1)^3} n_s + \frac{n_f}{2} \right), \\ \hat{c} &= -\frac{d^2}{8\Omega_{d-1}^3} \left(\frac{(d-2)^2}{(d-1)^3} n_s + n_f \right). \end{aligned} \quad (3.24)$$

Hence, with the above expressions, eq. (2.21) takes the following simple form

$$h_{n,1} = \frac{d\Gamma(d/2)^3}{2\pi^{(d-2)/2}\Gamma(d+2)} \left(\frac{n_s}{d-1} + \frac{n_f}{2} \right), \quad (3.25)$$

while eq. (2.45) becomes

$$h_{n,2} = -\frac{d\Gamma(d/2)^3}{4\pi^{(d-2)/2}\Gamma(d+3)} \left(\frac{11d^4 - 33d^3 + 16d^2 + 28d - 16}{(d-1)^3} n_s + 2(2d^2 - d - 2) n_f \right). \quad (3.26)$$

On the other hand, in the case of free fields, h_n can be evaluated in full generality by means of heat kernel methods, as we describe in appendix B. By explicit calculations from this approach then, we have verified that first derivative of the scaling dimension so obtained agrees with eq. (3.25) in various spacetime dimensions up to $d = 14$. We were also able to explicitly verify that this approach reproduces eq. (3.26) for massless fermions in dimensions up to $d = 12$. Unfortunately, we must report that for the conformally coupled scalar, there was a discrepancy between $h_{n,2}$ as evaluated using the heat kernel results and as given in eq. (3.26). We return to this issue in section 6.

4 OPE of spherical twist operators

In section 2, we described how the conformal mapping (2.2) between $S^1 \times H^{d-1}$ and the n -fold cover of R^d could be applied to evaluate the expectation value of the stress tensor in the presence of a spherical twist operator as in eq. (2.16), i.e.,

$$\langle T_{\alpha\beta} \sigma_n \rangle = \Omega^{d-2} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} \left(\langle T_{\mu\nu}(T_0/n) \rangle - \langle T_{\mu\nu}(T_0) \rangle \right). \quad (4.1)$$

where the conformal factor Ω is given by eq. (2.4). We examined this result in the limit where the stress tensor was brought very close to the twist operator in order to determine the conformal dimension of twist operators in the CFT in terms of the thermal energy density in the hyperbolic background, as given in eq. (2.20). In this section, we consider the opposite limit where the stress tensor is taken very far from the twist operator, i.e., $T_{\alpha\beta}$ is inserted at a large radius r with $r \gg R$.

In investigating any twist operator enclosing some finite region with long wavelength probes, such as in the correlator described above, the twist operator σ_n can be approxi-

¹⁴Here, $\lfloor d/2 \rfloor$ denotes the integer part of $d/2$.

mated by a sum of local operators \mathcal{O}_p and their descendants (indexed by k below) [41–43]¹⁵

$$\sigma_n = \langle \sigma_n \rangle \left(1 + \sum_{p,k} R^{\Delta_{p,k}} c_{p,k}^n \mathcal{O}_p^k \right), \quad (4.2)$$

where R is some (macroscopic) scale characterizing the size of σ_n and the $\Delta_{p,k}$ are the conformal dimensions of the operators \mathcal{O}_p^k . Two comments on this expansion are: the operators \mathcal{O}_p may be conformal primaries in a single copy of the CFT, but in general they will be products of two or more such operators inserted at the same point but in different copies of the CFT, i.e., on different sheets of the n -fold cover. In $d = 2$ dimensions, the twist operators are themselves local operators but calculating the Rényi entropy of an interval requires the insertion of two twist operators, one at each end of the interval. Hence in this case, i.e., $d = 2$, the above expansion corresponds to the operator product expansion coming from the fusion of the two twist operators.¹⁶ Hence following the common nomenclature, e.g., [41–43], we refer to eq. (4.2) as the ‘operator product expansion’ (OPE) of a single twist operator on a closed surface in higher dimensions.

Now we can use the expectation value (4.1) of the stress tensor in the presence of a spherical twist operator in a CFT to learn something about the corresponding OPE (4.2). If the OPE is used to replace σ_n in this expectation value, the only nonvanishing contribution will come from the descendants of the identity, i.e., conformal invariance dictates that $\langle T_{\alpha\beta} \mathcal{O}_p^k \rangle = 0$ for other operators. Hence for large separations, the leading contribution is¹⁷

$$\langle T_{\mu\nu}(x) \sigma_n \rangle = R^d \varepsilon_n^{\alpha\beta} \langle T_{\mu\nu}(x) T_{\alpha\beta}(0) \rangle + \dots \quad (4.3)$$

where the OPE coefficient takes the form $\varepsilon_n^{\alpha\beta}$, some (traceless and symmetric) polarization tensor that in general depends on the geometry of the surface operator. The ellipsis above denotes the higher descendants with more insertions of the stress tensor. Hence the leading long-distance behaviour in $\langle T_{\mu\nu}(x) \sigma_n \rangle$ is controlled by the two-point correlator of the stress tensor, given in eq. (2.29).

To determine the precise form of $\varepsilon_n^{\alpha\beta}$ for spherical twist operators, we first examine the long-distance behaviour of the expectation value in eq. (4.1). It will be sufficient to only evaluate the latter at $t_E = 0$ and some fixed angles. From eqs. (2.4) and (2.17), it follows then that in the limit $r \gg R$

$$\Omega \simeq \frac{2R^2}{r^2}, \quad \frac{\partial u}{\partial t_E} = -\frac{1}{R} \frac{\partial \tau_E}{\partial r} = 0, \quad \frac{\partial u}{\partial r} = \frac{1}{R} \frac{\partial \tau_E}{\partial t_E} \simeq -\frac{2R}{r^2}. \quad (4.4)$$

¹⁵Since σ_n is a nonlocal operator, there are certain ambiguities here. That is, the precise form of the expansion coefficients $c_{p,k}^n$ will depend on the choice of the scale R and of the reference point at which the operators \mathcal{O}_p^k are inserted. In eq. (4.3), we implicitly choose R is the radius of the sphere and the reference point as the center of the spherical region enclosed by the twist operator.

¹⁶The fusion rules of a pair twist operators have been computed in specific models, such as the Ising model, for example in [44, 45].

¹⁷Recall that the correlator $\langle T_{\alpha\beta} \sigma_n \rangle$ is implicitly normalized by dividing by $\langle \sigma_n \rangle$ but we have been leaving this normalization implicit to avoid further clutter. In any event, this normalization removes the factor of $\langle \sigma_n \rangle$ appearing in eq. (4.2) from the right-hand side of eq. (4.3).

Substituting these expressions into eq. (4.1), we obtain

$$\begin{aligned} \langle T_{t_E t_E} \sigma_n \rangle &= \left(\frac{2R^2}{r^2} \right)^d \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n) \right) = \frac{d-1}{2\pi n} \left(\frac{2R}{r^2} \right)^d h_n, \\ \langle T_{ij} \sigma_n \rangle &= -\frac{g_{ij}}{d-1} \left(\frac{2R^2}{r^2} \right)^d \left(\mathcal{E}(T_0) - \mathcal{E}(T_0/n) \right) = -\frac{g_{ij}}{2\pi n} \left(\frac{2R}{r^2} \right)^d h_n, \end{aligned} \quad (4.5)$$

where we have used eqs. (2.13), (2.14) and (2.20), as well as $\sinh u \simeq 2R/r$. Here, the indices i, j run over all of the directions orthogonal to t_E , i.e., r and $d-2$ angular coordinates, and g_{ij} is the flat space metric on these directions, i.e., $g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta, \dots)$. To simplify the comparison with eq. (4.3), we adopt Cartesian coordinates on these directions at this point, in which case we have simply $g_{ij} = \delta_{ij}$. Finally, we note that in eq. (4.5), $T_{\mu\nu}$ is inserted on a single sheet of the n -fold cover and therefore we must multiply it by n to in order to produce $\langle T_{\mu\nu} \sigma_n \rangle$ with the full stress tensor of the n copies of the CFT. Hence our final result is

$$\langle T_{t_E t_E} \sigma_n \rangle = \frac{d-1}{2\pi} \left(\frac{2R}{r^2} \right)^d h_n, \quad \langle T_{ij} \sigma_n \rangle = -\frac{\delta_{ij}}{2\pi} \left(\frac{2R}{r^2} \right)^d h_n, \quad (4.6)$$

whereas all other components of $\langle T_{\mu\nu} \sigma_n \rangle$ vanish (when we set $t_E = 0$). Of course, we see here that the long-distance behaviour of the expectation value (4.1) can be expressed in terms of the conformal dimension h_n of the twist operators.¹⁸

Now we return to eq. (4.3) and substitute eq. (2.29) for the two-point correlator of the stress tensor. However, as above, we will restrict our attention to $t_E = 0$ in which case the result may be written as

$$\langle T_{\mu\nu}(t_E = 0, x^i) \sigma_n \rangle \simeq R^d \varepsilon_n^{\alpha\beta} \frac{C_T}{r^{2d}} (\delta_{\alpha\mu} - 2\hat{r}_\alpha \hat{r}_\mu) (\delta_{\beta\nu} - 2\hat{r}_\beta \hat{r}_\nu) \quad (4.7)$$

where we have used the tracelessness of $\varepsilon_n^{\alpha\beta}$ to simplify the above expression. We have also defined here the radial unit vector in the x^i directions: $\hat{r}^\mu \equiv (0, x^i/r)$. Below, it will also be useful to define the unit vector point along the t_E direction: $\hat{t}^\mu \equiv (1, \vec{0})$.

Let us consider the possible form for $\varepsilon_n^{\alpha\beta}$. The geometry of the spherical entangling surface demands this polarization tensor must be rotationally symmetric in the x^i directions. Hence most general allowed tensor can be written as

$$\varepsilon_n^{\alpha\beta} = \alpha_1 \delta^{\alpha\beta} + \alpha_2 \hat{t}^\alpha \hat{t}^\beta + \alpha_3 \hat{r}^\alpha \hat{r}^\beta + \alpha_4 \left(\hat{r}^\alpha \hat{t}^\beta + \hat{t}^\alpha \hat{r}^\beta \right), \quad (4.8)$$

where $\alpha_{1,2,3,4}$ are all only functions of r . Further the tracelessness of the polarization tensor requires that

$$0 = d\alpha_1 + \alpha_2 + \alpha_3. \quad (4.9)$$

¹⁸Actually, we might use this long-distance correlator with a spherical twist operator as an alternate definition of the conformal dimension. Further, let us add that while we have explicitly shown that h_n (as well as R) controls the expectation value $\langle T_{\mu\nu} \sigma_n \rangle$ at large and small separations, in fact, one can easily verify from eq. (2.16) that the same is true of the entire correlator at any separation, e.g., see eq. (5.12).

Now substituting eq. (4.8) into eq. (4.7) yields

$$\langle T_{\mu\nu}(t_E = 0, x^i) \sigma_n \rangle \simeq \frac{C_T R^d}{r^{2d}} (\alpha_1 \delta_{\mu\nu} + \alpha_2 \hat{t}_\mu \hat{t}_\nu + \alpha_3 \hat{r}_\mu \hat{r}_\nu - \alpha_4 (\hat{r}_\mu \hat{t}_\nu + \hat{t}_\mu \hat{r}_\nu)) \quad (4.10)$$

Therefore comparing this expression with the results from the conformal mapping in eq. (4.6), we find

$$\alpha_1 = -\frac{2^{d-1} h_n}{\pi C_T}, \quad \alpha_2 = -d \alpha_1, \quad \alpha_3 = 0 = \alpha_4. \quad (4.11)$$

That is, we can write the polarization tensor $\varepsilon_n^{\alpha\beta}$ appearing in the OPE of a spherical twist operator as

$$\varepsilon_n^{\alpha\beta} = \frac{2^{d-1} h_n}{\pi C_T} (d \hat{t}^\alpha \hat{t}^\beta - \delta^{\alpha\beta}). \quad (4.12)$$

Finally we note that the term proportional to $\delta^{\alpha\beta}$ can be dropped because the trace of the stress tensor vanishes. Hence the contribution of the stress tensor to the OPE (4.2) reduces to

$$\sigma_n(0) = \langle \sigma_n \rangle \left(1 + \gamma h_n R^d T_{t_E t_E}(0) + \dots \right) \quad \text{with } \gamma = \frac{2^{d-1} d}{\pi C_T}, \quad (4.13)$$

for a spherical twist operator of radius R positioned as the origin. We note that the above OPE coefficient depends on the order of the twist operator only through the appearance of h_n , i.e., the factor γ is independent of n .

Of course, a similar analysis using the conformal mapping in section 2 could be made to evaluate other terms in the OPE of a spherical twist operator — see section 6.

5 Twist operators near $n = 1$

Twist operators σ_n are originally defined for integer $n \geq 2$, however, formally we can consider these operators for arbitrary n . Such a continuation was already implicit in section 2.2, where the conformal dimension h_n was expanded near $n = 1$. In this section, we consider a similar expansion for small $(n - 1)$ of the twist operators themselves. Again, this is a formal expansion which indicates that any correlators involving σ_n will behave in a universal manner in the limit $n \rightarrow 1$. Note that our discussion here is quite general, i.e., it is not limited to spherical entangling surfaces or to CFT's. Rather our main result in eq. (5.7) applies for general entangling geometries and for any quantum field theory.

Our approach will be to begin by considering the correlator of the twist operator σ_n with some collection of operators, which we denote collectively as \mathcal{X} .¹⁹ However, we will restrict our attention to the case where all of the operators comprising \mathcal{X} are in a single copy of the QFT, say, the first of the n copies. That is, generally the correlators $\langle \sigma_n \mathcal{X} \rangle$ are defined by inserting the operators \mathcal{X} in the path integral of the QFT on the n -fold covering space but we limiting our considerations to the situation where all of the insertions are

¹⁹In general, \mathcal{X} may include both local and nonlocal operator insertions, e.g., when the underlying QFT is a gauge theory, \mathcal{X} may include Wilson line operators.

made on the first sheet. Next, we construct a new ‘effective twist operator’ $\tilde{\sigma}_n$ which only acts within the first QFT but reproduces any such correlator, i.e.,

$$\langle \sigma_n \mathcal{X} \rangle = \langle \tilde{\sigma}_n \mathcal{X} \rangle_1 \tag{5.1}$$

where the subscript on the second correlator indicates that this expression is evaluated in the first copy of the QFT. Formally, it is straightforward to define this new operator by integrating out the other copies of the QFT for which there are no operator insertions in the above correlators. That is,

$$\tilde{\sigma}_n \equiv \langle \sigma_n \rangle_{\{2, \dots, n\}}, \tag{5.2}$$

where the subscript on the right-hand side indicates that we are performing the path integral over the $(n-1)$ copies of the QFT other than the first copy. Now let us consider the Euclidean path integral representation of the correlator $\langle \sigma_n \mathcal{X} \rangle$ with an n -fold covering geometry, as illustrated for a simple example in the figure 1a. Recall that the n -fold cover was formulated to give a path integral construction of (integer) powers of the reduced density matrix, as in eq. (1.1). Hence when eq. (5.2) instructs us to perform the path integral over the $(n-1)$ empty sheets, we can interpret this portion as the path integral representation of the operator ρ_A^{n-1} . Therefore the same correlator can be evaluated within a single copy of the QFT by including an insertion of the latter operator in the region A, as shown in figure 1b. Hence we are led to conclude that the effective twist operator corresponds to²⁰

$$\tilde{\sigma}_n = \rho_A^{n-1}. \tag{5.4}$$

Let us re-iterate that it is essential for this argument that all of the operators in \mathcal{X} were from a single copy of the QFT or alternatively, that they are all inserted on a single sheet of the n -fold covering geometry.

Finally let us recall that the reduced density matrix appearing in the calculation of the Rényi entropy (1.1) can be expressed as

$$\rho_A = e^{-H_m} \tag{5.5}$$

for some Hermitian operator H_m . The latter is known as the modular Hamiltonian in the literature on axiomatic quantum field theory, e.g., [46], while it is referred to as the entanglement Hamiltonian in the condensed matter theory literature.²¹ However, we emphasize that generically the entanglement Hamiltonian is not a local operator and the evolution generated by H_m would not correspond to a local (geometric) flow. However, a CFT reduced to a spherical region provides an exception to this general rule. In this case, we can

²⁰The same conclusion can be reached with the following formal manipulations:

$$\langle \sigma_n \mathcal{X} \rangle = \text{Tr} [\rho_A^n \mathcal{X}] = \text{Tr} [\rho_A \rho_A^{n-1} \mathcal{X}] = \langle \rho_A^{n-1} \mathcal{X} \rangle_1, \tag{5.3}$$

where again the subscript on the last correlator indicates that this expression is evaluated in the first copy of the QFT. However, for the two intermediate expressions, i.e., $\text{Tr} [\rho_A^n \mathcal{X}]$ and $\text{Tr} [\rho_A \rho_A^{n-1} \mathcal{X}]$, the operator insertions are implicitly limited to be within the region A. Of course, the general discussion above had no such restriction.

²¹See for example ref. [47–50].

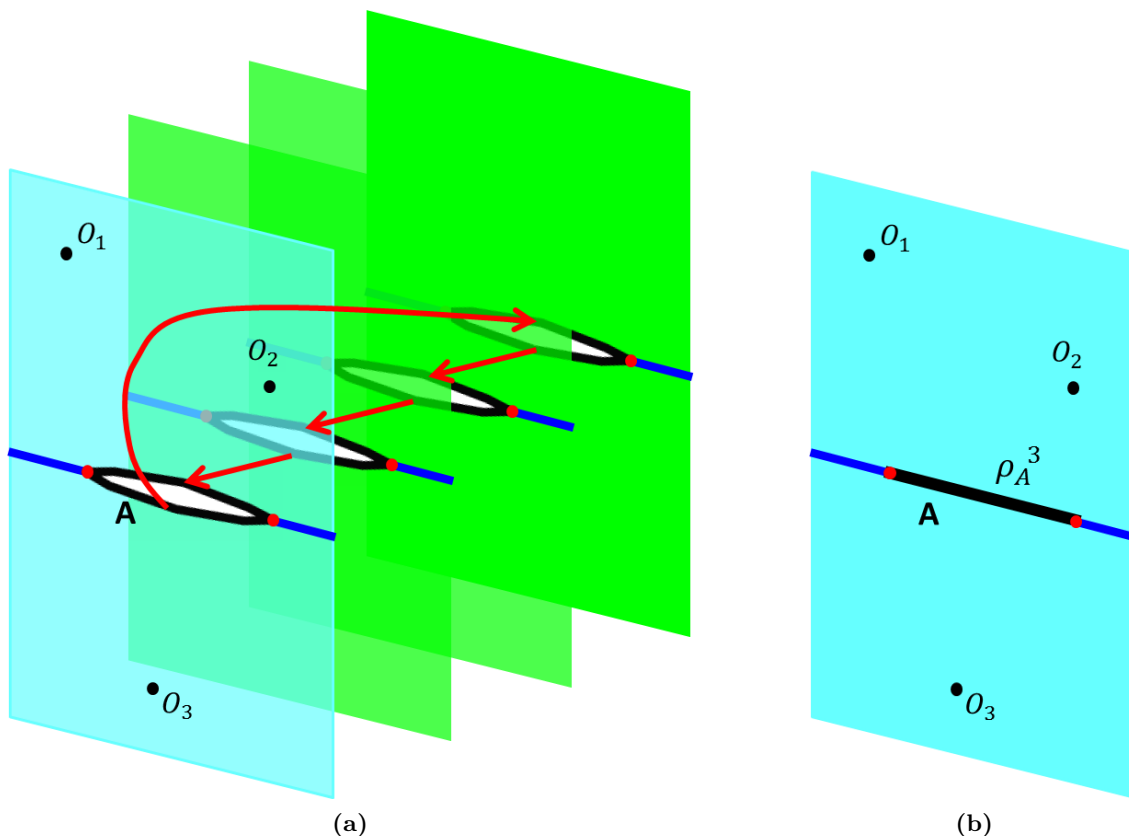


Figure 1. (Colour online) Panel (a) shows the four-fold geometry which would appear in the evaluation of the correlator $\langle \sigma_4 O_1 O_2 O_3 \rangle$, where the operator insertions all lie on the first sheet. The path integral on the next three sheets (shaded green) provides a representation of ρ_A^3 . Hence in (b), the same correlator is evaluated as a correlator on a single sheet with an operator insertion ρ_A^3 in the region A.

write H_m as [18]

$$H_m = -2\pi \int_{r \leq R} d^{d-1}x \frac{R^2 - r^2}{2R} T_{t_E t_E} + c'. \quad (5.6)$$

where the constant c' is fixed by demanding that the corresponding density matrix is normalized with unit trace. We return to this specific example in a moment but first continue with our general considerations.

In particular, given eq. (5.4), we may now write the effective twist operator as

$$\tilde{\sigma}_n = e^{-(n-1)H_m}. \quad (5.7)$$

In fact, this result precisely matches a conjecture presented recently in [57]. With eq. (5.7), we have produced an expression for the (effective) twist operator itself where we can easily consider the limit $n \rightarrow 1$ by simply expanding the right-hand side above in powers of $(n - 1)$. Again, the purpose of this expansion is to investigate the (universal) behaviour of correlators involving σ_n in the limit $n \rightarrow 1$. Of course, the above expression (5.7) will only prove useful in situations where the modular Hamiltonian is known and so the case of

interest here, i.e., a CFT reduced to a spherical region, is one such situation. Hence as we will see below, we can provide evidence supporting eq. (5.7) using the explicit expression for the modular Hamiltonian in eq. (5.6).

5.1 Consistency checks

In the following, we provide evidence confirming eq. (5.7) by focussing on the special situation of a spherical entangling surface in a CFT, for which the modular Hamiltonian is given by eq. (5.6). In particular, we perform two consistency checks of eq. (5.7) using this latter expression to evaluate the correlator $\langle T_{\alpha\beta} \sigma_n \rangle$ and a certain contribution to $\langle \sigma_n \sigma_n \rangle$, both in the limit $n \rightarrow 1$.

Correlator with the stress tensor: here, we evaluate the correlator $\langle T_{\alpha\beta} \sigma_n \rangle$ using eq. (5.7) and then compare the result to eq. (2.16), when taking the limit $n \rightarrow 1$. With eq. (5.7), the desired correlator becomes

$$\begin{aligned} \langle T_{\alpha\beta}(t_E = 0, x) \sigma_n \rangle &= \langle T_{\alpha\beta}(t_E = 0, x) \tilde{\sigma}_n \rangle_1 = \left\langle T_{\alpha\beta}(t_E = 0, x) e^{-(n-1)H_m} \right\rangle_1 \\ &= -(n-1) \langle T_{\alpha\beta}(t_E = 0, x) H_m \rangle_1 + \dots \\ &= 2\pi (n-1) \int_{y^2 \leq R^2} d^{d-1}y \left(\frac{R^2 - y^2}{2R} \right) \langle T_{\alpha\beta}(t_E = 0, x) T_{t_E t_E}(t_E = 0, y) \rangle_1 + \dots \end{aligned} \quad (5.8)$$

where the ellipsis indicates terms with higher powers of $(n-1)$. Note that we have restricted our attention to the case where both the stress tensor and the twist operator lie in the hyperplane at $t_E = 0$, which simplifies the subsequent calculations somewhat. Now, substituting eq. (2.29) into eq. (5.8) above yields,

$$\begin{aligned} \langle T_{\alpha\beta}(t_E = 0, x) \sigma_n \rangle &= 2\pi (n-1) C_T \int_{y^2 \leq R^2} d^{d-1}y \left(\frac{R^2 - y^2}{2R} \right) \frac{\delta_{\alpha t_E} \delta_{\beta t_E} - \frac{1}{d} \delta_{\alpha\beta}}{|x-y|^{2d}} + \dots \\ &= \frac{(n-1)\Omega_d}{(d+1)} \frac{R^d}{|r^2 - R^2|^d} C_T \left(\delta_{\alpha t_E} \delta_{\beta t_E} - \frac{1}{d} \delta_{\alpha\beta} \right) + \dots \end{aligned} \quad (5.9)$$

where $r = |x|$. The final result is written so as to accommodate both situations where $r > R$ and $r < R$ — in the latter case, the integral in the first line must be regulated along the lines of the discussion around eq. (2.33). Hence our final result for this correlator can be written as

$$\begin{aligned} \langle T_{t_E t_E}(t_E = 0, x) \sigma_n \rangle &= (n-1) \frac{(d-1)\Omega_d}{d(d+1)} \frac{R^d}{|r^2 - R^2|^d} C_T + \dots, \\ \langle T_{ij}(t_E = 0, x) \sigma_n \rangle &= -(n-1) \delta_{ij} \frac{\Omega_d}{d(d+1)} \frac{R^d}{|r^2 - R^2|^d} C_T + \dots, \\ \langle T_{it_E}(t_E = 0, x) \sigma_n \rangle &= 0, \end{aligned} \quad (5.10)$$

where we have adopted Cartesian coordinates in the $(d-1)$ directions orthogonal to t_E .

Next we turn to evaluating the same correlators using eq. (2.16). We will focus on reproducing the first line of eq. (5.10) since the remaining components follow from the

tracelessness of the stress tensor and the spherical symmetry of the twist operator. Using eqs. (2.4) and (2.17), we find

$$\Omega(t_E = 0, x) = \frac{2R^2}{|r^2 - R^2|}, \quad \left. \frac{\partial \tau_E}{\partial t_E} \right|_{t_E=0, x} = \frac{2R^2}{R^2 - r^2}. \quad (5.11)$$

Substituting these expressions into the first line of eq. (2.18) and replacing the difference of the energy densities using eq. (2.20), we find

$$\langle T_{t_E t_E}(t_E = 0, x) \sigma_n \rangle = \frac{d-1}{2\pi n} \left(\frac{2R}{|r^2 - R^2|} \right)^d h_n. \quad (5.12)$$

Now recall that h_n vanishes in the limit $n \rightarrow 1$ and hence when $n \simeq 1$, we can approximate the above expression as

$$\langle T_{t_E t_E}(t_E = 0, x) \sigma_n \rangle = (n-1) \frac{d-1}{2\pi} \left(\frac{2R}{|r^2 - R^2|} \right)^d \partial_n h_n|_{n=1} + \dots. \quad (5.13)$$

Of course, the next step is to replace $\partial_n h_n|_{n=1}$ in the above expression using eq. (2.21). For the purposes of the present comparison, we rewrite the latter equation as

$$\partial_n h_n|_{n=1} = \frac{2\pi \Omega_d}{2^d d(d+1)} C_T. \quad (5.14)$$

Then substituting this expression into eq. (5.13) yields precisely the correlator given in the first line of eq. (5.10).

Hence we have shown that to leading order in $(n-1)$, eq. (5.7) reproduces the correct correlator $\langle T_{\alpha\beta} \sigma_n \rangle$. The above calculations were limited to the case where both operators lie in the hypersurface $t_E = 0$ but it would be straightforward to extend the comparison for operator insertions at arbitrary relative positions. It would also be interesting to extend this comparison to higher orders in $(n-1)$, but a comment is in order on this point. As discussed earlier in this section, in eq. (5.8), we are inserting the stress tensor into a single copy of the CFT. In contrast, the correlator in eq. (4.6), and hence eq. (5.12), involves an insertion of the full stress tensor of all n copies of the CFT. However, it is straightforward to verify that in the limit $n \rightarrow 1$, differences between the two cases only arise at order $(n-1)^2$. Of course, for this comparison to succeed at higher orders, one must be careful to keep track of these differences.

Correlator of two twist operators: as in the previous section, we can combine eqs. (5.6) and (5.7) to produce a fairly explicit expression for the effective twist operator for single spherical region. Now one might consider whether this can be used to give useful information when we deal with multiple spherical regions, as considered in in [45, 64], but it turns out that this is a subtle issue, as we will discuss in section 6. In any event, as a step in this direction, we will examine the correlator $\langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle_1$ for two spherical regions, in the following.

To produce a tractable calculation, we look for the leading contribution to $\langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle_1$ in the limit $n \rightarrow 1$ and we also focus on the leading large-distance behaviour, i.e., if the

two spheres have radii R_1 and R_2 and their centers are separated by a distance r , then we consider $r \gg R_{1,2}$. Further, we position both spheres in the hyperplane $t_E = 0$ with the first sphere centered at $\vec{x}_{c,1}$ while the second is positioned at $\vec{x}_{c,2}$ with $|\vec{x}_{c,2} - \vec{x}_{c,1}| = r$. Adapting eq. (5.6) to these positions yields

$$H_{m,i} = -2\pi \int_{|\vec{x}_i - \vec{x}_{c,i}| < R_i} d^{d-1}x_i \frac{R_i^2 - |\vec{x}_i - \vec{x}_{c,i}|^2}{2R_i} T_{t_E t_E}(t_E = 0, \vec{x}_i) + c'. \quad (5.15)$$

To determine the leading $n \rightarrow 1$ behaviour, we replace the effective twist operators with eq. (5.7) and expand each of the two exponentials $\exp[-(n-1)H_{m,i}]$ to leading order in $(n-1)$. This yields

$$\begin{aligned} \langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle_1 &\simeq 4\pi^2 (n-1)^2 \prod_{i=1}^2 \int_{|\vec{x}_i - \vec{x}_{c,i}| < R_i} d^{d-1}x_i \frac{R_i^2 - |\vec{x}_i - \vec{x}_{c,i}|^2}{2R_i} \\ &\quad \times \langle T_{t_E t_E}(t_E = 0, \vec{x}_1) T_{t_E t_E}(t_E = 0, \vec{x}_2) \rangle \\ &\simeq 4\pi^2 (n-1)^2 \Omega_{d-2}^2 \prod_{i=1}^2 \int_{r' < R_i} dr' r'^{d-2} \frac{R_i^2 - r'^2}{2R_i} \frac{d-1}{d} \frac{C_T}{r^{2d}} + \dots, \quad (5.16) \\ &= \frac{4\pi^2 \Omega_{d-2}^2}{d(d-1)(d+1)^2} (n-1)^2 \left(\frac{R_1 R_2}{r^2} \right)^d C_T + \dots \end{aligned}$$

where we evaluated the correlator of the two stress tensors using eq. (2.31). However, since we only wish to determine the leading behaviour at large separations, we have approximated $|\vec{x}_2 - \vec{x}_1| \simeq r$ in this expression.

This limit of the correlator can also be calculated independently using our results from section 4 for the OPE expansion of the spherical twist operators. In particular, eq. (4.13) gives the leading contribution of the stress tensor, which suggests that the desired correlator is given by

$$\langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle_1 = \gamma^2 R_1^d R_2^d [(n-1) \partial_n h_n|_{n=1}]^2 \langle T_{t_E t_E}(t_E = 0, \vec{x}_{c,1}) T_{t_E t_E}(t_E = 0, \vec{x}_{c,2}) \rangle. \quad (5.17)$$

Hence we see the appearance of the familiar coefficient $\partial_n h_n|_{n=1}$ here and further that the correlator controlling this interaction term is precisely the same as appears in eq. (5.16). To verify that the numerical coefficient indeed coincides with that in eq. (5.16), first recall that $\gamma = 2^{d-1}d/(\pi C_T)$ as given in eq. (4.13). Then substituting in eqs. (2.21) and (2.31), we find

$$\langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle_1 = \frac{2^{2d} \pi^d d(d-1) \Gamma(d/2)^2}{\Gamma(d+2)^2} (n-1)^2 \left(\frac{R_1 R_2}{r^2} \right)^d C_T \quad (5.18)$$

and one can verify that the numerical prefactor here precisely matches that in eq. (5.16). Hence this agreement provides further evidence supporting our expression for the effective twist operator in eq. (5.7).

We note that the above agreement readily extends to the case where one of the operators is inserted away from the $t_E = 0$ hyperplane. Above, we saw that the same correlator of two stress tensors controls the long-distance and $n \rightarrow 1$ limit in both calculations. Further we already verified that the overall coefficients match as well.

Therefore we will continue to find agreement when the two effective twist operators are inserted at arbitrary positions. Interestingly, it seems this kind of correlator where the two insertions are not both at $t_E = 0$ is not something that one would naturally consider in evaluating the Rényi entropy with $\langle \sigma_{n,1} \sigma_{n,2} \rangle$.

6 Discussion

In this paper, we investigated various properties of twist operators in higher dimensional CFT's. In particular, we made use of the construction in [17, 18] where the entanglement entropy, as well as the Rényi entropies, of a spherical region in the flat space vacuum were related to the thermal entropy of the CFT on the hyperbolic background $S^1 \times H^{d-1}$. This conformal mapping allows one to evaluate the scaling dimension of the twist operators σ_n in terms of the energy density in the thermal ensemble [17], as described in section 2. While it was originally motivated by holographic studies of entanglement entropy, this construction makes no reference to the AdS/CFT correspondence and in particular then, the resulting expression for the conformal dimension (2.20) applies for any CFT. Further, we might note that while the radius of the sphere in flat space appears with an explicit factor of R^d in eq. (2.20), this is the only scale in the calculation and so the same scale also appears implicitly in the temperature (2.7) and as the curvature scale of the hyperbolic space. Since the underlying field theory is a CFT, the energy density in eq. (2.20) must produce a factor of $1/R^d$ leaving h_n to be a pure number which characterizes the conformal dimension of all twist operators in the theory.

Of course, eq. (2.20) may not provide a very practical approach to determining h_n , i.e., we must evaluate the energy density of a higher dimensional CFT in a curved background with a curvature scale R at temperatures of order $1/R$. However, we were able to use this expression to construct an expansion (2.35) of the conformal dimension in power series around $n = 1$ (where n is the order of the twist operator).²² Further, $h_{n,k} = \partial_n^k h_n|_{n=1}$, i.e., the coefficient of the term proportional to $(n-1)^k$ in eq. (2.35), is completely determined by the $(k+1)$ - and k -point correlation functions of the stress tensor in flat space. Hence, we showed that the first derivative of the conformal dimension had a simple universal form (2.21) which was fixed by C_T , the central charge appearing in the two-point correlator (2.29) of the stress tensor. This universal form was first discovered in [17] where it was found to apply to a variety of holographic CFT's but here, we established that it is a completely general result that applies in any higher dimensional CFT. We also showed that $\partial_n^2 h_n|_{n=1}$ has a similar universal form (2.45) which can be expressed in terms of \hat{a} , \hat{b} , \hat{c} , the three parameters which determine the three-point function of the stress tensor [24, 25].

In section 3 and appendix B, we verified the universal expressions in eqs. (2.21) and (2.45) with explicit calculations in a variety of holographic models, as well as for a free massless fermion and for a free conformally coupled scalar. However, we must remind the reader that for the free conformally coupled scalar in $d \geq 3$, the heat kernel calculations in appendix B produced a result for $h_{n,2}$ which was not in agreement with eq. (3.26), i.e., our

²²We note that this expansion was recently extended to include a chemical potential in discussing a new class of 'charged' Rényi entropies [51].

general formula (2.45) with the free field values for \hat{a} , \hat{b} and \hat{c} substituted in. It remains a challenge to explain this discrepancy at this point and we remind the reader that eq. (2.45) successfully passed all of our holographic tests, as well as agreeing with the heat kernel computations for free massless fermions. Addressing this challenge may in fact uncover some new perspectives on Rényi entropies and twist operators. Further, we might also point out that similar discrepancies appears in applying the approach of [18] to evaluate the entanglement entropy of a Maxwell field in four dimensions, e.g., [52–54]. We note that recently, [29] has proposed a possible resolution to the discrepancy.

In section 4, we considered the ‘operator product expansion’ of spherical twist operators in higher dimensional CFT’s. In particular, at an intermediate step, the calculation in section 2 of the scaling dimension produced the correlation function $\langle T_{\alpha\beta} \sigma_n \rangle$ in eq. (2.16). By examining this correlator in the limit where the separation of the two operators was much larger than the radius of the sphere, we were able to evaluate the coefficient of the stress tensor in the OPE of the twist operator, with the result given in eq. (4.13). Again, this result applies for general CFT’s with the coefficient being determined by the ratio h_n/C_T .

In principle, analogous calculations using the conformal mapping in section 2 could be made to evaluate other terms in the OPE of a spherical twist operator. In particular, if one could evaluate various thermal correlators in the background $S^1 \times H^{d-1}$, then they can be mapped to the corresponding correlators in the n -fold cover of flat space. By carefully examining the latter in the limit of large separations, one should be able to interpret them as flat space correlators with local operators inserted at the position of the twist operator, i.e., with the OPE of the spherical twist operator. Again, the necessity to first evaluate the thermal correlators may make this an impractical approach for determining the OPE coefficients except in special cases. However, one observation is that generally we do not expect any local operators apart from the stress tensor to acquire an expectation value in the thermal bath. Hence the only terms in the OPE with a single local operator in a single copy of the CFT would involve (normal ordered) products of the stress tensor, i.e., descendants of the identity operator. However, this does not preclude the appearance of terms involving the tensor product of operators in multiple copies of the n -fold replicated CFT [55] — see also [44, 45, 56]. Such contributions to the OPE would be revealed in the calculation described above by thermal correlators with several local operators suitably spaced along the thermal circle. It would be interesting to explicitly carry out such calculations in a holographic framework or with free fields.

In eq. (5.1), we proposed the construction of a new ‘effective twist operator’ $\tilde{\sigma}_n$ which acts within a single copy of the QFT to reproduce correlators with the twist operator. We also provided a simple representation of this effective twist operator in terms of the modular Hamiltonian in eq. (5.7). Further, a few preliminary consistency checks of eq. (5.7) were given in section 5.1. Our arguments in section 5 are quite general and so eq. (5.7) will apply for any quantum field theory, not just a CFT, and for any entangling geometry, not just a spherical entangling surface. One conclusion that is drawn from eq. (5.7) is that the reduced density matrix is fully determined by the twist field σ_2 , i.e., $\tilde{\sigma}_2 = \rho_A$. At first sight, this result may seem surprising because one needs at least all the Rényi entropies to get the entanglement spectrum, e.g., [58]. However, the Rényi entropies

provide a single number from the expectation value of each twist operator and so it should be expected that reconstructing the density matrix requires evaluating an infinite number of such expectation values. In contrast, $\tilde{\sigma}_2$ (or any single $\tilde{\sigma}_n$) is an operator with which in principle, one can calculate an infinite number of correlators. So given all of this available information, it is less surprising that one can reconstruct the full density matrix.

Eq. (5.7) exhibits an apparently curious feature: on the one hand, the twist operator is assumed to be an object which is independent of the quantum state of the underlying QFT but there will be a distinct modular Hamiltonian for each different state on a fixed region A , i.e., the modular Hamiltonian is defined with $\rho_A = \exp[-H_m]$ in eq. (5.5). However, the origin of this apparent disparity is straightforward. Recall that the effective twist operator is constructed from the original twist operator, as in eq. (5.2), by integrating out the $(n-1)$ copies of the QFT apart from the first copy. Certainly performing this path integral will produce an operator that depends on the quantum state since these other copies of the QFT will be in the same state as the final QFT in which $\tilde{\sigma}_n$ operates. Hence it would be interesting to test eq. (5.7) in a situation where the region under consideration was fixed but the density matrix was changed. One might observe a similar discrepancy in dimensionality: the twist operator is a $(d-2)$ -dimensional surface operator inserted along the entangling surface at the boundary of the region on which the density matrix is defined. In contrast, the modular Hamiltonian is in general a nonlocal object but certainly it involves integrals of operators over the entire region — for example, recall eq. (5.6). However, it is again clear from the path integral construction (5.2) of the effective twist operator that it is a nonlocal object with support across the entire region A .

Eq. (5.7) allows us to examine the behaviour of correlators of the twist operator with other operators in the limit $n \rightarrow 1$. Alternatively, we can consider an expansion for small $(n - 1)$ of the twist operators themselves. In particular then, the derivatives of this expression at $n = 1$ yield:

$$\partial_n \sigma_n|_{n=1} = -H_m, \quad \partial_n^2 \sigma_n|_{n=1} = H_m^2, \quad \partial_n^k \sigma_n|_{n=1} = (-)^k H_m^k. \quad (6.1)$$

Note that we are writing these expressions for the twist operators themselves, rather than the effective twist operators. To illustrate the sense in which these equalities hold, we consider applying the first derivative to one of the correlators²³ discussed in section 5,

$$\begin{aligned} \lim_{n \rightarrow 1} \partial_n \langle \sigma_n \mathcal{X} \rangle &= \lim_{n \rightarrow 1} \partial_n \langle \tilde{\sigma}_n \mathcal{X} \rangle_1 \\ &= \lim_{n \rightarrow 1} \partial_n \langle e^{-(n-1)H_m} \mathcal{X} \rangle_1 \\ &= -\langle H_m \mathcal{X} \rangle_1 \end{aligned} \quad (6.2)$$

Of course, this result for the first derivative is essentially equivalent to the recent result of [57]. Using techniques developed in [59, 60], the latter argues that evaluating correlator on a manifold with an infinitesimal conical defect along a certain codimension two surface is equivalent to the same correlator in flat space but with an extra insertion of the entanglement Hamiltonian. The correspondence of this result with the first derivative in eq. (6.1)

²³That is, we consider the correlator of the twist operator σ_n with some collection of operators \mathcal{X} , all of which act in a single copy of the QFT or are inserted on a single sheet of the n -fold covering geometry.

comes from the geometric approach to the replica trick, where one first analytically continues the background geometry to non-integer n [61] — see also the discussion in [62].²⁴ It is interesting that eq. (6.1) suggests that higher derivatives also produce a universal effect on correlators in terms of insertions of higher powers of the modular Hamiltonian.

To close, we return to the question of the modular Hamiltonian for regions with several simply-connected components. Of course, this discussion is closely related to the work appearing in [45, 64, 65]. Recall that in section 5, an explicit expression for the effective twist operator for single spherical region was constructed by combining eqs. (5.6) and (5.7). Naïvely, one may think this result can be used to give information about the entanglement for multiple spherical regions, at least in the limit where the separations between the various regions are large compared to the size of each sphere. For example, one might think that in evaluating the corresponding Rényi entropy, the following provides a good approximation

$$Z_n(N \text{ spherical regions}) \simeq \left\langle \prod_{i=1}^N \tilde{\sigma}_{n,i} \right\rangle_1 = \left\langle \prod_{i=1}^N \exp [-(n-1)H_{m,i}] \right\rangle_1, \quad (6.3)$$

where $H_{m,i}$ denotes the modular Hamiltonian (5.6) for the individual spherical region i . The basic assumption in writing eq. (6.3) is that at large separations, the full modular Hamiltonian $H_m^{(N)}$ for the N spherical regions is well approximated by the sum of the modular Hamiltonians derived for each of the individual regions, i.e., $H_m^{(N)} \simeq \sum_{i=1}^N H_{m,i}$.

Strictly speaking, this split of the modular Hamiltonian into a sum of terms for the individual regions cannot be true because it would imply that the mutual information between these regions vanishes. However, in fact, it is not even a reasonable approximation since it misses important leading order contributions. For example, in eq. (5.16), we found the leading behaviour in the correlator $\langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle$ to two spherical regions in a general CFT decayed as $(R_1 R_2 / r^2)^d$. However, it has been shown that the corresponding decay for $\langle \sigma_{n,1} \sigma_{n,2} \rangle$ is given by $(R_1 R_2 / r^2)^{d-2}$ for a free massless scalar field [64, 66–68]²⁵ and by $(R_1 R_2 / r^2)^{d-1}$ for a free massless fermion [66, 67]. These results explicitly demonstrate that in general the leading long-distance behaviour in correlator of two (spherical) twist operators is not controlled by the stress tensor, but rather by operators with a lower conformal dimension. In particular, this arises if the CFT contains primary operators \mathcal{O}_Δ with dimension $\Delta \leq d/2$ [55]. These operators can appear in the OPE of the twist operators in terms of the form $\sum_{i \neq j} \mathcal{O}_{\Delta,i} \otimes \mathcal{O}_{\Delta,j}$ where i, j indicate the copy of the CFT. If $\Delta \leq d/2$, these terms will dominate over the stress tensor in contributing to the long-distance decay in the correlator of the twist operators. Since the individual operators $\mathcal{O}_{\Delta,i}$ appear in different copies of the CFT in these terms, these contributions are not captured by the correlator of the effective twist operator $\langle \tilde{\sigma}_{n,1} \tilde{\sigma}_{n,2} \rangle$. The implicit assumption here is that $\langle \sigma_n \sigma_n \rangle$ is the standard correlator as would appear in the evaluation of $\text{Tr}[\rho_A^n]$ for a region with two separated components. That is, both twist operators are interlacing the

²⁴We note that this continuation is only straightforward for cases where there is a rotational symmetry about the entangling surfaces but that recent progress [63] also allows one to consider infinitesimal variations of the geometry around $n = 1$ for general entangling surfaces.

²⁵These references investigate the decay in the mutual information but their results imply an analogous decay in the correlator of the corresponding twist operators.

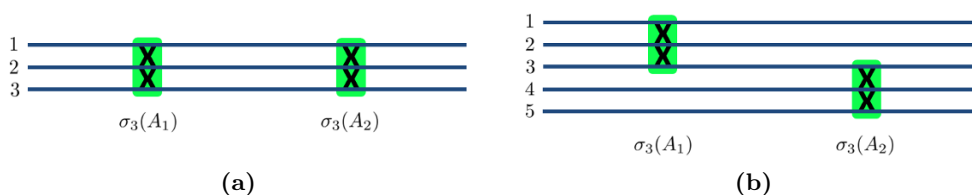


Figure 2. (Colour online) (a) The standard three-fold geometry which would appear in the evaluation of $\text{Tr}[\rho_A^3]$ for a region with two well separated components, A_1 and A_2 . Alternatively, each component A_i is delineated by a twist operator $\sigma_3(A_i)$, which connects the same three copies of the CFT. (b) An unconventional five-fold geometry where there is a cut through A_1 on sheets 1, 2 and 3 while the cut at A_2 runs through sheets 3, 4 and 5. The corresponding twist operators only have the third copy of the CFT in common.

same n copies of the CFT. One could consider more ‘exotic’ correlators where the two twist operators each connect n copies of the CFT but only one of these copies is common to both σ_n . In this situation, we would in fact expect that eq. (5.16) properly describes the leading long-distance behaviour of the correlator. These two different situations are illustrated in figure 2. It would be interesting to test these ideas by explicitly evaluating an example of the latter correlator in, e.g., a free field theory.

As an extension of the above discussion, we would like to consider the appearance of so-called ‘teleportation’ terms in the modular Hamiltonian of regions with several separated components. As a specific example, the modular Hamiltonian for a massless fermion in two dimensions for a region consisting of several disjoint intervals was constructed in [69] and was observed to contain ‘teleportation’ terms, which connected the fermion field at points within the causal diamonds of separate intervals. While the modular Hamiltonian is expected to be nonlocal in general, we would like to argue that this nonlocality will generically extend to the appearance of such ‘teleportation’ contributions. In fact, our discussion above implies that the long-distance behaviour of the correlator $\langle \sigma_n \sigma_n \rangle$ is controlled by such teleportation terms when the theory contains operators with $\Delta \leq d/2$. However, let us generalize these discussions as follows: first, we observe that eq. (6.1) indicates that the full modular Hamiltonian of the multicomponent region will be given by

$$H_m^{(N)} = -\partial_n \sum_i \sigma_n(A_i)|_{n=1} \tag{6.4}$$

where $\sigma_n(A_i)$ indicates the twist operator enclosing the component A_i . Next, let us consider the limit of very large separations between the different components, so that each of the individual $\sigma_n(A_i)$ can be represented by their OPE expansion. Further as discussed above and in section 4, the OPE of these individual twist operators will typically contain terms involving operators in several different copies of the underlying QFT. However, eq. (6.4) should only be considered as an equality in correlators within a single copy of the QFT. That is, implicitly the last $(n-1)$ copies of the QFT are trivially integrated out on the right-hand side of eq. (6.4) leaving an operator within the first copy. However, this implicit step of performing the path integral for the other copies of the QFT will convert

the contributions which connect multiple copies of the QFT in the individual OPE's into teleportation terms in the full modular Hamiltonian of the multi-component region. Of course, the appearance of such teleportation contribution is perhaps not very surprising as they simply reflect the fact that the density matrix ρ_A encodes nontrivial correlations between the different regions A_i . What is perhaps surprising in the two-dimensional example considered in [69] is that the modular Hamiltonian *is* local apart from these teleportation terms. It would be interesting to see if this behaviour extends higher dimensional CFT's for regions including several spherical components.

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A A useful integral

In this appendix, we evaluate the integral defined in eq. (2.43), which was essential for the computation of the second derivative of the scaling dimension

$$I = \prod_{i=1}^2 \left[\int d\Omega_i \int_0^1 dx_i x_i^{d-2} (1-x_i^2) \right] \frac{1}{|\vec{x}_1 - \vec{x}_2|^d} . \quad (\text{A.1})$$

First we note that the following relation holds within dimensional regularization

$$\frac{1}{|\vec{x}_1 - \vec{x}_2|^{d-1-2\alpha}} = \frac{4^\alpha \pi^{(d-1)/2} \Gamma(\alpha)}{\Gamma((d-1)/2 - \alpha)} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)}}{(p^2)^\alpha} \quad \text{with } \vec{x}_{1,2} \in R^{d-1} . \quad (\text{A.2})$$

Now, we replace $1/|\vec{x}_1 - \vec{x}_2|^d$ in the original expression for I by the above Fourier transform with $\alpha = -1/2$. This substitution decouples the integrals for $i = 1$ and 2 in eq. (A.1) and writing $\vec{p} \cdot \vec{x}_i = p x_i \cos \theta_i$, we can perform the polar integrals using the following identity

$$\int_0^\pi d\theta e^{\pm i p x \cos \theta} \sin^{d-3} \theta d\theta = \sqrt{\pi} \left(\frac{2}{px} \right)^{\frac{d-3}{2}} \Gamma\left(\frac{d-2}{2}\right) J_{\frac{d-3}{2}}(px) . \quad (\text{A.3})$$

As a result, we obtain

$$I = -\frac{\pi^{d/2}}{\Gamma(d/2)} \int \frac{d^{d-1}p}{p^{d-4}} \left[\int_0^1 dx x^{\frac{d-1}{2}} (1-x^2) J_{\frac{d-3}{2}}(px) \right]^2 . \quad (\text{A.4})$$

The integral in the square brackets can be readily evaluated

$$\int_0^1 dx x^{\frac{d-1}{2}} (1-x^2) J_{\frac{d-3}{2}}(px) = \frac{2}{p^2} J_{\frac{d+1}{2}}(p), \tag{A.5}$$

while the final integral over p finally yields

$$I = -\frac{2^{d+3}\pi^{d-2}}{d(d+2)\Gamma(d-1)}. \tag{A.6}$$

Note that there is no contradiction between the sign of I and the fact that the integrand in eq. (A.1) is positive definite. Indeed, the integral in eq. (A.1) is power law divergent, and we implicitly utilize dimensional regularization to evaluate it here. Such regularization amounts to dropping the divergent term which is positive in this case whereas the subleading correction happens to be negative.

B Free fields on $S^1 \times H^{d-1}$

In this appendix, we use heat kernel techniques to evaluate the scaling dimension h_n in the case of massless Dirac fermion ψ and conformally coupled scalar ϕ on the Euclidean manifold $\mathcal{M} = S^1 \times H^{d-1}$, where S^1 corresponds to Euclidean time compactified on a circle with period β . Their actions are given respectively by

$$S(\phi) = \int_{\mathcal{M}} d^d x \sqrt{g} \frac{1}{2} \left((\nabla\phi)^2 + \frac{d-2}{4(d-1)} \mathcal{R} \phi^2 \right), \tag{B.1}$$

$$S(\psi) = \int_{\mathcal{M}} d^d x \sqrt{g} \bar{\psi} \not{\nabla} \psi, \tag{B.2}$$

where \mathcal{R} is the Ricci scalar of the background geometry and we explain our spinor conventions in what follows.

For either of the above classes of theories, the partition function is Gaussian and can be exactly evaluated using the heat kernel approach, e.g., [70]

$$\log Z^{(s)} = \frac{e^{i2\pi s}}{2} \int \frac{dt}{t} \text{Tr} K_{\mathcal{M}}^{(s)} e^{-tm_s^2}, \tag{B.3}$$

where $K_{\mathcal{M}}^{(s)}(t, x, y)$ with $x, y \in \mathcal{M}$ is the heat kernel of the corresponding *massless* wave operator on \mathcal{M} . The trace of the heat kernel involves taking the limit of coincident points, i.e., $y \rightarrow x$, and integrating over the remaining position x . Of course, a trace is also taken over the spinor indices in the case of the spin-1/2 field — see below. In the above expression and throughout the following, we use $s = 0$ or $1/2$ to indicate the scalar or fermion cases, respectively. Finally m_s denotes the ‘effective’ mass of the field under study. For the fermion, we have simply $m_{s=1/2} = 0$, however, given the non-minimal coupling of the scalar, we have

$$m_{s=0}^2 = -\frac{(d-2)^2}{4R^2}, \tag{B.4}$$

where we used $\mathcal{R}(H^{d-1}) = -(d-2)(d-1)/R^2$ for a hyperbolic space of radius R .

The wave operators are separable on the product manifold under consideration and hence the heat kernel on \mathcal{M} can be expressed as the product of the two individual heat kernels on S^1 and H^{d-1} , i.e.,

$$K_{\mathcal{M}}^{(s)} = K_{S^1}^{(s)} K_{H^{d-1}}^{(s)}, \tag{B.5}$$

where for brevity we have suppressed the arguments of the heat kernels here. This separation of variables is less evident in the case of the spin- $\frac{1}{2}$ field due to spinor structure of the heat kernel and we justify it later on. Given eq. (B.5), one can write

$$\text{Tr} K_{\mathcal{M}}^{(s)} = \text{Tr} K_{S^1}^{(s)} \text{Tr} K_{H^{d-1}}^{(s)}, \tag{B.6}$$

where each trace on the right-hand side involves an integration over the corresponding component of the product manifold. In the case of spin- $\frac{1}{2}$ field, there is an additional trace over spinor indices.

Finally, the partition function can be used to evaluate thermal energy density

$$\mathcal{E}(\beta) = -\frac{1}{R^{d-1}V_{\Sigma}} \frac{\partial}{\partial \beta} \log Z^{(s)}(\beta), \tag{B.7}$$

where, as in the main text, $R^{d-1}V_{\Sigma}$ is the (regulated) volume of H^{d-1} — see [17]. The energy density is an essential ingredient in the computation of the scaling dimension of the spherical twist operator using eq. (2.20).

B.1 Conformally coupled scalar

The heat kernel on a circle can be evaluated using the method of images. It is given by an infinite sum of heat kernels on an infinite line shifted by an integer times the inverse temperature, i.e., $n\beta$, with respect to each other. The latter is necessary to maintain periodic boundary conditions for scalar field on a circle. As a result, we get

$$\int_{S^1} K_{S^1}^{(1/2)}(t, x, x) = \frac{2\beta}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} e^{-\frac{k^2 \beta^2}{4t}}, \tag{B.8}$$

where the $k = 0$ term has been suppressed since it represents zero temperature limit and simply shifts the free energy by a constant.

The scalar heat kernel on the hyperbolic space can be found in a vast literature, e.g., see [71–73]

$$K_{H^{2\ell+1}}^{(0)}(t, x, y) = \frac{1}{(4\pi t)^{1/2}} \left(\frac{-1}{2\pi R^2 \sinh \rho} \frac{\partial}{\partial \rho} \right)^{\ell} e^{-\frac{\ell^2 t}{R^2} - \frac{\rho^2 R^2}{4t}}, \tag{B.9}$$

$$K_{H^{2\ell+2}}^{(0)}(t, x, y) = e^{-\frac{(2\ell+1)^2}{4R^2} t} \left(\frac{-1}{2\pi R^2 \sinh \rho} \frac{\partial}{\partial \rho} \right)^{\ell} f_{H^2}^{(0)}(\rho, t), \tag{B.10}$$

where ℓ is 0 or a positive integer, ρ is the geodesic distance between x and y measured in units of R , and

$$f_{H^2}^{(0)}(\rho, t) = \frac{\sqrt{2}R}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{\tilde{\rho} e^{-\frac{R^2 \tilde{\rho}^2}{4t}}}{\sqrt{\cosh \tilde{\rho} - \cosh \rho}} d\tilde{\rho}. \tag{B.11}$$

We now turn to consider even and odd d separately.

Even d

Let us assume that $d = 2\ell + 2$ and take the limit of coincident points in eq. (B.9), then $K_{H^{2\ell+1}}^{(0)}(t, x, x)$ takes the following general form [74]

$$K_{H^{2\ell+1}}^{(0)}(t, x, x) = \frac{P_{\ell-1}^{(0)}(t/R^2)}{(4\pi t)^{\ell+1/2}} e^{-\frac{\ell^2 t}{R^2}} . \tag{B.12}$$

From (B.9), it follows that for $\ell = 0$, $P_{-1}^{(0)}(x) = 1$ while for $\ell > 0$, $P_{\ell-1}^{(0)}(x)$ is polynomial of degree $\ell - 1$ with rational coefficients

$$P_{\ell-1}^{(0)}(x) = \sum_{j=0}^{\ell-1} a_{j,\ell-1}^{(0)} x^j . \tag{B.13}$$

For example, the first few polynomials are given by

$$\begin{aligned} P_0^{(0)}(x) &= 1, \\ P_1^{(0)}(x) &= 1 + \frac{2}{3}x, \\ P_2^{(0)}(x) &= 1 + 2x + \frac{16}{15}x^2, \\ P_3^{(0)}(x) &= 1 + 4x + \frac{28}{5}x^2 + \frac{96}{35}x^3, \\ P_4^{(0)}(x) &= 1 + \frac{20}{3}x + \frac{52}{3}x^2 + \frac{1312}{63}x^3 + \frac{1024}{105}x^4, \\ P_5^{(0)}(x) &= 1 + 10x + \frac{124}{3}x^2 + \frac{5560}{63}x^3 + \frac{30656}{315}x^4 + \frac{10240}{231}x^5 . \end{aligned} \tag{B.14}$$

As one may surmise from these examples, $a_{0,\ell-1}^{(0)} \equiv 1$ for $\ell \geq 0$.

Substituting eqs. (B.8) and (B.12) into eqs. (B.3) and (B.6), yields

$$\log Z^{(0)}(\beta) = \frac{V_\Sigma \beta^{1-d}}{\pi^{d/2}} \sum_{j=0}^{(d-4)/2} a_{j,\ell-1}^{(0)} \left(\frac{\beta}{2R}\right)^{2j} \Gamma\left(\frac{d}{2} - j\right) \zeta(d - 2j), \tag{B.15}$$

Finally using eq. (B.7), the scaling dimension (2.20) takes the following form

$$h_n = \frac{(2\pi)^{1-d}}{d-1} \sum_{j=0}^{(d-4)/2} a_{j,\ell-1}^{(0)} (2j - d + 1) \pi^{2j-d/2} (n^{2j-d+1} - n) \Gamma\left(\frac{d}{2} - j\right) \zeta(d - 2j). \tag{B.16}$$

Differentiating this expression with respect to n and comparing with eq. (2.33), yields

$$C_T = \frac{d}{d-1} \frac{2}{\pi^d \Omega_{d+2}} \sum_{j=0}^{(d-4)/2} a_{j,\ell-1}^{(0)} (2j - d + 1)(2j - d) \pi^{2j-d/2+1} \Gamma\left(\frac{d}{2} - j\right) \zeta(d - 2j). \tag{B.17}$$

However, C_T is also given by eq. (3.23). The two results agree provided that

$$1 = \frac{2 \Omega_{d-1}^2}{\pi^{3d/2} \Omega_{d+2}} \sum_{j=0}^{(d-4)/2} a_{j,\ell-1}^{(0)} (2j - d + 1)(2j - d) \pi^{2j+1} \Gamma\left(\frac{d}{2} - j\right) \zeta(d - 2j). \tag{B.18}$$

Using eq. (B.14) we verified that this identity holds up to $d = 14$.

$d = 3$

While heat kernel computation for odd d is straightforward, it is much more tedious than for even d since trace of the heat kernel over even dimensional hyperbolic space cannot be represented in terms of elementary functions. Therefore, as an example, we consider $d = 3$ only. Generalizations to higher odd dimensions are straightforward. Combining eqs. (B.3), (B.6), (B.8) and (B.10), yields

$$\log Z^{(0)}(\beta) = \frac{V_\Sigma \beta R^3}{16\pi^2} \sum_{k=1}^{\infty} \int_0^\infty d\tilde{\rho} \frac{\tilde{\rho}}{\sinh \frac{\tilde{\rho}}{2}} \int_0^\infty \frac{dt}{t^3} e^{-\frac{R^2 \tilde{\rho}^2}{4t} - \frac{k^2 \beta^2}{4t}} . \quad (\text{B.19})$$

Carrying out integration over t and summation over k , yields

$$\log Z^{(0)}(\beta) = \frac{V_\Sigma}{4\pi^2 \beta R} \int_0^\infty \frac{d\tilde{\rho}}{\sinh \frac{\tilde{\rho}}{2}} \frac{\pi^2 R^2 \rho^2 + \beta \sinh^2 \frac{\pi R \tilde{\rho}}{\beta} (\pi R \tilde{\rho} \coth \frac{\pi R \tilde{\rho}}{\beta} - 2\beta)}{\tilde{\rho}^3 \sinh^2 \frac{\pi R \tilde{\rho}}{\beta}} . \quad (\text{B.20})$$

Note that this integral converges. Combining now eq. (2.20) with eq. (B.7), we obtain

$$\partial_n h_n|_{n=1} = \frac{2\pi^2 R^2}{V_\Sigma} \frac{\partial^2}{\partial \beta^2} \log Z^{(1/2)}(\beta) \Big|_{\beta=2\pi R} , \quad (\text{B.21})$$

Hence,

$$\partial_n h_n|_{n=1} = \frac{\pi}{256} , \quad (\text{B.22})$$

Comparing this result with eq. (2.33) leads to $C_T = \frac{3}{32\pi^2}$ in $d = 3$. The latter agrees with eq. (3.23).

$\partial_n^2 h_n|_{n=1}$

It is natural to use the heat kernel results to also consider the second derivative of the scaling dimension. In particular, using eq. (B.16), it is straightforward to evaluate $h_{n,2}$ for a conformally coupled scalar field. When we substitute the corresponding values for \hat{a} , \hat{b} and \hat{c} (see eq. (3.24)) into our general formula (2.45) for the second derivative, we find the expression given in eq. (3.26). Unfortunately, it turns out that the two expressions only agree for $d = 2$ and they differ in higher dimensions. We do not have a clear understanding of this discrepancy. We find it very challenging since eq. (2.45) successfully passed all of our holographic tests and further we show below that it agrees with heat kernel computations for free massless fermions.

B.2 Dirac fermion

We start from reviewing our spinor notation. The spinors are associated with an orthonormal frame, e_a^μ , on $\mathcal{M} = S^1 \times H^{d-1}$ satisfying

$$e_a^\mu e_b^\nu g_{\mu\nu} = \delta_{ab} . \quad (\text{B.23})$$

The Clifford algebra in the orthonormal frame is generated by d matrices γ^a , satisfying the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} . \quad (\text{B.24})$$

The dimension of these matrices is $2^{\lfloor \frac{d}{2} \rfloor}$, and the associated $d(d-1)/2$ generators of $\text{SO}(d)$ are

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]. \quad (\text{B.25})$$

They satisfy the standard $\text{SO}(d)$ commutation rules

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{bc}\sigma^{ad} - \delta^{ac}\sigma^{bd} - \delta^{bd}\sigma^{ac} + \delta^{ad}\sigma^{bc}, \quad (\text{B.26})$$

and the commutator of σ^{ab} with γ^c is

$$[\sigma^{ab}, \gamma^c] = \delta^{bc}\gamma^a - \delta^{ac}\gamma^b. \quad (\text{B.27})$$

The covariant derivative of a spinor may be written in terms of e_a^μ as follows

$$\nabla_a = e_a^\mu \nabla_\mu, \quad \nabla_\mu = \partial_\mu + \frac{1}{2}\sigma^{bc}\omega_{\mu bc}, \quad \omega_{\mu bc} = e_b^\nu(\partial_\mu e_{c\nu} - \Gamma_{\nu\mu}^\alpha e_{c\alpha}). \quad (\text{B.28})$$

It satisfies the following anticommutation relations [75]

$$[\nabla_a, \nabla_b]\psi = -\frac{1}{2}R_{abcd}\sigma^{cd}\psi. \quad (\text{B.29})$$

In the case of free massless fermions on \mathcal{M} , we have

$$Z^{(1/2)}(\beta) = \det(\not{\nabla}), \quad (\text{B.30})$$

where we used eq. (B.2) and $\not{\nabla} = \gamma^a \nabla_a$. Since the γ^a matrices are covariantly constant, one can use eqs. (B.24) and (B.29) to verify the following identity

$$\not{\nabla}^2 = (\gamma^a \nabla_a)^2 = \delta^{ab}\nabla_a \nabla_b - \frac{R}{4}. \quad (\text{B.31})$$

Hence, the partition function for free massless fermions can be written in the following form

$$\log Z^{(1/2)} = \frac{1}{2} \log \det(\not{\nabla} \cdot \not{\nabla}^\dagger) = \frac{1}{2} \text{Tr} \log(-\not{\nabla}^2), \quad (\text{B.32})$$

and can be evaluated using the heat kernel approach (B.3) where $K_{\mathcal{M}}^{(1/2)}$ is associated with operator $(-\not{\nabla}^2)$. Note that in the case of $\mathcal{M} = S^1 \times H^{d-1}$, $\nabla_0 = \partial_\tau$ and therefore from eq. (B.31) $\not{\nabla}^2 = \partial_\tau^2 + \not{\nabla}^2 \Big|_{H^{d-1}}$. As a result, one can separate the Euclidean time from the coordinates on H^{d-1} and get eqs. (B.5) and (B.6).

$K_{S^1}^{(1/2)}$ can be readily evaluated. Similarly to the scalar case, it is given by an infinite sum of heat kernels on an infinite line shifted by an integer times the inverse temperature, $n\beta$, with respect to each other and weighted by $(-1)^n$ to maintain the antiperiodic boundary conditions for the fermions on a circle

$$\int_{S^1} K_{S^1}^{(1/2)}(t, x, x) = \frac{2\beta}{\sqrt{4\pi t}} \sum_{k=1}^{\infty} (-1)^k e^{-\frac{k^2\beta^2}{4t}} \mathbb{I}_{\lfloor \frac{d}{2} \rfloor}. \quad (\text{B.33})$$

where $\mathbb{I}_{\lfloor \frac{d}{2} \rfloor}$ is the unit matrix in $2^{\lfloor \frac{d}{2} \rfloor}$ dimensions and the $k=0$ term has been dropped from the above expression, as it corresponds to $\beta \rightarrow \infty$ (zero temperature) limit and simply shifts the free energy by a constant.

Furthermore, if $d = 2\ell + 2$ with $\ell = 0, 1, 2, \dots$, i.e., odd dimensional hyperbolic space, then heat kernel is given by [76]

$$K_{H^{2\ell+1}}^{(1/2)}(t, x, y) = U(x, y) \cosh \frac{\rho}{2} \left(\frac{-1}{2\pi R^2} \frac{\partial}{\partial \cosh \rho} \right)^\ell \left(\cosh \frac{\rho}{2} \right)^{-1} \frac{e^{-\frac{\rho^2 R^2}{4t}}}{(4\pi t)^{1/2}}, \quad (\text{B.34})$$

where ρ is the geodesic distance between x and y in units of R and $U(x, y)$ is the parallel spinor propagator from x to y .

On the other hand, for odd $d = 2\ell + 3$ with $\ell = 0, 1, 2, \dots$, we have [76]

$$K_{H^{2\ell+2}}^{(1/2)}(t, x, y) = U(x, y) \cosh \frac{\rho}{2} \left(\frac{-1}{2\pi R^2} \frac{\partial}{\partial \cosh \rho} \right)^\ell \left(\cosh \frac{\rho}{2} \right)^{-1} f_2(\rho, t), \quad (\text{B.35})$$

where

$$f_2(\rho, t) = \frac{R\sqrt{2} \left(\cosh \frac{\rho}{2} \right)^{-1}}{(4\pi t)^{3/2}} \int_\rho^\infty \frac{\tilde{\rho} \cosh \frac{\tilde{\rho}}{2} e^{-\frac{R^2 \tilde{\rho}^2}{4t}}}{\sqrt{\cosh \tilde{\rho} - \cosh \rho}} d\tilde{\rho}. \quad (\text{B.36})$$

The structure of $U(x, y)$ is not important for our needs, as we are interested in the limit of coincident points in which case $U(x, y)$ reduces to an identity matrix on the $2^{\lfloor \frac{d}{2} \rfloor}$ -dimensional spinor space. We should note here that according to [76], the dimension of the spinor space associated with eq. (B.34) is twice smaller and thus a modification of eq. (B.34) might be expected. However, the same reasoning presented in [76] which leads to eq. (B.34) can be equally well applied to the case considered here without necessity to introduce any changes.

We turn now to use the above results to evaluate the scaling dimension h_n in various dimensions. We consider separately even and odd d .

Even d

It follows from eq. (B.34) that for $d = 2\ell + 2$, $K_{H^{d-1}}(x, x, t)$ takes the following general form

$$K_{H^{d-1}}^{(1/2)}(t, x, x) = \frac{P_\ell^{(1/2)}(t/R^2)}{(4\pi t)^{\ell+1/2}} \mathbb{I}_{\ell+1}, \quad (\text{B.37})$$

where $\mathbb{I}_{\ell+1}$ is an identity matrix on a $2^{\ell+1}$ -dimensional spinor space, and $P_\ell^{(1/2)}(x)$ is a polynomial of degree ℓ with rational coefficients

$$P_\ell(x) = \sum_{j=0}^{\ell} a_{j,\ell}^{(1/2)} x^j. \quad (\text{B.38})$$

In particular $a_{0,\ell}^{(1/2)} = 1$, and

$$\begin{aligned} P_0^{(1/2)}(x) &= 1, \\ P_1^{(1/2)}(x) &= 1 + \frac{1}{2}x, \\ P_2^{(1/2)}(x) &= 1 + \frac{5}{3}x + \frac{3}{4}x^2, \end{aligned}$$

$$\begin{aligned}
 P_3^{(1/2)}(x) &= 1 + \frac{7}{2}x + \frac{259}{60}x^2 + \frac{15}{8}x^3, \\
 P_4^{(1/2)}(x) &= 1 + 6x + \frac{141}{10}x^2 + \frac{3229}{210}x^3 + \frac{105}{16}x^4, \\
 P_5^{(1/2)}(x) &= 1 + \frac{55}{6}x + \frac{209}{6}x^2 + \frac{17281}{252}x^3 + \frac{117469}{1680}x^4 + \frac{945}{32}x^5.
 \end{aligned}
 \tag{B.39}$$

Substituting eqs. (B.33) and (B.37) into eq. (B.6) and then into eq. (B.3), yields

$$\log Z^{(1/2)}(\beta) = V_\Sigma \beta^{1-d} \frac{2^{d/2}}{\pi^{d/2}} \sum_{j=0}^{(d-2)/2} a_{j,\ell}^{(1/2)} \left(\frac{\beta}{2R}\right)^{2j} (1-2^{2j+1-d}) \Gamma\left(\frac{d}{2}-j\right) \zeta(d-2j),
 \tag{B.40}$$

Using now eq. (B.7), the scaling dimension (2.20) can be evaluated as

$$h_n = \frac{(2\pi)^{1-d/2}}{d-1} \sum_{j=0}^{(d-2)/2} a_{j,\ell}^{(1/2)} (2j-d+1) \pi^{2j-d} (n^{2j-d+1} - n) (1-2^{2j-d+1}) \Gamma\left(\frac{d}{2}-j\right) \zeta(d-2j).
 \tag{B.41}$$

Differentiating this expression with respect to n and comparing with (2.33), yields

$$C_T = \frac{d}{d-1} \frac{2^{d/2+1}}{\pi^{d/2} \Omega_{d+2}} \sum_{j=0}^{(d-2)/2} a_{j,\ell}^{(1/2)} (2j-d+1)(2j-d) \pi^{2j-d+1} (1-2^{2j-d+1}) \Gamma\left(\frac{d}{2}-j\right) \zeta(d-2j).
 \tag{B.42}$$

On the other hand, according to eq. (3.23)

$$C_T = \frac{d}{2} \frac{2^{\lfloor \frac{d}{2} \rfloor}}{\Omega_{d-1}^2},
 \tag{B.43}$$

where $\lfloor d/2 \rfloor$ is the integer part of $d/2$. Of course, when d is even $\lfloor d/2 \rfloor = d/2$. These two expressions for C_T agree provided that

$$1 = \frac{4}{d-1} \frac{\Omega_{d-1}^2}{\pi^{d/2} \Omega_{d+2}} \sum_{j=0}^{(d-2)/2} a_{j,\ell}^{(1/2)} (2j-d+1)(2j-d) \pi^{2j-d+1} (1-2^{2j-d+1}) \Gamma\left(\frac{d}{2}-j\right) \zeta(d-2j).
 \tag{B.44}$$

Using eq. (B.39), we explicitly verified that this identity indeed holds up to $d = 12$.

Let us now consider the second derivative of the scaling dimension given by eq. (B.41)

$$\partial_n^2 h_n|_{n=1} = \frac{(2\pi)^{1-d/2}}{d-1} \sum_{j=0}^{(d-2)/2} a_{j,\ell}^{(1/2)} (2j-d)(2j-d+1)^2 \pi^{2j-d} (1-2^{2j-d+1}) \Gamma\left(\frac{d}{2}-j\right) \zeta(d-2j).
 \tag{B.45}$$

Using eq. (B.39), we can then evaluate the second derivative in various dimensions. Table 1 summarizes final results up to $d = 12$. These results precisely match the expected expression (3.26) for $h_{n,2}$, which was derived by substituting the free field values for \hat{a} , \hat{b} and \hat{c} in eq. (3.24) into our general formula (2.45).

d	2	4	6	8	10	12
$h_{n,2}$	$-\frac{1}{6}$	$-\frac{13}{180\pi}$	$-\frac{16}{315\pi^2}$	$-\frac{59}{1050\pi^3}$	$-\frac{1504}{17325\pi^4}$	$-\frac{10960}{63063\pi^5}$

Table 1. $\partial_n^2 h_n|_{n=1}$ for fermions in various dimensions.

d=3

In the case of odd d the computation is more sophisticated than for even d since trace of the heat kernel over H^{d-1} cannot be represented in terms of elementary functions as in eq. (B.37). Therefore we illustrate a special case $d = 3$ only. Generalizations to higher odd dimensions are straightforward. Combining eqs. (B.3), (B.6), (B.33) and (B.35), yields

$$\log Z^{(1/2)}(\beta) = -\frac{V_\Sigma \beta R^3}{8\pi^2} \sum_{k=1}^{\infty} (-1)^k \int_0^\infty d\tilde{\rho} \tilde{\rho} \coth \frac{\tilde{\rho}}{2} \int_0^\infty \frac{dt}{t^3} e^{-\frac{R^2 \tilde{\rho}^2}{4t} - \frac{k^2 \beta^2}{4t}} . \quad (\text{B.46})$$

Carrying out integration over t and then summation over k gives

$$\log Z^{(1/2)}(\beta) = \frac{V_\Sigma}{2\pi^2 \beta R} \int_0^\infty d\tilde{\rho} \coth \frac{\tilde{\rho}}{2} \frac{2\beta^2 \sinh \frac{\pi R \tilde{\rho}}{\beta} - \pi R \tilde{\rho} (\beta + \pi R \tilde{\rho} \coth \frac{\pi R \tilde{\rho}}{\beta})}{\tilde{\rho}^3 \sinh \frac{\pi R \tilde{\rho}}{\beta}} . \quad (\text{B.47})$$

Note that the integral converges. Combining eq. (2.20) with eq. (B.7), yields

$$\begin{aligned} \partial_n h_n|_{n=1} &= \frac{2\pi^2 R^2}{V_\Sigma} \frac{\partial^2}{\partial \beta^2} \log Z^{(1/2)}(\beta) \Big|_{\beta=2\pi R} , \\ \partial_n^2 h_n|_{n=1} &= \frac{2\pi^2 R^2}{V_\Sigma} \left(\beta \frac{\partial^3}{\partial \beta^3} + 2 \frac{\partial^2}{\partial \beta^2} \right) \log Z^{(1/2)}(\beta) \Big|_{\beta=2\pi R} . \end{aligned} \quad (\text{B.48})$$

Hence, in our case we obtain

$$\begin{aligned} \partial_n h_n|_{n=1} &= \frac{\pi}{128} , \\ \partial_n^2 h_n|_{n=1} &= -\frac{13\pi}{960} . \end{aligned} \quad (\text{B.49})$$

These two results can be compared to eqs. (3.25) and (3.26) with $d = 3$ and again we find perfect agreement.

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