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Twisted Alexander polynomials for irreducible $SL(2, \mathbb{C})$-representations of torus knots

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Abstract. We prove that the twisted Alexander polynomial of a torus knot with an irreducible $SL(2, \mathbb{C})$-representation is locally constant. In the case of a $(2, q)$ torus knot, we can give an explicit formula for the twisted Alexander polynomial and deduce Hirasawa-Murasugi's formula for the total twisted Alexander polynomial. We also give examples which address a mis-statement in a paper of Silver and Williams.

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1. Introduction

Let K be a knot in the 3-sphere S^3 and $G(K) = \pi_1(S^3 - K)$ its knot group. In this paper, we consider the twisted Alexander polynomial $\Delta_{K,\rho}(t)$, which is defined as a rational expression over $\mathbb{C}$ with one variable t, for a knot K associated with an irreducible representation $\rho : G(K) \to SL(2, \mathbb{C})$. The twisted Alexander polynomial for a knot with a linear representation was originally introduced by Lin in [9]. It was generalized and developed by Wada in [12] for finitely presentable groups which include link groups. If we put $t = 1$, it is known that $\Delta_{K,\rho}(1)$ equals the Reidemeister torsion of the exterior of a knot K for the same representation ρ, under the acyclicity condition [6].

When ρ is a nonabelian representation, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ becomes a Laurent polynomial over $\mathbb{C}$ (see [7]). Since an irreducible representation is nonabelian, $\Delta_{K,\rho}(t)$ is a Laurent polynomial and all the coefficients of $\Delta_{K,\rho}(t)$ are complex valued functions on the space of irreducible representations in $SL(2, \mathbb{C})$. We then obtain the following.

Theorem 1.1. *If K is a torus knot, then every coefficient of $\Delta_{K,\rho}(t)$ is a locally constant function, that is, a constant function on each connected component of the space of irreducible $SL(2, \mathbb{C})$-representations.*

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Remark 1.2. (1) Johnson [3] proved that the Reidemeister torsion of a torus knot is a locally constant function on the space of irreducible $SL(2, \mathbb{C})$-representations. More generally, it is known that the Reidemeister torsion is locally constant for a Seifert fibered manifold [5].

(2) Kitayama observed in [8, Example 5.11] that every coefficient of the twisted Alexander polynomial of a torus knot is locally constant for $SU(2)$-representations.

This paper is organized as follows. In the next section, we review the definition for the twisted Alexander polynomial associated with $SL(2, \mathbb{C})$-representations. In Section 3, we describe the representation space of a torus knot (Proposition 3.7) according to Johnson's lecture note [3]. In the last section, we give two kinds of proofs for Theorem 1.1 and an explicit formula for the twisted Alexander polynomial for $(2, q)$ torus knots (Theorem 4.2). We also discuss the total twisted Alexander polynomial, due to Silver-Williams [11]. Hirasawa-Murasugi's formula [2] for the total twisted Alexander polynomial corresponding to parabolic representations of a $(2, q)$ torus knot is shown very easily (Corollary 4.5). In particular, we present an example for the twisted Alexander polynomial which cannot be written as a product of cyclotomic polynomials (Example 4.7). The example addresses a mis-statement in a paper of Silver and Williams [11].

We shall give a self-contained description through the paper, so we determine the representation space of a torus knot in detail (although it seems to be known to experts).

2. Twisted Alexander polynomials

In this section, we review the definition of $\Delta_{K,\rho}(t)$ for an $SL(2, \mathbb{C})$-representation ρ. There are several versions for the twisted Alexander polynomial, but in this paper we adopt the one due to Wada [12].

For a given knot K, we fix a presentation of its knot group $G(K)$:

$$P = \langle x_1, \dots, x_n \,|\, u_1, \dots, u_{n-1} \rangle.$$

We may assume its deficiency is one, but it might not be a Wirtinger presentation. We take the abelianization homomorphism $\alpha : G(K) \to \mathbb{Z} = \langle t \rangle$.

Given representations $\rho : G(K) \to SL(2, \mathbb{C})$ and $\alpha : G(K) \to \langle t \rangle$, they naturally induce two ring homomorphisms $\widetilde{\rho}$ and $\widetilde{\alpha}$ from the group ring $\mathbb{Z}G(K)$ to $M(2, \mathbb{C})$ and $\mathbb{Z}[t, t^{-1}]$ respectively, where $M(2, \mathbb{C})$ is the matrix algebra of 2×2 matrices over $\mathbb{C}$. Then $\widetilde{\rho} \otimes \widetilde{\alpha}$ defines a ring homomorphism $\mathbb{Z}G(K) \to M\left(2, \mathbb{C}[t, t^{-1}]\right)$. Let F_n denote the free group on generators $x_1, \dots, x_n$ and

$$\Phi : \mathbb{Z}F_n \to M\left(2, \mathbb{C}[t, t^{-1}]\right)$$

the composite of the surjection $\mathbb{Z}F_n \to \mathbb{Z}G(K)$ induced by the presentation P and the ring homomorphism $\widetilde{\rho} \otimes \widetilde{\alpha}$.

Let us consider the $(n-1) \times n$ matrix A whose (i, j) component is the 2×2 matrix

$$\Phi\left(\frac{\partial u_i}{\partial x_j}\right) \in M\left(2, \mathbb{C}[t, t^{-1}]\right),$$

where $\frac{\partial}{\partial x_j}$ $(j = 1, \ldots, n)$ denotes the free differential calculus (see [1]). This matrix A is called the Alexander matrix of the presentation P associated with ρ.

For $1 \leq j \leq n$, let us denote by A_j the $(n-1) \times (n-1)$ matrix obtained from A by removing the jth column. We regard A_j as a $2(n-1) \times 2(n-1)$ matrix with coefficients in $\mathbb{C}[t, t^{-1}]$.

The following two lemmas are the foundations of the definition for the twisted Alexander polynomial (see [12] for the proof).

Lemma 2.1. $\det \Phi(x_j - 1) \neq 0$ *for some* j.

Lemma 2.2. $\det A_j \det \Phi(x_k - 1) = \det A_k \det \Phi(x_j - 1)$ *for* $1 \leq j < k \leq n$.

From the above two lemmas, we can define the twisted Alexander polynomial of $G(K)$ associated with the representation $\rho : G(K) \to SL(2, \mathbb{C})$ to be a rational expression

$$\Delta_{K,\rho}(t) = \frac{\det A_j}{\det \Phi(x_j - 1)}$$

provided $\det \Phi(x_j - 1) \neq 0$.

Remark 2.3. Up to a factor of t^k $(k \in \mathbb{Z})$, this is an invariant of $G(K)$ with ρ (see [12, Theorem 1]). Namely, it does not depend on the choices of a presentation P. Hence we can consider it as a knot invariant.

In general, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ depends on ρ. However the following proposition is known.

Proposition 2.4. *If* ρ *and* ρ' *are conjugate as an* $SL(2, \mathbb{C})$*-representation, then* $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$.

Here a representation ρ is conjugate to a representation ρ' if there exists $S \in SL(2, \mathbb{C})$ such that $\rho(g) = S\rho'(g)S^{-1}$ for any $g \in G(K)$.

Under a generic assumption on ρ, the twisted Alexander polynomial becomes a Laurent polynomial (see [7, Theorem 3.1]).

Proposition 2.5. *If* $\rho : G(K) \to SL(2, \mathbb{C})$ *is a nonabelian representation, then* $\Delta_{K,\rho}(t)$ *is a Laurent polynomial with coefficients in* $\mathbb{C}$.

In this paper, we consider that each coefficient of $\Delta_{K,\rho}(t)$ is a complex valued function on the space of conjugacy classes of irreducible $SL(2, \mathbb{C})$-representations.

3. Representation space of a (p, q) torus knot

In this section, we recall a parametrization of the space of conjugacy classes of irreducible $SL(2, \mathbb{C})$-representations of a torus knot. This was demonstrated in the unpublished lecture notes [3] by Johnson.

Let (p, q) be a pair of coprime natural numbers. Hereafter let $K = T(p, q)$ be the (p, q) torus knot and $G(p, q)$ be its knot group. We take the following presentation of $G(p, q)$:

$$G(p, q) = \langle x, y \,|\, x^p y^{-q} \rangle.$$

First we quickly review some terminologies of a linear representation in $SL(2, \mathbb{C})$. A representation $\rho : G(p, q) \to SL(2, \mathbb{C})$ is called irreducible if there does not exist a nontrivial proper invariant subspace of $\mathbb{C}^2$ under the natural action of $\rho(G(p, q))$. A representation $\rho : G(p, q) \to SL(2, \mathbb{C})$ is called reducible if ρ is not irreducible. That is, there is an invariant 1-dimensional subspace of $\mathbb{C}^2$. A representation ρ is called abelian if $\rho(G(p, q))$ is an abelian subgroup of $SL(2, \mathbb{C})$. It is easy to see that an abelian representation is reducible.

Let R be the set of irreducible $SL(2, \mathbb{C})$-representations of $G(p, q)$. Fixing the generators x and y, R can be embedded into $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by the map $R \ni \rho \mapsto (\rho(x), \rho(y)) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. From this embedding, the topology of R can be induced from $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Let $\hat{R}$ be the space of conjugacy classes of irreducible representations, that is, the quotient space of R by conjugate action of $SL(2, \mathbb{C})$. In general $\hat{R}$ has some connected components. For a given representation ρ, we write $\hat{\rho}$ for its conjugacy class.

From now on, we start to describe the structure of $\hat{R}$. Choosing a pair (r, s) of natural numbers satisfying $ps - qr = 1$, then $m = x^{-r} y^s \in G(p, q)$ represents a meridian of $T(p, q)$. Let $\rho : G(p, q) \to SL(2, \mathbb{C})$ be an irreducible representation. For simplicity, we write a capital letter X for the image $\rho(x)$ of x, Y for $\rho(y)$ and so on.

Now we put $z = x^p = y^q \in G(p, q)$ which lies in the center of $G(p, q)$. Recall that the center of $SL(2, \mathbb{C})$ is $\{\pm E\}$, where E is the identity matrix of degree 2.

Lemma 3.1. $Z = \pm E$.

Proof. Assume that $Z \neq \pm E$. We take an eigenvalue λ of Z and its eigenspace $V_\lambda \subset \mathbb{C}^2$. Because z is a center element of $G(p, q)$, Z can be commuted with any matrix $S \in \rho(G(p, q))$. For any vector $v \in V_\lambda$,

$$Z(Sv) = S(Zv) = \lambda Sv.$$

Hence $Sv \in V_\lambda$ and it implies V_λ is an invariant subspace of ρ. By the irreducibility of ρ, V_λ is the full space $\mathbb{C}^2$. Therefore $\lambda = \lambda^{-1} = \pm 1$. Here we may put $Z = \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix}$ up to conjugation. If $t \neq 0$, then the above eigenspace V_λ is not the full space $\mathbb{C}^2$. This contradicts the irreducibility of ρ and then $Z = \pm E$. $\square$

Since $Z = X^p = Y^q = \pm E$, it holds that $X^{2p} = Y^{2q} = E$. On the other hand, we have the following.

Lemma 3.2. $X^r \neq \pm E,\ Y^s \neq \pm E.$

Proof. Assuming $X^r = \pm E$, we have $X^{2r} = E$. Since $ps - qr = 1$, $2ps = 2qr + 2$ holds. Thus $X^{2ps} = X^{2qr+2}$. Hence we have $E = X^2$ and then $X = \pm E$. It means that the representation ρ is abelian, but this is a contradiction. It is similarly proved that $Y^s \neq \pm E$. □

Here we let

$$\alpha^{\pm 1} = \exp(\pm\sqrt{-1}\pi a/p) \quad \text{and} \quad \beta^{\pm 1} = \exp(\pm\sqrt{-1}\pi b/q)$$

to be the eigenvalues of X and Y respectively, where we can assume that $0 < a < p$ and $0 < b < q$. Since

$$X^p = (-E)^a = Y^q = (-E)^b,$$

it holds that

$$a \equiv b \bmod 2.$$

From now on, let us fix $\operatorname{tr} X$ and $\operatorname{tr} Y$. We consider a conjugacy class of the representation ρ, so that we may assume $X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and Y is conjugate to $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ in $SL(2, \mathbb{C})$.

Remark 3.3. We remark that a is fixed but b is not. In fact, there are two choices of b $(0 < b < q)$, namely b or $-b \mod q$. Both of them give the same trace $\operatorname{tr} Y = 2\cos(\pi b/q)$.

If Y is an upper triangle matrix, then ρ is a reducible representation. In this case, the trace of the meridian image

$$\begin{aligned} M &= X^{-r} Y^s \\ &= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-r} \begin{pmatrix} \beta & * \\ 0 & \beta^{-1} \end{pmatrix}^{s} \text{ or } \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-r} \begin{pmatrix} \beta^{-1} & * \\ 0 & \beta \end{pmatrix}^{s} \end{aligned}$$

is given by

$$\operatorname{tr} M = \alpha^{-r}\beta^{\pm s} + \alpha^r \beta^{\mp s} = 2\cos\pi(ra/p \pm sb/q).$$

Namely we obtain the following lemma.

Lemma 3.4. *ρ is an irreducible representation if* $\operatorname{tr} M \neq \alpha^{-r}\beta^{\pm s} + \alpha^{r}\beta^{\mp s}$.

Now Y^s is conjugate to $\begin{pmatrix} \beta^s & 0 \\ 0 & \beta^{-s} \end{pmatrix}$ in $SL(2,\mathbb{C})$, so that Y^s has the form of

$$\begin{pmatrix} \beta^s+\delta & * \\ * & \beta^{-s}-\delta \end{pmatrix} \text{ or } \begin{pmatrix} \beta^{-s}+\delta & * \\ * & \beta^{s}-\delta \end{pmatrix},$$

where δ is any complex number. Therefore

$$\begin{aligned} \operatorname{tr} M &= \operatorname{tr}(X^{-r}Y^s) \\ &= \alpha^{-r}\beta^{\pm s} + \alpha^{r}\beta^{\mp s} + \delta(\alpha^{-r} - \alpha^{r}). \end{aligned}$$

Hence we can assume

$$\operatorname{tr} M = \alpha^{-r}\beta^{s} + \alpha^{r}\beta^{-s} + \delta(\alpha^{-r} - \alpha^{r})$$

by replacing β if necessary. We note that b has been fixed. This value of $\operatorname{tr} M$ can be any complex number because δ can be so.

Lemma 3.5. *If we put $U = X^{-r}$ and $V = Y^s$, then $X = Z^sU^q$ and $Y = Z^{-r}V^p$.*

Proof. Direct calculations. □

Lemma 3.6. *For any irreducible representation $\rho : G(p,q) \to SL(2,\mathbb{C})$, if* $\operatorname{tr} X$, $\operatorname{tr} Y$ *and* $\operatorname{tr} M$ *are fixed, then ρ is uniquely determined up to conjugation.*

Proof. We fix $\operatorname{tr} X$, $\operatorname{tr} Y$ and $\operatorname{tr} M$. Then we prove that X and Y are uniquely determined in $SL(2,\mathbb{C})$ up to mutual conjugation. First the value of $\operatorname{tr} X$ determines $\operatorname{tr} Z$ and $\operatorname{tr} U$, because $Z = X^p$ and $U = X^{-r}$. Hence Z can be determined since $Z = \pm E$. Similarly $\operatorname{tr} Y$ determines $\operatorname{tr} V$. Here $\operatorname{tr} M = \operatorname{tr} UV$ is fixed and U, V do not commute, so that U and V are determined in $SL(2,\mathbb{C})$ up to mutual conjugation. Therefore X and Y are uniquely determined up to conjugation by Lemma 3.5. □

Proposition 3.7 (Johnson [3]). *Each connected component $\hat{R}_{a,b}$ of $\hat{R}$ is determined by the following data*:

(1) $0 < a < p, 0 < b < q$.
(2) $a \equiv b \mod 2$.
(3) $\operatorname{tr} X = 2\cos(\pi a/p)$, $\operatorname{tr} Y = 2\cos(\pi b/q)$ *and* $Z = (-E)^a$.
(4) $\operatorname{tr} M \neq 2\cos\pi(ra/p \pm sb/q)$.

In particular, $\hat{R}_{a,b}$ is parametrized by $\operatorname{tr} M$ *and has complex dimension one.*

4. A formula for the $(2, q)$ torus knot

We start to compute the twisted Alexander polynomial of the (p, q) torus knot from the presentation

$$G(p,q) = \langle x, y \mid x^p y^{-q} \rangle.$$

Let us denote the relator by $u = x^p y^{-q}$. In this case, we easily see that

$$\frac{\partial u}{\partial x} = 1 + x + \cdots + x^{p-1}$$

holds. Then by definition we have

$$\begin{aligned}
\Delta_{K,\rho}(t) &= \frac{\det\Phi\left(\dfrac{\partial u}{\partial x}\right)}{\det\Phi(y-1)} \\
&= \frac{\det(E + t^q X + t^{2q} X^2 + \cdots + t^{(p-1)q} X^{p-1})}{\det(t^p Y - E)} \\
&= \frac{\left(1 + \alpha t^q + \cdots + \alpha^{p-1} t^{(p-1)q}\right)\left(1 + \alpha^{-1} t^q + \cdots + \alpha^{-(p-1)} t^{(p-1)q}\right)}{1 - \left(\beta + \beta^{-1}\right) t^p + t^{2p}},
\end{aligned}$$

where we have assumed that $X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and Y is conjugate to $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ for $\alpha = \exp\left(\sqrt{-1}\pi a/p\right)$, $\beta = \exp\left(\sqrt{-1}\pi b/q\right)$. From the above description, it is easy to see that $\Delta_{K,\rho}(t)$ can be determined by the fixed a and b. This completes the proof of Theorem 1.1.

Remark 4.1. For a reducible nonabelian representation $\rho : G(p,q) \to SL(2,\mathbb{C})$, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is expressed via the classical Alexander polynomial. More precisely, the following holds:

$$\begin{aligned}
\Delta_{K,\rho}(t) &= \frac{\Delta_K(\mu t)\Delta_K(\mu^{-1} t)}{t^2 - (\operatorname{tr} M)\, t + 1} \\
&= \frac{(t^{pq} - \mu^{pq})\left(t^{pq} - \mu^{-pq}\right)}{(t^p - \mu^p)\left(t^p - \mu^{-p}\right)(t^q - \mu^q)\left(t^q - \mu^{-q}\right)},
\end{aligned}$$

where $\mu \in \mathbb{C}$ satisfies $\Delta_K(\mu^2) = 0$ and $\mu + \mu^{-1} = \operatorname{tr} M$ (see [7, Theorem 3.1]).

Theorem 1.1 also can be shown by the following argument. We now put $K = T(p,q)$ on the standard torus T^2 in S^3. Here S^3 cut along T^2 consists of two solid tori U_1 and U_2. Let $\pi : (S^3 - K)_\infty \to S^3 - K$ be the infinite cyclic covering associated with $\alpha : G(p,q) \to \mathbb{Z} = \langle t \rangle$. For simplicity, we write U_i' to $U_i - K$, and set $\tilde{U}_i' = \pi^{-1}(U_i')$ for $i = 1, 2$. Then we have $(S^3 - K)_\infty = \tilde{U}_1' \cup \tilde{U}_2'$. For the union we obtain the Mayer-Vietoris exact sequence with twisted coefficients:

$$\to H_1(\tilde{U}_1' \cup \tilde{U}_2'; \mathbb{C}^2_\rho) \to H_0(\tilde{U}_1' \cap \tilde{U}_2'; \mathbb{C}^2_\rho) \to \oplus_i H_0(\tilde{U}_i'; \mathbb{C}^2_\rho) \to H_0(\tilde{U}_1' \cup \tilde{U}_2'; \mathbb{C}^2_\rho) \to 0,$$

where $\mathbb{C}^2_\rho$ is $\mathbb{Z}G(p,q)$-module defined by the representation $\rho : G(p,q) \to SL(2,\mathbb{C})$. The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is given by the ratio of the orders of

$$H_1(\tilde{U}'_1 \cup \tilde{U}'_2; \mathbb{C}^2_\rho) = H_1(S^3 - K; \mathbb{C}[t, t^{-1}]^2_{\rho\otimes\alpha})$$

and

$$H_0(\tilde{U}'_1 \cup \tilde{U}'_2; \mathbb{C}^2_\rho) = H_0(S^3 - K; \mathbb{C}[t, t^{-1}]^2_{\rho\otimes\alpha}),$$

so that it is determined by $H_0(\tilde{U}'_1 \cap \tilde{U}'_2; \mathbb{C}^2_\rho)$, $H_0(\tilde{U}'_i; \mathbb{C}^2_\rho)$ and $H_0(\tilde{U}'_1 \cup \tilde{U}'_2; \mathbb{C}^2_\rho)$. However these twisted homology groups depend only on the traces of X, Y and X^pY^q, because all the spaces U'_1, U'_2 and $U'_1 \cap U'_2$ are homotopic to S^1 and the core curves are corresponding to x, y and x^py^q respectively. Namely the twisted Alexander polynomial is locally constant.

Now in the case of $p = 2$, we can give an explicit formula for the twisted Alexander polynomial. In this case, a must be 1 and then $\hat{R}$ consists of $\frac{q-1}{2}$ components $\hat{R}_{1,b}$ $(0 < b < q,\ b$ is odd).

Theorem 4.2. *Let K be the $(2,q)$ torus knot and ρ_b an irreducible representation with $\hat{\rho}_b \in \hat{R}_{1,b}$, Then the twisted Alexander polynomial is given by*

$$\Delta_{K,\rho_b}(t) = \left(t^2+1\right) \prod_{0<k<q,\ k:\text{odd},\ k\neq b} \left(t^2 - \xi_k\right)\left(t^2 - \bar{\xi}_k\right),$$

where $\xi_k = \exp(\sqrt{-1}\pi k/q)$.

Proof. Here we have $\alpha = \sqrt{-1}$. The numerator of $\Delta_{K,\rho_b}(t)$ is $1 + (\alpha + \alpha^{-1})t^q + t^{2q} = 1 + t^{2q}$. On the other hand, the denominator is $(t^2 - \xi_b)(t^2 - \bar{\xi}_b)$, because $\beta = \exp(\sqrt{-1}\pi b/q)$. The polynomial $t^{2q} + 1$ has the factorization

$$t^{2q} + 1 = \left(t^2+1\right) \prod_{0<k<q,\ k:\text{odd}} (t^2 - \xi_k)(t^2 - \bar{\xi}_k)$$

over $\mathbb{C}[t]$, so that we can obtain the desired formula. □

Example 4.3. Let $K = T(2,3)$, the trefoil knot. In this case, there is just one connected component $\hat{R}_{1,1}$ and we see that

$$\Delta_{K,\rho}(t) = \frac{t^6+1}{t^4 - t^2 + 1} = t^2 + 1$$

holds for any ρ with $\hat{\rho} \in \hat{R}_{1,1}$ (see [10, Theorem 4.1]).

A representation $\rho : G(K) \to SL(2,\mathbb{C})$ is called parabolic if the image of any meridian is a matrix with trace 2. For a torus knot $T(p,q)$, we can show the following.

Proposition 4.4. *There exists uniquely a conjugacy class of a parabolic representation on any connected component $\hat{R}_{a,b}$.*

Proof. This is a straightforward consequence of Proposition 3.7, namely 2 is a value allowed by Proposition 3.7 (4). In fact we can easily show that $2\cos\pi(ra/p \pm sb/q)$ never coincides with 2. □

The $(2, q)$ torus knot is one of 2-bridge knots. For a parabolic representation of a 2-bridge knot, Silver-Williams introduced the total twisted Alexander polynomial, which is denoted by $D_{K,\rho}(t)$. It is defined by taking the product of $\Delta_{K,\rho}(t)$ over parabolic representations corresponding to the roots of the Riley polynomial (see [11] for details).

As an immediate corollary of Theorem 4.2 and Proposition 4.4, we have Hirasawa-Murasugi's formula of $D_{K,\rho}(t)$ for the $(2, q)$ torus knot.

Corollary 4.5 (Hirasawa-Murasugi [2]). *For the $(2,q)$ torus knot, the total twisted Alexander polynomial $D_{K,\rho}(t)$ is given by*

$$\begin{aligned} D_{K,\rho}(t) &= \prod_{0<b<q,\ b:\mathrm{odd}} \Delta_{K,\rho_b}(t) \\ &= \left(t^2+1\right)\left(t^{2q}+1\right)^{\frac{q-3}{2}}, \end{aligned}$$

where $\hat{\rho}_b \in \hat{R}_{1,b}$.

Proof. Since each connected component $\hat{R}_{1,b}$ contains just one class of a parabolic representation, we can calculate the total twisted Alexander polynomial as follows.

$$\begin{aligned} D_{K,\rho}(t) &= \prod_{0<b<q,\ b:\mathrm{odd}} \Delta_{K,\rho_b}(t) \\ &= \frac{t^{2q}+1}{(t^2-\xi_1)(t^2-\bar{\xi}_1)} \cdot \frac{t^{2q}+1}{(t^2-\xi_3)(t^2-\bar{\xi}_3)} \cdots \frac{t^{2q}+1}{(t^2-\xi_{q-2})(t^2-\bar{\xi}_{q-2})} \\ &= \frac{(t^{2q}+1)^{\frac{q-1}{2}}}{\dfrac{t^{2q}+1}{t^2+1}} = \left(t^2+1\right)\left(t^{2q}+1\right)^{\frac{q-3}{2}}. \end{aligned}$$

This completes the proof. □

Example 4.6. Let $K = T(2, 5)$. Then there exist two connected components $\hat{R}_{1,1}$ and $\hat{R}_{1,3}$ in the irreducible $SL(2, \mathbb{C})$-representation space of $G(2, 5)$. A direct calculation shows that

$$\Delta_{K,\rho_\pm}(t) = t^6 + \frac{1 \pm \sqrt{5}}{2}t^4 + \frac{1 \pm \sqrt{5}}{2}t^2 + 1$$

holds for any $\hat{\rho}_+ \in \hat{R}_{1,1}$ and $\hat{\rho}_- \in \hat{R}_{1,3}$. If we take the product of them, we obtain the total twisted Alexander polynomial

$$D_{K,\rho}(t) = \Delta_{K,\rho_+}(t) \cdot \Delta_{K,\rho_-}(t) = \left(t^2+1\right)\left(t^{10}+1\right).$$

The result reveals that $D_{K,\rho}(t)$ is a product of cyclotomic polynomials, although the twisted Alexander polynomial is not (see [2, Proposition 10.4] and [11, Theorem 6.1]).

Finally, let us consider the ρ-twisted Alexander polynomial Δ_1^ρ defined in [11, Section 3] (see also [4, Theorem 4.1]). It is related to our twisted Alexander polynomial $\Delta_{K,\rho}(t)$ as follows:

$$\Delta_1^\rho = \Delta_{K,\rho}(t) \cdot \Delta_0^\rho,$$

where Δ_0^ρ is the order of the cokernel of ∂_1 for the chain complex

$$0 \longrightarrow \Lambda^2 \xrightarrow{\partial_2} \left(\Lambda^2\right)^2 \xrightarrow{\partial_1} \Lambda^2 \longrightarrow 0.$$

Here $\Lambda = \mathbb{C}[t, t^{-1}]$ and the differentials are given by

$$\partial_2 = \left(\Phi\left(\frac{\partial u}{\partial x}\right) \ \Phi\left(\frac{\partial u}{\partial y}\right)\right), \quad \partial_1 = \begin{pmatrix} \Phi(x-1) \\ \Phi(y-1) \end{pmatrix}.$$

Then Δ_0^ρ equals the greatest common divisor of the 2×2 subdeterminants of the matrix representing ∂_1.

In [11, Corollary 6.3] Silver and Williams stated that the ρ-twisted Alexander polynomial corresponding to a parabolic representation of a torus knot is a product of cyclotomic polynomials. The next example shows that it is a false statement.

Example 4.7. Let $K = T(4,3)$. There are three connected components $\hat{R}_{1,1}$, $\hat{R}_{2,2}$ and $\hat{R}_{3,1}$. Let us focus on the component $\hat{R}_{1,1}$. In this case, $\alpha = \exp(\sqrt{-1}\pi/4)$ and $\beta = \exp(\sqrt{-1}\pi/3)$. Thus for any representation $\hat{\rho} \in \hat{R}_{1,1}$, we obtain

$$\begin{aligned} \Delta_{K,\rho}(t) &= \frac{1+\sqrt{2}t^3+t^6+t^{12}+\sqrt{2}t^{15}+t^{18}}{1-t^4+t^8} \\ &= \left(1+t^4\right)\left(1+\sqrt{2}t^3+t^6\right). \end{aligned}$$

To get Δ_1^ρ for the knot $K = T(4,3)$, we calculate Δ_0^ρ for a representation $\rho : G(4,3) \to SL(2,\mathbb{C})$ defined by

$$\rho(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} \beta + \sqrt{-1} & -\gamma \\ \gamma & \beta^{-1} - \sqrt{-1} \end{pmatrix},$$

where $\gamma = \sqrt{-1-\sqrt{3}}$. We then obtain

$$\partial_1 = \begin{pmatrix} \rho(x)t^3 - E \\ \rho(y)t^4 - E \end{pmatrix}$$

and $\Delta_0^\rho = 1$ by a direct calculation. In fact, there are two subdeterminants

$$f_{13}(t) = -\gamma t^4 \left(\alpha t^3 - 1\right) \quad \text{and} \quad f_{24}(t) = -\gamma t^4 \left(\alpha^{-1} t^3 - 1\right)$$

such that $\gcd(f_{13}, f_{24}) = 1$, where $f_{ij}(t)$ is the determinant of the 2×2 matrix consisting of the ith and the jth rows of ∂_1.

Therefore the ρ-twisted Alexander polynomial is given by

$$\begin{aligned} \Delta_1^\rho &= \Delta_{K,\rho}(t) \cdot \Delta_0^\rho \\ &= \left(1 + t^4\right)\left(1 + \sqrt{2}t^3 + t^6\right) \end{aligned}$$

and not an integral polynomial. In particular, it is not a product of cyclotomic polynomials. Of course this formula is valid for a parabolic $SL(2,\mathbb{C})$-representation in $\hat{R}_{1,1}$, because of Theorem 1.1 and Proposition 4.4.

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