

Twisted compactifications of 3d $\mathcal{N} = 4$ theories and conformal blocks

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ABSTRACT: Three-dimensional $\mathcal{N} = 4$ supersymmetric quantum field theories admit two topological twists, the Rozansky-Witten twist and its mirror. Either twist can be used to define a supersymmetric compactification on a Riemann surface and a corresponding space of supersymmetric ground states. These spaces of ground states can play an interesting role in the Geometric Langlands program. We propose a description of these spaces as conformal blocks for certain non-unitary Vertex Operator Algebras and test our conjecture in some important examples. The two VOAs can be constructed respectively from a UV Lagrangian description of the $\mathcal{N} = 4$ theory or of its mirror. We further conjecture that the VOAs associated to an $\mathcal{N} = 4$ SQFT inherit properties of the theory which only emerge in the IR, such as enhanced global symmetries. Thus knowledge of the VOAs should allow one to compute the spaces of supersymmetric ground states for a theory coupled to supersymmetric background connections for the full symmetry group of the IR SCFT. In particular, we propose a conformal field theory description of the spaces of ground states for the $T[\text{SU}(N)]$ theories. These theories play a role of S-duality kernel in maximally supersymmetric $\text{SU}(N)$ gauge theory and thus the corresponding spaces of supersymmetric ground states should provide a kernel for the Geometric Langlands duality for special unitary groups.

KEYWORDS: Conformal Field Theory, Duality in Gauge Field Theories, Extended Supersymmetry, Topological Field Theories

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1 Introduction

This is a companion paper to [1]. The main subject of this paper are $\mathcal{N} = 4$ SQFT in three dimensions, equipped with interesting global symmetry groups. Such three-dimensional SQFTs can appear as boundary degrees of freedom for half-BPS boundary conditions in $\mathcal{N} = 4$ Super Yang Mills theory [2].

In turns, half-BPS boundary conditions in $\mathcal{N} = 4$ SYM descend, upon compactification on a Riemann surface C , to branes which play an important role in the gauge theory interpretation of the Geometric Langlands duality [3–6]. When the original boundary conditions involve boundary degrees of freedom in the form of 3d $\mathcal{N} = 4$ SQFT, the Chan-Paton bundles for the corresponding branes arise as spaces of super-symmetric ground states on C for the corresponding 3d SQFTs [1].

The objective of this paper is to find a characterization of these Chan-Paton bundles which is flexible enough to overcome a crucial problem: the $\mathcal{N} = 4$ SQFTs we are interested in often have important low energy symmetry groups which are not fully visible in any known UV Lagrangian descriptions.

Our strategy is to associate to each $\mathcal{N} = 4$ SQFT a Vertex Operator Algebra whose conformal blocks on C match the desired Chan-Paton spaces and whose symmetries match the low energy symmetries of the SQFT.

Some elements of our proposal are somewhat conjectural and the Vertex Operator Algebra which occur in practice in our calculations are rather intricate and non-unitary. The main focus of this paper is to analyze some important basic examples and to collect evidence that the VOA associated to linear quiver gauge theories of unitary groups have hidden symmetries which match the known IR symmetry enhancement of the gauge theories.

For the very simplest examples, we will also describe in some detail the interpretation of conformal blocks as Chan-Paton bundles for branes. The definition of conformal blocks for the non-unitary VOA which occur in our setup have subtleties which may be somewhat unfamiliar to physicists (including this author) and may require a somewhat refined mathematical treatment, possibly involving notions in Derived Algebraic Geometry. This is particularly the case if one wants to map the conformal blocks to objects in the derived categories of D-module used to describe BAA branes in Geometric Langlands or of quasi-coherent sheaves used to describe BBB branes.

A full mathematical treatment of these examples goes beyond the scope of this paper, but we will at least attempt to provide physical motivations for these subtleties. Appendix A describes in some detail some finite-dimensional examples of D-modules as

Chan-Paton bundles for BAA branes in \mathbb{C}^{2n} defined by simple choices of boundary degrees of freedom in the UV. These examples capture some of the subtleties which arise in the construction of conformal blocks.

Finally, we suspect that the VOAs discussed in this paper can actually occur as algebras of BPS local operators on certain deformed supersymmetric boundary conditions for the 3d $\mathcal{N} = 4$ gauge theories. This is the case for theories of free hypermultiplets [7, 8] and should remain true when gauge fields are added to the mix. This would give a direct physical motivation for the relationship between the space of conformal blocks of the VOAs and the space of super-symmetric ground states for the corresponding 3d SQFTs.

2 3d $\mathcal{N} = 4$ gauge theories on a Riemann surface

Recall that the three-dimensional $\mathcal{N} = 4$ supersymmetry algebra admits an $SU(2)_H \times SU(2)_C$ R-symmetry group. In standard Lagrangian theories the $SU(2)_H$ R-symmetry group acts on the hyper-multiplet scalar fields while $SU(2)_C$ acts on the vectormultiplet scalar fields. Three-dimensional mirror symmetry exchange the $SU(2)_{H,C}$ subgroups.

In order to define a super-symmetric compactification on a Riemann surface, we have two natural choices: we can twist by the Cartan subgroup of $SU(2)_H$ or by the Cartan subgroup of $SU(2)_C$. We denote the two possibilities as “H-twist” or “C-twist” respectively. When the 3d theories are used to define enriched Neumann boundary conditions for four-dimensional SYM, as in [1], these twists can also be denoted respectively as a “BAA twist” and “BBB twist”, according to the type of branes they give rise to.

2.1 The H -twist

Consider a three-dimensional $\mathcal{N} = 4$ SQFT T with an unbroken $SO(2)_H$ Cartan subgroup of the $SO(3)_H$ R-symmetry which rotates the three complex structures on the Higgs branch. We are mainly interested in renormalizable $\mathcal{N} = 4$ gauge theories, which satisfy this requirement automatically in the absence of complex FI parameters.

The $SO(2)_H$ symmetry can be used to compactify the theory on a Riemann surface C while preserving four scalar supercharges. The result is an effective $\mathcal{N} = 4$ supersymmetric quantum mechanics which we can denote as the H -twist of T . We denote as $\mathcal{H}_H[T, C]$ the space of supersymmetric ground states of this quantum mechanics.

Our objective is to justify and test the following conjectures:

- The space $\mathcal{H}_H[T, C]$ can be identified with the space of conformal blocks on C for a vertex algebra $\mathbb{A}_H[T]$.
- If G_H is the group of global symmetries acting on the Higgs branch of T , $\mathcal{H}_H[T, C]$ can be promoted to a (twisted) D-module over the space $\text{Bun}[C, G_H]$ of G_H bundles on C . The D-module structure is associated to the presence of a G_H current subalgebra in $\mathbb{A}_H[T]$.

- If G_C is the group of global symmetries acting on the Coulomb branch of T , $\mathcal{H}_H[T, C]$ can be promoted to a sheaf over the space $\text{Loc}[C, G_C]$ of (complexified) G_C flat connections on C . The sheaf structure is associated to an outer G_C automorphism of $\mathbb{A}_H[T]$ which allows one to couple it to G_C flat connection.¹

The theory $T[G]$ which appears in the study of S -duality in four-dimensional $\mathcal{N} = 4$ G gauge theory has $G_H = G$ and $G_C = {}^\vee G$, Geometric Langlands dual groups. We expect the algebra $\mathbb{A}_H[T[G]]$ to contain a G current algebra and have a ${}^\vee G$ outer automorphism. Correspondingly, the space of ground states $\mathcal{H}_H[T[G], C]$ should be simultaneously a (twisted) D-module over $\text{Bun}[C, G]$ and a sheaf over $\text{Loc}[C, {}^\vee G]$. The relation between S -duality and Geometric Langlands duality suggests that $\mathcal{H}_H[T[G], C]$ should play the role of a “duality kernel” in the Geometric Langlands program.

We propose an explicit construction of the algebra $\mathbb{A}_H[T]$ when T is a gauge theory with gauge group G and matter hypermultiplets in a symplectic representation M of G : we define $\mathbb{A}_H[T]$ as an (extended) coset model given by the quotient of a theory of free symplectic bosons valued in M by the G current subalgebra generated by the moment maps in M .

Concretely, the symplectic bosons vertex algebra $\text{Sb}[M]$ is generated by holomorphic fields Z_a of spin $1/2$ valued in M , with OPE controlled by the symplectic form ω_{ab} on M :

$$Z_a(z)Z_b(w) \sim \frac{\omega_{ab}}{z-w} \tag{2.1}$$

The vertex algebra contains WZW currents valued in the Lie algebra of G :

$$J_I(z) =: \mu_G^I(Z) :=: Z_a(z)T_I^{ab}Z_b(z) : \tag{2.2}$$

where T_I^{ab} generate the symplectic action of G on M .

The vertex algebra $S[M]$ decomposes into a direct sum of representations of the \hat{G} current algebra generated by the WZW currents $J_I(z)$. In the absence of Abelian factors in the gauge group, we define the coset vertex algebra $\mathbb{A}_H[T]$ as the coefficient of the vacuum Verma module in the sum:

$$\text{Sb}[M] = [\mathbb{A}_H[T] \otimes V_0] \oplus \dots \tag{2.3}$$

This is what is usually called a coset in the physics literature and denoted as

$$\mathbb{A}_H[T] = \frac{\text{Sb}[M]}{\hat{G}} \tag{2.4}$$

In unitary theories, the coset is usually computed by looking at vertex operators in the original theory with trivial OPE with the WZW currents $J_I(z)$. Because the symplectic

¹Here we identified the space of complexified G_C flat connections with the space of local systems $\text{Loc}[C, G_C]$. The two spaces are topologically the same but not algebraically. The difference is important in the mathematical treatment of the Geometric Langlands program. The conformal blocks for $\mathbb{A}_H[T]$ are ultimately defined as solutions of Ward identities which depend polynomially on the complexified G_C flat connection on C , seen as a holomorphic connection on a bundle. As far as we understand, that means that the conformal blocks are naturally associated to the space of complexified G_C flat connections on C rather than the space of local systems.

boson vertex algebra is not unitary, it is possible for the vacuum module to appear a submodule of a larger indecomposable module and thus our definition of coset turns out to be a bit more restrictive than that. We will see an explicit example later on.

In the presence of Abelian factors in the gauge group, the vertex algebra $\mathbb{A}_H[T]$ will be graded by characters for the Abelian gauge symmetry. The degree 0 part is defined as before. The degree p part is defined as the coefficient of a Verma module of momentum p for the Abelian currents, vacuum for the non-Abelian currents.

$$\text{Sb}[M] = \oplus_p \left[\mathbb{A}_H^{(p)}[T] \otimes V_p \right] \oplus \dots \tag{2.5}$$

In order to get a standard (fermionic) VOA, we should restrict the momentum p to values for which the conformal dimensions of the coset fields are (half) integral. Typically, this will mean p lies in some full rank sublattice of the charge lattice, as the level of Abelian WZW currents in the symplectic boson theory will be integral.

In any case, we expect the coset algebra to contain WZW currents $:\mu_{G_F}(Z):$ valued in the Lie algebra of the Higgs branch flavor symmetry G_F . Furthermore, the p grading of the algebra gives the expected action of the Abelian subgroup of G_C which is visible in the UV description of $\mathcal{N} = 4$ gauge theories: each $U(1)$ factor in G contributes a $U(1)$ factor to G_C .

In many cases, the G_C global symmetry of $\mathcal{N} = 4$ gauge theories is enhanced in the IR to a larger non-Abelian symmetry. A crucial check of our conjecture will be the existence of a corresponding enhancement of the global symmetry of $\mathbb{A}_H[T]$.

In many important situations, where the level of the \hat{G} current algebra is sufficiently negative and integral, we have found that a certain operation of BRST reduction provides similar results as the coset operation and may even be better motivated conceptually. It will allow us to make contact with the work of [9, 10] and borrow very useful results about hidden symmetries of certain VOAs.

2.2 The C -twist

Mirror symmetry exchanges the role of $SO(3)_H$ and $SO(3)_C$, the R-symmetry which rotates the three complex structures on the Coulomb branch. A twisted compactification of T on C which employs the Cartan subgroup $SO(2)_C$ of $SO(3)_C$ gives an effective $\mathcal{N} = 4$ supersymmetric quantum mechanics which we can denote as the C -twist of T . We denote as $\mathcal{H}_C[T, C]$ the space of supersymmetric ground states of this quantum mechanics.

The following statements should hold true:

- The space $\mathcal{H}_C[T, C]$ can be identified with the space of conformal blocks on C for a vertex algebra $\mathbb{A}_C[T]$.
- If G_C is the group of global symmetries acting on the Higgs branch of T , $\mathcal{H}_C[T, C]$ can be promoted to a (twisted) D-module over the space $\text{Bun}[C, G_C]$ of G_C bundles on C . The D-module structure is associated to the presence of a G_C current subalgebra in $\mathbb{A}_C[T]$.
- If G_H is the group of global symmetries acting on the Coulomb branch of T , $\mathcal{H}_C[T, C]$ can be promoted to a sheaf over the space $\text{Loc}[C, G_H]$ of (complexified) G_H connections on C . The sheaf structure is associated to outer G_H automorphism of $\mathbb{A}_C[T]$ which allows one to couple it to G_H flat connections.

For theories of free hypermultiplets, we have a simple prescription for $\mathbb{A}_C[T]$ as an algebra $\text{Fc}[M]$ of *fermionic currents*, described by an OPE

$$j_a(z)j_b(w) \sim \frac{\omega_{ab}}{(z-w)^2} \tag{2.6}$$

We do not have a prescription for computing $\mathbb{A}_C[T]$ for general gauge theories. An obvious strategy, when possible, is to look for a mirror gauge theory description T^\dagger of T and compute $\mathbb{A}_H[T^\dagger]$ instead.

2.3 Sheafs and D-modules from tt^* geometry

Here we would like to briefly explain the reason for the appearance of $\text{Bun}[C, G_H]$ and $\text{Loc}[C, G_C]$ in our story. We refer the reader to [1] for a more detailed discussion.

The four supercharges we consider can be better understood by referring to the mirror Rozansky-Witten (mRW) twist of the 3d theory, which is a topological twist in three dimensions. In our twisted compactification on C , the mRW supercharge can be decomposed into two parts with opposite $\text{SO}(2)_H$ charge:

$$Q_{\text{mRW}} = Q_H + \bar{Q}_H \tag{2.7}$$

It belong to a general family of nilpotent supercharges

$$Q_H^\zeta = Q_H + \zeta \bar{Q}_H \tag{2.8}$$

The existence of this family of supercharges in an $\mathcal{N} = 4$ supersymmetric quantum mechanics constrains how the space of ground states is fibered over certain parameter spaces of supersymmetric deformations of quantum mechanics. These constraints on the Berry connection were first discussed in the study of the tt^* geometry of $(2, 2)$ two-dimensional sigma models [11].

The specific form of the Berry connection constraints depends on the specific form of the super-multiplet to which the deformation of the supercharges and Hamiltonian belongs [12]. The supermultiplet may include several deformations and/or protected operators.

In general, the Berry connection constraints can be expressed in terms of a ‘‘Lax connection’’, a family of differential operators D^ζ on the parameter space which depends holomorphically and linearly in ζ and commute with each other at any given value of ζ . Depending on the specific deformation super-multiplet, the differential operators can take different forms. They are always built from the Berry connections associated to deformations in the supermultiplet and from the expectation values of protected operators in the same supermultiplet.

The original work on tt^* geometry involved supermultiplets which contain a complex deformation parameter and an extra chiral operator, such that the differential operators D^ζ are the Lax connection for a Hitchin system on the parameter space:

$$D_u^\zeta = D_u + \frac{\Phi_u}{\zeta} \qquad D_{\bar{u}}^\zeta = D_{\bar{u}} + \zeta \bar{\Phi}_{\bar{u}} \tag{2.9}$$

where u is a complex structure deformation, D_u the associated Berry connection on the space of ground states and Φ_u the expectation value between ground states of the chiral operator associated to the u deformation. We denote these deformation parameters as “BAA-type” deformations as the data can be employed to define BAA branes in 2d (4, 4) sigma models.

Another important possibility are Cauchy-Riemann equations for tri-holomorphic bundles on hyper-Kähler parameter spaces. For example, for an R^4 parameter space they would look like

$$D_u^\zeta = D_u + \frac{D_{\bar{v}}}{\zeta} \quad D_{\bar{u}}^\zeta = D_{\bar{u}} - \zeta D_v \quad (2.10)$$

When the theory has several supermultiplets of deformations, the D^ζ operators all commute with each other at fixed ζ . We denote these deformation parameters as “BBB-type” deformations as the data can be employed to define BBB branes in 2d (4, 4) sigma models.

There are two natural way to deform our compactification of T on C , by coupling to background connections for G_H or G_C on C . We can identify supersymmetric deformations by looking at the BPS equations for these background fields.

There are no constraints on G_H connections, but the dependence on the holomorphic part of the connection is Q -exact. The topological theory is thus coupled only to a G_H bundle and we get a $\text{Bun}[C, G_H]$ factor in the parameter space. This type of deformation is analogous to the complex structure deformations in the original tt^* work. The corresponding chiral operator is one of the moment map operators on the Higgs branch of the 3d theory. The BAA-type structure associated to G_H connections is thus a connection on the sheaf of ground states on $\text{Bun}[C, G_H]$ together with a Higgs field, the expectation value of the moment map operator, which together satisfy the higher-dimensional version of Hitchin equations on $\text{Bun}[C, G_H]$.

For generic ζ , and in particular for $\zeta = 1$, the Lax connection for such Hitchin system is a flat connection on $\text{Bun}[C, G_H]$. It equips the cohomology of Q_H^ζ with the structure of a D-module on $\text{Bun}[C, G_H]$. More precisely, we expect that one should be able to identify the output of the supersymmetric quantum mechanics with an object in some appropriate derived category of D-modules on $\text{Bun}[C, G_H]$.

The second possibility is a bit more intricate. The BPS equations require us to turn on both a G_C connection A_C and a background complex adjoint scalar Φ_C in the G_C twisted vectormultiplet (i.e. a complex FI parameter). The twist by $\text{SO}(2)_H$ makes the complex FI parameters Φ_C of T into one forms on C , valued in the Lie algebra of G_C . A pair (A_C, Φ_C) of background G_C connection A_C and scalar Φ_C preserves Q_H^ζ if the auxiliary Lax connection

$$D_z^\zeta = D_z[A_C] + \frac{\Phi_{C,z}}{\zeta} \quad D_{\bar{z}}^\zeta = D_{\bar{z}}[A_C] + \zeta \Phi_{C,\bar{z}} \quad (2.11)$$

is flat for all ζ . It preserves all four scalar supercharges if (A_C, Φ_C) is a solution of G_C Hitchin’s equations on C .

Thus this factor of the parameter space is the Hitchin moduli space $\mathcal{M}[G_C, C]$. The corresponding BBB-type structure is a tri-holomorphic sheaf on $\mathcal{M}[G_C, C]$: the sheaf of

ground states for the $\mathcal{N} = 4$ quantum mechanics has a Berry connection which is holomorphic in all complex structures of $\mathcal{M}[G_C, C]$. The choice of ζ is a choice of complex structure on $\mathcal{M}[G_C, C]$.

For generic ζ , and in particular for $\zeta = 1$, we can identify the parameter space with $\text{Loc}[C, G_C]$. For any given ζ , it gives the cohomology of Q_H^ζ the structure of an holomorphic sheaf on $\mathcal{M}[G_C, C]$ in complex structure ζ . More precisely, we expect that one should be able to identify the output of the supersymmetric quantum mechanics with an object in some appropriate derived category of sheaves on $\text{Loc}[C, G_C]$.

As the Lax connections on the two factors of the parameter space commute, the sheaf structure on $\text{Loc}[C, G_C]$ and the D-module structure on $\text{Bun}[C, G_H]$ are compatible, i.e. the flat connection on $\text{Bun}[C, G_H]$ with parameter ζ commutes with the anti-holomorphic derivatives in complex structure ζ for the sheaf on $\text{Loc}[C, G_C]$.

The BAA and BBB structures on the supersymmetric ground states can be encoded as BBB and BAA branes on $\mathcal{M}[G_C, C]$ and $\mathcal{M}[G_H, C]$ respectively. Physically, this arises from the promotion of a three-dimensional $\mathcal{N} = 4$ SQFT to a half-BPS interface for four-dimensional G_H and G_C $\mathcal{N} = 4$ gauge theories: compactification on a Riemann surface C reduces the four-dimensional gauge theories to $\mathcal{M}[G_C, C]$ and $\mathcal{M}[G_H, C]$ sigma-models and the 3d interface to a BPS interface between the two sigma models, which is of BBB type on one side and BAA on the other side.

In particular, the sheaf of ground states for the $T[G]$ theory gives a BPS interface between the $\mathcal{M}[G, C]$ and $\mathcal{M}[G^\vee, C]$ sigma models which should implement the mirror symmetry relation between the two sigma models, i.e. the Geometric Langlands duality.

We refer to [1] for more details and for a description of the geometric structures which emerge at $\zeta = 0$.

2.4 From hypermultiplets to symplectic bosons and fermionic currents

In the absence of gauge fields, there is a simple way to understand the algebras $\mathbb{A}_H[M]$ and $\mathbb{A}_C[M]$ we associate to hypermultiplets valued in M .

2.4.1 H-twist

As standard hypermultiplet scalars transform in a doublet of $\text{SO}(3)_H$, the H-twist makes them into spinors on C . The hypermultiplet fermions are already spinors on C to start with.

The three-dimensional action can be recast as a supersymmetric quantum mechanics akin to a Landau-Ginzburg theory with a Kähler target manifold [13]. The target of the quantum mechanics is the space of sections Z of the bundle $K^{1/2} \otimes M$ on C . The superpotential is the symplectic boson action:

$$W = \int_C \langle Z, D_{\bar{z}} Z \rangle \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ is the symplectic pairing on M and $D_{\bar{z}}$ the anti-holomorphic covariant derivative associated to the bundle.

Notice that if we pick a global symmetry group G_H acting symplectically on the hypermultiplets, we can take M to be a non-trivial G_H bundle rather than the constant bundle.

Thus $\text{Bun}[C, G_H]$ is a parameter space of complex structure/superpotential deformations for the LG quantum mechanics. The variation of W along $\text{Bun}[C, G_H]$, which is the integral over C of the moment map $\mu(Z)$ contracted with the variation of the anti-holomorphic connection,

$$\delta W = \int_C \mu(Z) \cdot \delta A_{\bar{z}}^{G_H} \tag{2.13}$$

gives a local operator in the quantum mechanics which combines with the Berry connection to give the tt^* structure mentioned above.

The space of ground states of an $\mathcal{N} = 4$ Landau-Ginzburg quantum mechanics with finite-dimensional target space U is the cohomology of U relative to the locus where $\text{Re } W \ll 0$, i.e. the space of integration cycles for forms which behave as e^W . This cohomology has an integral basis and it is a locally constant sheaf on the space of complex structure/superpotential deformations for the quantum mechanics: the parallel transport is defined by continuous deformations of the integration contours.

This structure can be recast as a D-module: the D-module associated to the Picard-Fuchs equations satisfied by integrals of the form

$$\oint_{\gamma} \omega e^W \tag{2.14}$$

where ω lies in an appropriate dW -deformation of De Rham cohomology. See appendix A for several examples.

The finite-dimensional model suggests that the space $\mathcal{H}_H[M, C]$ should coincide with the space of conformal blocks for a theory of chiral symplectic bosons, defined by the path integral

$$\int DZ e^{\int_C \langle Z, D_{\bar{z}} Z \rangle} \tag{2.15}$$

with a (twisted) D-module action given by the WZW current subalgebra defined by the moment maps

$$J_{G_H} =: \mu(Z) : \tag{2.16}$$

This path integral gives a free vertex algebra $\text{Sb}[M]$ with OPE

$$Z_a(z) Z_b(w) \sim \frac{\omega_{ab}}{z - w} \tag{2.17}$$

where ω is the symplectic form on M . A simple way to understand why symplectic bosons can be coupled to a gauge bundle is to observe that this OPE is invariant under holomorphic gauge transformations of the $Z_a(z)$.

There is an alternative perspective which supports this proposal: the theory of free hypermultiplets admits a boundary condition which preserves a $(0, 4)$ two-dimensional subalgebra of the supersymmetry algebra. A mRW twist of the theory is known to lead to a theory of holomorphic symplectic bosons on the boundary [7, 8]. This construction thus gives a map from the space of conformal blocks for symplectic bosons to the space of states of mRW-twisted free hypermultiplets.

The conformal blocks for symplectic bosons on a Riemann surface C in the absence of a background gauge bundle depend on a choice of spin structure $K^{1/2}$. In the presence of a background gauge bundle E , Z_a transforms as a section of the associated bundle $E_M \otimes K^{1/2}$.²

As long as $E_M \otimes K^{1/2}$ has no global sections, so that the symplectic boson has no zero-modes on C , the path integral (2.15) has an obvious meaning and gives a single conformal block, i.e. a unique solution of the Ward identities for correlation functions of the Z_a . The partition function is the inverse of the square root of the determinant of the $\bar{\partial}$ operator on $E_M \otimes K^{1/2}$. As one approaches the locus where $E_M \otimes K^{1/2}$ has global sections, the partition function will diverge.

In a component of the space of bundles where the symplectic bosons have generically no zero-modes, a naive description of the space of conformal blocks is a rank 1 D-module with a regular singularity at the locus where $E_M \otimes K^{1/2}$ has global sections. The finite-dimensional examples in appendix A make it clear that this description is incomplete and additional conformal blocks are hidden at special loci in the space of bundles. Such hidden conformal blocks are even more important in components of the space of bundles where zero-modes exist generically.

These additional conformal blocks are important in matching and improving the classical description [1] of the BAA brane as a complex Lagrangian submanifold of the space of Higgs bundles (E, φ) : the Lagrangian has a component wrapping the $\varphi = 0$ locus and extra components which sit on the co-normal bundle to the locus where $E_M \otimes K^{1/2}$ has global sections.

A general description of the space of conformal blocks is that of a complex of D-modules, with a differential which imposes the Ward identities on correlation functions. This is described in appendix A. We expect that this complex can be systematically simplified, at least locally on the space of bundles, but we leave that to future work and focus on concrete examples.

2.4.2 C-twist

The RW twist of a theory of free hypermultiplets (i.e. the mRW twist of a theory of free twisted hypermultiplets) leaves the hypermultiplet scalars unaffected, but changes the quantum numbers of the fermions: part of the fermions become spin-zero superpartners of the bosonic scalar fields and the other half become one-forms. The spin zero fields are rather boring, but the fermionic one forms have an interesting Chern-Simons action built from the symplectic pairing on M . It is natural to expect that the space of ground states on C will be the space of conformal blocks of fermionic WZW currents $\text{Fc}[M]$ valued in M , with OPE

$$j_a(z)j_b(w) \sim \frac{\omega_{ab}}{(z-w)^2} \tag{2.18}$$

²Rather than considering this as a choice of a bundle E and spin structure $K^{1/2}$ it is more natural to take $E_M \otimes K^{1/2}$ to be some sort of generalized $\text{Spin}_\mathbb{C}$ structure and the space of conformal blocks as a D-module over the moduli space of such structures.

Notice that if we pick a global symmetry group G_C acting symplectically on the hypermultiplets, this system has no G_C -valued WZW subalgebra: we can couple the system to a flat G_C connection, but there is no holomorphic current to encode the infinitesimal changes in the connection. The conformal blocks form a sheaf over $\text{Loc}[C, G_C]$, as expected.

A simple way to understand this fact is to observe that the OPE (2.18) is not invariant under holomorphic gauge transformations, because of the double pole. It can be made invariant by adding a dependence on an holomorphic G_C connection \mathcal{A}_{ab} on C

$$j_a(z)j_b(w) \sim \frac{\omega_{ab}}{(z-w)^2} + \frac{\mathcal{A}_{ab}(w)}{z-w} \tag{2.19}$$

which combined with the bundle data into a holomorphic description of a G_C local system, i.e. a bundle equipped with a holomorphic connection on the Riemann surface C .

The spin zero fields in the hypermultiplets are expected to be completely trivial as long as the G_C local system has no scalar global sections [1]. If scalar global sections exist, the system becomes more complicated, in a manner we now describe. Notice that if the 3d theory is coupled to four-dimensional gauge theory, the vevs for the spin zero fields in the hypermultiplets trigger vevs for the four-dimensional scalar fields which are not included in the picture of a sigma model on the Hitchin moduli space.

The calculations in [1] predict that the sheaf of supersymmetric ground states should arise from the quantization of a phase space given by the de Rahm cohomology of forms on C valued in M . This is known to coincide with the sheaf of (derived) conformal blocks for fermionic currents valued in M [14].

In order to understand the relationship, we can pick a polarization in the phase space which splits into $(*, 0)$ forms and $(*, 1)$ forms and build a Fock space out of $\Omega_{*,0}$. The Fock space can be identified with a collection of potential correlation functions

$$\langle j_{a_1}(z_1) \cdots j_{a_n}(z_n) \phi_{b_1}(w_1) \cdots \phi_{b_m}(w_m) \rangle \tag{2.20}$$

and the BRST differential takes the schematic form

$$\begin{aligned} Q\langle \cdots \rangle &= \int dzd\bar{z} \left\langle \left(\bar{\partial} j_a(z) - \omega_{ab} \partial \frac{\delta}{\delta j_b(z)} \right) \frac{\delta}{\delta \phi_a(z)} \cdots \right\rangle + \\ &+ \int dzd\bar{z} \left\langle \left(j_a(z) \omega^{ab} \bar{\partial} \phi_b(z) + \partial \phi_b(z) \frac{\delta}{\delta j_b(z)} \right) \cdots \right\rangle \end{aligned} \tag{2.21}$$

This seems a reasonable definition for a space of (derived) conformal blocks for the fermionic currents. The Q cohomology in cohomological degree 0 consists of correlation functions for the j_a currents satisfying the Ward identities of fermionic currents

$$\left\langle \left(\bar{\partial} j_a(z) - \omega_{ab} \partial \frac{\delta}{\delta j_b(z)} \right) \cdots \right\rangle = 0 \tag{2.22}$$

If there are no zeromodes for the scalars, we expect this to exhaust the cohomology.

2.5 Coset versus BRST reduction

The vertex algebra of symplectic bosons is a crucial ingredient of another construction which associates vertex algebras to gauge theories with eight supercharges: the construction of vertex algebras for $\mathcal{N} = 2$ four-dimensional SCFTs [9, 10]. The requirement of conformal symmetry in four dimensions imposes strong constraints on the gauge theory matter content: the level of the \hat{G} current algebra in $\text{Sb}[M]$ should be twice the critical level.

At this particular value of the level, it is possible to pair up the $S[M]$ algebra with a system of b, c ghosts valued in the gauge Lie algebra and write down a BRST operator of the schematic form $Q_{\text{BRST}} = cJ + bcc$. The BRST cohomology produces the vertex algebras for the $\mathcal{N} = 2$ four-dimensional gauge theory.

It is quite obvious that if we build a 3d theory T with the same matter content and gauge group as a 4d SCFT, operators in our coset will belong also to the Q_{BRST} cohomology. Moreover, the central charge of the resulting VOAs also coincide: the central charge for the ghosts precisely cancels the central charge of the WZW currents at twice the critical level:

$$\left. \frac{k \dim G}{k+h} \right|_{k=-2h} + (-2) \times \dim G = 0 \tag{2.23}$$

Inspection of examples will strongly suggest that $\mathbb{A}_H[T]$ coincides in this situation with the 4d chiral algebra. We conjecture

$$\mathbb{A}_H[T] = \frac{\text{Sb}[M]}{\hat{G}_{-2h}} = \{ \text{Sb}[M] \times (b, c), Q_{\text{BRST}} \} \tag{2.24}$$

Assuming that this correspondence holds will be rather useful later in the paper: the 4d chiral algebra of theories of class S has unexpected symmetries which are thus inherited by our coset and which will be instrumental in demonstrating the Coulomb branch symmetry enhancements for unitary quiver gauge theories.

If the matter content of the three-dimensional theory is beyond the amount allowed in four-dimensions, so that the level of the WZW currents is more negative than twice the critical level, by an integral amount $-n$, we can still *add* a standard \hat{G}_n WZW model to the symplectic bosons and then apply the BRST reduction.

Again, the resulting VOA seems closely related to the one obtained by a direct coset of the symplectic bosons. For example, the central charge of the \hat{G}_n WZW model combines with the central charge of the ghosts to cancel the central charge of the \hat{G} WZW currents in the symplectic boson theory:

$$\left. \frac{k \dim G}{k+h} \right|_{k=-2h-n} + \left. \frac{k \dim G}{k+h} \right|_{k=n} + (-2) \times \dim G = 0 \tag{2.25}$$

Again, we expect this BRST construction to give an alternative definition of $\mathbb{A}_H[T]$. We conjecture

$$\mathbb{A}_H[T] = \frac{\text{Sb}[M]}{\hat{G}_{-2h-n}} = \left\{ \text{Sb}[M] \times \hat{G}_n \times (b, c), Q_{\text{BRST}} \right\} \tag{2.26}$$

3 Free hypermultiplets

We will discuss now some examples of VOA associated to free hypermultiplets in various representations.

3.1 H-twist of a single hypermultiplet

The vertex algebra $\text{Sb}[\mathbb{C}^2]$ of a single symplectic boson has two bosonic generators, $X(z)$ and $Y(z)$, with OPE

$$X(z)Y(w) \sim \frac{1}{z-w} \tag{3.1}$$

and conformal dimension $1/2$. Several of the features we discuss below can be found discussed at length in [15].

The stress tensor can be written as

$$T = \frac{1}{2}X\partial Y - \frac{1}{2}Y\partial X \tag{3.2}$$

and gives a central charge of $c_{XY} = -1$.³

The basic vacuum module of the symplectic boson VOA is generated by half-integral modes in the expansion

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{X_{n-\frac{1}{2}}}{z^n} \quad Y(z) = \sum_{n=-\infty}^{\infty} \frac{Y_{n-\frac{1}{2}}}{z^n} \tag{3.5}$$

with $[X_{n-\frac{1}{2}}, Y_{m-\frac{1}{2}}] = \delta_{n+m,1}$. The module is generated from the identity by the action of the negative modes in the expansion.

The vacuum module belongs to the sector with Neveu-Schwarz boundary conditions for X and Y . The sector with Ramond boundary conditions is somewhat more subtle, because of the existence of zeromodes which satisfy an Heisenberg algebra:

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{X_n}{z^{n+\frac{1}{2}}} \quad Y(z) = \sum_{n=-\infty}^{\infty} \frac{Y_n}{z^{n+\frac{1}{2}}} \tag{3.6}$$

with $[X_n, Y_m] = \delta_{n+m,0}$ and in particular $[X_0, Y_0] = 1$. Useful Ramond modules can be induced from any modules for the Heisenberg algebra of zeromodes.

Obvious choices are modules generated from vectors $|R, \pm\rangle$ which are annihilated either by X_0 or Y_0 and all positive modes. A less obvious choice is a module generated by vectors $|R, \lambda + n\rangle$ annihilated by positive modes, with n an integer and $0 < \lambda < 1$ and

$$Y_0|R, \lambda + n\rangle = |R, \lambda + n + 1\rangle \quad X_0|R, \lambda + n\rangle = (n + \lambda)|R, \lambda + n - 1\rangle \tag{3.7}$$

As our focus in this paper is on Riemann surfaces with no punctures, vertex operators for general modules of the symplectic boson algebra will play a limited role.

³We can check that this is the correct stress tensor

$$T(z)X(w) \sim \frac{1}{2} \frac{X(w)}{(z-w)^2} + \frac{\partial X(w)}{z-w} \quad T(z)Y(w) \sim \frac{1}{2} \frac{Y(w)}{(z-w)^2} + \frac{\partial Y(w)}{z-w} \tag{3.3}$$

and compute the central charge

$$T(z)T(w) \sim -\frac{1}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{3.4}$$

As the symplectic boson CFT can be described by a free chiral action $\int_C X \bar{\partial} Y$, we expect that as long as the action has no zeromodes the space of conformal blocks will be one-dimensional, generated by the Gaussian path integral with that action. In particular, the partition function should be just the inverse of the determinant of the $\bar{\partial}$ operator acting on sections of $K^{\frac{1}{2}}$. If we do not couple the symplectic boson to a background gauge bundle, we need to select an even spin structure $K^{\frac{1}{2}}$ in order to avoid zeromodes. It is more natural, though, to couple the system to background gauge fields. We will come back to that momentarily.

This expectation can be verified by directly solving on the Riemann surface C the Ward identities of the symplectic boson VOA or by assembling the conformal block by sewing up punctured spheres. The Ward identities express the correlation functions of X and Y fields in terms of the Green's function for the $\bar{\partial}$ operator. Concretely, $\frac{\langle X(z)Y(w) \rangle_C}{\langle 1 \rangle_C}$ is the unique meromorphic section of $K^{\frac{1}{2}}$ with a single pole of residue 1 at w . The overall normalization is determined by computing from the Green's function the stress tensor one-point function $\frac{\langle T(z) \rangle_C}{\langle 1 \rangle_C}$ and thus the dependence on the complex structure of C .

Similarly, the Ward identities allow one to reduce any sphere three-point function of vacuum descendants to the sphere partition function. Conformal blocks on a generic Riemann surface equipped with an even spin structure can be computed, say, by sewing together pairs of punctures from a sphere with $2g$ NS punctures by inserting complete sets of descendants of the identity, possibly with a twist acting as -1 on X and Y in order to select a specific spin structure. It is also possible to reproduce the answers by sewing along Ramond sector channels, but there are important subtleties associated to the zeromodes.

If we consider odd spin structures, instead, we cannot define a partition function unless we remove the zeromodes. It is possible to remove the zeromodes without locally interfering with the Ward identities, but the global behaviour of correlation functions is spoiled by logarithmic monodromies. The situation is improved by introducing a $U(1)$ gauge bundle. We will do that now.

3.2 Current subalgebras: $U(1)_{-1}$

The symplectic boson theory has an obvious $U(1)_{-1}$ WZW current

$$J = XY \tag{3.8}$$

rotating X and Y with charge ± 1 . This allows one to couple the symplectic boson to a $U(1)$ bundle L on the Riemann surface.

Concretely, this statement is related to the observation that the OPE of symplectic bosons is well-behaved under holomorphic gauge transformations: as the OPE has a single pole, the replacement $X(z) \rightarrow g(z)X(z)$ and $Y(z) \rightarrow g(z)^{-1}Y(z)$ does not change the singular part of the OPE, and shifts J by the expected $g^{-1}\partial g$ determined by the anomaly.

Notice that it is natural to think about the $L \otimes K^{\frac{1}{2}}$ bundle as a Spin_C structure on the Riemann surface, rather than choosing a spin structure and then a line bundle. Correspondingly, the conformal blocks are best defined over the moduli space of Spin_C structures on the Riemann surface.

If we take the line bundle to have degree 0, the path integral produces a partition function

$$Z_{C,L} = \frac{1}{\det \bar{\partial}_{L \otimes K^{\frac{1}{2}}}} \tag{3.9}$$

which has a pole along the Θ -divisor in the space of line bundles, where zeromodes appear.

In the first approximation, we can envision the conformal blocks in degree 0 as a one-dimensional line bundle on the space of $U(1)$ bundles $\text{Bun}_0(U(1), C)$ equipped with the structure of a (twisted) D-module with a regular singularity at the Θ -divisor. The partition function plays the role of a flat section of that D-module. Recall that the D-module structure on conformal blocks is simply the statement that we can change the $U(1)$ bundle infinitesimally by inserting a $U(1)$ current in the partition function, decomposing it into the OPE of two symplectic bosons and use the Ward identities to re-express that in terms of the original partition function.

The discussion in appendix A makes it clear that such a description, though, is dangerously simplistic. The space of conformal blocks should really be thought of as a complex of (infinite-dimensional) vector bundles with a D-module structure. The cohomology of that complex away from the Θ -divisor is the naive one-dimensional space of conformal blocks, but a lot of structure and hidden components may be present at the Θ -divisor itself.

There is a simple trick which produces examples of such non-trivial components of the space of conformal blocks: start from the standard partition function and correlation functions and take a discontinuity across the Θ -divisor, transforming the poles into delta functions. Equivalently, we can act on the standard conformal blocks with a ∂^\dagger operator along $\text{Bun}_0(U(1), C)$, which again transforms poles into delta functions. This agrees with the expectation from finite-dimensional analogue systems in appendix A that extra non-trivial components may be found in cohomological degree -1 at loci where a pair of dual zeromodes appear.

The example of genus 1 conformal blocks is already rather instructive. For a generic point in $\text{Bun}_0(U(1), E_\tau)$ parameterized by the variable x , the partition function and correlation functions take the form⁴

$$\begin{aligned} \langle 1 \rangle &= \frac{\eta(\tau)}{\theta(x, \tau)} \\ \langle X(z)Y(w) \rangle &= 2\pi i \frac{\eta(\tau)^4}{\theta(x, \tau)^2} \frac{\theta(z-w+x, \tau)}{\theta(z-w, \tau)} \\ &\dots \end{aligned} \tag{3.10}$$

Taking the discontinuity at $x = 0$ we get our candidate hidden conformal block:

$$\begin{aligned} \langle 1 \rangle &= \frac{1}{\eta^2(\tau)} \delta(x) \\ \langle X(z)Y(w) \rangle &= \frac{1}{\eta^2(\tau)} \delta'(x) + \frac{1}{\eta^2(\tau)} \frac{\theta'(z-w, \tau)}{\theta(z-w, \tau)} \delta(x) \\ &\dots \end{aligned} \tag{3.11}$$

⁴To check this formulae, observe that $\langle X(z)Y(w) \rangle$ is the unique meromorphic section with a single pole or residue $\langle 1 \rangle$ and that it gives the correct stress-tensor 1-pt function proportional to $\partial_\tau \langle 1 \rangle$.

The “partition function” is a natural regularization of the naive path integral, with zero-modes removed. The correlation function may appear worrisome because $\frac{\theta'(z-w,\tau)}{\theta(z-w,\tau)}$ shifts by a constant as $z \rightarrow z - \tau$. This compensates, though, the fact that $e^{2\pi i x} \delta'(x) = \delta'(x) - 2\pi i \delta(x)$. Thus the correlation function is still a section of the correct bundle.

Similar considerations apply in higher genus, though new components in even lower cohomological degree may appear at special loci in the Θ divisor.

If the $U(1)$ line bundle has degree greater than 0, $Y(z)$ will generically have d zero-modes while the equations of motion for $X(z)$ will be obstructed. The opposite occurs in negative degree. Solving the Ward identities will simply be generically impossible and the (cohomology of the complex of) conformal blocks will be generically trivial. The D-module of conformal blocks, though is still non-trivial: the finite-dimensional example in appendix A suggests that non-trivial solutions of Ward identities appear at the co-dimension $d+1$ locus Θ_d where $X(z)$ acquires at least one zero-modes and thus $Y(z)$ has $d+1$ zero-modes.

Concretely, the correlation functions are expected to vanish unless we have d more X insertions than Y insertions. At the co-dimension $d+1$ locus Θ_d in $\text{Bun}_d(U(1), E_\tau)$ where $X(z)$ has some zero-modes $\rho(z)$, we can postulate

$$\langle X(z_1) \cdots X(z_d) \rangle = \rho(z_1) \cdots \rho(z_d) \delta_{\Theta_d}^{(d)} \tag{3.12}$$

Because of the existence of $d+1$ obstructions for the equations of motion of X , we can only find the Green’s function if we allow for logarithmic monodromies in d directions, as on the degree 0 case. We expect to be able to compensate for that using the d normal derivatives of the δ_{Θ_d} , as before. Thus the next non-trivial correlation function will have a schematic form

$$\langle X(z_1) \cdots X(z_d) X(z_{d+1}) Y(w) \rangle = \sum_a \prod_{b \neq a} \rho(z_b) \left(G(z_a, w) \delta_{\Theta_d}^{(d)} + g(z_a, w) \cdot \partial \delta_{\Theta_d}^{(d)} \right) \tag{3.13}$$

etcetera.

Notice that both the Θ -divisor and the Θ_d loci for $g-1 > d > 0$ can be parameterized nicely by the divisor given by the $g-1-d$ zeroes of the $X(z)$ zero-modes. At degree $d = g-1$ the locus Θ_{g-1} consists of the trivial bundle only and the X zero-modes is constant. For d greater than $g-1$ we do not expect any interesting conformal blocks. This agrees with the classical picture described in [1]. Similar considerations apply for negative d .

3.2.1 Twisted modules

In the presence of a background $U(1)$ connection one can consider twisted sectors for the symplectic boson, where the mode expansion is shifted appropriately

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{X_{n+\alpha-\frac{1}{2}}}{z^{n+\alpha}} \quad Y(z) = \sum_{n=-\infty}^{\infty} \frac{Y_{n-\alpha-\frac{1}{2}}}{z^{n-\alpha}} \tag{3.14}$$

For $\alpha \neq \frac{1}{2}$ we have natural highest weight modules annihilated by the positive modes. These modules will appear, say, when we sew up a Riemann surface in a gauge where

X and Y have non-trivial periodicity around the handles. For future reference, we can compute

$$\langle \alpha | X(z)Y(w) | \alpha \rangle = \frac{w^\alpha}{z^\alpha} \frac{1}{z-w} \tag{3.15}$$

leading to the $U(1)$ charge and scaling dimension of the highest weight vectors

$$\langle \alpha | J(z) | \alpha \rangle = -\frac{\alpha}{z} \quad \langle \alpha | T(z) | \alpha \rangle = -\frac{\alpha^2}{2z^2} \tag{3.16}$$

These twisted modules can be obtained from the standard vacuum module by a singular gauge transformation.

At $\alpha = \frac{1}{2}$ we need to consider the various possible Ramond sector modules. Approaching $\alpha = \frac{1}{2}$ from above or below one gets the $|R, \pm\rangle$ modules. Instead the general Ramond modules give us

$$\langle R, \lambda | X(z)Y(w) | R, \lambda \rangle = \frac{w^{\frac{1}{2}}}{z^{\frac{1}{2}}} \frac{1}{z-w} + \frac{\lambda}{z^{\frac{1}{2}}w^{\frac{1}{2}}} \tag{3.17}$$

and thus

$$\langle R, \lambda | J(z) | R, \lambda \rangle = \left(\lambda - \frac{1}{2} \right) \frac{1}{z} \quad \langle R, \lambda | T(z) | R, \lambda \rangle = -\frac{1}{8z^2} \tag{3.18}$$

This module is not obtained by a singular gauge transformation of the vacuum module. We expect it to play an important role in the sewing construction of the non-standard conformal blocks described above. It also plays an important role in the bosonization of the XY system, which will be a crucial ingredient in the study of Abelian mirror symmetry and S-duality.

For example, the characters and traces over the modules generated from $|\alpha\rangle$ or $|R, \pm\rangle$ all essentially give the same $\frac{\eta(\tau)}{\theta(x,\tau)}$ torus partition function and associated correlation functions, with $x = \tau(\alpha - \frac{1}{2}) + \beta$ and β being the $U(1)$ fugacity. On the other hand, the characters and traces of the modules generated from $|R, \lambda\rangle$ give the $\frac{1}{\eta^2(\tau)}\delta(x)$ torus partition function and associated correlation functions.

3.2.2 Current subalgebras: $SU(2)_{-\frac{1}{2}}$

The symplectic boson vertex algebra actually contains a full set of $SU(2)_{-\frac{1}{2}}$ WZW currents:⁵

$$J^- = \frac{1}{2}X^2 \quad J^3 = \frac{1}{2}XY \quad J^+ = \frac{1}{2}Y^2 \tag{3.20}$$

⁵We can verify the level from the OPEs:

$$\begin{aligned} J^3(z)J^3(w) &\sim -\frac{1}{4} \frac{1}{(z-w)^2} \\ J^3(z)J^\pm(w) &\sim \pm \frac{J^\pm}{z-w} \\ J^-(z)J^+(w) &\sim \frac{1}{2} \frac{1}{(z-w)^2} + \frac{2J^3}{(z-w)}. \end{aligned} \tag{3.19}$$

Notice that X and Y can be identified with the spin $\frac{1}{2}$ primaries Z_α for the $SU(2)$ current algebra: the dimension of a spin $\frac{1}{2}$ primary is precisely $\frac{1}{2}$. Furthermore, the Sugawara stress tensor can be computed from

$$: J^3 J^3 := \frac{1}{4} X^2 Y^2 + \frac{1}{4} X \partial Y - \frac{1}{4} Y \partial X \quad (3.21)$$

and

$$: J^- J^+ := \frac{1}{4} X^2 Y^2 + Y \partial X \quad : J^+ J^- := \frac{1}{4} X^2 Y^2 - X \partial Y \quad (3.22)$$

so that

$$: J^3 J^3 : - \frac{1}{2} : J^- J^+ : - \frac{1}{2} : J^+ J^- : = \frac{3}{2} T \quad (3.23)$$

Thus T coincides with the Sugawara stress tensor for $SU(2)_{-\frac{1}{2}}$.

The currents of integral spin in the symplectic boson current algebra can be organized into the vacuum module of $SU(2)_{-\frac{1}{2}}$, while the currents of half-integral spin can be organized into the spin $\frac{1}{2}$ module of $SU(2)_{-\frac{1}{2}}$. Thus we can envision the symplectic boson VOA as an extension of the $SU(2)_{-\frac{1}{2}}$ WZW VOA.

We can use the $SU(2)_{-\frac{1}{2}}$ WZW symmetry to couple the symplectic boson system to $SU(2)$ bundles. Again, it is actually most natural to couple the symplectic boson to an $SU(2)$ version of a $\text{Spin}_{\mathbb{C}}$ structure: rather than picking an $SU(2)$ bundle E and combining it with a spin structure, we can give the product $E \otimes K^{\frac{1}{2}}$ an intrinsic meaning. This should correspond to E being a section of a certain gerbe.

From the four-dimensional perspective, this is due to the Z_2 anomaly of a single half-hypermultiplet coupled to $SU(2)$ gauge fields, which is cancelled by anomaly inflow from a non-trivial discrete theta angle in the four-dimensional bulk. The bulk theory with such a discrete theta angle is conventionally denoted as $\text{Sp}(1)'$ and is mapped to itself by S-duality. Correspondingly, the space of solutions of Hitchin equations twisted by that gerbe should be self-mirror.

The partition function of the symplectic boson coupled to the twisted $SU(2)$ bundle is

$$Z_{C,E} = \frac{1}{\sqrt{\det \bar{\partial}_{E \otimes K^{\frac{1}{2}}}}} \quad (3.24)$$

and has square-root singularities at the co-dimension 1 locus $\Theta_{SU(2)}$ where a zero mode appears. Notice that there is a Z_2 symmetry mapping $Z_\alpha(z) \rightarrow -Z_\alpha(z)$ and solutions of Ward identities built from this partition function involve correlation functions with an even number of Z_α insertions.

At that locus $\Theta_{SU(2)}$ we expect to also find a second conformal block (in cohomological degree 0, see examples in appendix A) which has zero partition function, but non-zero 1-point function

$$\langle Z_\alpha(z) \rangle = \rho_\alpha(z) \delta_{\Theta_{SU(2)}} \quad (3.25)$$

proportional to the zero mode $\rho_\alpha(z)$ and more general correlation functions of an odd number of fields involving the δ function at $\Theta_{SU(2)}$ and its derivatives.

3.3 Free hypermultiplets in a fundamental representation, H-twist

As a preparation for later sections, we should discuss briefly some features of the VOA $\text{Sb}[\mathbb{C}^{2N}]$ obtained as the product of N copies symplectic boson VOAs.

The vertex algebra has $2N$ bosonic generators, $X_a(z)$ and $Y^a(z)$, with OPE

$$X_a(z)Y^b(w) \sim \frac{\delta_a^b}{z-w} \quad (3.26)$$

and all other OPE trivial. All fields have conformal dimension $1/2$.

The stress tensor can be taken to be

$$T = \frac{1}{2}X_a\partial Y^a - \frac{1}{2}Y^a\partial X_a \quad (3.27)$$

with central charge $-N$.

The current algebra contains a WZW $\text{Sp}(N)_{-\frac{1}{2}}$ current subalgebra:

$$J_{ab}^- = \frac{1}{2}X_aX_b \quad J_a^b = \frac{1}{2}X_aY^b \quad J_+^{ab} = \frac{1}{2}Y^aY^b \quad (3.28)$$

Here J_a^b are the currents for an $\text{U}(N)_{-1}$ current subalgebra and J_{ab}^- , J_+^{ab} the remaining currents in $\text{Sp}(N)_{-\frac{1}{2}}$. Furthermore, T coincides with the Sugawara stress tensor for $\text{Sp}(N)_{-\frac{1}{2}}$.⁶ The central charge matches as well. Thus $\text{Sb}[\mathbb{C}^{2N}]$ can be interpreted as an extension of an $\text{Sp}(N)_{-\frac{1}{2}}$ VOA. The (X_a, Y^a) fields can be identified with the $\text{Sp}(N)_{-\frac{1}{2}}$ primaries in the fundamental representation.

We can also focus on the $\text{U}(N)_{-1}$ currents Notice also that

$$: J_a^a J_b^b := \frac{1}{4}X_aX_bY^aY^b + \frac{1}{4}X_a\partial Y^a - \frac{1}{4}Y^a\partial X_a \quad (3.32)$$

and thus

$$T = \frac{2}{N-1} : J_a^b J_b^a : - \frac{2}{N-1} : J_a^a J_b^b := T_{\text{SU}(N)_{-1}} + T_{\text{U}(1)} \quad (3.33)$$

Thus T also coincides with the Sugawara stress tensor for $\text{U}(N)_{-1}$. The central charge matches as well.

The X_a and Y^a fields can be identified with the $\text{U}(N)_{-1}$ primaries in the fundamental or anti-fundamental representation. Notice that the dimension $1/2$ receives a contribution $1/2 + 1/(2N)$ from $\text{SU}(N)_{-1}$ and $-1/(2N)$ from $\text{U}(1)$.

⁶In detail,

$$: J_a^b J_b^a := \frac{1}{4}X_aX_bY^aY^b + \frac{N}{4}X_a\partial Y^a - \frac{N}{4}Y^a\partial X_a \quad (3.29)$$

and

$$: J_{ab}^- J_+^{ab} := \frac{1}{4}X_aX_bY^aY^b + \frac{N+1}{2}Y^a\partial X_a \quad : J_+^{ab} J_{ab}^- := \frac{1}{4}X_aX_bY^aY^b - \frac{N+1}{2}X_a\partial Y^a \quad (3.30)$$

so that

$$: J_a^b J_b^a : - \frac{1}{2} : J_{ab}^- J_+^{ab} : - \frac{1}{2} : J_+^{ab} J_{ab}^- := \left(N + \frac{1}{2}\right) T. \quad (3.31)$$

The vertex algebra of N symplectic bosons should contain infinitely many $U(N)_{-1}$ primaries. For example, the symmetric polynomials $X_{a_1} \cdots X_{a_n}$ should be $U(N)_{-1}$ primaries labelled by the symmetric powers of the fundamental representation, and $Y^{a_1} \cdots Y^{a_n}$ should be $U(N)_{-1}$ primaries labelled by the symmetric powers of the anti-fundamental representation. There may be other primaries as well, hidden deeper into the symplectic bosons Verma module.

The current algebras we identified imply that the VOA of N symplectic bosons will give D-modules on $\text{Bun}(\text{Sp}(N), C)$ (or better, the modification of that which parameterizes bundles of the form $E_{\text{Sp}(N)} \otimes K^{\frac{1}{2}}$) or on $\text{Bun}(U(N), C)$ (or better, the modification of that which parameterizes bundles of the form $E_{U(N)} \otimes K^{\frac{1}{2}}$).

These D-modules encode the BAA branes associated to certain boundary conditions for the corresponding four-dimensional gauge theories.

The S-dual of these boundary conditions is known. For the $\text{Sp}(N)'$ boundary condition (here the prime indicates the presence of a discrete θ angle, which makes the $\text{Sp}(N)'$ theory self-S-dual) that is a maximal Nahm pole. For the $U(N)$ boundary condition that is a sub-regular Nahm pole, breaking the gauge group to a $U(1)$ subgroup, which is gauged at the boundary. The BBB image of Nahm pole boundary conditions is poorly understood, though. It would be very interesting to test this expectation, say by computing Hecke modifications of the symplectic bosons D-module.

Another setup involving fundamental hypermultiplets is that of a D5 interface between two $U(N)$ theories or a half-D5 between an $\text{Sp}(N)$ and an $\text{Sp}(N)'$ theory. The BAA image of that is a D-module on the product of two copies, say, of $\text{Bun}(U(N), C)$ localized on the diagonal. It can also be interpreted as a functor mapping D-modules on, say, $\text{Bun}(U(N), C)$ to D-modules on the same space. The functor consists simply of taking a tensor product with the D-module defined by the VOA of N symplectic bosons.

The S-dual of (half-)D5 interfaces are (half-)NS5 interfaces, which we will discuss momentarily [2, 16].

More general D5 interfaces can be defined between $U(N)$ and $U(M)$ gauge groups with different ranks $M < N$ (or symplectic groups or orthogonal with different ranks) but they are simpler and do not involve boundary degrees of freedom, only certain Nahm poles. They give D-modules on the product of $\text{Bun}(U(N), C)$ and $\text{Bun}(U(M), C)$ localized on the image of the block-diagonal embedding of $U(M)$ bundles into $U(N)$ bundles.

3.4 Bi-fundamental free hypermultiplets, H-twist

We now organize NM symplectic bosons into two $N \times M$ blocks, $X_a^i(z)$ and $Y_i^a(z)$, with OPE

$$X_a^i(z)Y_j^b(w) \sim \frac{\delta_a^b \delta_j^i}{z-w} \tag{3.34}$$

and all other OPE trivial. All fields have conformal dimension $1/2$.

The stress tensor can be taken to be

$$T = \frac{1}{2} X_a^i \partial Y_i^a - \frac{1}{2} Y_i^a \partial X_a^i \tag{3.35}$$

with central charge $-NM$.

We can define $SU(N)_{-M} \times SU(M)_{-N} \times U(1)$ currents

$$J_a^b = \frac{1}{2} X_a^i Y_i^b - \frac{\delta_a^b}{2N} X_c^i Y_i^c \quad J_j^i = \frac{1}{2} X_a^i Y_j^a - \frac{\delta_j^i}{2N} X_a^k Y_k^a \quad J = \frac{1}{2} X_a^i Y_i^a \quad (3.36)$$

and denote for convenience as \tilde{J}_a^b and \tilde{J}_j^i the bilinear currents without traces removed, which generate $U(N)_{-M}$ and $U(M)_{-N}$ current algebras.

We can compute

$$: \tilde{J}_a^b \tilde{J}_b^a := \frac{1}{4} X_a^i X_b^j Y_j^a Y_i^b + \frac{N}{4} X_a^i \partial Y_i^a - \frac{N}{4} Y_i^a \partial X_a^i \quad (3.37)$$

and

$$: \tilde{J}_j^i \tilde{J}_i^j := \frac{1}{4} X_a^i X_b^j Y_j^a Y_i^b + \frac{M}{4} X_a^i \partial Y_i^a - \frac{M}{4} Y_i^a \partial X_a^i \quad (3.38)$$

and thus

$$T = \frac{2}{N-M} : \tilde{J}_a^b \tilde{J}_b^a : - \frac{2}{N-M} : \tilde{J}_j^i \tilde{J}_i^j := T_{SU(N)_{-M}} + T_{SU(M)_{-N}} + T_{U(1)} \quad (3.39)$$

Thus T also coincides with the Sugawara stress tensor for the $SU(N)_{-M} \times SU(M)_{-N} \times U(1)$ currents. The central charge matches as well. This is a non-unitary analogue of level-rank duality.

The X_a^i and Y_i^a fields can be identified with the $SU(N)_{-M} \times SU(M)_{-N} \times U(1)$ primaries in the bi-fundamental representation. The symmetric polynomials $X_{a_1}^{i_1} \cdots X_{a_n}^{i_n}$ can be decomposed into sums of products of irreducible irreps of the permutation group, which will be primaries of $SU(N)_{-M} \times SU(M)_{-N} \times U(1)$ in the corresponding representations, and so on.

In a similar manner, the $N \times M$ hypermultiplets can be re-organized in terms of $Sp(N)_{-\frac{M}{2}}$ and $SO(M)_{-N}$ WZW current sub-algebra, i.e. transforming as the orthosymplectic version of a bi-fundamental field. We will discuss the $N = 1$ case momentarily.

Bi-fundamental hypermultiplets are the basic building blocks for NS5 and half-NS5 interfaces. They can give us D-modules on products of spaces of bundles or functors mapping D-modules on a space of bundles to another. These BAA objects will be dual to the BBB objects built from D5 and half-D5 interfaces.

3.5 N free hypermultiplets as $SU(2)$ doublets, H-twist

It is interesting to take N copies of the symplectic boson and look at the properties of the $SU(2)_{-N/2}$ current algebra which acts diagonally on them. We can organize the fields into $SO(N)$ fundamentals:

$$X_i(z) Y_j(w) \sim \frac{\delta_{ij}}{z-w} \quad (3.40)$$

The $SU(2)_{-N/2}$ currents take the form

$$J^- = \frac{1}{2} X_i X_i \quad J^3 = \frac{1}{2} X_i Y_i \quad J^+ = \frac{1}{2} Y_i Y_i \quad (3.41)$$

We also have an $SO(N)_{-2}$ current algebra (we normalize them in the same way as a level 1 currents $\psi_i\psi_j$ in a theory of N free fermions).

$$J_{ij} = \frac{1}{2}X_iY_j - \frac{1}{2}X_jY_i \tag{3.42}$$

The central charge at such level is $c_{SO(N)_{-2}} = -\frac{N(N-1)}{N-4}$, which combines with the central charge of the $SU(2)$ algebra $c_{SU(2)_{-N/2}} = \frac{3N}{N-4}$ to give the total central charge $-N$ of the symplectic bosons. The total stress tensor is actually the sum of the Sugawara stress tensors for the two current algebras.⁷

3.6 C-twist of a single hypermultiplet

The vertex algebra $\text{Fc}[\mathbb{C}^2]$ has two fermionic generators, $x(z)$ and $y(z)$, with OPE

$$x(z)y(w) \sim \frac{1}{(z-w)^2} \tag{3.47}$$

and conformal dimension 1. We can denote them as “fermionic currents”. They can be also thought of a $PSU(1|1)$ current algebra.

The stress tensor can be taken to be

$$T = -xy \tag{3.48}$$

and gives $c_{xy} = -2$.⁸

This vertex algebra can be found in several free CFTs, but in these realizations either x or y or both are derivatives of some dimension 0 operator. These realizations clearly produce some (sections of the sheaf of) conformal blocks, but not necessarily all of them.

⁷Indeed

$$: J^3 J^3 := \frac{1}{4}(X \cdot Y)^2 + \frac{1}{4}X \cdot \partial Y - \frac{1}{4}Y \cdot \partial X \tag{3.43}$$

and

$$: J^- J^+ := \frac{1}{4}X^2 Y^2 + Y \cdot \partial X \quad : J^+ J^- := \frac{1}{4}X^2 Y^2 - X \cdot \partial Y \tag{3.44}$$

On the other hand,

$$: J_{ij} J^{ij} := \frac{1}{2}X^2 Y^2 - \frac{1}{2}(X \cdot Y)^2 - \frac{N-1}{2}X \cdot \partial Y + \frac{N-1}{2}Y \cdot \partial X \tag{3.45}$$

so that

$$: J^3 J^3 : - \frac{1}{2} : J^- J^+ : - \frac{1}{2} : J^+ J^- : + \frac{1}{2} : J_{ij} J^{ij} := \frac{4-N}{2} T. \tag{3.46}$$

⁸Indeed, we have OPE

$$T(z)x(w) \sim \frac{x(w)}{(z-w)^2} + \frac{\partial x(w)}{(z-w)} \quad T(z)y(w) \sim \frac{y(w)}{(z-w)^2} + \frac{\partial y(w)}{(z-w)} \tag{3.49}$$

and

$$T(z)T(w) \sim -\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{3.50}$$

3.6.1 Coupling to flat bundles

Although the algebra has an $SU(2)_o$ outer automorphisms rotating x and y , it has no corresponding current algebra. When working in an $SU(2)_o$ covariant way, we can denote the currents as z_α .

The vertex algebra can be coupled to an $SU(2)_o$ complexified local system and the dependence on the holomorphic part of the connection will not drop out. We expect conformal blocks to define a sheaf on $\text{Loc}(SU(2), C)$. (We denote the group with the compact form, but we refer to local systems for the complexified group).

It is convenient to represent the local system as a D-module, i.e. prescribe an $SU(2)_o$ bundle E equipped with an holomorphic $SU(2)_o$ connection $A_{\alpha\beta}(z)$. The connection modifies the OPE to

$$z_\alpha(z)z_\beta(w) \sim \frac{\epsilon_{\alpha\beta}}{(z-w)^2} + \frac{A_{\alpha\beta}}{z-w} \tag{3.51}$$

In order to find conformal blocks we need to solve the Ward identities associated to these OPE with currents which are sections of E on C .

The space of conformal blocks for a generic local system has dimension 2^{2g-2} and can be identified with the Fock space built from the $(2g-2)$ holomorphic sections $\omega_\alpha^a(z)$ of $E \otimes K$. Essentially, we can postulate that correlation functions with less than n insertion vanish and correlation functions with exactly n insertions are

$$\langle Z_{\alpha_1}(z_1) \cdots Z_{\alpha_n}(z_n) \rangle \sim \omega_{\alpha_1}^{[a_1]}(z_1) \cdots \omega_{\alpha_n}^{[a_n]}(z_n) \tag{3.52}$$

Other correlation functions are determined from the Ward identities.

3.7 Free hypermultiplets in fundamental or bi-fundamental representations, C-twist

A collection of N fermionic currents $x^a(z)$ and $y_a(z)$, with OPE

$$x^a(z)y_b(w) \sim \frac{\delta_b^a}{(z-w)^2} \tag{3.53}$$

give a VOA $\text{Fc}[\mathbb{C}^{2N}]$ with an $\text{Sp}(N)_o$ group of outer automorphisms, with an obvious $U(N)_o$ subgroup. It can be coupled to an $\text{Sp}(N)$ or $U(N)$ bundles equipped with a holomorphic connection.

A collection of $N \times M$ fermionic currents $x_i^a(z)$ and $y_a^i(z)$, with OPE

$$x_i^a(z)y_b^j(w) \sim \frac{\delta_b^a \delta_i^j}{(z-w)^2} \tag{3.54}$$

has an obvious action of $U(N) \times U(M)$. It can be coupled to $U(N) \times U(M)$ bundles equipped with a holomorphic connection. Similar considerations apply to $\text{Sp}(N) \times \text{SO}(M)$ actions.

These VOA will appear when one studies the BBB images of D5 and NS5 interfaces.

4 Abelian examples

Mirror symmetry is well understood for Abelian gauge theories. This provides us with some important checks of our proposal.

Notably, a U(1) gauge theory (SQED) coupled to a single hypermultiplet of charge 1 is mirror to a single free hypermultiplet. All other Abelian mirror symmetries follow from repeated applications of this simple duality relationship. Another important example is SQED coupled to two hypermultiplets of charge 1, which gives a UV description of $T[\text{SU}(2)]$ and is self-mirror.

Furthermore, S-duality for a 4d U(1) gauge theory acts in a very simple manner on boundary degrees of freedom: a 3d theory T with a U(1) factor in G_H considered as a boundary condition for a 4d U(1) gauge theory is mapped to a S-dual theory T' obtained from T by gauging the U(1). The theory T' has an obvious $\text{U}(1)_C$ factor in G_C . Applying mirror symmetry to T' we obtain a new theory ${}^\vee T$ with a $\text{U}(1)_H$ factor in G_H , the S-dual to T .

If T is associated to a VOA \mathbb{A} , then T' is associated to the coset $\frac{\mathbb{A}}{\text{U}(1)}$. It should be possible to argue in general that the conformal blocks for \mathbb{A} and $\frac{\mathbb{A}}{\text{U}(1)}$ give Geometric Langland dual objects for a U(1) gauge group, by matching the coset construction (possibly in the BRTS formalism) with an appropriate Fourier-Mukai transformation.

4.1 SQED with one flavor, H-twist

Following our prescription, we need to take the coset of the XY system by the U(1) current algebra generated by J^3 . The coset will be endowed with a $G_C = \text{U}(1)_o$ global symmetry. If our prescription is correct, we should obtain the same VOA as in the C-twist of a single free hypermultiplet.

Taking cosets by Abelian current algebras is a relatively simple procedure: we take primary operators of charge q under J_3 and strip off a U(1) vertex operator of charge q . For the XY model, this is essentially the standard bosonization of a $\beta\gamma$ system: we write

$$J^3 = \frac{1}{2}\partial\phi \quad X = e^{-\phi}x \quad Y = e^{\phi}y \tag{4.1}$$

The notation x and y is completely intentional: x and y are fermionic currents of conformal dimension 1, charge ± 1 under $\text{U}(1)_o$ and free OPE

$$x(z)y(w) \sim \frac{1}{(z-w)^2} \tag{4.2}$$

Notice that the central charges match: $c_{XY} = c_{xy} + c_{J^3} = -2 + 1$.

Taking the coset of the XY system by the algebra generated by J_3 leaves us with the algebra of x and y . This is beautifully consistent with the mirror symmetry relation between SQED with one flavor and a theory of a free hypermultiplet. Notice that the $\text{U}(1)_o$ global symmetry of the coset coincides with the $\text{U}(1)_o$ global symmetry of the x and y fermionic currents. The bosonization relation can be stated as

$$\text{Fc}[\mathbb{C}^2] = \frac{\text{Sb}[\mathbb{C}^2]}{\widehat{\text{U}}(1)_{-1}} \tag{4.3}$$

i.e.

$$\mathbb{A}_H[\text{SQED}_1] = \mathbb{A}_C[\text{Free hyper}] \tag{4.4}$$

It is also straightforward, but rather non-trivial, to verify that the characters of the vacuum module for the symplectic bosons decomposes appropriately into free bosons characters and characters for the fermionic currents: we can expand

$$\chi^{XY} = \frac{1}{\prod_{n=0}^{\infty} (1 - q^{n+1})^2} \sum_{n=0}^{\infty} \sum_{m=-n}^n z^m (-1)^{n-m} q^{\frac{n(n+1)-m^2}{2}} \tag{4.5}$$

The replacement

$$\chi_m^{U(1)}(q, z) = \frac{z^m q^{-\frac{m^2}{2}}}{\prod_{n=0}^{\infty} (1 - q^{n+1})} \rightarrow t^m \tag{4.6}$$

corresponds to stripping off the free boson Verma module, while keeping track of the $U(1)_o$ charge. It would map

$$\chi^{XY} \rightarrow \frac{1}{\prod_{n=0}^{\infty} (1 - q^{n+1})} \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \frac{t^{n+1/2} + t^{-n-1/2}}{t^{1/2} + t^{-1/2}} \tag{4.7}$$

which can be rewritten as

$$\chi^{XY} \rightarrow \frac{1}{\prod_{n=0}^{\infty} (1 - q^{n+1})} \frac{1}{t^{1/2} + t^{-1/2}} \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} t^{n+1/2} \tag{4.8}$$

and then

$$\chi^{XY} \rightarrow \prod_{n=0}^{\infty} (1 - tq^{n+1})(1 - t^{-1}q^{n+1}) \tag{4.9}$$

The right hand side is the character for the fermionic current algebra, graded by the $U(1)_o$ charge. Thus we can write

$$\chi^{XY} = \sum_{n=-\infty}^{\infty} \chi_n^{U(1)} \chi_n^{xy} \tag{4.10}$$

where χ_n^{xy} is the charge n part of the character of the xy fermionic current algebra.⁹

⁹The bosonization relation between X, Y and x, y is simple, but already for correlation functions on the sphere it leads to intricate identities between rational functions. As an example, we can look at a four-point function:

$$\langle X(z)X(z')Y(w)Y(w') \rangle = \frac{1}{(z-w)(z'-w')} + \frac{1}{(z-w')(z'-w)} \tag{4.11}$$

The $U(1)$ part of the correlation function is $\frac{(z-w)(z-w')(z'-w)(z'-w')}{(z-z')(w-w')}$. Stripping it off we get the rational function

$$\frac{(z-z')(w-w')}{(z-w)^2(z'-w')^2(z-w')(z'-w)} + \frac{(z-z')(w-w')}{(z-w)(z'-w')(z-w')^2(z'-w)^2} \tag{4.12}$$

which can be reorganized to

$$\langle x(z)x(z')y(w)y(w') \rangle = \frac{1}{(z-w')^2(z'-w)^2} - \frac{1}{(z-w)^2(z'-w')^2} \tag{4.13}$$

4.1.1 Twisted sector

We have observed that generic $U(1)$ twisted sectors $|\alpha\rangle$ for the symplectic bosons have a $U(1)$ charge proportional to the $U(1)$ twist. Indeed, it is well-known that such twisted sectors can be bosonized to the basic vertex operators $e^{\alpha\phi}$. The conformal dimensions match and these vertex operators induce the correct monodromy in $e^{\pm\phi}$. In particular, they are mapped back to the vacuum module under bosonization.

In order to find twisted sectors for the fermionic current VOA, we need to look at the $|R, \lambda\rangle$ general Ramond modules for the symplectic bosons. Indeed, these have $U(1)$ charge $\lambda - \frac{1}{2}$ and the corresponding $U(1)$ vertex operator $e^{(\lambda - \frac{1}{2})\phi}$ would induce a monodromy $-e^{\pm 2\pi\lambda}$ on the $e^{\pm\phi}$ vertex operators which appear in the symplectic bosons. Thus the $|R, \lambda\rangle$ general Ramond modules should contain a twisted sector with monodromy $e^{\pm 2\pi\lambda}$ for the fermionic currents.

Because of the role the $|R, \lambda\rangle$ general Ramond modules play in defining the hidden conformal blocks of the symplectic boson VOA, this fact also suggests that the hidden blocks should bear some relationship to the conformal blocks for the fermionic currents VOA coupled to general $U(1)$ local systems.

4.1.2 Coset vs BRST reduction

If we are given a vertex algebra which has a level 0 $U(1)$ current J , a nice BRST construction becomes available: we can add a system of bc ghosts of dimensions $(1, 0)$ and define the BRST charge

$$Q = \oint cJ \tag{4.14}$$

This will have the effect of removing from the theory J and all operators which are charged under J . Operators in the original VOA which have trivial OPE with J will remain as BRST-closed operators.

In our setup, the symplectic bosons have a $U(1)$ current of level -1 . We can add to them a standard system of complex fermions

$$\psi(z)\chi(w) \sim \frac{1}{z-w} \tag{4.15}$$

which have a single conformal block and a $U(1)$ current at level 1 and then take the BRST reduction with respect to $J = XY + \psi\chi$.

The result of this BRST reduction appears to be the same as the coset we discuss in this section, including the sectors of non-trivial $U(1)_o$ charge: operators of charge n in the theory of symplectic bosons can be dressed with charge n primaries of the free fermion VOA in order to give BRST cohomology classes, which have the same dimension and properties as the corresponding operators in the coset.

For example, we would identify the basic BRSt closed operators with the fermionic currents

$$X(z)\psi(z) \rightarrow x(z) \qquad Y(z)\chi(z) \rightarrow y(z) \tag{4.16}$$

The BRST reduction is a bit more systematic than the coset. In particular, it gives a more precise way to built conformal blocks of the coset theory, rather than attempting an

expansion of symplectic boson correlation functions into a product of $U(1)_{-1}$ conformal blocks and coset blocks.

Notice the OPE

$$X(z)\psi(z)Y(w)\chi(w) \sim \frac{1}{(z-w)^2} + \frac{J(w)}{z-w} \tag{4.17}$$

As the $U(1)_o$ symmetry is identified with the global symmetry of the ψ and χ fermions, we can couple the system naturally to an $U(1)_o$ bundle by coupling the fermions themselves. Furthermore, we can add a coupling to an $U(1)_o$ holomorphic connection $A_0(z)$ by deforming the BRST operator to

$$Q = \oint dz c(z) (J(z) - A_o(z)) \tag{4.18}$$

This allows us to couple the coset theory to a full $U(1)_o$ local system.

The bc system has an anomaly, which forces us to introduce at least g insertions of $b(z)$ and one insertion of $c(z)$. The corresponding correlation function is

$$\langle b(z_1) \cdots b(z_g) c(w) \rangle \sim \det_{ij} \omega^{a_i}(z_j) \tag{4.19}$$

where $\omega^{a_i}(z_j)$ is a basis of holomorphic differentials.

The $b(z)$ insertions are not BRST closed! Rather,

$$\{Q, b(z)\} = J(z) - A_o(z) \tag{4.20}$$

If we contract the $b(z)$ insertion with an anti-holomorphic differential $\delta\bar{A}$ then the first term in the right hand side $\int dz d\bar{z} J(z) \delta\bar{A}$ is a total derivative along $Bun(U(1), C)$.

In the absence of $A_o(z)$, that means that we can identify the correlation functions of the combined system of symplectic bosons, complex fermions and ghosts as a top holomorphic form in $Bun(U(1), C)$, mapped to an exact form by Q . Integrating the correlation function over a middle-dimensional cycle $Bun(U(1), C)$ gives a BRST-invariant answer, which we plan to identify with a conformal block for the coset theory, the fermionic currents:

$$\begin{aligned} & \langle x(z_1) \cdots y(w_1) \cdots \rangle_{\text{Fc}[C^2], \bar{A}_0; \Gamma} \\ &= \oint_{\Gamma \in Bun(U(1), C)} D\bar{A} \langle X(z_1) \cdots Y(w_1) \cdots \rangle_{\text{Sb}[C^2], \bar{A}} \langle \psi(z_1) \cdots \chi(w_1) \cdots \rangle_{\psi\chi, \bar{A} + \bar{A}_o} \end{aligned} \tag{4.21}$$

where the measure $D\bar{A}$ is given by the bc system correlation function (4.19).

In the presence of $A_o(z)$, the BRST transformation of the measure involves an extra constant 1-form $\int dz d\bar{z} A_o(z) \delta\bar{A}$ on the $Bun(U(1), C)$ torus. That means that the correlation function becomes BRST closed when the measure (4.19) is multiplied by an appropriate Fourier kernel $e^{S[A_o, \bar{A}]}$ such that $\frac{\delta S}{\delta \bar{A}} = A_o(z)$.

This seems a rather reasonable way to do a Fourier-Mukai-like transformation mapping the D-module of conformal blocks for symplectic bosons to the sheaf of conformal blocks for the fermionic currents. This should be a direct manifestation of the fact that S-duality for

U(1) gauge theory maps the boundary condition associated to a single free hypermultiplet back to itself.

It would be nice to mimic in this setup the classical mirror symmetry relationship described in [1]. A crucial role there was played by the $2g - 2$ points on the surface where the U(1) Higgs field vanishes and by the 2^{2g-2} ways they could be distributed between the X and Y classical sections.

The symplectic boson theory has conformal blocks which are localized on the Θ_d locus in $\text{Bun}_d(\text{U}(1), C)$ where zeromodes appear, which is parameterized by the $g - 1 - d$ zeroes of the negative charge zeromode for positive d and by the $g - 1 + d$ zeroes of the positive charge zeromode for non-positive d .

We expect these conformal blocks to map to the conformal blocks of fermionic currents which have a $\text{U}(1)_o$ anomaly d . A possible explanation would be that for a general point on $\text{Loc}(C, \text{U}(1))$ the integrand of (4.21) is not single-valued on $\text{Bun}(\text{U}(1), C)$. Good integration cycles would consist of a small loop around Θ_d times a contour integral over Θ_d . It would be nice to give a detailed derivation of this relationship.

We can give a toy demonstration of this for a torus partition function. The contour integral for a partition function is

$$\oint d a \eta^2(\tau) e^{2\pi i a b_0} \frac{\eta(\tau)}{\theta(a, \tau)} \frac{\theta(a + a_o, \tau)}{\eta(\tau)} \tag{4.22}$$

Here the first factor is the bc partition function, followed by the Fourier kernel, the XY partition function and the $\psi\chi$ partition functions.

If we take the contour to run around the pole at $a = 0$ we get

$$\oint_0 d a \eta^2(\tau) e^{2\pi i a b_0} \frac{\eta(\tau)}{\theta(a, \tau)} \frac{\theta(a + a_o, \tau)}{\eta(\tau)} = \frac{\theta(a_o, \tau)}{\eta(\tau)} \tag{4.23}$$

which is the character for the fermionic currents in a general $\text{U}(1)_o$ background, with no zeromodes.

On the other hand, if b_0 is 0 (or an integer n) we can take a contour integral on the A cycle of the torus and the calculation mimics the character computations done earlier in the section, yielding the vacuum character $\chi_{xy}(a_0, \tau)$. Contour integrals along other cycles of the torus impose other linear constraints on b_0 and a_0 and give other modular images of $\chi_{xy}(a_0, \tau)$.

4.2 SQED with N flavors, H-twist

We start from N copies of the symplectic bosons and take a coset by the diagonal U(1) current. According to our prescription

$$\mathbb{A}_H[\text{SQED}_N] = \frac{\text{Sb}[\mathbb{C}^{2N}]}{\widehat{\text{U}}(1)_{-N}} \tag{4.24}$$

The coset operation will strip U(1) vertex operators of appropriate charge from the symplectic bosons X_a and Y^a . The resulting fields A_a, B^a can be thought as $\text{SU}(N)_{-1}$ primaries in

the fundamental or anti-fundamental representation, of conformal dimension $1/2 + 1/(2N)$ and charge ± 1 under $U(1)_o$.

More generally, symmetric polynomials of X_a or symmetric polynomials of Y^a will become WZW primaries of conformal dimension $n/2 + n^2/(2N)$ with charge $\pm n$ under $U(1)_o$.

The coset vertex algebra contains an $SU(N)_{-1}$ current algebra and the stress tensor is the Sugawara stress tensor. The coset current algebra, though, is larger than $SU(N)_{-1}$. It includes, for example, the fields which arises from the N -th symmetric powers of X_a and Y_a , which have dimension N and $U(1)_o$ charge 1. It also includes operators with no $U(1)_o$ charge, such as an operator O_a^b of dimension 2 in the adjoint of $SU(N)_{-1}$, built by removing the $U(1)$ contributions from $X_a \partial Y^b - Y^b \partial X_a$.

As the dimension 3 operators with no $U(1)_o$ charge seem to be accounted fully by currents or derivatives of O_a^b , it would appear that the $SU(N)_{-1}$ currents, together with O_a^b , have a closed set of OPE, with the OO OPE involving bilinears and derivatives of the currents themselves. The existence of such a self-consistent, associative OPE is typically rather non-trivial and could be taken perhaps as an alternative definition of the subsector with no $U(1)_o$ charge in the coset VOA.

It is interesting to mimic the basic trick of Abelian mirror symmetry: apply the basic mirror symmetry operation to all hypermultiplets. We can apply the bosonization formula to each symplectic boson:

$$X_a Y^a = \partial \phi_a \quad X_a = e^{-\phi_a} x_a \quad Y^a = e^{\phi_a} y^a \tag{4.25}$$

This bosonization hides the $U(N)_{-1}$ current algebra, leaving only a Cartan subalgebra manifest.

The coset by the diagonal $U(1)$ current is now elementary: we simply impose the constraint $\sum_a \phi_a = 0$. This gives a free field realization of the coset vertex algebra in terms of $\text{Fc}[\mathbb{C}^{2N}]$ and $N - 1$ free bosons.

The $N = 2$ example is rather special and deserves a separate treatment. We will come back to that momentarily.

Finally, we can consider a BRST reduction which should reproduce the coset VOA: we couple the N symplectic bosons to a $U(1)_N$ current algebra in order to get a level 0 current J . Adding a set of bc ghosts gives us a BRST current cJ and we can pass to BRST cohomology. Again, the extra $U(1)$ current, the ghosts and the $U(1)_{-N}$ currents will cancel out and leave the coset fields behind.

The conformal blocks for the coset VOA will define a D-module on $\text{Bun}(SU(N), C)$ which is also a sheaf on $\text{Loc}(U(1)_o, C)$. If this theory is used to define a boundary condition for a 4d $SU(N)$ gauge theory, the result is the S-dual of the sub-regular Nahm pole boundary condition breaking the gauge group to a $U(1)$ subgroup, which has Dirichlet b.c. and whose background connection is the point in $\text{Loc}(U(1)_o, C)$. This statement should have a direct interpretation in terms of Hecke modifications of the D-module on $\text{Bun}(SU(N), C)$. It would be nice to understand this better.

4.3 The $T[\text{SU}(2)]$ theory, H-twist

This is a self-mirror theory with an enhancement to $\text{SU}(2)_o$ of the naive $\text{U}(1)_o$ tri-holomorphic isometry of the Coulomb branch. We would like to see if the symmetry enhancement is manifest in our proposed current algebra. If so, this will be a strong test of our proposal. Furthermore, this theory plays a role in S-duality for four-dimensional $\mathcal{N} = 4$ SYM with an $\text{SU}(2)$ gauge group. The corresponding vertex algebra should be a duality kernel in Geometric Langlands for the group $\text{SU}(2)$.

A computation of the character for the coset of two symplectic bosons by the diagonal $\text{U}(1)$ current algebra produces a very pleasing result:

$$\chi_{\mathbb{A}_H[T[\text{SU}(2)]]} = \sum_{j=0}^{\infty} (t^j + t^{j-1} + \dots + t^{-j}) \chi_j[\text{SU}(2)_{-1}] \quad (4.26)$$

where t is the fugacity for the $\text{U}(1)_o$ global symmetry.

We can recast the character as

$$\chi_{\mathbb{A}_H[T[\text{SU}(2)]]} = \sum_{j=0}^{\infty} \chi_j[\text{SU}(2)_o] \chi_j[\text{SU}(2)_{-1}] \quad (4.27)$$

This is clearly compatible with the idea that the outer $\text{U}(1)$ global symmetry has been promoted to an outer $\text{SU}(2)_o$ global symmetry, in such a way that the spin j primaries of the $\text{SU}(2)_{-1}$ WZW current algebra transform in a spin j representation of $\text{SU}(2)_o$. The $\text{SU}(2)_o$ enhancement seems to be previously known to experts in the subject.

We can give a bosonized description of the coset as follows:

$$\begin{aligned} J &= X_a Y^a = \partial\phi \\ X_a &= e^{-\phi} A_a \quad Y^a = e^{\phi} B^a \end{aligned} \quad (4.28)$$

The fields A_a and B^a have dimension $3/4$. The outer $\text{SU}(2)_o$ global symmetry will rotate among each other A_a and $B_a \equiv \epsilon_{ab} B^b$.

We can make the $\text{SU}(2)_o$ global symmetry manifest if we look at the free field realization:

$$\begin{aligned} A_1 &= e^{-\tilde{\phi}} x_1 & B^1 &= e^{\tilde{\phi}} y^1 \\ A_2 &= e^{\tilde{\phi}} x_2 & B^2 &= e^{-\tilde{\phi}} y^2 \end{aligned} \quad (4.29)$$

The $\text{SU}(2)_o$ global symmetry is a subgroup of the automorphism group of the four fermionic currents x_a, y^a , which rotates the doublets (x_1, y^2) and $(x_2, -y^1)$.

In a better notation, we can write the parameterization as

$$C_1^\alpha = e^{-\tilde{\phi}} z_1^\alpha \quad C_2^\alpha = e^{\tilde{\phi}} z_2^\alpha \quad (4.30)$$

with OPE

$$z_1^\alpha(z) z_2^\beta(w) \sim \frac{\epsilon^{\alpha\beta}}{(z-w)^2} \quad (4.31)$$

with $SU(2)_o$ -invariant WZW currents

$$\begin{aligned} J^3 &= \frac{1}{2} \partial \tilde{\phi} \\ J^- &= e^{-2\tilde{\phi}} z_1^\alpha z_1^\beta \epsilon_{\alpha\beta} \\ J^+ &= e^{2\tilde{\phi}} z_2^\alpha z_2^\beta \epsilon_{\alpha\beta} \end{aligned} \tag{4.32}$$

Going back to the symplectic bosons, we can use a notation

$$X_a(z) Y_b(w) \sim \frac{\epsilon_{ab}}{z-w} \tag{4.33}$$

The coset current algebra includes $SU(2)_{-1}$ WZW currents and also a $SU(2)_o$ triplet of operators of dimension 2. The triplet consists of the operator O_{ab}^0 built from $X_{(a} \partial Y_{b)} - Y_{(a} \partial X_{b)}$ and the two operators

$$O_{ab}^+ = (X_a X_b) \quad O_{ab}^- = (Y_a Y_b) \tag{4.34}$$

where the parenthesis denotes removing the $U(1)$ primary.

Again, it seems likely that the O_{ab}^I operators form a closed OPE with the current algebra fields. The OPE is actually manifestly $SU(2)_o$ invariant [17]. This offers a potential route to study conformal blocks coupled to $SU(2)_o$ local systems by algebraic methods. It would be very interesting to do so.

Thus our proposal is that the kernel for Geometric Langlands with gauge group $SU(2)$:

$$\mathbb{A}_H[T[SU(2)]] = \frac{\text{Sb}[\mathbb{C}^4]}{\hat{U}(1)_{-2}} \tag{4.35}$$

4.4 The A_{k-1} theory, H-twist

Another natural example to consider is the linear quiver of $k-1$ $U(1)$ nodes, with a single flavor at each end. This theory is expected to have an enhancement of the Coulomb branch symmetry from $U(1)_C^{k-1}$ to $SU(k)_C$: it is the mirror of SQED with k flavors.

If we use bosonization of the k symplectic bosons, the enhancement is automatic: the coset removes all $U(1)$ currents except for the diagonal combination, leaving behind

$$A_a = e^{-\phi} x_a \quad B^a = e^{\phi} y^a \tag{4.36}$$

with $SU(k)_o$ global symmetry and $U(1)$ current algebra, as expected

4.5 The C-twist of general Abelian gauge theories

Abelian mirror symmetry is rather well understood and reduces to the basic mirror symmetry of SQED with one flavor.

If we start from a general Abelian theory, build the mirror Abelian theory and H -twist it, we should have the same result as if we C -twist the original theory. Furthermore, if we bosonize all the symplectic bosons in the mirror theory, the bosonized coset algebra can be expressed directly in terms of the original Abelian theory: we take a set of fermionic

currents x_a, y_a for each of the original hypermultiplets, add a free boson for each factor in the gauge group and dress the fermionic currents with gauge symmetry q by a free boson vertex operator of charge q .

The resulting candidate $\mathbb{A}_C[T]$ vertex algebra has an Abelian current algebra generated by the free bosons, which we identify with the current algebra for G_C , and a global symmetry algebra G_H .

It would be very interesting to find a non-Abelian generalization of this construction. Perhaps one can associate WZW currents to vectormultiplets and build some VOA by dressing WZW primary fields with the fermionic currents associated to the matter hypermultiplets.

5 Non-Abelian examples: SU(2) gauge theories

5.1 The SU(2) SQCD theory with four flavors ($G_H = \text{SO}(8)$), H-twist

This theory is a building block for several other examples in this paper. It is the dimensional reduction of a four-dimensional superconformal theory, a fact which allows us do some comparisons with the four-dimensional chiral algebras. It also inherits from four dimensions a non-trivial triality symmetry: in the IR, it becomes invariant under discrete symmetry transformations which act as triality on $\text{SO}(8)_H$. This symmetry is not a manifest symmetry in the UV gauge theory description.

The coset vertex algebra must consist of a collection of modules for the $\text{SO}(8)_{-2}$ current algebra. Remember that the coset stress tensor coincides with the Sugawara tensor for the $\text{SO}(8)_{-2}$ current algebra. The BRST procedure used in the 4d setup produces directly the irreducible vacuum module for the $\text{SO}(8)_{-2}$ current algebra itself, which is triality invariant.

In order to understand the coset procedure, we can begin to experiment with characters, decomposing the character of the symplectic bosons into characters for Verma modules of the $\text{SU}(2)_{-4}$ current algebra. This is likely too naive, but it is a reasonable starting point. We can write

$$\chi_{Sb[\mathbb{C}^{16}]} = \sum_j \chi_{\text{SU}(2)}[V_j] \chi_j \tag{5.1}$$

The tentative character for the vacuum representation χ_0 is *not* invariant under triality acting on the $\text{SO}(8)$ fugacities. Thus it is likely not the right answer for the coset vacuum character. In particular, it does not coincide with the $\text{SO}(8)_{-2}$ vacuum character.

On the other hand, something surprising happens: the combination

$$\tilde{\chi}_0 = \chi_0 - \chi_1 + \chi_2 - \dots \tag{5.2}$$

appears to be triality invariant and to coincide with the $\text{SO}(8)_{-2}$ vacuum character. It is natural to expect this is the correct answer for the \mathbb{A}_H vertex algebra for this theory.

A possible justification of this answer is that the $\text{SU}(2)_{-4}$ modules inside the symplectic bosons current algebra may be larger than Verma modules. For example, if the symplectic bosons current algebra involves the spin 0 module and an extension built out of the spin 0 and spin 1 $\text{SU}(2)_{-4}$ Verma modules, the coefficient of the spin 0 module would have to be smaller than the naive answer and an expression such as $\tilde{\chi}_0$ may appear.

In order to seek illumination, we can look at concrete expressions for operators in the coset. We may start from SU(2) invariant operators. Denoting the symplectic bosons as Z_i^α with i being the SO(8) index and α the SU(2) index, the SO(8)₋₂ currents are $\epsilon_{\alpha\beta} : Z_i^\alpha Z_j^\beta \text{ :}$.

At level 2, we have $\epsilon_{\alpha\beta} : Z_i^\alpha \partial Z_j^\beta \text{ :}$, which includes both a symmetric and an anti-symmetric tensors of SO(8), and $\epsilon_{\alpha\beta\epsilon\gamma\delta} : Z_i^\alpha Z_j^\beta Z_k^\gamma Z_l^\delta \text{ :}$. The symmetric traceless tensor is mapped under triality to the self-dual and anti-selfdual 4-forms, which are not present in the field built from four Z s. Thus the symmetric traceless tensor at level 2 is potentially problematic. This is also the representation which appears in the leading term in χ_1 , from $: Z_i^{(\alpha} Z_j^{\beta)} \text{ :}$.

Acting with SU(2)₋₄ current algebra operators onto $: Z_i^{(\alpha} Z_j^{\beta)} \text{ :}$ we find indeed that we can produce $\epsilon_{\alpha\beta} : Z_i^\alpha \partial Z_j^\beta \text{ :}$, which is thus secretly an SU(2)₋₄ descendant. this verifies the presence of indecomposable representations built from the vacuum and symmetric traceless tensor Verma modules.

This supports the conjecture that

$$\frac{\text{Sb}[\mathbb{C}^{16}]}{\widehat{SU}(2)_{-4}} = \widehat{SO}(8)_{-2} \tag{5.3}$$

5.1.1 The T[SU(3)] theory, H-twist

This theory is a two node quiver, with U(1) × U(2) gauge group and 3 extra flavors at the U(2) gauge node.

This is a self-mirror theory with an enhancement to SU(3)_C of the naive U(1)_C² tri-holomorphic isometry of the Coulomb branch. We would like to see if the symmetry enhancement is manifest in our proposed current algebra. If so, this will be a very strong test of our proposal. Furthermore, this theory plays a role in S-duality for four-dimensional $\mathcal{N} = 4$ SYM with an SU(3) gauge group. The corresponding vertex algebra should be a duality kernel in Geometric Langlands for the group SU(3).

Conveniently, this theory is obtained by gauging two U(1) symmetries of the SU(2) SQCD theory with four flavors. This will help our analysis. A naive computation of the character runs into the same type of trouble we encountered with this ancestor theory: if we decompose

$$\chi_{SB} = \sum_{j,n,m} \chi_{SU(2)}[V_j] \chi_n^{U(1)} \chi_m^{U(1)'} \chi_{j,n,m} \tag{5.4}$$

then the naive characters

$$\chi_j = \sum_{n,m} t_1^n t_2^m \chi_{j,n,m} \tag{5.5}$$

do *not* manifest any enhancement of the U(1)_o² naive global symmetry of the coset.

On the other hand, if we take the same combination of naive characters as before,

$$\tilde{\chi}_0 = \chi_0 - \chi_1 + \chi_2 - \dots \tag{5.6}$$

we get a striking result:

$$\tilde{\chi}_0 = \sum_{\lambda} \chi_{\lambda}[SU(3)^o](t_1, t_2) \chi_{\lambda}[SU(3)_{-2}] \tag{5.7}$$

where $\chi_\lambda[\text{SU}(3)^o]$ are the characters of finite-dimensional irreps of the $\text{SU}(3)^o$ global symmetry group and $\chi_\lambda[\text{SU}(3)_{-2}]$ are the characters of the Verma modules for the $\text{SU}(3)_{-2}$ current algebra.

This is clearly compatible with the idea that the outer $\text{U}(1)_o^2$ global symmetry has been promoted to an outer $\text{SU}(3)^o$ global symmetry, in such a way that the primaries of the $\text{SU}(3)_{-2}$ WZW current algebra transform in the corresponding representation of $\text{SU}(3)^o$.

A simple way to build currents for the coset is to start from $\text{SU}(2)$ -invariant expressions. In terms of elementary fields $X_a, Y_a, X_{i,a}, Y_a^i$, for example, we can write the three fields in the 3 representation as

$$X_{i,a}X_b\epsilon^{ab} \quad X_{i,a}Y_b\epsilon^{ab} \quad \epsilon_{ijk}Y_a^jY_b^k\epsilon^{ab} \tag{5.8}$$

of dimension 4/3.

In order to find the conjectural $\text{SU}(3)^o$ octet of adjoint currents, we need to go to level 3. We can combine pairs of the fields above and their conjugate. Again, these octet currents should form a closed current algebra OPE.

According to the discussion of the previous section, we can describe the coset as a $\text{U}(1)^2$ coset of the $\text{SO}(8)_{-2}$ vacuum module. These are the Abelian factors in the subgroup $\text{U}(1) \times \text{U}(3) \subset \text{U}(4) \subset \text{SO}(8)$. The triality symmetry leaves the $\text{SU}(3)$ subgroup unaffected, while rotating the two $\text{U}(1)$ into each other.

This description is very invaluable, as it makes the promotion of $\text{U}(1)_o^2$ to $\text{SU}(3)_o$ evident: triality acts on the $\text{U}(1)_o^2$ charges of coset operators as the Weyl group, and the promotion of the $\text{U}(1)_o$ associated to the $\text{U}(1)$ node of the quiver to an $\text{SU}(2)_o$ was made manifest by bosonization.

This supports the conjecture that

$$\mathbb{A}_H[T[\text{SU}(3)]] = \frac{\widehat{\text{SO}}(8)_{-2}}{\text{U}(1)_{-2} \otimes \text{U}(1)_{-6}} \tag{5.9}$$

where $\text{U}(1) \otimes \text{U}(1)$ is the subgroup of $\text{SO}(8)$ which commutes with an $\text{SU}(3)$ subgroup under which the fundamental of $\text{SO}(8)$ decomposes as $8 = 3 + \bar{3} + 1 + 1$.

6 Non-Abelian examples: unitary quivers

6.1 $T[\text{SU}(N)]$, H-twist

This theory is defined by a linear quiver of $\text{U}(1) \times \text{U}(2) \times \dots \times \text{U}(N-1)$ gauge groups, with N flavors for the last node.

$$\mathbb{A}_H[T[\text{SU}(N)]] = \frac{\text{Sb}[\mathbb{C}^{2N(N-1)}] \times \text{Sb}[\mathbb{C}^{2(N-1)(N-2)}] \dots \times \text{Sb}[\mathbb{C}^2]}{\hat{\text{U}}(N-1)_{2-2N} \dots \hat{\text{U}}(1)_{-2}} \tag{6.1}$$

The G_H current algebra is thus $\text{SU}(N)_{1-N}$. Notice that all nodes have level twice the critical level for the non-Abelian gauge fields. If we remove the Abelian factors, the resulting theory has a four-dimensional superconformal ancestor. Let's denote the quiver without Abelian factors as $\tilde{T}[\text{SU}(N)]$.

Both physically and at the level of the four-dimensional chiral algebra, the $\tilde{T}[\text{SU}(N)]$ quiver has an enhanced symmetry which generalizes the triality relation we encountered for $N = 2$. The enhanced symmetry acts on the $U(1)_H^{N-1}$ symmetries in the same way as the Weyl group acts on the Cartan torus of $\text{SU}(N)$. This is inherited from the class S description. Furthermore, the four-dimensional current algebra consists of a sum or products of modules for the $\text{SU}(N)_{1-N}$ WZW currents tensored with modules for the $U(1)_H^{N-1}$ WZW currents.

Under the assumption that our coset operation produces the same answer as the BRST reduction which gives the four-dimensional chiral algebra, we immediately learn that $\mathbb{A}_H[T[\text{SU}(N)]]$ has an enhanced $\text{SU}(N)_o$ global symmetry! Indeed, the triality-like global symmetry of $\mathbb{A}_H[\tilde{T}[\text{SU}(N)]]$ will persist in $\mathbb{A}_H[T[\text{SU}(N)]]$, acting on the $U(1)_o^{N-1}$ charges as a Weyl group. This symmetry, combined with the known symmetry enhancement at the $U(1)$ node of $\mathbb{A}_H[T[\text{SU}(N)]]$, implies the enhancement of $U(1)_o^{N-1}$ to $\text{SU}(N)_o$.

We furthermore conjecture a character

$$\chi_{\mathbb{A}_H[T[\text{SU}(N)]]} = \sum_{\lambda} \chi_{\lambda}[\text{SU}(N)_o](t) \chi_{\lambda}[\text{SU}(N)_{1-N}] \tag{6.2}$$

6.2 General global symmetry enhancement

We can now formulate a general strategy to argue that the H-twist current algebra for a general unitary quiver gauge theory has the expected enhancement of G_C .

In the UV, G_C consists of a product of $U(1)$ factors, one for each unitary group in the quiver. At “balanced” nodes where the total number of flavors, including bifundamental hypers to nearby nodes and fundamental hypers to a framing node, equals twice the rank of the unitary group, the $U(1)$ global symmetry is enhanced to $\text{SU}(2)$. The $\text{SU}(2)$ symmetry groups at nearby balanced nodes combine into larger groups: $\text{SU}(k + 1)$ for a chain of k balanced nodes and more generally an ADE group G_{Γ} for an ADE sub-graph Γ of balanced nodes.

The potential symmetry enhancement of the current algebra associated to a quiver gauge theory can also be studied “node by node”: we can first take the coset by the unitary group at a balanced node, and then by the remaining gauge groups. If the first step produces VOA with enhanced global symmetry $\text{SU}(2)_o$ commuting with the WZW symmetry used in the next step of the coset, that $\text{SU}(2)_o$ will persist at the next step of the calculation.

We do not know how to demonstrate directly that the VOA for the $U(N)$ gauge theory with $2N$ flavors has an outer $\text{SU}(2)_o$, though the conjectural relation with 4d calculations would make the Z_2 Weyl group of $\text{SU}(2)_o$ manifest. Perhaps a direct bootstrap of the VOA from a finite set of generators demonstrating $\text{SU}(2)_o$ would be possible.

On the other hand, the VOA for the $U(N)$ gauge theory with $2N$ flavors can be used in the calculation of the VOA for $T[\text{SU}(N + k)]$ quivers. There the $U(1) \times Z_2$ symmetry of the $U(N)$ node is embedded into an $\text{SU}(2)_o$ after the coset, which involves WZW currents which commute with $U(1) \times Z_2$. This makes it at least very plausible that the VOA for the $U(N)$ gauge theory with $2N$ flavors does indeed have $\text{SU}(2)_o$ global symmetry.

With that assumption, the S_{k+1} Weyl group symmetry for chain of k balanced nodes follow from the conjectural relation to the 4d calculations and combines with the $SU(2)_o$ at individual nodes to give the expected Coulomb branch symmetry enhancement.

A D-modules and Ward identities

A.1 Examples in T^*C

As a first toy example, consider the contour integral defining an Airy function

$$\text{Ai}(x) = \oint e^{\frac{z^3}{3} + xz} dz \tag{A.1}$$

The differential equation for the Airy function follows from some integration by parts:

$$\partial_x \text{Ai}(x) = \oint z e^{\frac{z^3}{3} + xz} dz \quad \partial_x^2 \text{Ai}(x) = \oint z^2 e^{\frac{z^3}{3} + xz} dz = -x \text{Ai}(x) \tag{A.2}$$

The two solutions to the differential equation can be obtained by selecting different integration contours. The differential equation can be cast as an holomorphic connection

$$\partial_x - \begin{pmatrix} 0 & 1 \\ -x & 0 \end{pmatrix} \tag{A.3}$$

One could consider the whole vector space of correlation functions

$$f_n(x) = \oint z^n e^{\frac{z^3}{3} + xz} dz \tag{A.4}$$

subject to the Ward identity

$$f_{n+2}(x) + x f_n(x) = \oint z^n \partial_z \left(e^{\frac{z^3}{3} + xz} \right) dz = -n f_{n-1}(x) \tag{A.5}$$

with the differential acting as $\partial_x f_n(x) = f_{n+1}(x)$. Of course, the Ward identity allows one to reduce the whole tower to $f_0(x)$ and $f_1(x)$ and the action of the differential to the 2×2 connection above.

Finally, in order to package the Ward identity in a better format, we can think about it as the differential on a complex. The complex consists of holomorphic differential forms on the complex plane parameterized by z . The differential is

$$d : \omega \rightarrow \partial^{(z)} \omega + (z^2 + x) \omega \tag{A.6}$$

where $\partial^{(z)}$ acts on the z direction only. Forms closed under the differential can be used in contours integrals of the form

$$f_\omega(x) = \oint \omega e^{\frac{z^3}{3} + xz} dz \tag{A.7}$$

and exact forms integrate to zero. The differential commutes with the holomorphic connection

$$\partial_x f_\omega(x) = f_{\partial_x \omega + z \omega}(x) \tag{A.8}$$

The description of the Airy function D-module as an infinite-dimensional complex may seem rather redundant compared to the simple 2×2 holomorphic connection we started with. It may be better suited, though, if one needs to describe the D-module as an object in some derived category of D-modules.

This may appear to be only a matter of mathematical formalization, but it is likely to become a bit more physical if we want to use this setup in order to define a complex Lagrangian (BAA) brane in $T^*\mathbb{C} \simeq \mathbb{C}^2$ supported on $p^2 + x = 0$ as a deformation of elementary BAA branes of the form $p = z$.

Turning z into a dynamical 1d chiral multiplet with a $\frac{z^3}{3}$ boundary superpotential defines a BAA boundary condition supported $p^2 + x = 0$ which we expect to precisely correspond to the above D-module. The infinite-dimensional complex is simply the Chan-Paton bundle defined by these auxiliary 1d degrees of freedom.

As the support of the D-brane is smooth, we could of course directly define the corresponding $p^2 + x = 0$ BPS boundary condition in the (4,4) sigma model with target \mathbb{C}^2 . Converting the physical boundary condition into the data of a D-module, would then require extra work, such as computing the space of A-type morphisms from the brane to the elementary BAA branes supported on the constant x fibers of $T^*\mathbb{C}$ and the parallel transport along the space of fibers.

In order to appreciate better the relative usefulness of different D-module descriptions, we can look at more singular examples. It is convenient to include a formal \hbar quantization variable in our formulae to help with semi-classical limits.

We will look at D-modules modelled on general integrals of the form

$$\oint \omega(z, x) e^{\frac{W(x, z)}{\hbar}} \tag{A.9}$$

with $\omega(z)$ being a holomorphic form on some auxiliary space Z parameterized by z .

The differential on the complex of holomorphic forms on Z will be

$$d : \omega \rightarrow \hbar \partial^Z \omega + \partial^Z W \wedge \omega \tag{A.10}$$

and the holomorphic connection

$$\hat{p} = \hbar \partial_x \omega + \partial_x W \omega \tag{A.11}$$

It is useful to shift the degrees of forms so that the top form has degree 0.

Recall that the brane wrapping $p = 0$ is described by the trivial D-module consisting of polynomials of x acted upon by $\hat{p} = \hbar \partial_x$. The brane wrapping $x = 0$ is described by the Fourier transform of that, modelled on the contour integral

$$\delta(x) = \oint e^{xz} dz \tag{A.12}$$

Here the complex is generated by the forms $x^n z^m$ and $x^n z^m dz$, with differential

$$x^n z^m \xrightarrow{d} (\hbar m x^n z^{m-1} + x^{n+1} z^m) dz \tag{A.13}$$

The cohomology consists of vectors of the form $z^n dz$ with multiplication by x acting as $-\hbar \partial_z$ and \hat{p} acting as multiplication by z , as expected.

A simple but non-trivial example is modelled on the Gaussian integral

$$\oint \omega(z, x) e^{\frac{1}{\hbar} x z^2} \quad (\text{A.14})$$

The complex is generated by the forms $x^n z^m$ and $x^n z^m dz$, with differential

$$x^n z^m \xrightarrow{d} (\hbar m x^n z^{m-1} + 2x^{n+1} z^{m+1}) dz \quad (\text{A.15})$$

and action (with or without an overall dz)

$$\hat{p}(x^n z^m) = \hbar n x^{n-1} z^m + x^n z^{m+2} \quad (\text{A.16})$$

The cohomology consists of vectors of the form $x^n dz$ and vectors of the form $z^m dz$. The multiplication by x acts as

$$\hat{x}(x^n dz) = x^{n+1} dz \quad \hat{x}(z^m dz) = -\frac{\hbar}{2}(m-1)z^{m-2} dz \quad (\text{A.17})$$

and \hat{p} acts as

$$\hat{p}(x^n dz) = \hbar \left(n - \frac{1}{2} \right) x^{n-1} dz \quad \hat{p}(z^m dz) = z^{m+2} dz \quad (\text{A.18})$$

We thus find the direct sum of two modules for the \hat{x}, \hat{p} Heisemberg algebra: one consisting of vectors of the form $x^n dz$ and $z^{2n} dz$ and one consisting of the vectors of the form $z^{2n+1} dz$. The latter summand represents a copy of the trivial brane wrapping $x = 0$. This is evident under the change of variables $z^2 \rightarrow z$. The former summand represents a single brane supported classically on $xp = 0$, distinct from the simple sum of the two components.

If we were to avoid looking too closely at $x = 0$, we may describe the D-module as a meromorphic connection with a regular singularity at the origin, something like $\hbar \left(\partial_x + \frac{1}{2x} \right)$. This would hide a whole extra brane sitting at $x = 0$!

A simple way to understand the existence of the extra component is to observe that the measure allows for a change of variables

$$\oint z^{2n+1} x^m e^{\frac{1}{\hbar} x z^2} = \frac{1}{2} \oint d(z^2) (z^2)^n x^m e^{\frac{1}{\hbar} x (z^2)} \quad (\text{A.19})$$

making the identification with a $x = 0$ brane obvious.

The system admits an interesting deformation, modelled on

$$\oint \omega(z, x) e^{\frac{1}{\hbar} (x z^2 + 2az)} \quad (\text{A.20})$$

This integral maps dz to $x^{-\frac{1}{2}} e^{-\frac{a^2}{\hbar x}}$ which suggests a single smooth brane supported on $p = \frac{a^2}{x^2}$. The deformation makes it quite clear that in the $a \rightarrow 0$ limit the brane wraps twice the $x = 0$ plane and once the $p = 0$ plane.

Next, consider the following example with two auxiliary fields, modelled on

$$\oint \omega(u, v, x) e^{\frac{1}{\hbar} x uv} \quad (\text{A.21})$$

Notice immediately that the system has an additional $U(1)_{uv}$ symmetry rotating u and v in opposite directions. The symmetry implies that the cohomology is $U(1)_{uv}$ -invariant, as the Lie derivative of a form along the $U(1)_{uv}$ vector field equals the anti-commutator of d with the operation of contraction with the $U(1)_{uv}$ vector field. Concretely, the integrand must preserve the $U(1)_{uv}$ for the integral to be non-zero.

The cohomology in degree 0 is intuitive: it consists of forms $x^n dudv$ and $(uv)^n dudv$. The multiplication by x acts as

$$\hat{x}(x^n dudv) = x^{n+1} dudv \quad \hat{x}(u^n v^n dudv) = \hbar n (uv)^{n-1} dudv \quad (\text{A.22})$$

and \hat{p} acts as

$$\hat{p}(x^n dudv) = \hbar(n-1)x^{n-1} dudv \quad \hat{p}(u^n v^n dudv) = (uv)^{n+1} dudv \quad (\text{A.23})$$

This module is a non-trivial extension of the basic $x = 0$ and $p = 0$ modules: the forms with positive powers of x form a sub-module, but $dudv$ is mapped to $xdudv$ by multiplication by x .

It is important to observe that there is also cohomology in degree -1 , generated by $udv + vdu$. It corresponds to the integral

$$\oint d(uv) e^{\frac{1}{\hbar} xuv} \quad (\text{A.24})$$

which is related to an $x = 0$ brane by the obvious change of variable. Indeed, the module consists of forms $(uv)^n d(uv)$ with \hat{p} acting as multiplication by uv and multiplication by x as derivative by uv . Thus the full system involves two branes, one in degree 0 and one in degree -1 . Although the two branes live in different degrees, they can communicate by morphisms of degree 1, i.e. extensions.

Next, we can consider a system involving N auxiliary variables z_i , modelled on

$$\oint \omega(z, x) e^{\frac{1}{2\hbar} \sum_i (x - a_i) z_i^2} \quad (\text{A.25})$$

If we assume that the a_i constants are all different, the cohomology appears to consist of $N + 1$ separate modules. One module is generated by the top form $\prod_i dz_i$. It is a complicated module, supported on $p \prod_i (x - a_i) = 0$. The other modules are generated by $z_k \prod_i dz_i$ and are isomorphic to the basic module associated to $x = a_i$.

This system is a particular case of the general

$$\oint \omega(z, x) e^{\frac{1}{2\hbar} z^t (x - A) z} \quad (\text{A.26})$$

where the $N \times N$ constant symmetric matrix A is taken to have N distinct eigenvalues.

This general model, perhaps re-written as

$$\oint \omega(z, x) e^{\frac{1}{2\hbar} z^t M(x) z} \quad (\text{A.27})$$

for some $N \times N$ matrix $M(x)$ which depends linearly on x , is a reasonable toy model to describe the D-modules one can encounter along one-dimensional slices of a parameter

space. It shows how one can get interesting extra cohomology supported on the locus where $M(x)$ has zeromodes.

In the $U(1)$ -invariant version of this problem, i.e.

$$\oint \omega(u, v, x) e^{\frac{1}{2\hbar} u^t M(x) v} \tag{A.28}$$

we can model the effect of an action which has k zeromodes for generic values of x , simply by taking M to be an $N \times N + k$ matrix. Notice that the $U(1)$ symmetry requires the integrand to have k more u 's than v 's

A.2 An extra examples in $T^*\mathbb{C}^2$

Consider the D-module on \mathbb{C}^2 modelled on

$$\oint \omega(u, v, w, x, y) e^{\frac{1}{\hbar}(xuv+yuw)} \tag{A.29}$$

with x, y coordinates on \mathbb{C}^2 in $T^*\mathbb{C}^2$.

This is a toy model for a situation where generically the path integral has a bosonic zeromode of charge 1 (here $xv + yw$) but at a special co-dimension 2 locus has 2 zeromodes of charge 1 and one of charge -1 .

Because of the generic zeromode, $dudvdw$ is not a good measure of integration:

$$dudvdwe^{\frac{1}{\hbar}(xuv+yuw)} = d\left((-udvdw - vdudw + wdudv)e^{\frac{1}{\hbar}(xuv+yuw)}\right) \tag{A.30}$$

Instead, we can find non-trivial cohomology in degree -1 :

$$\oint d(uv)d(uw)e^{\frac{1}{\hbar}(xuv+yuw)} \tag{A.31}$$

simply represents the standard brane at $x = y = 0$.

This example has obvious higher dimensional generalizations.

A.3 Matrix examples

It is instructive to look at a more general family of problems with bosonic zeromodes.

A.4 Symplectic bosons conformal blocks

The simple finite-dimensional examples we considered in the previous section are actually quite close to our main subject of interest: conformal blocks for symplectic bosons transforming in a representation M of some group G , coupled to some G -bundle E with associated bundle M_E .

The path integral is holomorphic, with action $\langle Z, \bar{\partial}_E Z \rangle$. General correlation functions can be written as

$$\oint \omega[Z] e^{\langle Z, \bar{\partial}_E Z \rangle} \tag{A.32}$$

with $\omega[Z]$ including the path integral measure and the fields inserted in the correlation function.

In analogy to the finite-dimensional case, one can consider a D-module on $\text{Bun}(G, C)$ built as a complex of possible $\omega[Z]$, with a differential

$$d : \omega \rightarrow \hbar \partial^Z \omega + \int_C \langle \delta Z, \bar{\partial}_E Z \rangle \wedge \omega \tag{A.33}$$

which guarantees that the overall path integral is at least formally a closed form and Ward identities are satisfied.

In other words, the cohomology of the complex is what we would usually call a conformal block: a collection of correlation functions which solve the Ward identities. As in the finite-dimensional examples, the space of conformal blocks will jump in complicated ways at loci where the number of zero-modes of the $\bar{\partial}_E$ operator changes.

Because of that, rather than passing to the cohomology it is natural to define formally the space of conformal blocks and its D-module structure in a derived sense as the whole complex, at least formally.

In order to give a somewhat less formal definition, one could seek some sort of intermediate description, which integrates out most of the degrees of freedom in the Z fields but still leaves a finite-dimensional path integral undone and describes conformal blocks as a non-trivial complex which does not jump wildly as we move along $\text{Bun}(G, C)$.

A suggestion may come from the standard sewing construction of Riemann surfaces and conformal blocks. In a gauge where the symplectic bosons are single-valued around the sewing fixtures, the conformal block is assembled from n -point functions of vacuum descendants on spheres. The latter are fully and easily determined by the Ward identities. The data of the bundle and all the subtleties which concern us arise in the gluing operation.

The whole data of the three-point functions can be expressed as a Gaussian functional of the Fourier modes $Z_{-n-\frac{1}{2}}^{(i)}$ of the symplectic bosons at the punctures. The gluing maps are also encoded into Gaussian functionals of the modes at the pairs of punctures being sewn together. Thus the whole conformal block becomes a Gaussian integral over the modes of the symplectic bosons at the punctures being sewn together.

This Gaussian integral is still infinite-dimensional, but it is simpler than the original path integral: effectively, we have integrated out all modes which are not holomorphic away from the sewing fixtures and we are only left with the work of imposing the gluing constraints. The D-module of conformal block should be well represented by this reduced Gaussian integral.

In order to simplify the problem further, we could try to integrate out non-holomorphic modes everywhere except than at a few selected points on the Riemann surface. That means adding sources to the equations of motion at these points and then requiring these sources to vanish by adding Lagrange multipliers.

Alternatively, it should be possible to give local descriptions in the neighbourhood of a point in $\text{Bun}(G, C)$ by computing the action at nearby points for the modes which are zero-modes at the point and modelling the conformal blocks on the resulting zero-mode path integral.

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