Twisted cubic and plane-line incidence matrix in PG(3,q)

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Abstract. We consider the structure of the plane-line incidence matrix of the projective space PG(3,q) with respect to the orbits of planes and lines under the stabilizer group of the twisted cubic. Structures of submatrices with incidences between a union of line orbits and an orbit of planes are investigated. For the unions consisting of two or three line orbits, the original submatrices are split into new ones, in which the incidences are also considered. For each submatrix (apart from the ones corresponding to a special type of lines), the numbers of lines in every plane and planes through every line are obtained. This corresponds to the numbers of ones in columns and rows of the submatrices.

Keywords: Twisted cubic, Projective space, Incidence matrix Mathematics Subject Classification (2010). 51E21, 51E22

1 Introduction

Let \mathbb{F}_q be the Galois field with q elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$. Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over \mathbb{F}_q . For an introduction to projective spaces over finite fields see [16,18,19].

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An *n*-arc in PG(N, q), with $n \ge N+1 \ge 3$, is a set of *n* points such that no N+1 points belong to the same hyperplane of PG(N, q). An *n*-arc is complete if it is not contained in an (n+1)-arc, see [1, 18, 19] and the references therein.

In PG(N,q), $2 \le N \le q-2$, a normal rational curve is any (q+1)-arc projectively equivalent to the arc $\{(t^N, t^{N-1}, \ldots, t^2, t, 1) : t \in \mathbb{F}_q\} \cup \{(1, 0, \ldots, 0)\}$. In PG(3,q), the normal rational curve is called a *twisted cubic* [17,19].

Twisted cubics in PG(3,q) and its connections with a number of other objects have been widely studied; see [17], the references therein, and [3,4,6–10,14,18,19,22,24]. In [17], the orbits of planes, lines and points in PG(3,q) under the group G_q of the projectivities fixing the twisted cubic are considered. In [13], the unions of line orbits considered in [17] are investigated in detail and split in separate orbits. In [3], the structure of the *point-plane* incidence matrix of PG(3,q) using orbits under G_q is described.

In this paper, we consider the structure of the plane-line incidence matrix of PG(3,q) with respect to G_q . We use the partitions of planes and lines into orbits and unions of orbits under G_q described in [13, 17]. We research the structures of the submatrices of incidences between an orbit of planes and a union of line orbits. For the unions consisting of two or three line orbits, the original submatrices are split into new ones, in which the incidences are also considered. For each submatrix (apart from the ones corresponding to a special type of lines), the numbers of lines in every plane and planes through every line are obtained. This corresponds to the numbers of ones in columns and rows of the submatrices.

Many submatrices considered are configurations in the sense of [15], see Definition 2.4 in Section 2.2. Such configurations are useful in several distinct areas, in particular, to construct bipartite graph codes without the so-called 4-cycles, see e.g. [2, 12, 20] and the references therein.

The theoretic results hold for $q \ge 5$. For q = 2, 3, 4, we describe the incidence matrices by computer search.

The results obtained increase our knowledge on the structure, properties, and incidences of planes and lines of PG(3, q).

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, the main results of this paper are summarized. Some useful relations are given in Section 4. The numbers of distinct planes in PG(3,q) through distinct lines and vice versa are obtained in Sections 5–7.

2 Preliminaries

2.1 Twisted cubic

We summarize the results on the twisted cubic of [17] useful in this paper.

Let $\mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of $\mathrm{PG}(3, q)$ with homogeneous coordinates $x_i \in \mathbb{F}_q$. For $t \in \mathbb{F}_q^+$, let P(t) be a point such that

$$P(t) = \mathbf{P}(t^3, t^2, t, 1) \text{ if } t \in \mathbb{F}_q; \quad P(\infty) = \mathbf{P}(1, 0, 0, 0).$$
 (2.1)

Let $\mathscr{C} \subset \mathrm{PG}(3,q)$ be the *twisted cubic* consisting of q+1 points P_1,\ldots,P_{q+1} no four of which are coplanar. We consider \mathscr{C} in the canonical form

$$\mathscr{C} = \{ P_1, P_2, \dots, P_{q+1} \} = \{ P(t) \mid t \in \mathbb{F}_q^+ \}.$$
(2.2)

A chord of \mathscr{C} is a line through a pair of real points of \mathscr{C} or a pair of complex conjugate points. In the last case it is an *imaginary chord*. If the real points are distinct, it is a *real chord*; if they coincide with each other, it is a *tangent*.

Let $\pi(c_0, c_1, c_2, c_3) \subset PG(3, q)$, be the plane with equation

$$c_0 x_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 = 0, \ c_i \in \mathbb{F}_q.$$

$$(2.3)$$

The osculating plane in the point $P(t) \in \mathcal{C}$ is as follows:

$$\pi_{\text{osc}}(t) = \boldsymbol{\pi}(1, -3t, 3t^2, -t^3) \text{ if } t \in \mathbb{F}_q; \ \pi_{\text{osc}}(\infty) = \boldsymbol{\pi}(0, 0, 0, 1).$$
(2.4)

The q+1 osculating planes form the osculating developable Γ to \mathscr{C} , that is a pencil of planes for $q \equiv 0 \pmod 3$ or a cubic developable for $q \not\equiv 0 \pmod 3$.

An axis of Γ is a line of PG(3,q) which is the intersection of a pair of real planes or complex conjugate planes of Γ . In the last case it is a generator or an imaginary axis. If the real planes are distinct it is a real axis; if they coincide with each other it is a tangent to \mathscr{C} .

For $q \not\equiv 0 \pmod{3}$, the null polarity \mathfrak{A} [16, Sections 2.1.5, 5.3], [17, Theorem 21.1.2] is given by

$$\mathbf{P}(x_0, x_1, x_2, x_3)\mathfrak{A} = \boldsymbol{\pi}(x_3, -3x_2, 3x_1, -x_0). \tag{2.5}$$

Notation 2.1. In future, we consider $q \equiv \xi \pmod{3}$ with $\xi \in \{-1,0,1\}$. Many values depend of ξ or have sense only for specific ξ . We note this by remarks or by superscripts " (ξ) ". If a value is the same for all q or a property holds for all q, or it is not relevant, or it is clear by the context, the remarks and superscripts " (ξ) " are not used. If a value is the same for $\xi = -1, 1$, then one may use " $\neq 0$ ". In superscripts, instead of " \bullet ", one can write "od" for odd q or "ev" for even q. If a value is the same for odd and even q, one may omit " \bullet ".

The following notation is used.

 G_q the group of projectivities in PG(3,q) fixing \mathscr{C} ;

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\mathbf{Z}_n
                     cyclic group of order n;
                     symmetric group of degree n;
\mathbf{S}_n
A^{tr}
                     the transposed matrix of A;
\#S
                     the cardinality of a set S;
\overline{AB}
                     the line through the points A and B.
                     Types \pi of planes:
                     an osculating plane of \Gamma;
\Gamma-plane
                     a plane containing exactly d distinct points of \mathscr{C}, d = 0, 2, 3;
d_{\mathscr{C}}-plane
\overline{1_{\mathscr{C}}}-plane
                     a plane not in \Gamma containing exactly 1 point of \mathscr{C};
                     the list of possible types \pi of planes, \mathfrak{P} \triangleq \{\Gamma, 2_{\mathscr{C}}, 3_{\mathscr{C}}, \overline{1_{\mathscr{C}}}, 0_{\mathscr{C}}\};
\mathfrak{P}
                     a plane of the type \pi \in \mathfrak{P};
\pi-plane
                     the orbit of \pi-planes under G_q, \pi \in \mathfrak{P}.
\mathscr{N}_{\pi}
                     Types \lambda of lines with respect to cubic \mathscr{C}:
RC-line
                     a real chord of \mathscr{C};
RA-line
                     a real axis of \Gamma for \xi \neq 0;
T-line
                     a tangent to \mathscr{C};
IC-line
                     an imaginary chord of \mathscr{C};
                     an imaginary axis of \Gamma for \xi \neq 0;
IA-line
U\Gamma
                     a non-tangent unisecant in a \Gamma-plane;
Un\Gamma-line
                     a unisecant not in a \Gamma-plane (it is always non-tangent);
E\Gamma-line
                     an external line in a \Gamma-plane (it cannot be a chord);
\mathrm{En}\Gamma-line
                     an external line, other than a chord, not in a \Gamma-plane;
A-line
                     the axis of \Gamma for \xi = 0
                     (it is the single line of intersection of all the q + 1 \Gamma-planes);
EA-line
                     an external line meeting the axis of \Gamma for \xi = 0;
\mathcal{L}^{(\xi)}
                     the list of possible types \lambda of lines,
                     \mathfrak{L}^{(\neq 0)} \triangleq \{ RC, RA, T, IC, IA, U\Gamma, Un\Gamma, E\Gamma, En\Gamma \} \text{ for } \xi \neq 0,
                     \mathfrak{L}^{(0)} \triangleq \{ RC, T, IC, U\Gamma, Un\Gamma, En\Gamma, A, EA \} \text{ for } \xi = 0;
                     a line of the type \lambda \in \mathfrak{L}^{(\xi)};
\lambda-line
L_{\Sigma}^{(\xi)}
                     the total number of orbits of lines in PG(3, q);
L_{\lambda\Sigma}^{(\xi)\bullet}
                     the total number of orbits of \lambda-lines, \lambda \in \mathfrak{L}^{(\xi)};
                     the union (class) of all L_{\lambda\Sigma}^{(\xi)\bullet} orbits of \lambda-lines under G_q, \lambda \in \mathfrak{L}^{(\xi)}.
\mathcal{O}_{\lambda}
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The following theorem summarizes results from [17] useful in this paper.

Theorem 2.2. [17, Chapter 21] The following properties of the twisted cubic \mathscr{C} of (2.2) hold:

- (i) The group G_q acts triply transitively on \mathscr{C} . Also,
- (a) $G_q \cong PGL(2,q)$ for $q \geq 5$; $G_4 \cong \mathbf{S}_5 \cong P\Gamma L(2,4) \cong \mathbf{Z}_2 PGL(2,4)$, $\#G_4 = 2 \cdot \#PGL(2,4) = 120$; $G_3 \cong \mathbf{S}_4 \mathbf{Z}_2^3$, $\#G_3 = 8 \cdot \#PGL(2,3) = 192$; $G_2 \cong \mathbf{S}_3 \mathbf{Z}_2^3$, $\#G_2 = 8 \cdot \#PGL(2,2) = 48$.
- (b) The matrix M corresponding to a projectivity of G_q has the general form

$$\mathbf{M} = \begin{bmatrix} a^{3} & a^{2}c & ac^{2} & c^{3} \\ 3a^{2}b & a^{2}d + 2abc & bc^{2} + 2acd & 3c^{2}d \\ 3ab^{2} & b^{2}c + 2abd & ad^{2} + 2bcd & 3cd^{2} \\ b^{3} & b^{2}d & bd^{2} & d^{3} \end{bmatrix}, \ a, b, c, d \in \mathbb{F}_{q},$$

$$ad - bc \neq 0.$$
(2.6)

(ii) Under G_q , $q \geq 5$, there are the following five orbits \mathcal{N}_{π} of planes:

$$\mathcal{N}_{1} = \mathcal{N}_{\Gamma} = \{\Gamma \text{-}planes\}, \quad \#\mathcal{N}_{\Gamma} = q + 1;$$

$$\mathcal{N}_{2} = \mathcal{N}_{2_{\mathscr{C}}} = \{2_{\mathscr{C}} \text{-}planes\}, \quad \#\mathcal{N}_{2_{\mathscr{C}}} = q^{2} + q;$$

$$\mathcal{N}_{3} = \mathcal{N}_{3_{\mathscr{C}}} = \{3_{\mathscr{C}} \text{-}planes\}, \quad \#\mathcal{N}_{3_{\mathscr{C}}} = (q^{3} - q)/6;$$

$$\mathcal{N}_{4} = \mathcal{N}_{\overline{1_{\mathscr{C}}}} = \{\overline{1_{\mathscr{C}}} \text{-}planes\}, \quad \#\mathcal{N}_{\overline{1_{\mathscr{C}}}} = (q^{3} - q)/2;$$

$$\mathcal{N}_{5} = \mathcal{N}_{0_{\mathscr{C}}} = \{0_{\mathscr{C}} \text{-}planes\}, \quad \#\mathcal{N}_{0_{\mathscr{C}}} = (q^{3} - q)/3.$$

$$(2.7)$$

- (iii) For $q \not\equiv 0 \pmod{3}$, the null polarity \mathfrak{A} (2.5) interchanges \mathscr{C} and Γ and their corresponding chords and axes.
- (iv) The lines of PG(3,q) can be partitioned into classes called \mathcal{O}_i and \mathcal{O}'_i , each of which is a union of orbits under G_q .

(a)
$$q \not\equiv 0 \pmod{3}$$
, $q \geq 5$, $\mathcal{O}'_i = \mathcal{O}_i \mathfrak{A}$, $\#\mathcal{O}'_i = \#\mathcal{O}_i$, $i = 1, ..., 6$.
 $\mathcal{O}_1 = \mathcal{O}_{RC} = \{RC\text{-}lines\}, \ \mathcal{O}'_1 = \mathcal{O}_{RA} = \{RA\text{-}lines\},$ (2.8)
 $\#\mathcal{O}_{RC} = \#\mathcal{O}_{RA} = (q^2 + q)/2;$
 $\mathcal{O}_2 = \mathcal{O}'_2 = \mathcal{O}_T = \{T\text{-}lines\}, \ \#\mathcal{O}_T = q + 1;$
 $\mathcal{O}_3 = \mathcal{O}_{IC} = \{IC\text{-}lines\}, \ \mathcal{O}'_3 = \mathcal{O}_{IA} = \{IA\text{-}lines\}, \ \#\mathcal{O}_{IC} = \#\mathcal{O}_{IA} = (q^2 - q)/2;$
 $\mathcal{O}_4 = \mathcal{O}'_4 = \mathcal{O}_{U\Gamma} = \{U\Gamma\text{-}lines\}, \ \#\mathcal{O}_{U\Gamma} = q^2 + q;$
 $\mathcal{O}_5 = \mathcal{O}_{Un\Gamma} = \{Un\Gamma\text{-}lines\}, \mathcal{O}'_5 = \mathcal{O}_{E\Gamma} = \{E\Gamma\text{-}lines\}, \#\mathcal{O}_{Un\Gamma} = \#\mathcal{O}_{E\Gamma} = q^3 - q;$

$$\mathcal{O}_6 = \mathcal{O}_6' = \mathcal{O}_{\operatorname{En}\Gamma} = \{\operatorname{En}\Gamma \text{-}lines\}, \ \#\mathcal{O}_{\operatorname{En}\Gamma} = (q^2 - q)(q^2 - 1).$$

For q > 4 even, the lines in the regulus complementary to that of the tangents form an orbit of size q + 1 contained in $\mathcal{O}_4 = \mathcal{O}_{U\Gamma}$.

(b)
$$q \equiv 0 \pmod{3}$$
, $q > 3$.
Classes $\mathcal{O}_1, \dots, \mathcal{O}_6$ are as in (2.8); $\mathcal{O}_7 = \mathcal{O}_A = \{A\text{-line}\}, \#\mathcal{O}_A = 1;$ (2.9)
 $\mathcal{O}_8 = \mathcal{O}_{EA} = \{EA\text{-lines}\}, \#\mathcal{O}_{EA} = (q+1)(q^2-1).$

- (v) The following properties of chords and axes hold.
- (a) For all q, no two chords of C meet off C.
 Every point off C lies on exactly one chord of C.
- (b) Let $q \not\equiv 0 \pmod{3}$. No two axes of Γ meet unless they lie in the same plane of Γ . Every plane not in Γ contains exactly one axis of Γ .
- (vi) For q > 2, the unisecants of \mathscr{C} such that every plane through such a unisecant meets \mathscr{C} in at most one point other than the point of contact are, for q odd, the tangents, while for q even, the tangents and the unisecants in the complementary regulus.

The following theorem summarizes results from [13] useful in this paper.

Theorem 2.3. For line orbits under G_q the following holds.

- (i) The following classes of lines consist of a single orbit:
 - $\mathcal{O}_1 = \mathcal{O}_{RC} = \{RC\text{-}lines\}, \ \mathcal{O}_2 = \mathcal{O}_T = \{T\text{-}lines\}, \ and \}$
 - $\mathcal{O}_3 = \mathcal{O}_{IC} = \{IC\text{-}lines\}, for all q;$
 - $\mathcal{O}_4 = \mathcal{O}_{\mathrm{U}\Gamma} = \{\mathrm{U}\Gamma\text{-lines}\}, \text{ for odd } q;$
 - $\mathcal{O}_5 = \mathcal{O}_{\mathrm{Un}\Gamma} = \{\mathrm{Un}\Gamma\text{-lines}\}\ and\ \mathcal{O}_5' = \mathcal{O}_{\mathrm{E}\Gamma} = \{\mathrm{E}\Gamma\text{-lines}\},\ for\ even\ q;$
 - $\mathcal{O}_1' = \mathcal{O}_{RA} = \{RA\text{-lines}\} \text{ and } \mathcal{O}_3' = \mathcal{O}_{IA} = \{IA\text{-lines}\}, \text{ for } \xi \neq 0;$
 - $\mathcal{O}_7 = \mathcal{O}_A = \{A\text{-lines}\}, \text{ for } \xi = 0.$
- (ii) Let $q \geq 8$ be even. The non-tangent unisecants in a Γ -plane (i.e. $U\Gamma$ -lines, class $\mathcal{O}_4 = \mathcal{O}_{U\Gamma}$) form two orbits of size q+1 and q^2-1 . The orbit of size q+1 consists of the lines in the regulus complementary to that of the tangents. Also, the (q+1)-orbit and (q^2-1) -orbit can be represented in the form $\{\ell_1\varphi|\varphi\in G_q\}$ and $\{\ell_2\varphi|\varphi\in G_q\}$, respectively, where ℓ_j is a line such that $\ell_1 = \overline{P_0\mathbf{P}(0,1,0,0)}$, $\ell_2 = \overline{P_0\mathbf{P}(0,1,1,0)}$, $P_0 = \mathbf{P}(0,0,0,1) \in \mathscr{C}$.
- (iii) Let $q \geq 5$ be odd. The non-tangent uniscants not in a Γ -plane (i.e. $\operatorname{Un}\Gamma$ -lines, class $\mathcal{O}_5 = \mathcal{O}_{\operatorname{Un}\Gamma}$) form two orbits of size $\frac{1}{2}(q^3 q)$. These orbits can be represented in the form $\{\ell_j\varphi|\varphi\in G_q\}$, j=1,2, where ℓ_j is a line such that $\ell_1=\overline{P_0\mathbf{P}(1,0,1,0)}$, $\ell_2=\overline{P_0\mathbf{P}(1,0,\rho,0)}$, $P_0=\mathbf{P}(0,0,0,1)\in\mathscr{C}$, ρ is not a square.

- (iv) Let $q \geq 5$ be odd. Let $q \not\equiv 0 \pmod{3}$. The external lines in a Γ -plane (class $\mathcal{O}'_5 = \mathcal{O}_{\mathrm{E}\Gamma}$) form two orbits of size $(q^3 q)/2$. These orbits can be represented in the form $\{\ell_j \varphi | \varphi \in G_q\}$, j = 1, 2, where $\ell_j = \mathfrak{p}_0 \cap \mathfrak{p}_j$ is the intersection line of planes \mathfrak{p}_0 and \mathfrak{p}_j such that $\mathfrak{p}_0 = \pi(1,0,0,0) = \pi_{\mathrm{osc}}(0)$, $\mathfrak{p}_1 = \pi(0,-3,0,-1)$, $\mathfrak{p}_2 = \pi(0,-3\rho,0,-1)$, ρ is not a square, cf. (2.3), (2.4).
- (v) Let $q \equiv 0 \pmod{3}$, $q \geq 9$. The external lines meeting the axis of Γ (i.e. EA-lines, class $\mathcal{O}_8 = \mathcal{O}_{EA}$) form three orbits of size $q^3 q$, $(q^2 1)/2$, $(q^2 1)/2$. The $(q^3 q)$ -orbit and the two $(q^2 1)/2$ -orbits can be represented in the form $\{\ell_1 \varphi | \varphi \in G_q\}$ and $\{\ell_j \varphi | \varphi \in G_q\}$, j = 2, 3, respectively, where ℓ_j are lines such that $\ell_1 = \overline{P_0^A \mathbf{P}(0, 0, 1, 1)}$, $\ell_2 = \overline{P_0^A \mathbf{P}(1, 0, 1, 0)}$, $\ell_3 = \overline{P_0^A \mathbf{P}(1, 0, \rho, 0)}$, $P_0^A = \mathbf{P}(0, 1, 0, 0)$, ρ is not a square.

2.2 The plane-line incidence matrix of PG(3,q)

The space PG(N,q) contains $\theta_{N,q}$ points and hyperplanes, and $\beta_{N,q}$ lines;

$$\theta_{N,q} = \frac{q^{N+1} - 1}{q - 1}, \ \beta_{N,q} = \frac{(q^{N+1} - 1)(q^{N+1} - q)}{(q^2 - 1)(q^2 - q)}.$$
 (2.10)

Let $\mathcal{I}^{\Pi\Lambda}$ be the $\beta_{3,q} \times \theta_{3,q}$ plane-line incidence matrix of PG(3, q) in which columns correspond to planes, rows correspond to lines, and there is an entry "1" if the corresponding line lies in the corresponding plane. Every column and every row of $\mathcal{I}^{\Pi\Lambda}$ contains $\theta_{2,q}$ and $\theta_{1,q}$ ones, respectively. Thus, $\mathcal{I}^{\Pi\Lambda}$ is a tactical configuration [16, Chapter 2.3], [21, Chapter 7, Section 2]. Moreover, $\mathcal{I}^{\Pi\Lambda}$ gives a 2-($\theta_{3,q}, \theta_{1,q}, 1$) design [23] since there is exactly one line as the intersection of any two planes.

Definition 2.4. [15] A configuration (v_r, b_k) is an incidence structure of v points and b lines such that each line contains k points, each point lies on r lines, and two different points are connected by at most one line. If v = b and, hence, r = k, the configuration is symmetric, denoted by v_k .

For an introduction to the configurations see [11,15] and the references therein.

The transposition $(\mathcal{I}^{\Pi\Lambda})^{tr}$ gives the $\theta_{3,q} \times \beta_{3,q}$ line-plane incidence matrix. It can be viewed as a (v_r, b_k) configuration with $v = \beta_{3,q}$, $b = \theta_{3,q}$, $r = \theta_{1,q}$, $k = \theta_{2,q}$, as there is at most one plane through two different lines.

3 The main results

Throughout the paper, we consider orbits of lines and planes under G_q .

Notation 3.1. In addition to Notation 2.1, the following notation is used:

$$\mathcal{O}_{\lambda_j} \qquad \text{the j-th orbit of the class } \mathcal{O}_{\lambda}, \ j=1,\dots,L_{\lambda\Sigma}^{(\xi)\bullet}, \ \mathcal{O}_{\lambda}=\bigcup_{j=1}^{L_{\lambda\Sigma}^{(\xi)\bullet}}\mathcal{O}_{\lambda_j};$$

$$\lambda_j\text{-lines} \qquad \lambda\text{-lines forming the j-th orbit } \mathcal{O}_{\lambda_j} \text{ of the class } \mathcal{O}_{\lambda}, \ \lambda \in \mathfrak{L}^{(\xi)};$$

$$\Lambda_{\lambda_j,\pi}^{(\xi)\bullet} \qquad \text{the number of lines from an orbit } \mathcal{O}_{\lambda_j} \text{ in a π-plane};$$

$$\Lambda_{\lambda,\pi}^{(\xi)\bullet} \qquad \text{the total number of λ-lines in a π-plane};$$

$$\Pi_{\pi,\lambda_j}^{(\xi)\bullet} \qquad \text{the exact number of π-planes through a line of an orbit } \mathcal{O}_{\lambda_j};$$

$$\Pi_{\pi,\lambda_j}^{(\xi)\bullet} \qquad \text{the average number of π-planes through a λ-line over all the } \lambda\text{-lines}; \text{ if the union (class) } \mathcal{O}_{\lambda} \text{ consists of a single orbit then,}$$

$$\text{in fact, } \Pi_{\pi,\lambda}^{(\xi)} \text{ is the exact number of π-planes through each λ-line};$$

$$\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda} \qquad \text{the $\#\mathcal{O}_{\lambda}\times\#\mathcal{N}_{\pi}$ submatrix of the plane-line incidence matrix $\mathcal{I}^{\Pi\Lambda}$ with incidencies between π-planes and λ-lines; }$$

$$\mathcal{I}_{\pi,\lambda_j}^{\Pi\Lambda} \qquad \text{the $\#\mathcal{O}_{\lambda_j}\times\#\mathcal{N}_{\pi}$ submatrix of $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ with incidencies between π-planes and λ_j-lines.}$$

Remark 3.2. If $L_{\lambda\Sigma}^{(\xi)\bullet} = 1$ then $\Pi_{\pi,\lambda}^{(\xi)\bullet}$ certainly is an integer. If λ -lines form two or more orbits, i.e. $L_{\lambda\Sigma}^{(\xi)\bullet} \geq 2$, then $\Pi_{\pi,\lambda}^{(\xi)\bullet}$ may be not integer as well as integer.

On the other end, for all pairs (π, λ) , we always have the same total number of λ -lines in each π -plane, i.e. $\Lambda_{\lambda,\pi}^{(\xi)\bullet}$ always is an integer, see Lemma 4.1.

From now on, we consider $q \geq 5$ apart from Theorem 3.4.

Tables 1 and 2 and Theorem 3.3 summarize the results of Sections 4–6. Theorem 3.4 is obtained by an exhaustive computer search using the symbol calculation system Magma [5].

For the plane-line incidence matrix $\mathcal{I}^{\Pi\Lambda}$ of $\operatorname{PG}(3,q), q \equiv \xi \pmod 3$, Table 1 shows the values $\Pi_{\pi,\lambda}^{(\xi)}$ (top entry) and $\Lambda_{\lambda,\pi}^{(\xi)}$ (bottom entry) for each pair $(\pi,\lambda), \pi \in \mathfrak{P}$, where $\Pi_{\pi,\lambda}^{(\xi)}$ is the exact (if $L_{\lambda\Sigma}^{(\xi)\bullet}=1$) or average (if $L_{\lambda\Sigma}^{(\xi)\bullet}\geq 2$) number of π -planes through every λ -line, whereas $\Lambda_{\lambda,\pi}^{(\xi)}$ always is the exact number of λ -lines in every π -plane. In other words, $\Pi_{\pi,\lambda}^{(\xi)}$ is the exact or average number of ones in every row of the submatrix $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of $\mathcal{I}^{\Pi\Lambda}$, whereas $\Lambda_{\lambda,\pi}^{(\xi)}$ always is the exact number of ones in every column of $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$. The superscript (ξ) is omitted for $\lambda \in \{\operatorname{RC}, \operatorname{T}, \operatorname{IC}, \operatorname{U}\Gamma, \operatorname{Un}\Gamma\}$ where the values $\Pi_{\pi,\lambda}^{(\xi)}, \Lambda_{\lambda,\pi}^{(\xi)}$ are the same for all q.

Table 1: Values $\Pi_{\pi,\lambda}^{(\xi)}$ (top entry) and $\Lambda_{\lambda,\pi}^{(\xi)}$ (bottom entry) for submatrices $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of the plane-line incidence matrix of PG(3,q), $q \equiv \xi \pmod{3}$, $\xi \in \{1,-1,0\}$, $q \geq 5$, $\pi \in \mathfrak{P}$, $\lambda \in \mathfrak{L}^{(\neq 0)} \cup \mathfrak{L}^{(0)}$. The superscript (ξ) is omitted if the values $\Pi_{\pi,\lambda}^{(\xi)}$ and $\Lambda_{\lambda,\pi}^{(\xi)}$ are the same for all q

101 0	4						
	\mathcal{O}_{λ}	$\mathscr{N}_{\pi} \to$	Γ -	$2_{\mathscr{C}}$ -	$3_{\mathscr{C}}$ -	$\overline{1_{\mathscr{C}}}$ -	$0_{\mathscr{C}}$ -
$L_{\lambda\Sigma}^{\mathrm{od}}$	\downarrow		planes	planes		planes	
$L_{\lambda\Sigma}^{\mathrm{ev}}$	λ -lines		q+1	$q^2 + q$	$\frac{1}{6}(q^3-q)$	$\frac{1}{2}(q^3 - q)$	$\frac{1}{3}(q^3-q)$
1	RC-lines	$\Pi_{\pi, RC}$	0	2	q-1	0	0
1	$\frac{1}{2}(q^2+q)$	$\Lambda{\mathrm{RC},\pi}$	0	1	3	0	0
1	T-lines	$\Pi_{\pi,\mathrm{T}}$	1	q	0	0	0
1	q+1	$\Lambda{\mathrm{T},\pi}$	1	1	0	0	0
1	IC-lines	$\Pi_{\pi, \mathrm{IC}}$	0	0	0	q+1	0
1	$\frac{1}{2}(q^2 - q)$	$\Lambda{{ m IC},\pi}$	0	0	0	1	0
1	$U\Gamma$ -lines	$\Pi_{\pi,\mathrm{U}\Gamma}$	1	1	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	0
2	$q^2 + q$	$\Lambda_{\mathrm{U}\Gamma,\pi}$	q	1	3	1	0
2	$Un\Gamma$ -lines	$\Pi_{\pi,\mathrm{Un}\Gamma}$	0	2 (1)	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$	0
1	$q^3 - q$	$\Lambda{\mathrm{Un}\Gamma,\pi}$	0	2(q-1)	3(q-2)	q	0
1	RA-lines	$\Pi_{\pi,\mathrm{RA}}^{(1)}$	2	0	$\frac{1}{3}(q-1)$	0	$\frac{2}{3}(q-1)$
1	$\frac{1}{2}(q^2+q)$	$\Lambda{ ext{RA},\pi}^{(1)}$	q	0	1	0	1
1	RA-lines	$\Pi_{\pi,\mathrm{RA}}^{(-1)}$	2	0	0	q-1	0
1	$\frac{1}{2}(q^2+q)$	$\Lambda_{{ m RA},\pi}^{(-1)}$	q	0	0	1	0
1	IA-lines	$\Pi_{\pi,\mathrm{IA}}^{(1)}$	0	0	0	q+1	0
1	$\frac{1}{2}(q^2-q)$	$\Lambda^{(1)}_{{ m IA},\pi}$	0	0	0	1	0
1	IA-lines	$\Pi_{\pi,\mathrm{IA}}^{(-1)}$	0	0	$\frac{1}{3}(q+1)$	0	$\frac{2}{3}(q+1)$
1	$\frac{1}{2}(q^2 - q)$	$\Lambda{{\rm IA},\pi}^{(-1)}$	0	0	1	0	1
2	$E\Gamma$ -lines	$\Pi_{\pi,\mathrm{E}\Gamma}^{(1)}$	1	1	$\frac{1}{6}(q-4)$	$\frac{1}{2}q$	$\frac{1}{3}(q-1)$
1	$q^3 - q$	$\Lambda^{(1)}{{ m E}\Gamma,\pi}$	$q^2 - q$	q-1	q-4	q	q-1
2	$E\Gamma$ -lines	$\Pi_{\pi,\mathrm{E}\Gamma}^{(-1)}$	1	1	$\frac{1}{6}(q-2)$	$\frac{1}{2}(q-2)$	$\frac{1}{3}(q+1)$
1	$q^3 - q$	$\Lambda^{(-1)}_{{ m E}\Gamma,\pi}$	$q^2 - q$	q-1	q-2	q-2	q+1
≥ 2	$\mathrm{En}\Gamma$ -lines	$\Pi_{\pi, \operatorname{En}\Gamma}^{(1)}$	0	1	$\frac{q^2 - 3q + 4}{6(q - 1)}$	$\frac{q^2 - q - 2}{2(q - 1)}$	$\frac{q^2+1}{3(q-1)}$
≥ 2	$q^4 - q^3 - q^2 + q$	$\Lambda^{(1)}_{{ m En}\Gamma,\pi}$	0	$\frac{(q-1)^2}{1}$	$q^2 - 3q + 4$	$q^2 - q - 2$	$q^2 + 1$
≥ 2	$\mathrm{En}\Gamma$ -lines	$\Pi_{\pi, \operatorname{En}\Gamma}^{(-1)}$	0	1	$\frac{1}{6}(q-2)$	$\frac{1}{2}q$	$\frac{\frac{1}{3}(q+1)}{}$
≥ 2	$q^4 - q^3 - q^2 + q$	$\Lambda^{(-1)}_{{ m En}\Gamma,\pi}$	0	$(q-1)^2$	$q^2 - 3q + 2$	$q^2 - q$	$q^2 - 1$
≥ 2	$\mathrm{En}\Gamma$ -lines	$\Pi_{\pi, \operatorname{En}\Gamma}^{(0)}$	0	1	$\frac{q^2 - 3q + 3}{6(q - 1)}$	$\frac{q^2 - q - 1}{2(q - 1)}$	$\frac{q^2}{3(q-1)}$
	$q^4 - q^3 - q^2 + q$	$\Lambda^{(0)}_{{\rm En}\Gamma,\pi}$	0	$(q-1)^2$		$q^2 - q - 1$	q^2

Table 1: continue Values $\Pi_{\pi,\lambda}^{(\xi)}$ (top entry) and $\Lambda_{\lambda,\pi}^{(\xi)}$ (bottom entry) for submatrices $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of the plane-line incidence matrix of $\mathrm{PG}(3,q), q \equiv \xi \pmod 3, \xi \in \{1,-1,0\}, q \geq 5, \pi \in \mathfrak{P}, \lambda \in \mathfrak{L}^{(\neq 0)} \cup \mathfrak{L}^{(0)}$. The superscript (ξ) is omitted if the values $\Pi_{\pi,\lambda}^{(\xi)}$ and $\Lambda_{\lambda,\pi}^{(\xi)}$ are the same for all q

	\mathcal{O}_{λ}	$\mathscr{N}_{\pi} \to$	Γ -	$2_{\mathscr{C}}$ -	$3_{\mathscr{C}}$ -	$\overline{1_{\mathscr{C}}}$ -	$0_{\mathscr{C}}$ -
$L_{\lambda\Sigma}^{\mathrm{od}}$	\downarrow		planes	planes	planes	planes	planes
$L_{\lambda\Sigma}^{\mathrm{ev}}$	λ -lines		q+1	$q^2 + q$	$\frac{1}{6}(q^3 - q)$	$\frac{1}{2}(q^3-q)$	$\frac{1}{3}(q^3-q)$
1	A-lines	$\Pi_{\pi,\mathrm{A}}^{(0)}$	q+1	0	0	0	0
	1	$\Lambda_{{ m A},\pi}^{(0)}$	1	0	0	0	0
3	EA-lines	$\Pi_{\pi,\mathrm{EA}}^{(0)}$	1	$\frac{q}{q+1}$	$\frac{q(q-2)}{6(q+1)}$	$\frac{q^2}{2(q+1)}$	$\frac{1}{3}q$
	$q^3 + q^2 - q - 1$	$\Lambda^{(0)}_{{ m EA},\pi}$	$q^2 - 1$	q-1	q-2	q	q+1

The 1-st column of Table 1 shows the total number of orbits of λ -lines for q odd $(L_{\lambda\Sigma}^{\text{od}},$ top entry) and q even $(L_{\lambda\Sigma}^{\text{ev}},$ bottom entry).

In Table 2, the values $\Pi_{\pi,\lambda_j}^{(\xi)\bullet}$ and $\Lambda_{\pi,\lambda_j}^{(\xi)\bullet}$ are given for the following cases: even $q \not\equiv 0 \pmod{3}$ with $\lambda = \mathrm{U}\Gamma$; odd q with $\lambda = \mathrm{U}\Gamma$; odd $q \equiv \xi \pmod{3}$, $\xi \in \{1,-1\}$, with $\lambda = \mathrm{E}\Gamma$; and $q \equiv 0 \pmod{3}$ with $\lambda = \mathrm{E}A$.

Theorem 3.3. Let $q \ge 5$, $q \equiv \xi \pmod{3}$. Let notations be as in Section 2 and Notations 2.1, 3.1. The following holds:

- (i) In PG(3, q), for the submatrices $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of the plane-line incidence matrix $\mathcal{I}^{\Pi\Lambda}$, the values $\Pi_{\pi,\lambda}^{(\xi)}$ (i.e. the exact or average number of π -planes through a λ -line) and $\Lambda_{\lambda,\pi}^{(\xi)}$ (i.e. the exact number of λ -lines in a π -plane) are given in Table 1. The numbers $L_{\lambda\Sigma}^{(\xi)\bullet}$ of line orbits under G_q in classes \mathcal{O}_{λ} are also collected in the tables. For the submatrices $\mathcal{I}_{\pi,\lambda_j}^{\Pi\Lambda}$ corresponding to each of two orbits of the classes $\mathcal{O}_4 = \mathcal{O}_{U\Gamma}$, $\mathcal{O}_5 = \mathcal{O}_{Un\Gamma}$, $\mathcal{O}_5' = \mathcal{O}_{E\Gamma}$, and three orbits of the class $\mathcal{O}_8 = \mathcal{O}_{EA}$, the values $\Pi_{\pi,\lambda_j}^{(\xi)\bullet}$, $\Lambda_{\pi,\lambda_j}^{(\xi)\bullet}$ are given in Table 2.
- (ii) Let a class \mathcal{O}_{λ} consist of a single orbit according to Theorem 2.3(i). Then, in Table 1, the value of $\Pi_{\pi,\lambda}^{(\xi)}$, $\pi \in \mathfrak{P}$, is the exact number of π -planes through every λ -line.
- (iii) Let $\pi \in \mathfrak{P}$ for all q. Let a class \mathcal{O}_{λ} consist of a single orbit according to Theorem 2.3(i). Then the submatrix $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of $\mathcal{I}^{\Pi\Lambda}$ is a (v_r, b_k) configuration of Definition 2.4 with $v = \# \mathcal{N}_{\pi}$, $b = \# \mathcal{O}_{\lambda}$, $r = \Lambda_{\lambda,\pi}$, $k = \Pi_{\pi,\lambda}$. Also, up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ with $\Lambda_{\lambda,\pi}^{(\xi)} = 1$ can be viewed as a concatenation of $\Pi_{\pi,\lambda}^{(\xi)}$ identity matrices of order $\# \mathcal{O}_{\lambda}$. The same holds for the submatrices $\mathcal{I}_{\pi,\lambda_j}^{\Pi\Lambda}$.
- (iv) Let $\lambda \in \{U\Gamma, E\Gamma\}$ if $q \not\equiv 0 \pmod{3}$, $\lambda \in \{U\Gamma, EA\}$ if $q \equiv 0 \pmod{3}$. Then, independently of the number of orbits in the class \mathcal{O}_{λ} , we have exactly one Γ -plane through every λ -line. Up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\Gamma,\lambda}^{\Pi\Lambda}$ can be viewed as a vertical concatenation of $\Lambda_{\lambda,\Gamma}^{(\xi)}$ identity matrices of order $\#\mathcal{N}_{\Gamma} = q + 1$.
- (v) For $q \not\equiv 0 \pmod{3}$, the submatrix $\mathcal{I}_{\Gamma,RA}^{\Pi\Lambda}$ of $\mathcal{I}^{\Pi\Lambda}$ is a simple complete 2-(q+1,2,1) design in the sense of [23, Section 1.6].
- (vi) For all $q \geq 5$, all q + 1 planes through an imaginary chord are $\overline{1_{\mathscr{C}}}$ -planes forming a pencil. The $\binom{q}{2}(q+1)$ -orbit of all $\overline{1_{\mathscr{C}}}$ -planes can be partitioned into $\binom{q}{2}$ pencils of planes having an imaginary chord as axis. If $q \equiv 1 \pmod{3}$, a similar property holds for imaginary axes and $\overline{1_{\mathscr{C}}}$ -planes.

Table 2: Values $\Pi_{\pi,\lambda_j}^{(\xi)\bullet}$ (top entry) and $\Lambda_{\pi,\lambda_j}^{(\xi)\bullet}$ (bottom entry) for submatrices $\mathcal{I}_{\pi,\lambda_j}^{\Pi\Lambda}$ of the plane-line incidence matrix of $\mathrm{PG}(3,q), q \geq 5, \pi \in \mathfrak{P}$; even $q \not\equiv 0 \pmod 3$ for $\lambda = \mathrm{U}\Gamma$; odd q for $\lambda = \mathrm{U}\Gamma$; odd $q \equiv \xi \pmod 3$, $\xi \in \{1,-1\}$, for $\lambda = \mathrm{E}\Gamma$; $q \equiv 0 \pmod 3$ for $\lambda = \mathrm{E}\Lambda$

$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\lambda = EA$						
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	\mathcal{O}_{λ_j}	$\mathcal{N}_{\pi} \rightarrow$	Γ-	$2_{\mathscr{C}}$ -	$3_{\mathscr{C}}$ -	$\overline{1_{\mathscr{C}}}$ -	$0_{\mathscr{C}}$ -
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	↓		planes	planes			
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	λ_j -lines		q+1	$q^2 + q$	$\frac{1}{6}(q^3-q)$	$\frac{1}{2}(q^3-q)$	$\frac{1}{3}(q^3-q)$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$U\Gamma_1$ -lines	$\pi, \cup 1$	1	q	0	0	0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	q+1	$^{I1}{ m U}\Gamma_1.\pi$	1	1	0	0	0
$\begin{array}{ c c c c c }\hline Un\Gamma_1-lines & \Pi^{\rm od}_{\pi, Un\Gamma_1} & 0 & 1 & \frac{1}{2}(q-1) & \frac{1}{2}(q+1) & 0 \\ \frac{1}{2}(q^3-q) & \Lambda^{\rm od}_{Un\Gamma_1,\pi} & 0 & \frac{1}{2}(q-1) & \frac{3}{2}(q-1) & \frac{1}{2}(q+1) & 0 \\ \hline Un\Gamma_2-lines & \Pi^{\rm od}_{\pi, Un\Gamma_2} & 0 & 3 & \frac{1}{2}(q-3) & \frac{1}{2}(q-1) & 0 \\ \frac{1}{2}(q^3-q) & \Lambda^{\rm od}_{Un\Gamma_2,\pi} & 0 & \frac{3}{2}(q-1) & \frac{3}{2}(q-3) & \frac{1}{2}(q-1) & 0 \\ \hline E\Gamma_1-lines & \Pi^{(1)od}_{\pi, E\Gamma_1} & 1 & 0 & \frac{1}{6}(q-1) & \frac{1}{2}(q+1) & \frac{1}{3}(q-1) \\ \frac{1}{2}(q^3-q) & \Lambda^{(1)od}_{E\Gamma_1,\pi} & \frac{1}{2}(q^2-q) & 0 & \frac{1}{2}(q-1) & \frac{1}{2}(q+1) & \frac{1}{2}(q-1) \\ \hline E\Gamma_2-lines & \Pi^{(1)od}_{\pi, E\Gamma_2} & 1 & 2 & \frac{1}{6}(q-7) & \frac{1}{2}(q-1) & \frac{1}{3}(q-1) \\ \frac{1}{2}(q^3-q) & \Lambda^{(1)od}_{E\Gamma_2,\pi} & \frac{1}{2}(q^2-q) & q-1 & \frac{1}{2}(q-7) & \frac{1}{2}(q-1) & \frac{1}{2}(q-1) \\ \hline E\Gamma_1-lines & \Pi^{(-1)od}_{\pi, E\Gamma_2} & 1 & 0 & \frac{1}{6}(q+1) & \frac{1}{2}(q-1) & \frac{1}{3}(q+1) \\ \frac{1}{2}(q^3-q) & \Lambda^{(-1)od}_{E\Gamma_1,\pi} & \frac{1}{2}(q^2-q) & 0 & \frac{1}{2}(q+1) & \frac{1}{2}(q-1) & \frac{1}{2}(q+1) \\ \hline E\Gamma_2-lines & \Pi^{(-1)od}_{\pi, E\Gamma_2} & 1 & 2 & \frac{1}{6}(q-5) & \frac{1}{2}(q-3) & \frac{1}{3}(q+1) \\ \frac{1}{2}(q^3-q) & \Lambda^{(-1)od}_{E\Gamma_1,\pi} & \frac{1}{2}(q^2-q) & q-1 & \frac{1}{2}(q-5) & \frac{1}{2}(q-3) & \frac{1}{2}(q+1) \\ \hline EA_1-lines & \Pi^{(0)od}_{\pi, EA_1} & 1 & 1 & \frac{1}{6}(q-3) & \frac{1}{2}(q-1) & \frac{1}{3}q \\ q^3-q & \Lambda^{(0)od}_{EA_1,\pi} & q^2-q & q-1 & q-3 & q-1 & q \\ \hline EA_2-lines & \Pi^{(0)od}_{\pi, EA_2} & 1 & 0 & \frac{1}{3}q & 0 & \frac{2}{3}q \\ \frac{1}{2}(q^2-1) & \Lambda^{(0)od}_{EA_2,\pi} & \frac{1}{2}(q-1) & 0 & 1 & 0 & 1 \\ \hline EA_3-lines & \Pi^{(0)od}_{\pi, EA_2} & 1 & 0 & 0 & q & 0 \\ \hline \end{array}$	$U\Gamma_2$ -lines	$^{11}\pi.\mathrm{U}\Gamma_{2}$	1	0	$\frac{1}{2}q$	$\frac{1}{2}q$	0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		$\Lambda_{\mathrm{U}\Gamma_2,\pi}^{(\neq 0)\mathrm{ev}}$	q-1	0	3	1	0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		$\Pi^{\mathrm{od}}_{\pi,\mathrm{Un}\Gamma_1}$	0	_	4	Z \-	0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		$\Lambda_{\mathrm{Un}\Gamma_{1},\pi}^{\mathrm{od}}$	0	$\frac{1}{2}(q-1)$	$\frac{3}{2}(q-1)$		0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		$\Pi_{\pi,\mathrm{Un}\Gamma_2}^{\mathrm{od}}$	0	3	2 (1	Z (- /	0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{1}{2}(q^3-q)$	$\Lambda_{\mathrm{Un}\Gamma_2,\pi}^{\mathrm{od}}$	0	$\frac{3}{2}(q-1)$	$\frac{3}{2}(q-3)$	$\frac{1}{2}(q-1)$	0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$E\Gamma_1$ -lines	$\Pi_{\pi,\mathrm{E}\Gamma_1}^{(1)\mathrm{od}}$	1	0	$\frac{1}{6}(q-1)$	$\frac{1}{2}(q+1)$	$\frac{\frac{1}{3}(q-1)}{}$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\frac{1}{2}(q^3-q)$	$\Lambda^{(1)\mathrm{od}}_{\mathrm{E}\Gamma_1,\pi}$	$\frac{1}{2}(q^2 - q)$	0	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$E\Gamma_2$ -lines	$\Pi_{\pi,\mathrm{E}\Gamma_2}^{(1)\mathrm{od}}$	1	2	$\frac{1}{6}(q-7)$	$\frac{1}{2}(q-1)$	$\frac{1}{3}(q-1)$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{1}{2}(q^3-q)$	$\Lambda_{\mathrm{E}\Gamma_2,\pi}$	$\frac{1}{2}(q^2 - q)$	q-1	$\frac{1}{2}(q-7)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$E\Gamma_1$ -lines	$\Pi_{\pi \text{ EC}_1}^{(-1)\text{od}}$	1	0	$\frac{1}{6}(q+1)$	$\frac{1}{2}(q-1)$	$\frac{1}{3}(q+1)$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\frac{1}{2}(q^3-q)$	$\Lambda^{(-1)\mathrm{od}}_{\mathrm{E}\Gamma_1,\pi}$	$\frac{1}{2}(q^2-q)$	0	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q+1)$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$E\Gamma_2$ -lines	$\Pi_{\pi,\mathrm{E}\Gamma_2}^{(-1)\mathrm{od}}$	1	2	$\frac{1}{6}(q-5)$	$\frac{1}{2}(q-3)$	$\frac{1}{3}(q+1)$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\frac{1}{2}(q^3-q)$	$\Lambda_{\mathrm{E}\Gamma_2,\pi}$	$\frac{1}{2}(q^2 - q)$	q-1	$\frac{1}{2}(q-5)$	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q+1)$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	EA ₁ -lines	$\Pi_{\pi, \mathrm{EA}_1}^{(0)\mathrm{od}}$	1	1	$\frac{1}{6}(q-3)$	$\frac{1}{2}(q-1)$	$\frac{1}{3}q$
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$q^3 - q$	$\Lambda^{(0)\mathrm{od}}_{\mathrm{EA}_1,\pi}$	$q^2 - q$	q-1	q-3	q-1	_
$\frac{\frac{1}{2}(q^2-1)}{\text{EA}_3\text{-lines}} \frac{\Lambda_{\text{EA}_2,\pi}^{(0)\text{od}} \frac{1}{2}(q-1)}{1} 0 1 0 1$	EA ₂ -lines	$\Pi_{\pi, \mathrm{EA}_2}^{(0)\mathrm{od}}$	1		$\frac{1}{3}q$	0	$\frac{2}{3}q$
EA ₃ -lines $\Pi_{\pi, \text{EA}_3}^{(0) \text{od}}$ 1 0 0 q 0	$\frac{1}{2}(q^2-1)$	$\Lambda^{(0)\mathrm{od}}_{\mathrm{EA}_2,\pi}$	$\frac{1}{2}(q-1)$	0	1	0	
$\frac{1}{2}(q^2-1)$ $\Lambda_{\text{EA}_3,\pi}^{(0)\text{od}}$ $\frac{1}{2}(q-1)$ 0 0 1	EA ₃ -lines	$\Pi_{\pi, \mathrm{EA}_3}^{(0)\mathrm{od}}$		0	0	q	0
	$\frac{1}{2}(q^2-1)$	$\Lambda^{(0){ m od}}_{{ m EA}_3,\pi}$	$\frac{1}{2}(q-1)$	0	0	1	0

Theorem 3.4. Let the types of lines and planes be as in Table 1.

- (i) Let q = 2. The group G₂ ≅ S₃Z₂³ contains 8 subgroups isomorphic to PGL(2,2) divided into two conjugacy classes. For one of these subgroups, the matrices corresponding to the projectivities of the subgroup assume the form described by (2.6). For this subgroup (and only for it) the line-plane incidence matrix has the form of Table 1 for q ≡ -1 (mod 3).
- (ii) Let q = 3. The group $G_3 \cong \mathbf{S}_4 \mathbf{Z}_2^3$ contains 24 subgroups isomorphic to PGL(2,3) divided into four conjugacy classes. For one of these subgroups, the matrices corresponding to the projectivities of the subgroup assume the form described by (2.6). For this subgroup (and only for it) the line-plane incidence matrix has the form of Table 1 for $q \equiv 0 \pmod{3}$.
- (iii) Let q = 4. The group $G_4 \cong \mathbf{S}_5 \cong P\Gamma L(2,4)$ contains one subgroup isomorphic to PGL(2,4). The matrices corresponding to the projectivities of this subgroup assume the form described by (2.6) and for this subgroup the line-plane incidence matrix has the form of Table 1 for $q \equiv 1 \pmod{3}$.

4 Some useful relations

In this section, we omit the superscripts " (ξ) ", "od", and "'ev' as they are the same for all terms in a formula; in particular, we use \mathfrak{L} and $L_{\lambda\Sigma}$ instead of $\mathfrak{L}^{(\xi)}$ and $L_{\lambda\Sigma}$. In further, when relations of this section are applied, we add the superscripts if they are necessary by the context.

Lemma 4.1. The following holds:

- (i) The number $\Lambda_{\lambda_j,\pi}$ of lines from an orbit \mathcal{O}_{λ_j} in a plane of an orbit \mathcal{N}_{π} is the same for all planes of \mathcal{N}_{π} .
- (ii) The total number $\Lambda_{\lambda,\pi}$ of lines from an orbit union \mathcal{O}_{λ} in a plane of an orbit \mathcal{N}_{π} is the same for all planes of \mathcal{N}_{π} . We have

$$\Lambda_{\lambda,\pi} = \sum_{j=1}^{L_{\lambda\Sigma}} \Lambda_{\lambda_j,\pi}.$$
(4.1)

- (iii) The number Π_{π,λ_j} of planes from an orbit \mathcal{N}_{π} through a line of an orbit \mathcal{O}_{λ_j} is the same for all lines of \mathcal{O}_{λ_j} .
- (iv) The average number $\Pi_{\pi,\lambda}$ of planes from an orbit \mathcal{N}_{π} through a line of a union \mathcal{O}_{λ} over all lines of \mathcal{O}_{λ} satisfies the following relations:

$$\Lambda_{\lambda,\pi} \cdot \# \mathcal{N}_{\pi} = \Pi_{\pi,\lambda} \cdot \# \mathcal{O}_{\lambda}; \tag{4.2}$$

$$\Pi_{\pi,\lambda} = \frac{1}{\#\mathcal{O}_{\lambda}} \sum_{j=1}^{L_{\lambda\Sigma}} \left(\Pi_{\pi,\lambda_j} \cdot \#\mathcal{O}_{\lambda_j} \right). \tag{4.3}$$

- (v) If $L_{\lambda\Sigma} = 1$, then \mathcal{O}_{λ} is an orbit and the number of planes from \mathcal{N}_{π} through a line of \mathcal{O}_{λ} is the same for all lines of \mathcal{O}_{λ} . In this case $\Pi_{\pi,\lambda}$ is certainly an integer. If $\Pi_{\pi,\lambda}$ is not an integer then the union \mathcal{O}_{λ} contains more than one orbit, i.e. $L_{\lambda\Sigma} \geq 2$.
- Proof. (i) Consider planes \mathfrak{p}_1 and \mathfrak{p}_2 of \mathcal{N}_{π} . Denote by ℓ a line of \mathcal{O}_{λ_j} . Let $S(\mathfrak{p}_1)$ and $S(\mathfrak{p}_2)$ be subsets of \mathcal{O}_{λ_j} such that $S(\mathfrak{p}_1) = \{\ell \in \mathcal{O}_{\lambda_j} | \ell \in \mathfrak{p}_1\}$, $S(\mathfrak{p}_2) = \{\ell \in \mathcal{O}_{\lambda_j} | \ell \in \mathfrak{p}_2\}$. There exists $\varphi \in G_q$ such that $\mathfrak{p}_2 = \mathfrak{p}_1 \varphi$. Clearly, φ embeds $S(\mathfrak{p}_1)$ in $S(\mathfrak{p}_2)$, i.e. $S(\mathfrak{p}_1)\varphi \subseteq S(\mathfrak{p}_2)$ and $\#S(\mathfrak{p}_1) \leq \#S(\mathfrak{p}_2)$. In the same way, φ^{-1} embeds $S(\mathfrak{p}_2)$ in $S(\mathfrak{p}_1)$, i.e. $\#S(\mathfrak{p}_2) \leq \#S(\mathfrak{p}_1)$. Thus, $\#S(\mathfrak{p}_2) = \#S(\mathfrak{p}_1)$.
- (ii) For fixed λ , orbits \mathcal{O}_{λ_i} do not intersect each other.
- (iii) The assertion can be proved similarly to case (i).
- (iv) The cardinality C_1 of the multiset consisting of lines of \mathcal{O}_{λ} in all planes of \mathcal{N}_{π} is equal to $\Lambda_{\lambda,\pi} \cdot \# \mathcal{N}_{\pi}$. The cardinality C_2 of the multiset consisting of planes of \mathcal{N}_{π} through all lines of \mathcal{O}_{λ} is $\Pi_{\pi,\lambda} \cdot \# \mathcal{O}_{\lambda}$. Every C_i is the number of ones in the incidence submatrix $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of $\mathcal{I}^{\Pi\Lambda}$. Thus, $C_1 = C_2$.

The assertion (4.3) holds as \mathcal{O}_{λ} is partitioned by $L_{\lambda\Sigma}$ orbits \mathcal{O}_{λ_i} .

(v) The assertion follows from the case (iii).

Corollary 4.2. If $\Pi_{\pi,\lambda} = 0$ then $\Lambda_{\lambda,\pi} = 0$ and vice versa.

Proof. The assertions follow from
$$(4.2)$$
.

Theorem 4.3. Let the lines of PG(3,q) be partitioned under G_q into $\#\mathfrak{L}$ classes \mathcal{O}_{λ} where every class is a union of orbits of λ -lines, $\lambda \in \mathfrak{L}$. Also, let the planes of PG(3,q) be partitioned under G_q by $\#\mathfrak{P}$ orbits \mathscr{N}_{π} of π -planes, $\pi \in \mathfrak{P}$. The following holds:

$$\sum_{\pi \in \mathfrak{P}} \Pi_{\pi,\lambda} = q + 1, \ \lambda \ is \ fixed; \tag{4.4}$$

$$\sum_{\lambda \in \mathfrak{L}} \Lambda_{\lambda,\pi} = \beta_{2,q} = q^2 + q + 1, \ \pi \ is fixed. \tag{4.5}$$

Proof. Relations (4.4) and (4.5) hold as the lines and the planes of PG(3, q) are partitioned under G_q by unions of line orbits and by orbits of planes, respectively. In total, in PG(3, q), there are q + 1 planes through every line and $\beta_{2,q}$ lines in every plane.

Theorem 4.4. Let ℓ_{ext} be an external line with respect to \mathscr{C} . Let $\Pi_{\pi}(\ell_{\text{ext}})$ be the number of π -planes through ℓ_{ext} , $\pi \in \mathfrak{P}$. The following holds:

$$\Pi_{\Gamma}(\ell_{\text{ext}}) + \Pi_{\overline{1_{\mathscr{C}}}}(\ell_{\text{ext}}) + 2\Pi_{2_{\mathscr{C}}}(\ell_{\text{ext}}) + 3\Pi_{3_{\mathscr{C}}}(\ell_{\text{ext}}) = q + 1; \tag{4.6}$$

$$\Pi_{0_{\mathscr{C}}}(\ell_{\text{ext}}) = \Pi_{2_{\mathscr{C}}}(\ell_{\text{ext}}) + 2\Pi_{3_{\mathscr{C}}}(\ell_{\text{ext}}). \tag{4.7}$$

Proof. For (4.6), we consider q+1 planes through ℓ_{ext} and a point of \mathscr{C} . These planes cannot be $0_{\mathscr{C}}$ -planes. Also, every $2_{\mathscr{C}}$ - and $3_{\mathscr{C}}$ -plane appears two and three times, respectively. Finally, (4.7) follows from (4.4) and (4.6).

Corollary 4.5. The following holds:

$$\Pi_{\pi,\lambda} = \mathcal{P}_{\pi,\lambda} \triangleq \frac{\Lambda_{\lambda,\pi} \cdot \# \mathcal{N}_{\pi}}{\# \mathcal{O}_{\lambda}}; \tag{4.8}$$

$$\Lambda_{\lambda,\pi} = \mathcal{L}_{\lambda,\pi} \triangleq \frac{\Pi_{\pi,\lambda} \cdot \#\mathcal{O}_{\lambda}}{\#\mathcal{N}_{\pi}}; \tag{4.9}$$

$$\Pi_{\pi^*,\lambda} = \mathscr{P}_{\pi^*,\lambda} \stackrel{\triangle}{=} q + 1 - \sum_{\pi \in \mathfrak{P} \setminus \{\pi^*\}} \Pi_{\pi,\lambda}, \ \lambda \ is \ fixed, \ \pi^* \in \mathfrak{P};$$

$$(4.10)$$

$$\Lambda_{\lambda^*,\pi} = \mathcal{L}_{\lambda^*,\pi} \triangleq q^2 + q + 1 - \sum_{\lambda \in \mathfrak{L} \setminus \{\lambda^*\}} \Lambda_{\lambda,\pi}, \ \pi \ is \ fixed,, \ \lambda^* \in \mathfrak{L}.$$
 (4.11)

Proof. The assertions directly follow from (4.2), (4.4), (4.5).

5 The numbers of π -planes through λ -lines and λ -lines in π -planes, for PG(3, q), $q \not\equiv 0 \pmod{3}$

The values of $\#\mathcal{N}_{\pi}$, $\#\mathcal{O}_{\lambda}$, needed for (4.8), (4.9), are taken from (2.7)–(2.9). When we use (4.10), (4.11) the values $\Pi_{\pi,\lambda}$, $\Lambda_{\lambda,\pi}$ obtained above are summed up.

Theorem 5.1. For all q, the following holds:

(i) An RC-line cannot lie in Γ -, $\overline{1_{\mathscr{C}}}$ -, and $0_{\mathscr{C}}$ -planes. Thus,

$$\Pi_{\pi,RC} = \Lambda_{RC,\pi} = 0, \ \pi \in \{\Gamma, \overline{1_{\mathscr{C}}}, 0_{\mathscr{C}}\}.$$

(ii) The number of $2_{\mathscr{C}}$ -planes and $3_{\mathscr{C}}$ -planes through a real chord of \mathscr{C} is equal to 2 and q-1, respectively. Every $2_{\mathscr{C}}$ -plane (resp. $3_{\mathscr{C}}$ -plane) contains one (resp. three) real chords of \mathscr{C} . Thus,

$$\Pi_{2_{\mathscr{C}},RC} = 2, \ \Pi_{3_{\mathscr{C}},RC} = q - 1, \ \Lambda_{RC,2_{\mathscr{C}}} = 1, \ \Lambda_{RC,3_{\mathscr{C}}} = 3.$$

Proof. (i) An RC-line contains two points of \mathscr{C} ; it cannot lie in Γ -, $\overline{1_{\mathscr{C}}}$ -, $0_{\mathscr{C}}$ -planes as these planes have less than 2 points in common with the cubic \mathscr{C} .

(ii) We consider the real chord through points K, Q of \mathscr{C} . Every plane through a real chord is either a $2_{\mathscr{C}}$ -plane or a $3_{\mathscr{C}}$ -plane. Each of the q-1 points R of $\mathscr{C} \setminus \{K,Q\}$ gives rise to the $3_{\mathscr{C}}$ -plane through K,Q,R. Therefore, $\Pi_{3_{\mathscr{C}},RC} = q-1$. By (4.10), $\Pi_{2_{\mathscr{C}},RC} = q+1-\Pi_{3_{\mathscr{C}},RC} = 2$.

The assertions on $\Lambda_{RC,\pi}$ follow from the definitions of the planes.

Theorem 5.2. (i) For all q, a Γ -plane contains one tangent and q non-tangent unisecants. The tangent and non-tangent unisecants lying in a Γ -plane do not lie in other Γ -planes. Thus,

$$\Lambda_{T,\Gamma} = 1, \ \Lambda_{U\Gamma,\Gamma} = q, \ \Pi_{\Gamma,T} = \Pi_{\Gamma,U\Gamma} = 1.$$

(ii) Let $q \not\equiv 0 \pmod{3}$. Then a Γ -plane contains q real axes and $q^2 - q$ $E\Gamma$ -lines. The external lines lying in a Γ -plane do not lie in other Γ -planes. Also, there are two Γ -planes through a real axis of Γ . Thus,

$$\Lambda_{\mathrm{RA},\Gamma}^{(\neq 0)} = q, \ \Lambda_{\mathrm{E\Gamma},\Gamma}^{(\neq 0)} = q^2 - q, \ \Pi_{\Gamma,\mathrm{E\Gamma}}^{(\neq 0)} = 1, \ \Pi_{\Gamma,\mathrm{RA}}^{(\neq 0)} = 2.$$

(iii) For all q, a Γ -plane does not contain IA-, Un Γ -, En Γ -, and IC-lines. Thus,

$$\Lambda_{\lambda,\Gamma} = \Pi_{\Gamma,\lambda} = 0, \ \lambda \in \{IA, Un\Gamma, En\Gamma, IC\}.$$

- Proof. (i) The assertions follow from the definitions of the lines. In total, we have q+1 unisecants in the contact point of a Γ-plane. One of these unisecants is a tangent, the other ones are UΓ-lines. Also, the intersection of two Γ-planes is a real axis. By Theorem 2.2(iv), the unisecants and real axes are distinct non-intersecting classes of lines.
- (ii) In total there are q+1 Γ -planes. For $q \not\equiv 0 \pmod 3$, each Γ -plane intersects the remaining Γ -planes by distinct lines that provide q real axes in it. Thus, for any Γ -plane, we have $\Lambda_{\Gamma,\Gamma} + \Lambda_{U\Gamma,\Gamma} + \Lambda_{RA,\Gamma}^{(\not=0)} = 1 + q + q$. The remaining $\beta_{2,q} (2q+1) = q^2 q$ lines in the Γ -plane are $E\Gamma$ -lines. The intersection of two Γ -planes is a real axis; this provides $\Pi_{\Gamma,E\Gamma}^{(\not=0)} = 1$ and $\Pi_{\Gamma,RA}^{(\not=0)} = 2$.
- (iii) The assertions with respect to IA-, Un Γ and En Γ -lines follow form the definitions of the lines. Also, if an IC-line lies in a Γ -plane then the line intersects the tangent belonging to this plane; contradiction, as by Theorem 2.2(v) no two chords of $\mathscr C$ meet off $\mathscr C$.

Theorem 5.3. For all q, the following holds. All $d_{\mathscr{C}}$ -planes, d=0,2,3, and all Γ -planes contain no the imaginary chords. All q+1 planes through an imaginary chord are $\overline{1_{\mathscr{C}}}$ -planes forming a pencil. The $\binom{q}{2}(q+1)$ -orbit of all $\overline{1_{\mathscr{C}}}$ -planes can be partitioned into $\binom{q}{2}$ pencils of planes having an imaginary chord as axis. So,

$$\Lambda_{\mathrm{IC},\pi} = \Pi_{\pi,\mathrm{IC}} = 0, \ \pi \in \{\Gamma, 2_{\mathscr{C}}, 3_{\mathscr{C}}, 0_{\mathscr{C}}\}, \ \Lambda_{\mathrm{IC},\overline{1_{\mathscr{C}}}} = 1, \ \Pi_{\overline{1_{\mathscr{C}}},\mathrm{IC}} = q+1.$$

Proof. Any $2_{\mathscr{C}}$ -plane and $3_{\mathscr{C}}$ -plane contains a real chord. An osculating plane contains a tangent. If a $2_{\mathscr{C}}$, or a $3_{\mathscr{C}}$, or a Γ -plane contains an imaginary chord then it intersects the real chord or the tangent; contradiction, as by Theorem 2.2(v) no two chords of \mathscr{C} meet off \mathscr{C} . Thus, we have a $\overline{1_{\mathscr{C}}}$ -plane through an imaginary chord and any point of \mathscr{C} . In total, there are $\#\mathscr{C} = q + 1$ such $\overline{1_{\mathscr{C}}}$ -planes for every imaginary chord. Also, by Theorem 2.2(iv)(a), $\#\mathscr{O}_{\rm IC} = {q \choose 2}$.

Theorem 5.4. For all q, a U Γ - and Un Γ -line cannot lie in a $0_{\mathscr{C}}$ -plane, i.e.

$$\Lambda_{\mathrm{U}\Gamma,0_{\mathscr{C}}} = \Pi_{0_{\mathscr{C}},\mathrm{U}\Gamma} = \Lambda_{\mathrm{U}\mathrm{n}\Gamma,0_{\mathscr{C}}} = \Pi_{0_{\mathscr{C}},\mathrm{U}\mathrm{n}\Gamma} = 0.$$

Proof. A UΓ- or UnΓ-line have one point common with \mathscr{C} whereas a $0_{\mathscr{C}}$ -plane has no such points.

Theorem 5.5. For all q, the following holds:

(i) We consider a real chord \mathcal{RC} and two Γ -planes in its touch points. Every $2_{\mathscr{C}}$ -plane through \mathcal{RC} intersects one of these Γ -planes in its tangent and another in a non-tangent unisecant. Thus,

$$\Lambda_{T,2_{\mathscr{C}}} = \Lambda_{U\Gamma,2_{\mathscr{C}}} = 1.$$

Also,

$$\Pi_{2_{\mathscr{C}},U\Gamma} = \Pi_{2_{\mathscr{C}},E\Gamma} = \Lambda_{U\Gamma,\overline{1_{\mathscr{C}}}} = 1, \ \Pi_{\overline{1_{\mathscr{C}}},U\Gamma} = \Pi_{3_{\mathscr{C}},U\Gamma} = (q-1)/2,$$

$$\Lambda_{U\Gamma,3_{\mathscr{C}}} = 3, \ \Lambda_{Un\Gamma,2_{\mathscr{C}}} = 2(q-1), \ \Pi_{2_{\mathscr{C}},Un\Gamma} = 2, \ \Lambda_{E\Gamma,2_{\mathscr{C}}} = q-1.$$

(ii) Through a tangent, there are q $2_{\mathscr{C}}$ -planes. Also, a tangent cannot lie in a $3_{\mathscr{C}}$ - and $0_{\mathscr{C}}$ -plane; we have no $\overline{1_{\mathscr{C}}}$ -planes through a tangent. Thus,

$$\Pi_{2_{\mathscr{C}},T} = q, \ \Lambda_{T,\pi} = \Pi_{\pi,T} = 0, \ \pi \in \{3_{\mathscr{C}}, \overline{1_{\mathscr{C}}}, 0_{\mathscr{C}}\}.$$

Proof. (i) Let P_1, P_2 be the intersections points of \mathcal{RC} and \mathscr{C} and let $\mathcal{T}_1, \mathcal{T}_2$ and Γ_1, Γ_2 be the corresponding tangents and Γ-planes. The plane π_1 through \mathcal{T}_1 and \mathcal{RC} is a $2_{\mathscr{C}}$ -plane due to Theorem 2.2(vi); it intersects Γ_2 by a unisecant \mathscr{U}_2 . If \mathscr{U}_2 is \mathcal{T}_2 then \mathcal{T}_1 meets \mathcal{T}_2 off \mathscr{C} , contradiction, see Theorem 2.2(v). Thus, \mathscr{U}_2 is a non-tangent unisecant, i.e. an UΓ-line. Similar case holds for the plane π_2 through \mathcal{T}_2 and \mathcal{RC} . This gives $\Lambda_{T,2_{\mathscr{C}}} = \Lambda_{U\Gamma,2_{\mathscr{C}}} = 1$.

Each of q real chords touched in P_1 gives one $2_{\mathscr{C}}$ -plane through some U Γ -line also touched in P_1 ; distinct chords give $2_{\mathscr{C}}$ -planes for distinct U Γ -lines, i.e. we have $\Pi_{2_{\mathscr{C}},U\Gamma}=1$ as in a Γ -plane there are q U Γ -lines.

By Theorem 2.2(v), a $3_{\mathscr{C}}$ -plane through \mathcal{RC} cannot meet a Γ -plane in a tangent; it intersects Γ_1 and Γ_2 in U Γ -lines. Clearly, these U Γ -lines do not coincide with intersection lines of other planes through \mathcal{RC} . In total, a $3_{\mathscr{C}}$ -plane contains 3 real chords, see Theorem 5.1(ii); formally this gives (with repetitions) $2 \cdot 3 = 6$ U Γ -lines in a $3_{\mathscr{C}}$ -plane. Each of these six U Γ -lines is counted twice. So, $\Lambda_{U\Gamma,3_{\mathscr{C}}} = 3$. By (4.8), we obtain $\Pi_{3_{\mathscr{C}},U\Gamma} = \mathcal{P}_{3_{\mathscr{C}},U\Gamma} = (q-1)/2$. By (4.10), we have $\Pi_{\overline{1_{\mathscr{C}}},U\Gamma} = \mathcal{P}_{\overline{1_{\mathscr{C}}},U\Gamma} = (q-1)/2$. Now, by (4.9), $\Lambda_{U\Gamma,\overline{1_{\mathscr{C}}}} = \mathcal{L}_{U\Gamma,\overline{1_{\mathscr{C}}}} = 1$.

We consider the $2_{\mathscr{C}}$ -plane π_1 . Through each of the two touch points P_1 , P_2 of π_1 we have q unisecants of \mathscr{C} lying in π_1 . By above, one of these 2q unisecants is the tangent $\mathcal{T}_1 \in \Gamma_1$ whereas another is an U Γ -line $\mathscr{U}_2 \in \Gamma_2$; the other 2q-2 unisecants are Un Γ -lines. So, $\Lambda_{\text{Un}\Gamma,2_{\mathscr{C}}} = 2(q-1)$. Now, by (4.8), we obtain $\Pi_{2_{\mathscr{C}},\text{Un}\Gamma} = \mathcal{P}_{2_{\mathscr{C}},\text{Un}\Gamma} = 2$.

The $2_{\mathscr{C}}$ -plane π_1 intersects also all the q-1 Γ -planes of $\Gamma \setminus \{\Gamma_1, \Gamma_2\}$. An intersection line is not a unisecant, or an axis, or a chord. Really, we considered above all the unisecants of π_1 . By Theorem 2.2(v), every plane not in Γ contains exactly one axis of Γ . As π_1 contains the tangent $\mathcal{T}_1 \in \Gamma_1$, it cannot have another axis. Similarly, if an intersection line is a chord, it intersects \mathcal{T}_1 , contradiction, see Theorem 2.2(v). So, all the q-1 intersection lines are external lines in Γ -planes other than chords. Thus, $\Lambda_{\Gamma,2_{\mathscr{C}}} = q-1$. Now, by (4.8), we obtain $\Pi_{2_{\mathscr{C}},\Gamma} = \mathcal{P}_{2_{\mathscr{C}},\Gamma} = 1$.

(ii) We have q real chords through a point P of \mathscr{C} . Every real chord gives one $2_{\mathscr{C}}$ -plane through the tangent in P, see the case (i). In total, we have q $2_{\mathscr{C}}$ -planes through the tangent.

A tangent intersects \mathscr{C} in one point; it cannot lie in a $0_{\mathscr{C}}$ -plane having no points common with \mathscr{C} . Also, by Theorem 2.2(vi), every plane through a tangent meets \mathscr{C} in at most one point other than the point of contact; therefore, $\Pi_{3_{\mathscr{C}},T} = 0$. Finally, by (4.10), we have $\Pi_{\overline{1_{\mathscr{C}}},T} = \mathscr{P}_{\overline{1_{\mathscr{C}}},T} = 0$.

Theorem 5.6. For all q, the following holds:

$$\Lambda_{\mathrm{RA},2_\mathscr{C}} = \Pi_{2_\mathscr{C},\mathrm{RA}} = \Lambda_{\mathrm{IA},2_\mathscr{C}} = \Pi_{2_\mathscr{C},\mathrm{IA}} = 0, \ \Lambda_{\mathrm{En}\Gamma,2_\mathscr{C}} = (q-1)^2, \ \Pi_{2_\mathscr{C},\mathrm{En}\Gamma} = 1.$$

Proof. By Theorem 2.2(v), every plane not in Γ contains exactly one axis of Γ. By Theorem 5.5(i), $\Lambda_{T,2_{\mathscr{C}}} = 1$, i.e. a $2_{\mathscr{C}}$ plane contains a tangent. Therefore, a $2_{\mathscr{C}}$ plane cannot contain RA- and IA-lines. Now, by (4.11), we have $\Lambda_{En\Gamma,2_{\mathscr{C}}} = \mathscr{L}_{En\Gamma,2_{\mathscr{C}}} = (q-1)^2$. Finally, by (4.8), we obtain $\Pi_{2_{\mathscr{C}},En\Gamma} = \mathcal{P}_{2_{\mathscr{C}},En\Gamma} = 1$.

Theorem 5.7. For all q, the following holds:

$$\Lambda_{\mathrm{Un}\Gamma,3_{\mathscr{C}}}=3(q-2),\ \Pi_{3_{\mathscr{C}},\mathrm{Un}\Gamma}=(q-2)/2,\ \Lambda_{\mathrm{Un}\Gamma,\overline{1_{\mathscr{C}}}}=q,\ \Pi_{\overline{1_{\mathscr{C}}},\mathrm{Un}\Gamma}=q/2.$$

Proof. A 3_{\mathscr{C}}-plane contains q-1 unisecants in each of the points in common with \mathscr{C} ; in total, we have 3(q-1) unisecants. By Theorem 5.5(i), $\Lambda_{U\Gamma,3_{\mathscr{C}}}=3$, i.e. the three unisecants lie also in Γ-planes. The remaining unisecants are UnΓ-lines; we have $\Lambda_{Un\Gamma,3_{\mathscr{C}}}=3(q-1)-3$. Now, by (4.8), we obtain $\Pi_{3_{\mathscr{C}},Un\Gamma}=\mathcal{P}_{3_{\mathscr{C}},Un\Gamma}=(q-2)/2$.

A $\overline{1_{\mathscr{C}}}$ -plane contains q+1 unisecants. By Theorem 5.5(i), $\Lambda_{U\Gamma,\overline{1_{\mathscr{C}}}}=1$, i.e. one unisecant lies also in a Γ -plane. The remaining unisecants are $Un\Gamma$ -lines; we have $\Lambda_{Un\Gamma,\overline{1_{\mathscr{C}}}}=q$. Now, by (4.8), we obtain $\Pi_{\overline{1_{\mathscr{C}}},Un\Gamma}=\mathcal{P}_{\overline{1_{\mathscr{C}}},Un\Gamma}=q/2$.

Theorem 5.8. For $q \not\equiv 0 \pmod{3}$, the following holds:

$$\begin{split} &\Lambda_{\mathrm{RA},3_{\mathscr{C}}}^{(1)} = \Lambda_{\mathrm{RA},0_{\mathscr{C}}}^{(1)} = \Lambda_{\mathrm{IA},\overline{1_{\mathscr{C}}}}^{(1)} = \Lambda_{\mathrm{IA},3_{\mathscr{C}}}^{(-1)} = \Lambda_{\mathrm{IA},0_{\mathscr{C}}}^{(-1)} = \Lambda_{\mathrm{RA},\overline{1_{\mathscr{C}}}}^{(-1)} = 1, \\ &\Pi_{3_{\mathscr{C}},\mathrm{RA}}^{(1)} = (q-1)/3, \ \Pi_{0_{\mathscr{C}},\mathrm{RA}}^{(1)} = 2(q-1)/3, \ \Pi_{\overline{1_{\mathscr{C}}},\mathrm{IA}}^{(1)} = q+1, \\ &\Lambda_{\mathrm{IA},3_{\mathscr{C}}}^{(1)} = \Pi_{3_{\mathscr{C}},\mathrm{IA}}^{(1)} = \Lambda_{\mathrm{IA},0_{\mathscr{C}}}^{(1)} = \Pi_{0_{\mathscr{C}},\mathrm{IA}}^{(1)} = \Lambda_{\mathrm{RA},\overline{1_{\mathscr{C}}}}^{(1)} = \Pi_{\overline{1_{\mathscr{C}}},\mathrm{RA}}^{(1)} = 0, \\ &\Pi_{3_{\mathscr{C}},\mathrm{IA}}^{(-1)} = (q+1)/3, \ \Pi_{0_{\mathscr{C}},\mathrm{IA}}^{(-1)} = 2(q+1)/3, \ \Pi_{\overline{1_{\mathscr{C}}},\mathrm{RA}}^{(-1)} = q-1, \\ &\Lambda_{\mathrm{RA},3_{\mathscr{C}}}^{(-1)} = \Pi_{3_{\mathscr{C}},\mathrm{RA}}^{(-1)} = \Lambda_{\mathrm{RA},0_{\mathscr{C}}}^{(-1)} = \Pi_{0_{\mathscr{C}},\mathrm{RA}}^{(-1)} = \Lambda_{\mathrm{IA},\overline{1_{\mathscr{C}}}}^{(-1)} = \Pi_{\overline{1_{\mathscr{C}}},\mathrm{IA}}^{(-1)} = 0. \end{split}$$

Proof. By Theorem 2.2(v), for $q \not\equiv 0 \pmod{3}$, every plane not in Γ contains exactly one axis of Γ . By Theorem 5.5(ii), $\Lambda_{T,3_{\mathscr{C}}} = 0$, i.e. a $3_{\mathscr{C}}$ -plane does not contain a tangent. So, a $3_{\mathscr{C}}$ -plane must contain either an RA-line or an IA-line but not together. Thus, it is sufficient to consider only the following two variants for a $3_{\mathscr{C}}$ -plane:

- (a) $\Lambda_{RA,3_{\mathscr{C}}} = 1$ and $\Lambda_{IA,3_{\mathscr{C}}} = 0$; (b) $\Lambda_{RA,3_{\mathscr{C}}} = 0$ and $\Lambda_{IA,3_{\mathscr{C}}} = 1$.
- (a) Let $\Lambda_{RA,3_{\mathscr{C}}} = 1$, $\Lambda_{IA,3_{\mathscr{C}}} = 0$.

By Corollary 4.2, we obtain $\Pi_{3_{\mathscr{C}},IA} = 0$. By Theorem 5.6, $\Pi_{2_{\mathscr{C}},IA} = 0$. As an IA-line is external for \mathscr{C} , by (4.7), we obtain $\Pi_{0_{\mathscr{C}},IA} = \Pi_{2_{\mathscr{C}},IA} + 2\Pi_{3_{\mathscr{C}},IA} = 0$ whence $\Lambda_{IA,0_{\mathscr{C}}} = 0$. By Theorem 5.2(iii), $\Pi_{\Gamma,IA} = 0$. Now, by (4.10), we have $\Pi_{\overline{1_{\mathscr{C}}},IA} = \mathscr{P}_{\overline{1_{\mathscr{C}}},IA} = q + 1$. By (4.9), we obtain $\Lambda_{IA,\overline{1_{\mathscr{C}}}} = \mathcal{L}_{IA,\overline{1_{\mathscr{C}}}} = 1$. Therefore, by Theorem 2.2(v), it is necessary to put $\Lambda_{RA,\overline{1_{\mathscr{C}}}} = 0$ whence $\Pi_{\overline{1_{\mathscr{C}}},RA} = 0$.

By Theorem 5.5(ii), $\Lambda_{T,0_{\mathscr{C}}} = 0$. Above, we proved $\Lambda_{IA,0_{\mathscr{C}}} = 0$. So, by Theorem 2.2(v), we must put $\Lambda_{RA,0_{\mathscr{C}}} = 1$. Then, by (4.8), we obtain $\Pi_{0_{\mathscr{C}},RA} = \mathcal{P}_{0_{\mathscr{C}},RA} = 2(q-1)/3$. It is an integer if $q \equiv 1 \pmod{3}$ whereas for $q \equiv -1 \pmod{3}$ it is not integer. By Theorem 2.3(i), the class \mathcal{O}_{RA} is an orbit; therefore, $\Pi_{0_{\mathscr{C}},RA}$ must be integer. Thus, the case (a) is possible only for $q \equiv 1 \pmod{3}$.

- By (4.8), $\Pi_{3_{\mathscr{C}},RA} = \mathcal{P}_{3_{\mathscr{C}},RA} = (q-1)/3$. It is an integer if $q \equiv 1 \pmod{3}$.
- (b) Let $\Lambda_{RA,3_{\mathscr{C}}} = 0$, $\Lambda_{IA,3_{\mathscr{C}}} = 1$.

By Corollary 4.2, we obtain $\Pi_{3_{\mathscr{C}},RA}=0$. By Theorem 5.6, $\Pi_{2_{\mathscr{C}},RA}=0$. So, by (4.7), $\Pi_{0_{\mathscr{C}},RA}=0$ whence $\Lambda_{RA,0_{\mathscr{C}}}=0$. By Theorem 5.2(ii), $\Pi_{\Gamma,RA}^{(\neq 0)}=2$. Now, by (4.10), $\Pi_{\overline{1_{\mathscr{C}}},RA}=\mathscr{P}_{\overline{1_{\mathscr{C}}},RA}=q-1$. By (4.9), we obtain $\Lambda_{RA,\overline{1_{\mathscr{C}}}}=\mathcal{L}_{RA,\overline{1_{\mathscr{C}}}}=1$. Hence, by Theorem 2.2(v), it is necessary to put $\Lambda_{IA,\overline{1_{\mathscr{C}}}}=0$ whence $\Pi_{\overline{1_{\mathscr{C}}},IA}=0$.

By Theorem 5.5(ii), $\Lambda_{T,0_{\mathscr{C}}} = 0$. Above, we proved $\Lambda_{RA,0_{\mathscr{C}}} = 0$. So, by Theorem 2.2(v), we must put $\Lambda_{IA,0_{\mathscr{C}}} = 1$. By (4.8), we have $\Pi_{0_{\mathscr{C}},IA} = \mathcal{P}_{0_{\mathscr{C}},IA} = 2(q+1)/3$. It is an integer if $q \equiv -1 \pmod{3}$ but for $q \equiv 1 \pmod{3}$ it is not an integer. By Theorem 2.3(i), the class \mathcal{O}_{IA} is an orbit; therefore, $\Pi_{0_{\mathscr{C}},IA}$ must be an integer. Thus, the case (b) is possible only for $q \equiv -1 \pmod{3}$. By (4.8), we obtain $\Pi_{3_{\mathscr{C}},IA} = \mathcal{P}_{3_{\mathscr{C}},IA} = (q+1)/3$. It is an integer if $q \equiv -1 \pmod{3}$.

Corollary 5.9. Let $q \equiv 1 \pmod{3}$. Then all $d_{\mathscr{C}}$ -planes with d = 0, 2, 3 and all osculating planes contain no the imaginary axes (IA-lines). All the q+1 planes through an IA-line are $\overline{1_{\mathscr{C}}}$ -planes forming a pencil. The $\binom{q}{2}(q+1)$ -orbit of all $\overline{1_{\mathscr{C}}}$ -planes can be partitioned into $\binom{q}{2}$ pencils of planes having an IA-line as axis.

Proof. By above,
$$\Lambda_{\text{IA},\pi}^{(1)} = \Pi_{\pi,\text{IA}}^{(1)} = 0$$
, $\pi \in \{\Gamma, 2_{\mathscr{C}}, 3_{\mathscr{C}}, 0_{\mathscr{C}}\}$, $\Lambda_{\text{IA},\overline{1_{\mathscr{C}}}}^{(1)} = 1$, $\Pi_{\overline{1_{\mathscr{C}}},\text{IA}}^{(1)} = q+1$. Also, $\#\mathcal{O}_{\text{IA}} = \binom{q}{2}$.

Theorem 5.10. For $q \not\equiv 0 \pmod{3}$, the following holds:

$$\begin{split} &\Lambda_{\text{E}\Gamma,3_{\mathscr{C}}}^{(1)} = q-4, \ \Lambda_{\text{E}\Gamma,\overline{1_{\mathscr{C}}}}^{(1)} = q, \ \Lambda_{\text{E}\Gamma,0_{\mathscr{C}}}^{(1)} = q-1, \\ &\Pi_{3_{\mathscr{C}},\text{E}\Gamma}^{(1)} = (q-4)/6, \ \Pi_{\overline{1_{\mathscr{C}}},\text{E}\Gamma}^{(1)} = q/2, \ \Pi_{0_{\mathscr{C}},\text{E}\Gamma}^{(1)} = (q-1)/3; \\ &\Lambda_{\text{En}\Gamma,3_{\mathscr{C}}}^{(1)} = q^2 - 3q + 4, \ \Lambda_{\text{En}\Gamma,\overline{1_{\mathscr{C}}}}^{(1)} = q^2 - q - 2, \ \Lambda_{\text{En}\Gamma,0_{\mathscr{C}}}^{(1)} = q^2 + 1, \\ &\Pi_{3_{\mathscr{C}},\text{En}\Gamma}^{(1)} = \frac{q^2 - 3q + 4}{6(q-1)}, \ \Pi_{\overline{1_{\mathscr{C}}},\text{En}\Gamma}^{(1)} = \frac{q^2 - q - 2}{2(q-1)}, \ \Pi_{0_{\mathscr{C}},\text{En}\Gamma}^{(1)} = \frac{q^2 + 1}{3(q-1)}. \\ &\Lambda_{\text{E}\Gamma,3_{\mathscr{C}}}^{(-1)} = \Lambda_{\text{E}\Gamma,\overline{1_{\mathscr{C}}}}^{(-1)} = q - 2, \ \Lambda_{\text{E}\Gamma,0_{\mathscr{C}}}^{(-1)} = q + 1, \\ &\Pi_{3_{\mathscr{C}},\text{E}\Gamma}^{(-1)} = (q-2)/6, \ \Pi_{\overline{1_{\mathscr{C}}},\text{E}\Gamma}^{(-1)} = (q-2)/2, \ \Pi_{0_{\mathscr{C}},\text{E}\Gamma}^{(-1)} = (q+1)/3; \\ &\Lambda_{\text{En}\Gamma,3_{\mathscr{C}}}^{(-1)} = (q-1)(q-2), \ \Lambda_{\text{En}\Gamma,\overline{1_{\mathscr{C}}}}^{(-1)} = q^2 - q, \ \Lambda_{\text{En}\Gamma,0_{\mathscr{C}}}^{(-1)} = q^2 - 1, \\ &\Pi_{3_{\mathscr{C}},\text{En}\Gamma}^{(-1)} = (q-2)/6, \ \Pi_{\overline{1_{\mathscr{C}}},\text{En}\Gamma}^{(-1)} = q/2, \ \Pi_{0_{\mathscr{C}},\text{En}\Gamma}^{(-1)} = (q+1)/3. \end{split}$$

Proof. Let $q \equiv 1 \pmod{3}$. Each $3_{\mathscr{C}}$ -plane intersects all q+1 Γ -planes. By Theorem 5.5(i), $\Lambda_{U\Gamma,3_{\mathscr{C}}}=3$, i.e. the three intersections correspond to unisecants in Γ -planes. By Theorem 5.8, $\Lambda_{RA,3_{\mathscr{C}}}^{(1)} = 1$. An RA-line is the intersection of two Γ-planes. So, the two intersections of a $3_{\mathscr{C}}$ -plane with Γ -planes correspond to real axes. The remaining intersections correspond to external lines; thus, $\Lambda^{(1)}_{\mathrm{E}\Gamma,3_{\mathscr{C}}}=q+1-3-2$. By (4.11), we obtain $\Lambda^{(1)}_{\operatorname{En}\Gamma,3_{\mathscr{C}}} = \mathscr{L}^{(1)}_{\operatorname{En}\Gamma,3_{\mathscr{C}}} = q^2 - 3q + 4.$

By Theorem 5.5(i), $\Pi_{2_{\mathscr{C}},\text{E}\Gamma} = 1$. By (4.7), $\Pi_{0_{\mathscr{C}},\text{E}\Gamma}^{(1)} = (q-1)/3$. By (4.9), $\Lambda_{\text{E}\Gamma,0_{\mathscr{C}}}^{(1)} =$

 $\mathcal{L}_{\text{E}\Gamma,0_{\mathscr{C}}}^{(1)} = q - 1. \text{ By } (4.11), \ \Lambda_{\text{En}\Gamma,0_{\mathscr{C}}}^{(1)} = \mathcal{L}_{\text{En}\Gamma,0_{\mathscr{C}}}^{(1)} = q^2 + 1.$ By Theorem 5.2(ii), $\Pi_{\Gamma,\text{E}\Gamma}^{(\neq 0)} = 1. \text{ By } (4.10), \ (4.9), \ \Pi_{\overline{1_{\mathscr{C}}},\text{E}\Gamma}^{(1)} = \mathcal{P}_{\overline{1_{\mathscr{C}}},\text{E}\Gamma}^{(1)} = q/2, \ \Lambda_{\text{E}\Gamma,\overline{1_{\mathscr{C}}}}^{(1)} = q/2$ $\mathcal{L}_{\text{Er}\Gamma,\overline{1}_{\infty}}^{(1)} = q$. Using (4.11), we have $\Lambda_{\text{Err}\Gamma,\overline{1}_{\infty}}^{(1)} = \mathscr{L}_{\text{Err}\Gamma,\overline{1}_{\infty}}^{(1)} = q^2 - q - 2$.

Let $q \equiv -1 \pmod{3}$. Each $3_{\mathscr{C}}$ -plane intersects all q+1 Γ -planes. By Theorem 5.5(i), $\Lambda_{\mathrm{U}\Gamma,3_{\mathscr{C}}}=3$, i.e. the three intersections correspond to unisecants in Γ -planes. The remaining intersections correspond to external lines; so, $\Lambda_{\text{E}\Gamma,3_{\mathscr{C}}}^{(-1)} = q + 1 - 3$. Now, using (4.11), we obtain $\Lambda_{\operatorname{En}\Gamma,3_{\mathscr{C}}}^{(-1)}=\mathscr{L}_{\operatorname{En}\Gamma,3_{\mathscr{C}}}^{(-1)}=(q-1)(q-2).$

By Theorem 5.5(i), $\Pi_{2_{\mathscr{C}},E\Gamma} = 1$. By (4.7), $\Pi_{0_{\mathscr{C}},E\Gamma}^{(-1)} = (q+1)/3$. By (4.9), $\Lambda_{E\Gamma,0_{\mathscr{C}}}^{(-1)} = \mathcal{L}_{E\Gamma,0_{\mathscr{C}}}^{(-1)} = q+1$. By (4.11), $\Lambda_{En\Gamma,0_{\mathscr{C}}}^{(-1)} = \mathcal{L}_{En\Gamma,0_{\mathscr{C}}}^{(-1)} = q^2-1$. By Theorem 5.2(ii), $\Pi_{\Gamma,E\Gamma}^{(\neq 0)} = 1$. By (4.10), $\Pi_{\overline{1_{\mathscr{C}},E\Gamma}}^{(-1)} = \mathscr{P}_{\overline{1_{\mathscr{C}},E\Gamma}}^{(-1)} = (q-2)/2$. By (4.9), $\Lambda_{E\Gamma,\overline{1_{\mathscr{C}}}}^{(-1)} = \mathcal{L}_{E\Gamma,\overline{1_{\mathscr{C}}}}^{(-1)} = q-2$. By (4.11), $\Lambda_{En\Gamma,\overline{1_{\mathscr{C}}}}^{(-1)} = \mathcal{L}_{En\Gamma,\overline{1_{\mathscr{C}}}}^{(-1)} = q^2-q$. Finally, by (4.8), we obtain $\Pi_{3_{\mathscr{C}},E\Gamma}^{(1)}$, $\Pi_{3_{\mathscr{C}},E\Gamma}^{(1)}$, $\Pi_{0_{\mathscr{C}},En\Gamma}^{(1)}$, $\Pi_{1_{\mathscr{C}},En\Gamma}^{(1)}$, $\Pi_{3_{\mathscr{C}},E\Gamma}^{(-1)}$, $\Pi_{3_{\mathscr{C}},E\Gamma}^{(-1$

and $\Pi_{\overline{1_{\mathscr{C}}}, \operatorname{En}\Gamma}^{(-1)}$, using the values of $\Lambda_{\lambda, \pi}^{(\xi)}$ obtained above.

Corollary 5.11. For $q \equiv 1 \pmod{3}$ and odd $q \equiv -1 \pmod{3}$, the class $\mathcal{O}_6 = \mathcal{O}_{\operatorname{En}\Gamma} =$ $\{\operatorname{En}\Gamma\text{-lines}\}\ contains\ at\ least\ two\ line\ orbits\ under\ G_q.$

Proof. By Theorem 5.10, for $q \equiv 1 \pmod{3}$, $\Pi_{0_{\mathscr{C}}, \text{En}\Gamma}^{(1)} = (q^2+1)/3(q-1)$; it is not an integer as $q^2 + 1 \equiv 2 \pmod{3}$ but $3(q - 1) \equiv 0 \pmod{9}$. For $q \equiv -1 \pmod{3}$, $\Pi_{\frac{1}{4} \notin \text{En}\Gamma}^{(-1)} = q/2$; it is not an integer for odd q. Now we use Lemma 4.1(v).

Theorem 5.12. For $q \not\equiv 0 \pmod{3}$, the submatrix $\mathcal{I}_{\Gamma,RA}^{\Pi\Lambda}$ of $\mathcal{I}^{\Pi\Lambda}$ is a simple complete 2-(q+1,2,1) design in the sense of [23, Section 1.6].

Proof. Any two Γ -planes intersect each other in an RA-line. All these intersections correspond to $\mathcal{I}_{\Gamma,RA}^{\Pi\Lambda}$.

The numbers of π -planes through λ -lines and of λ -6 lines in π -planes, for PG(3,q), $q \equiv 0 \pmod{3}$

For RC-, T-, IC-, UΓ-, and UnΓ-lines we use the results of Section 5, see Table 1 where these results are written without superscripts. Also, we may use the values $\Lambda_{\text{En}\Gamma,\Gamma}$ $\Pi_{\Gamma, \text{En}\Gamma} = 0$, see Theorem 5.2(iii).

Theorem 6.1. For $q \equiv 0 \pmod{3}$, the following holds:

$$\begin{split} &\Lambda_{\mathrm{A},\Gamma}^{(0)} = 1, \ \Pi_{\Gamma,\mathrm{A}}^{(0)} = q+1, \ \Lambda_{\mathrm{A},\pi}^{(0)} = \Pi_{\pi,\mathrm{A}}^{(0)} = 0, \ \pi \in \{2_{\mathscr{C}}, 3_{\mathscr{C}}, \overline{1_{\mathscr{C}}}, 0_{\mathscr{C}}\}, \\ &\Lambda_{\mathrm{EA},\Gamma}^{(0)} = q^2 - 1, \ \Pi_{\Gamma,\mathrm{EA}}^{(0)} = 1. \end{split}$$

Proof. For $q \equiv 0 \pmod{3}$, Γ -planes form a pencil with axis A-line, see Section 2. This implies the first row of the assertion. By (4.11), we obtain $\Lambda_{\text{EA},\Gamma}^{(0)} = \mathcal{L}_{\text{EA},\Gamma}^{(0)} = q^2 - 1$. By $\label{eq:problem} (4.8),\,\Pi^{(0)}_{\Gamma,\mathrm{EA}} = \mathcal{P}^{(0)}_{\Gamma,\mathrm{EA}} = 1.$ **Theorem 6.2.** For $q \equiv 0 \pmod{3}$, we have

$$\Lambda^{(0)}_{\mathrm{EA},2_{\mathscr{C}}} = q-1, \ \Pi^{(0)}_{2_{\mathscr{C}},\mathrm{EA}} = \frac{q}{q+1}, \ \Lambda^{(0)}_{\mathrm{En}\Gamma,2_{\mathscr{C}}} = (q-1)^2, \ \Pi^{(0)}_{2_{\mathscr{C}},\mathrm{En}\Gamma} = 1.$$

Proof. In the proof of Theorem 5.5(i), it is shown that a $2_{\mathscr{C}}$ -plane intersects two Γ-planes, placed in its points in common with \mathscr{C} , by unisecants to \mathscr{C} and the other q-1 Γ-planes by lines external with respect to \mathscr{C} . As these external lines lie in Γ-planes, they intersect the axis (A-line), i.e. they are EA-lines. Thus, $\Lambda_{\text{EA},2_{\mathscr{C}}}^{(0)} = q-1$. Now, using (4.11), we obtain $\Lambda_{\text{En}\Gamma,2_{\mathscr{C}}}^{(0)} = \mathscr{L}_{\text{En}\Gamma,2_{\mathscr{C}}}^{(0)} = (q-1)^2$. Finally, by (4.8), we have $\Pi_{2_{\mathscr{C}},\text{EA}}^{(0)} = \mathcal{P}_{2_{\mathscr{C}},\text{EA}}^{(0)} = q/(q+1)$, $\Pi_{2_{\mathscr{C}},\text{En}\Gamma}^{(0)} = \mathcal{P}_{2_{\mathscr{C}},\text{En}\Gamma}^{(0)} = 1$.

Theorem 6.3. For $q \equiv 0 \pmod{3}$, the following holds:

$$\begin{split} &\Lambda_{\mathrm{EA},3_{\mathscr{C}}}^{(0)} = q - 2, \ \Lambda_{\mathrm{EA},\overline{1_{\mathscr{C}}}}^{(0)} = q, \ \Lambda_{\mathrm{EA},0_{\mathscr{C}}}^{(0)} = q + 1, \\ &\Lambda_{\mathrm{En}\Gamma,3_{\mathscr{C}}}^{(0)} = q^2 - 3q + 3, \ \Lambda_{\mathrm{En}\Gamma,\overline{1_{\mathscr{C}}}}^{(0)} = q^2 - q - 1, \ \Lambda_{\mathrm{En}\Gamma,0_{\mathscr{C}}}^{(0)} = q^2, \\ &\Pi_{3_{\mathscr{C}},\mathrm{EA}}^{(0)} = \frac{q(q-2)}{6(q+1)}, \ \Pi_{\overline{1_{\mathscr{C}}},\mathrm{EA}}^{(0)} = \frac{q^2}{2(q+1)}, \ \Pi_{0_{\mathscr{C}},\mathrm{EA}}^{(0)} = \frac{1}{3}q, \\ &\Pi_{3_{\mathscr{C}},\mathrm{En}\Gamma}^{(0)} = \frac{q^2 - 3q + 3}{6(q-1)}, \ \Pi_{\overline{1_{\mathscr{C}}},\mathrm{En}\Gamma}^{(0)} = \frac{q^2 - q - 1}{2(q-1)}, \ \Pi_{0_{\mathscr{C}},\mathrm{En}\Gamma}^{(0)} = \frac{q^2}{3(q-1)}. \end{split}$$

Proof. A 3_{\mathscr{C}}-plane intersects all q+1 Γ-planes. Exactly three of these intersections are unisecants of \mathscr{C} as $\Lambda_{\text{UΓ},3_{\mathscr{C}}}=3$, see Theorem 5.5(i). The other q-2 intersections correspond to lines external with respect to \mathscr{C} . As these external lines lie in Γ-planes they intersect the axis (A-line), i.e. they are EA-lines. So, $\Lambda_{\text{EA},3_{\mathscr{C}}}^{(0)}=q-2$. Similarly, a $\overline{1_{\mathscr{C}}}$ -plane intersects exactly one Γ-plane by a unisecant, see $\Lambda_{\text{UΓ},\overline{1_{\mathscr{C}}}}=1$ in Theorem 5.5(i); the intersections with the other q Γ-planes provide $\Lambda_{\text{EA},\overline{1_{\mathscr{C}}}}^{(0)}=q$. Finally, all q+1 intersections of a $0_{\mathscr{C}}$ -plane with Γ-planes are external lines by the definition of a $0_{\mathscr{C}}$ -plane. This gives $\Lambda_{\text{EA},0_{\mathscr{C}}}^{(0)}=q+1$.

Now, using (4.11), we obtain $\Lambda^{(0)}_{\operatorname{En}\Gamma,3_{\mathscr{C}}}$, $\Lambda^{(0)}_{\operatorname{En}\Gamma,\overline{1_{\mathscr{C}}}}$, and $\Lambda^{(0)}_{\operatorname{En}\Gamma,0_{\mathscr{C}}}$. Finally, by (4.8), we obtain $\Pi^{(0)}_{3_{\mathscr{C}},\operatorname{EA}}$, $\Pi^{(0)}_{\overline{1_{\mathscr{C}}},\operatorname{EA}}$, $\Pi^{(0)}_{0_{\mathscr{C}},\operatorname{En}\Gamma}$, $\Pi^{(0)}_{3_{\mathscr{C}},\operatorname{En}\Gamma}$, $\Pi^{(0)}_{\overline{1_{\mathscr{C}}},\operatorname{En}\Gamma}$, and $\Pi^{(0)}_{0_{\mathscr{C}},\operatorname{En}\Gamma}$, using the values of $\Lambda^{(0)}_{\lambda,\pi}$ obtained above.

Corollary 6.4. For $q \equiv 0 \pmod{3}$, the class $\mathcal{O}_6 = \mathcal{O}_{En\Gamma} = \{En\Gamma\text{-lines}\}\$ contains at least two line orbits under G_q .

Proof. By Theorem 6.3, $\Pi_{\overline{1_{\mathscr{C}}}, \operatorname{En}\Gamma}^{(0)} = (q^2 - q - 1)/2(q - 1)$; it is not an integer as the numerator is odd but the denominator is even. Now we use Lemma 4.1(v).

Theorem 6.5. Let $\pi \in \mathfrak{P}$. Let a class \mathcal{O}_{λ} consist of a single orbit. Then the submatrix $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ of $\mathcal{I}^{\Pi\Lambda}$ is a (v_r, b_k) configuration of Definition 2.4 with $v = \# \mathcal{N}_{\pi}$, $b = \# \mathcal{O}_{\lambda}$, $r = \Lambda_{\lambda,\pi}$, $k = \Pi_{\pi,\lambda}$. Also, up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ with $\Lambda_{\lambda,\pi}^{(\xi)} = 1$ can be viewed as a concatenation of $\Pi_{\pi,\lambda}^{(\xi)}$ identity matrices of order $\# \mathcal{O}_{\lambda}$. The same holds for the submatrices $\mathcal{I}_{\pi,\lambda_i}^{\Pi\Lambda}$.

Proof. As the class \mathcal{O}_{λ} is an orbit, $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ contains $\Pi_{\pi,\lambda}$ (resp. $\Lambda_{\lambda,\pi}$) ones in every row (resp. column), see Lemma 4.1. In PG(3, q), two planes intersect along a line. Therefore, two points of $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ are connected by at most one line. If $\Lambda_{\lambda,\pi}^{(\xi)} = 1$, $\mathcal{I}_{\pi,\lambda}^{\Pi\Lambda}$ contains $\Pi_{\pi,\lambda}^{(\xi)}$ (resp. 1) ones in every row (resp. column).

7 The numbers of π -planes through λ_j -lines and λ_j -lines in π -planes in the orbits forming classes $\mathcal{O}_{\mathrm{U}\Gamma}$, $\mathcal{O}_{\mathrm{E}\Gamma}$, and $\mathcal{O}_{\mathrm{E}\mathrm{A}}$

Theorem 7.1. Let $\lambda \in \{U\Gamma, E\Gamma\}$ if $q \not\equiv 0 \pmod{3}$; $\lambda \in \{U\Gamma, EA\}$ if $q \equiv 0 \pmod{3}$. Then, independently of the number of orbits in the class \mathcal{O}_{λ} , we have exactly one Γ -plane through every λ -line. Moreover, up to rearrangement of rows and columns, the submatrices $\mathcal{I}_{\Gamma,\lambda}^{\Pi\Lambda}$ can be viewed as a vertical concatenation of $\Lambda_{\lambda,\Gamma}^{(\xi)}$ identity matrices of order $\#\mathcal{N}_{\Gamma} = q + 1$.

Proof. By the definitions of the lines, one Γ -plane through a line always exists. If we have two Γ -planes through a line then it is an RA-line.

Corollary 7.2. We consider two $\frac{1}{2}(q^3-q)$ -orbits of $E\Gamma$ -lines, for odd $q \not\equiv 0 \pmod{3}$, and three orbits \mathcal{O}_{EA_1} , \mathcal{O}_{EA_2} , and \mathcal{O}_{EA_3} of EA-lines of sizes q^3-q , $\frac{1}{2}(q^2-1)$, and $\frac{1}{2}(q^2-1)$, respectively, for $q \equiv 0 \pmod{3}$. The following holds:

$$\Pi_{\Gamma, \text{E}\Gamma_{j}}^{(\neq 0)\text{od}} = \Pi_{\Gamma, \text{EA}_{i}}^{(0)\text{od}} = 1, \ j = 1, 2, \ i = 1, 2, 3;$$

$$\Lambda_{\text{E}\Gamma_{j}, \Gamma}^{(\neq 0)\text{od}} = \frac{1}{2}(q^{2} - q), \ j = 1, 2; \ \Lambda_{\text{EA}_{1}, \Gamma}^{(0)\text{od}} = q^{2} - q, \ \Lambda_{\text{EA}_{i}, \Gamma}^{(0)\text{od}} = \frac{1}{2}(q - 1), \ i = 2, 3.$$
(7.1)

Proof. We use Theorem 2.3(iv)(v). The 1-st row of (7.1) follows from Theorem 7.1. The values in the 2-nd row are obtained by (4.9).

Theorem 7.3. Let q be even. For the (q+1)-orbit $\mathcal{O}_{U\Gamma_1}$ and the (q^2-1) -orbit $\mathcal{O}_{U\Gamma_2}$ of the class $\mathcal{O}_4 = \mathcal{O}_{U\Gamma}$, the following holds, see Table 2:

$$\Pi^{(\neq 0)\mathrm{ev}}_{\Gamma, \mathrm{U}\Gamma_1} = \Pi^{(\neq 0)\mathrm{ev}}_{\Gamma, \mathrm{U}\Gamma_2} = \Lambda^{(\neq 0)\mathrm{ev}}_{\mathrm{U}\Gamma_1, \Gamma} = \Lambda^{(\neq 0)\mathrm{ev}}_{\mathrm{U}\Gamma_1, 2_\mathscr{C}} = \Lambda^{(\neq 0)\mathrm{ev}}_{\mathrm{U}\Gamma_2, \overline{1_\mathscr{C}}} = 1; \ \Pi^{(\neq 0)\mathrm{ev}}_{2_\mathscr{C}, \mathrm{U}\Gamma_1} = q;$$

$$\Pi_{\pi, \text{U}\Gamma_{1}}^{(\neq 0)\text{ev}} = \Lambda_{\text{U}\Gamma_{1}, \pi}^{(\neq 0)\text{ev}} = 0 \text{ if } \pi \in \{3_{\mathscr{C}}, \overline{1_{\mathscr{C}}}, 0_{\mathscr{C}}\}; \ \Pi_{3_{\mathscr{C}}, \text{U}\Gamma_{2}}^{(\neq 0)\text{ev}} = \Pi_{\overline{1_{\mathscr{C}}}, \text{U}\Gamma_{2}}^{(\neq 0)\text{ev}} = \frac{1}{2}q;$$

$$\Pi_{\pi, \text{U}\Gamma_{2}}^{(\neq 0)\text{ev}} = \Lambda_{\text{U}\Gamma_{2}, \pi}^{(\neq 0)\text{ev}} = 0 \text{ if } \pi \in \{2_{\mathscr{C}}, 0_{\mathscr{C}}\}; \ \Lambda_{\text{U}\Gamma_{2}, \Gamma}^{(\neq 0)\text{ev}} = q - 1; \ \Lambda_{\text{U}\Gamma_{2}, 3_{\mathscr{C}}}^{(\neq 0)\text{ev}} = 3.$$

Proof. By Theorem 7.1, $\Pi_{\Gamma, U\Gamma_j}^{(\neq 0)\text{ev}} = 1$. Through a UΓ-line, there are q+1 planes one of which is a Γ-plane. For a line of the (q+1)-orbit $\mathcal{O}_{U\Gamma_1}$, the remaining q planes are $2_{\mathscr{C}}$ -planes, see Theorems 2.2(vi) and 2.3(ii), So, $\Pi_{2_{\mathscr{C}}, U\Gamma_1}^{(\neq 0)\text{ev}} = q$. Now, by (4.4) and Corollary 4.2, we obtain $\Pi_{\pi, U\Gamma_1}^{(\neq 0)\text{ev}} = \Lambda_{U\Gamma_1, \pi}^{(\neq 0)\text{ev}} = 0$, $\pi \in \{3_{\mathscr{C}}, \overline{1_{\mathscr{C}}}, 0_{\mathscr{C}}\}$. By (4.9), we have $\Lambda_{U\Gamma_1, \Gamma}^{(\neq 0)\text{ev}} = \Lambda_{U\Gamma_1, 2_{\mathscr{C}}}^{(\neq 0)\text{ev}} = 1$. Now, by (4.1), using $\Lambda_{U\Gamma, \pi}$ and $\Lambda_{U\Gamma_1, \pi}^{(\neq 0)\text{ev}}$, we obtain all $\Lambda_{U\Gamma_2, \pi}^{(\neq 0)\text{ev}}$ and then, by (4.8), we calculate all $\Pi_{\pi, U\Gamma_2}^{(\neq 0)\text{ev}}$.

Remind that for $q \equiv -1 \pmod{4}$ (resp. $q \equiv 1 \pmod{4}$), -1 is a non-square (resp. square) in \mathbb{F}_q . Also, for $q \equiv -1 \pmod{3}$ (resp. $q \equiv 1 \pmod{3}$), -3 is a non-square (resp. square) in \mathbb{F}_q .

Theorem 7.4. Let q be odd. For the $\frac{1}{2}(q^3-q)$ -orbits $\mathcal{O}_{\mathrm{Un}\Gamma_1}$ and $\mathcal{O}_{\mathrm{Un}\Gamma_2}$ of the class $\mathcal{O}_5 = \mathcal{O}_{\mathrm{Un}\Gamma}$, the following holds, see Table 2:

$$\begin{split} &\Pi^{\text{od}}_{\pi,\text{Un}\Gamma_{j}} = \Lambda^{\text{od}}_{\text{Un}\Gamma_{j},\pi} = 0, \ \pi = \Gamma, 0_{\mathscr{C}}, \ j = 1, 2; \ \Pi^{\text{od}}_{2_{\mathscr{C}},\text{Un}\Gamma_{1}} = 1; \ \Pi^{\text{od}}_{2_{\mathscr{C}},\text{Un}\Gamma_{2}} = 3; \\ &\Lambda^{\text{od}}_{\text{Un}\Gamma_{1},2_{\mathscr{C}}} = \Pi^{\text{od}}_{3_{\mathscr{C}},\text{Un}\Gamma_{1}} = \Pi^{\text{od}}_{\overline{1_{\mathscr{C}}},\text{Un}\Gamma_{2}} = \Lambda^{\text{od}}_{\text{Un}\Gamma_{2},\overline{1_{\mathscr{C}}}} = \frac{1}{2}(q-1); \\ &\Pi^{\text{od}}_{3_{\mathscr{C}},\text{Un}\Gamma_{2}} = \frac{1}{2}(q-3); \ \Lambda^{\text{od}}_{\text{Un}\Gamma_{2},3_{\mathscr{C}}} = \frac{3}{2}(q-3); \ \Lambda^{\text{od}}_{\text{Un}\Gamma_{2},2_{\mathscr{C}}} = \Lambda^{\text{od}}_{\text{Un}\Gamma_{1},3_{C}} = \frac{3}{2}(q-1); \\ &\Pi^{\text{od}}_{\overline{1_{\mathscr{C}}},\text{Un}\Gamma_{1}} = \Lambda^{\text{od}}_{\text{Un}\Gamma_{1},\overline{1_{\mathscr{C}}}} = \frac{1}{2}(q+1). \end{split}$$

Proof. By the definition, Γ- and $0_{\mathscr{C}}$ -planes do no contain UnΓ-lines. So, $\Pi^{\text{od}}_{\Gamma,\text{Un}\Gamma_j} = \Pi^{\text{od}}_{0_{\mathscr{C}},\text{Un}\Gamma_j} = 0$, j = 1, 2.

Now, see Theorem 2.3(iii) and [13, Theorem 6.13, Proof], we consider a plane $\mathfrak{p} = \boldsymbol{\pi}(c_0, c_1, c_2, c_3), c_i \in \mathbb{F}_q$, through the line $\ell' = \overline{\mathbf{P}(0, 0, 0, 1)\mathbf{P}(1, 0, 1, 0)}$ of $\mathcal{O}_{\mathrm{Un}\Gamma_j}, j \in \{1, 2\}$. We find the number \mathbb{N} of points P(t) in \mathfrak{p} other than P(0), see (2.1). If and only if $\mathbb{N} = 1$, \mathfrak{p} is a $2_{\mathscr{C}}$ -plane. By (2.3), $c_0 + c_2 = c_3 = 0$ whence $\mathfrak{p} = \boldsymbol{\pi}(c_0, c_1, -c_0, 0)$.

If $c_0 = 0$ then $\mathbb{N} = 1$, $P(\infty) = \mathbf{P}(1, 0, 0, 0) \in \mathfrak{p}$.

Let $c_0 \neq 0$. Then $P(\infty) \notin \mathfrak{p}$. If $P(t) = \mathbf{P}(t^3, t^2, t, 1) \in \mathfrak{p}$, $t \in \mathbb{F}_q^*$, then $c_0 t^3 + c_1 t^2 - c_0 t = 0$ and $t^2 + c t - 1 = 0$, $c \in \mathbb{F}_q$, whence $t = -c/2 \pm \sqrt{(c/2)^2 + 1}$. If $q \equiv -1 \pmod{4}$, we have $\mathbb{N} \in \{0, 2\}$ when c runs over \mathbb{F}_q ; if $q \equiv 1 \pmod{4}$, we have $\mathbb{N} = 1$ exactly for two values of c with $\sqrt{(c/2)^2 + 1} = 0$.

So, when c_0, c_1 runs over \mathbb{F}_q , there are either one or three cases $\mathbb{N}=1$ that corresponds to $\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_j}\in\{1,3\}$. By Theorem 5.5(i), $\Pi_{2_{\mathscr{C}},\mathrm{Un}\Gamma}=2$, whence, by (4.3) and Theorem 2.3(iii), we have $\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_1}+\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_2}=4$. Therefore, if $\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_1}=1$ then $\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_2}=3$ and vice versa. We put $\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_1}=1$ w.l.o.g.

Consider q planes through a Un Γ -line and a point of \mathscr{C} . They are either $2_{\mathscr{C}}$ - or $3_{\mathscr{C}}$ -planes; in that, each $3_{\mathscr{C}}$ -plane appears two times. So, $\Pi^{\mathrm{od}}_{2_{\mathscr{C}},\mathrm{Un}\Gamma_j} + 2\Pi^{\mathrm{od}}_{3_{\mathscr{C}},\mathrm{Un}\Gamma_j} = q$ whence, by above, $\Pi^{\mathrm{od}}_{3_{\mathscr{C}},\mathrm{Un}\Gamma_1} = (q-1)/2$. Also, by (4.10), $\Pi^{\mathrm{od}}_{\overline{1_{\mathscr{C}}},\mathrm{Un}\Gamma_1} = (q+1)/2$. Now, by (4.9), we obtain all $\Lambda^{\mathrm{od}}_{\mathrm{Un}\Gamma_1,\pi}$ from $\Pi^{\mathrm{od}}_{\pi,\mathrm{Un}\Gamma_1}$. Then by (4.1) and (4.8), we calculate all $\Lambda^{\mathrm{od}}_{\mathrm{Un}\Gamma_2,\pi}$ and $\Pi^{\mathrm{od}}_{\pi,\mathrm{Un}\Gamma_2}$.

Lemma 7.5. Let $q \equiv \xi \pmod{3}$ be odd; let also $q \equiv \beta \pmod{4}$, $\xi, \beta \in \{1, -1\}$. Let $f(x) = -\frac{4}{3}x^2 - 3$. Let $V^{(\xi,\beta)} = \{c \in \mathbb{F}_q^* | f(c) \text{ is a non-square in } \mathbb{F}_q^*\}$, $R^{(\beta)} = \{c \in \mathbb{F}_q^* | f(c) = 0\}$. Then $\#R^{(\beta)} = \beta + 1$, $\#V^{(\xi,\beta)} = \frac{1}{2}(q + 2\xi - 2 - \beta)$.

Proof. The roots of f(x) are $\pm \frac{3}{2}\sqrt{-1}$. This explains $\#R^{(\beta)}$.

Let η be the quadratic character of \mathbb{F}_q . For $a \in \mathbb{F}_q^*$, $\eta(a) = 1$ if a is a square in \mathbb{F}_q^* and $\eta(a) = -1$ otherwise. By [21, Theorem 5.18],

$$\sum_{c \in \mathbb{F}_q \backslash R^{(\beta)}} \eta(f(c)) = -\eta\left(-\frac{4}{3}\right) = -\xi$$

where by $c \in \mathbb{F}_q \setminus R^{(\beta)}$ we note that $\eta(0)$ is not defined. As the number $q - \#R^{(\beta)}$ of summands in \sum_c is odd, $(q - \#R^{(\beta)} + 1)/2$ summands are equal to $-\xi$ while $(q - \#R^{(\beta)} - 1)/2$ ones are ξ . Also, $\eta(f(0)) = \eta(-3) = \xi$. Now $\#V^{(\xi,\beta)}$ can be obtained by straightforward calculation.

Theorem 7.6. Let $q \equiv \xi \pmod{3}$ be odd, $\xi \in \{1, -1\}$. For the $\frac{1}{2}(q^3 - q)$ -orbits $\mathcal{O}_{\mathrm{E}\Gamma_j}$, j = 1, 2, of the class $\mathcal{O}_5' = \mathcal{O}_{\mathrm{E}\Gamma}$, in addition to Corollary 7.2 the following holds, see Table 2:

$$\Pi_{2_{\mathscr{C}},\text{E}\Gamma_{1}}^{(\xi)\text{od}} = \Lambda_{\text{E}\Gamma_{1},2_{\mathscr{C}}}^{(\xi)\text{od}} = 0; \ \Pi_{2_{\mathscr{C}},\text{E}\Gamma_{2}}^{(\xi)\text{od}} = 2, \ \Lambda_{\text{E}\Gamma_{2},2_{\mathscr{C}}}^{(\xi)\text{od}} = q-1; \ \Pi_{3_{\mathscr{C}},\text{E}\Gamma_{1}}^{(\xi)\text{od}} = \frac{1}{6}(q-\xi);$$

$$\Lambda_{\text{E}\Gamma_{1},3_{\mathscr{C}}}^{(\xi)\text{od}} = \Lambda_{\text{E}\Gamma_{1},0_{\mathscr{C}}}^{(\xi)\text{od}} = \Lambda_{\text{E}\Gamma_{2},0_{\mathscr{C}}}^{(\xi)\text{od}} = \frac{1}{2}(q-\xi); \ \Lambda_{\text{E}\Gamma_{1},0_{\mathscr{C}}}^{(\xi)\text{od}} = \Lambda_{\text{E}\Gamma_{2},0_{\mathscr{C}}}^{(\xi)\text{od}} = \frac{1}{3}(q-\xi);$$

$$\Pi_{\overline{1_{\mathscr{C}}},\text{E}\Gamma_{1}}^{(\xi)\text{od}} = \Lambda_{\text{E}\Gamma_{1},\overline{1_{\mathscr{C}}}}^{(\xi)\text{od}} = \frac{1}{2}(q+\xi); \ \Pi_{\overline{1_{\mathscr{C}}},\text{E}\Gamma_{2}}^{(\xi)\text{od}} = \Lambda_{\text{E}\Gamma_{2},\overline{1_{\mathscr{C}}}}^{(\xi)\text{od}} = \frac{1}{2}(q+\xi-2);$$

$$\Pi_{3_{\mathscr{C}},\text{E}\Gamma_{2}}^{(\xi)\text{od}} = \frac{1}{6}(q-\xi-6), \ \Lambda_{\text{E}\Gamma_{2},3_{\mathscr{C}}}^{(\xi)\text{od}} = \frac{1}{2}(q-\xi-6).$$

Proof. We use Theorem 2.3(iii)(iv). The null polarity \mathfrak{A} (2.5) maps the points $P_0 = \mathbf{P}(0,0,0,1)$ and $P' = \mathbf{P}(1,0,1,0)$ of [13, Theorem 6.13, Proof] to the planes $\mathfrak{p}_0 = \boldsymbol{\pi}(1,0,0,0)$ and $\mathfrak{p}' = \boldsymbol{\pi}(0,-3,0,-1)$, respectively. The UnΓ-line $\ell' = \overline{P_0P'}$ is mapped to an EA-line $\overline{\ell}$ so that $\ell'\mathfrak{A} = \mathfrak{p}_0 \cap \mathfrak{p}' \triangleq \overline{\ell}$. Let $\overline{\pi} = \boldsymbol{\pi}(c_0,c_1,c_2,c_3)$, $c_i \in \mathbb{F}_q$, be a plane through $\overline{\ell}$. By [17, Section 15.2], the matrix associated with $\overline{\ell}$ is

$$\widehat{\Lambda} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

and $\overline{\pi}\hat{\Lambda} = 0$. It gives $-c_2 = c_1 - 3c_3 = 3c_2 = 0$ whence $\overline{\pi} = \pi(c_0, c_1, 0, c_1/3)$.

If $c_0 \neq 0$, $c_1 = 0$ then $\overline{\pi} = \pi(1, 0, 0, 0) = \mathfrak{p}_0 = \pi_{\rm osc}(0)$, $P(0) \in \overline{\pi} = \mathfrak{p}_0$. Thus, $\overline{\pi} = \mathfrak{p}_0$ is a Γ -plane.

If $c_0 = 0$ then $c_1 \neq 0$, $\overline{\pi} = \pi(0, 1, 0, 1/3) = \mathfrak{p}'$, $P(\infty) = \mathbf{P}(1, 0, 0, 0) \in \overline{\pi} = \mathfrak{p}'$, and $P(t) = \mathbf{P}(t^3, t^2, t, 1) \in \overline{\pi} = \mathfrak{p}'$ if and only if $t = \pm \sqrt{-1/3}$. So, $\overline{\pi} = \mathfrak{p}'$ is a $3_{\mathscr{C}}$ -plane for $\xi = 1$ and a $\overline{1_{\mathscr{C}}}$ -plane for $\xi = -1$.

Let $c_0 \neq 0, c_1 \neq 0$. Then $\overline{\pi} = \pi(1, c, 0, c/3) \triangleq \overline{\pi}(c), c \in \mathbb{F}_q^*$, $P(\infty) \notin \overline{\pi}(c)$, $P(t) = \mathbf{P}(t^3, t^2, t, 1) \in \overline{\pi}(c)$ if and only if t satisfies the equation F(t), see (2.3), with the discriminant $\Delta(F)$ obtained by [16, Lemma 1.18(ii)]. We have

$$F(t) = t^3 + ct^2 + \frac{c}{3} = 0, \ t \in \mathbb{F}_q, \ \Delta(F) = c^2 \left(-\frac{4}{3}c^2 - 3 \right), \ c \in \mathbb{F}_q^*.$$

Let $q \equiv \beta \pmod{4}$, $\beta \in \{1, -1\}$. Let $\mathbb{N}_j^{(\xi, \beta)}$ be the number of values of $c \in \mathbb{F}_q^*$ providing exactly j roots of F(t) in the corresponding \mathbb{F}_q .

The plane $\overline{\pi}(c)$ is a $0_{\mathscr{C}}$ -, $\overline{1_{\mathscr{C}}}$ -, $2_{\mathscr{C}}$ -, and $3_{\mathscr{C}}$ -plane according as F(t) has 0, 1, 2, and 3 roots in \mathbb{F}_q .

By above, $\Pi_{\Gamma, \text{E}\Gamma_j}^{(\xi)\text{od}} = 1$, $\Pi_{\overline{1_{\mathscr{C}}}, \text{E}\Gamma_j}^{(\xi)\text{od}} = \mathbb{N}_1^{(\xi,\beta)} + (1-\xi)/2$, $\Pi_{2_{\mathscr{C}}, \text{E}\Gamma_j}^{(\xi)\text{od}} = \mathbb{N}_2^{(\xi,\beta)}$, where we take into account the planes $\overline{\pi} = \mathfrak{p}_0$, $\overline{\pi} = \mathfrak{p}'$, and $\overline{\pi}(c)$.

By [16, Corollary 1.30], where all $A_i \neq 0$ in our case, F(t) has exactly two roots if and only if $\Delta(F) = 0$. Also, by [16, Corollary 1.15(ii)], if $\Delta(F) \neq 0$ then F(t) has exactly one root if and only if $\Delta(F)$ is a non-square in \mathbb{F}_q .

We put j=1, use Lemma 7.5, and obtain $\Pi_{\overline{1_{\mathscr{C}},\mathrm{E}\Gamma_{1}}}^{(\xi)\mathrm{od}}=\#V^{(\xi,\beta)}+(1-\xi)/2=(q+\xi-1-\beta)/2$, $\Pi_{2_{\mathscr{C}},\mathrm{E}\Gamma_{1}}^{(\xi)\mathrm{od}}=\#R^{(\beta)}=\beta+1$, whence, by (4.6), (4.7), with $\Pi_{\Gamma,\mathrm{E}\Gamma_{j}}^{(\xi)\mathrm{od}}=1$, we have $\Pi_{3_{\mathscr{C}},\mathrm{E}\Gamma_{1}}^{(\xi)\mathrm{od}}=(q-\xi-3-3\beta)/6$, $\Pi_{0_{\mathscr{C}},\mathrm{E}\Gamma_{1}}^{(\xi)\mathrm{od}}=(q-\xi)/3$. Then, using (4.3) with $\Pi_{\pi,\mathrm{E}\Gamma}^{(\xi)\mathrm{od}}$ obtained above, see Table 1, we obtain $\Pi_{2_{\mathscr{C}},\mathrm{E}\Gamma_{2}}^{(\xi)\mathrm{od}}=1-\beta$, $\Pi_{\overline{1_{\mathscr{C}},\mathrm{E}\Gamma_{2}}}^{(\xi)\mathrm{od}}=(q+\xi-1+\beta)/2$, $\Pi_{3_{\mathscr{C}},\mathrm{E}\Gamma_{2}}^{(\xi)\mathrm{od}}=(q-\xi-3+3\beta)/6$, $\Pi_{0_{\mathscr{C}},\mathrm{E}\Gamma_{2}}^{(\xi)\mathrm{od}}=(q-\xi)/3$.

For $\beta = -1$, the formulae above give the values $\Pi_{\pi, \mathrm{E}\Gamma_j}^{(\xi)\mathrm{od}}$, j = 1, 2, as in Table 2. Moreover, $\beta = 1$ provides the same values but the numbers j of orbits $\mathcal{O}_{\mathrm{E}\Gamma_j}$ change places, i.e. we have j = 2 instead of j = 1 and vice versa. Therefore, β does not appear in the final formulae.

In conclusion, by (4.9), from $\Pi_{\pi, \mathrm{E}\Gamma_i}^{(\xi)\mathrm{od}}$ we obtain $\Lambda_{\mathrm{E}\Gamma_i, \pi}^{(\xi)\mathrm{od}}$.

Theorem 7.7. Let $q \equiv 0 \pmod{3}$, $q \geq 9$. For the orbits \mathcal{O}_{EA_1} , \mathcal{O}_{EA_2} , and \mathcal{O}_{EA_3} of EAlines (class $\mathcal{O}_8 = \mathcal{O}_{EA}$) of sizes $q^3 - q$, $\frac{1}{2}(q^2 - 1)$, and $\frac{1}{2}(q^2 - 1)$, respectively, in addition

to Corollary 7.2 the following holds, see Table 2:

$$\begin{split} &\Pi^{(0)\text{od}}_{2\mathscr{C}, \text{EA}_1} = \Lambda^{(0)\text{od}}_{\text{EA}_2, 3\mathscr{C}} = \Lambda^{(0)\text{od}}_{\text{EA}_2, 0\mathscr{C}} = \Lambda^{(0)\text{od}}_{\text{EA}_3, \overline{1}_\mathscr{C}} = 1; \ \Lambda^{(0)\text{od}}_{\text{EA}_1, 2\mathscr{C}} = \Lambda^{(0)\text{od}}_{\text{EA}_1, \overline{1}_\mathscr{C}} = q-1; \\ &\Pi^{(0)\text{od}}_{\pi, \text{EA}_j} = \Lambda^{(0)\text{od}}_{\text{EA}_j, \pi} = 0, \ j=2 \ \textit{with} \ \pi = 2\mathscr{C}, \overline{1}_\mathscr{C}, \ j=3 \ \textit{with} \ \pi = 2\mathscr{C}, 3\mathscr{C}, 0\mathscr{C}; \\ &\Pi^{(0)\text{od}}_{\overline{1}_\mathscr{C}, \text{EA}_3} = \Lambda^{(0)\text{od}}_{\text{EA}_1, 0\mathscr{C}} = q, \ \Pi^{(0)\text{od}}_{3\mathscr{C}, \text{EA}_2} = \Pi^{(0)\text{od}}_{0\mathscr{C}, \text{EA}_1} = \frac{1}{3}q, \ \Pi^{(0)\text{od}}_{0\mathscr{C}, \text{EA}_2} = \frac{2}{3}q; \\ &\Pi^{(0)\text{od}}_{3\mathscr{C}, \text{EA}_1} = \frac{1}{6}(q-3), \ \Lambda^{(0)\text{od}}_{\text{EA}_1, 3\mathscr{C}} = q-3; \ \Pi^{(0)\text{od}}_{\overline{1}_\mathscr{C}, \text{EA}_1} = \frac{1}{2}(q-1). \end{split}$$

Proof. We denote $x_i = \Pi^{(0)\text{od}}_{2\mathscr{C}, \text{EA}_i}, \ \overline{x}_i = \Lambda^{(0)\text{od}}_{\text{EA}_i, 2\mathscr{C}}, \ y_i = \Pi^{(0)\text{od}}_{0\mathscr{C}, \text{EA}_i}, \ \overline{y}_i = \Lambda^{(0)\text{od}}_{\text{EA}_i, 0\mathscr{C}}, \ i = 1, 2, 3.$ Obviously, all the values must be integer. By Theorems 6.2, 6.3, $\Pi_{2\omega,EA}^{(0)} = q/(q+1)$, $\Pi_{0_{\infty},EA}^{(0)} = q/3$, whence, by (4.3), (4.4), (7.1), we have

$$qx_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = q, \ x_i \in \{0, 1, \dots, q\};$$
 (7.2)

$$6qy_1 + 3y_2 + 3y_3 = 2q^2 + 2q, \ y_i \in \{0, 1, \dots, q - x_i\}.$$

$$(7.3)$$

For (7.2), there are only two solutions $x_1 = 0, x_2 = x_3 = q$ and $x_1 = 1, x_2 = x_3 = 0$. Taking into account (4.4), (7.1), the 1-st solution implies $y_2 = y_3 = 0$, $y_1 = (q+1)/3$, contradiction as y_1 must be integer. So, $x_1 = 1, x_2 = x_3 = 0$.

It is easy to see that $y_1 = q/3$ is the only possibility to provide $2q^2$ in (7.3). Then, by

(4.9), $\overline{y}_1 = q$, and by (4.1), $\overline{y}_2 + \overline{y}_3 = \Lambda_{\mathrm{EA},0_{\mathscr{C}}}^{(0)\mathrm{od}} - \overline{y}_1 = 1$, see Theorem 6.3. We put $\overline{y}_2 = 1$, $\overline{y}_3 = 0$, w.l.o.g., whence, by (4.8), $y_2 = 2q/3$, $y_3 = 0$. Now, by (4.7), we obtain $\Pi_{3_{\mathscr{C}},\mathrm{EA}_i}^{(0)\mathrm{od}} = (y_i - x_i)/2$, i = 1, 2, 3, and then, by (4.6), we calculate $\Pi_{\overline{1_{\mathscr{C}}},\mathrm{EA}_i}^{(0)\mathrm{od}} = q + 1 - \Pi_{\Gamma,\mathrm{EA}_i}^{(0)\mathrm{od}} - 2x_i - 3\Pi_{3_{\mathscr{C}},\mathrm{EA}_i}^{(0)\mathrm{od}}$, i = 1, 2, 3. In conclusion, by (4.9), we obtain the remaining values of $\Lambda_{\mathrm{EA}_i,\pi}^{(0)\mathrm{od}}$ from $\Pi_{\pi,\mathrm{EA}_i}^{(0)\mathrm{od}}$.

Acknowledgments

The research of S. Marcugini, and F. Pambianco was supported in part by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INDAM) and by University of Perugia (Project: Curve, codici e configurazioni di punti, Base Research Fund 2018).

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