TWISTED K-THEORY AND K-THEORY OF BUNDLE GERBES

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ABSTRACT. In this note we introduce the notion of bundle gerbe K-theory and investigate the relation to twisted K-theory. We provide some examples. Possible applications of bundle gerbe K-theory to the classification of D-brane charges in nontrivial backgrounds are briefly discussed.

1. INTRODUCTION

Based on explicit calculations of D-brane charges and the analysis of brane creation-annihilation processes it has been argued that D-branes, in the absence of background B-fields, carry charges which take values in K-theory [25, 40, 15, 26]. (For background on D-branes see, e.g., [29].) This proposal has been extended to incorporate nontrivial background B-fields in [40, 17] for torsion B-fields, and in [5, 3] for general B-fields, in which case twisted K-theory [34] is needed. The picture of D-brane charges taking values in (twisted) K-theory has received further support from an analysis of M-theory [7], noncommutative tachyons [41, 13] and explicit examples (see, e.g., [10] and references therein).

On the other hand, since B-fields are most naturally described as connections over 1-gerbes, it has been clear for some time that gerbes are relevant to understanding the properties of D-branes in string theory. The occurrence of gerbes can, for instance, be inferred from the anomaly cancellation argument in [11] and is mentioned explicitly in [13].

We believe that gerbes play a role in string theory which is yet to be fully understood. The aim of this note is to argue that the twisted K-theory of a pair $(M, [H])$, where M is a manifold and $[H]$ is an integral Čech class, can be obtained from the K-theory of a special kind of gerbe over M, namely the bundle gerbes of [27]. In this paper, for the first time, we introduce the notion of a bundle gerbe module, which, in a sense, can also be thought of as a twisted vector bundle or non-abelian gerbe (see [16] for an earlier proposal), and define the K-theory of bundle gerbes as the Grothendieck group of the semi-group of bundle gerbe modules. We show that bundle gerbe K-theory is isomorphic to twisted K-theory, whenever $[H]$ is a torsion class in $H^3(M, \mathbb{Z})$. When $[H]$ is not a torsion class in $H^3(M, \mathbb{Z})$ we consider the lifting bundle gerbe associated to the $PU(\mathcal{H})$ bundle with Dixmier-Douady class $[H]$ and in this case we prove that twisted K-theory is the Grothendieck group of the semi-group of $U_{\mathcal{K}}$-bundle gerbe modules, which are the infinite dimensional cousins of bundle gerbe modules. It remains to understand how it might be used in string theory for example whether the analysis of [11]

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applies in the case where the background B-field does not define a torsion class in $H^3(M,\mathbb{Z})$ (related issues have recently been discussed in [22]).

This note is organised as follows. Section 2 summarises the theory of bundle gerbes. These are geometric objects that are associated with degree 3 integral Čech cohomology classes on M. The notion of stable equivalence of bundle gerbes, which is essential for the understanding of the sense in which the degree 3 class (known as the Dixmier-Douady class of the bundle gerbe) determines an associated bundle gerbe is the subject of Section 3. The K-theory of bundle gerbes is introduced in Sections 4 and 5 and in Section 6 we analyse characteristic classes of bundle gerbe modules. Twisted K-theory in its various manifestations is described in Section 6 where we prove that the bundle gerbe K-theory is isomorphic to twisted K-theory in the torsion case and analyse characteristic classes of bundle gerbe modules. In Section 7 we consider bundle gerbes with non-torsion Dixmier-Douady class and show that twisted K-theory is isomorphic to the $U_{\mathcal{K}}$ bundle gerbe K-theory of the lifting bundle gerbe. We extend our discussion of characteristic classes for bundle gerbe modules to the non-torsion case in Section 9, where we also discuss twisted cohomology. In Section 8 we calculate some examples of twisted K-theory, and we conclude with some remarks in Section 10.

While completing this note a preprint [19] appeared which uses similar ideas in the context of the K-theory of orbifolds and another [20] which introduces twisted vector and principal bundles which are the same as our bundle gerbe modules when the bundle gerbe arises from an open cover.

2. Bundle gerbes

2.1. **Bundle gerbes and Dixmier-Douady classes.** Before recalling the definition of bundle gerbe from [27] we need some notation for fibre products. Mostly we will be working with smooth manifolds and smooth maps but often these will need to be infinite-dimensional. In the interest of brevity we will just say map.

We will be interested in maps $\pi\colon Y \to M$ which admit local sections. That is, for every $x \in M$ there is an open set U containing x and a local section $s\colon U \to Y$. We call such maps locally split. Note that a locally split map is necessarily surjective. Locally trivial fibrations are, of course, locally split, but the converse is not true. Indeed one case of particular interest will be when M has an open cover $\{U_i\}_{i\in I}$ and

$$Y = \{(x,i) \mid x \in U_i\}$$

the disjoint union of all the open sets U_i with $\pi(x,i) = x$. This example is locally split by $s_i\colon U_i \to Y$, with $s_i(x) = (x,i)$ but it is rarely a fibration.

Let $\pi\colon Y \to M$ be locally split. Then we denote by $Y^{[2]} = Y \times_\pi Y$ the fibre product of Y with itself over π, that is the subset of pairs (y,y') in $Y \times Y$ such that $\pi(y) = \pi(y')$. More generally we denote the pth fold fibre product by $Y^{[p]}$.

Recall that a hermitian line bundle $L \to M$ is a complex line bundle with a fibrewise hermitian inner product. For such a line bundle the set of all vectors of norm 1 is a principal $U(1)$ bundle. Conversely if $P \to M$ is a principal $U(1)$ bundle then associated to it is a complex line bundle with fibrewise hermitian inner product. This is formed in the standard way as the quotient of $P\times\mathbb{C}$ by the action of $U(1)$ given by $(p,z)w = (pw, w^{-1}z)$ where $w \in U(1)$. The theory of bundle gerbes as developed in [27] used principal bundles (actually $\mathbb{C}^\times$ bundles) but it can be equivalently expressed in terms of hermitian line bundles. In the discussion below we will mostly adopt this perspective. All maps between hermitian line bundles will be assumed to preserve the inner product unless we explicitly comment otherwise.

A bundle gerbe[1] over M is a pair (L, Y) where $\pi\colon Y \to M$ is a locally split map and L is a hermitian line bundle $L \to Y^{[2]}$ with a product, that is, a hermitian isomorphism

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \to L_{(y_1,y_3)}$$

for every (y_1, y_2) and (y_2, y_3) in $Y^{[2]}$. We require the product to be smooth in y_1, y_2 and y_3 but in the interests of brevity we will not state the various definitions needed to make this requirement precise, they can be found in [27]. The product is required to be associative whenever triple products are defined. Also in [27] it is shown that the existence of the product and the associativity imply isomorphisms $L_{(y,y)} \simeq \mathbb{C}$ and $L_{(y_1,y_2)} \simeq L^*_{(y_2,y_1)}$. We shall often refer to a bundle gerbe (L, Y) as just L.

Various operations are possible on bundle gerbes. Let (L, Y) be a bundle gerbe over M. Let $\pi\colon Z \to N$ be another locally split map and let $\hat{\phi}\colon Z \to Y$ be a fibre map covering a map $\phi\colon N \to M$. Then there is an induced map $\hat{\phi}^{[2]}\colon Z^{[2]} \to Y^{[2]}$ which can be used to pull-back the bundle $L \to Y^{[2]}$ to a bundle $(\hat{\phi}^{[2]})^{-1}(L) \to Z^{[2]}$. This has an induced product on it and defines a bundle gerbe which we denote, for simplicity, by $(\phi^{-1}(L), Z)$ or $\phi^{-1}(L)$. Two special cases of this are important. The first is when we just have a map $f\colon N \to M$ and use this to pull-back $Y \to M$ to $f^{-1}(Y) \to N$. The second is when we have $M = N$ and ϕ the identity.

If (L, Y) is a bundle gerbe we can define a new bundle gerbe, (L^*, Y), the dual of (L, Y), by taking the dual of L. Also if (L, Y) and (J, Z) are two bundle gerbes we can define their product $(L \otimes J, Y \times_\pi Z)$ where $Y \times_\pi Z = \{(y, z)\colon \pi_Y(y) = \pi_Z(z)\}$ is the fibre product of Y and Z over their projection maps.

A morphism from a bundle gerbe (L, Y) to a bundle gerbe (J, Z) consists of a pair of maps (g, f) where $f\colon Y \to Z$ is a map commuting with the projection to M and $g\colon L \to J$ is a bundle map covering the induced map $f^{[2]}\colon Y^{[2]} \to Z^{[2]}$ and commuting with the bundle gerbe products on J and L respectively. If f and g are isomorphisms then we call (g, f) a bundle gerbe isomorphism.

If J is a hermitian line bundle over Y then we can define a bundle gerbe $\delta(J)$ by $\delta(J) = \pi_1^{-1}(J) \otimes \pi_2^{-1}(J)^*$, that is $\delta(J)_{(y_1,y_2)} = J_{y_2} \otimes J^*_{y_1}$, where $\pi_i : Y^{[2]} \to Y$ is the map which omits the ith element. The bundle gerbe product is induced by the natural pairing

$$J_{y_2} \otimes J^*_{y_1} \otimes J_{y_3} \otimes J^*_{y_2} \to J_{y_3} \otimes J^*_{y_1}.$$

A bundle gerbe which is isomorphic to a bundle gerbe of the form $\delta(J)$ is called *trivial*. A choice of J and a bundle gerbe isomorphism $\delta(J) \simeq L$ is called a *trivialisation*. If J and K are trivialisations of P then we have natural isomorphisms

$$J_{y_1} \otimes J^*_{y_2} \simeq K_{y_1} \otimes K^*_{y_2}$$

and hence

$$J^*_{y_1} \otimes K_{y_1} \simeq J^*_{y_2} \otimes K_{y_2}$$

so that the bundle $J \otimes K$ is the pull-back of a hermitian line bundle on M. Moreover if J is a trivialisation and L is a bundle on M then $J \otimes \pi^{-1}(L)$ is also a trivialisation. Hence the set of all trivialisations of a given bundle gerbe is naturally acted on by the set of all hermitian line bundles on M. This is analogous to the way in which

[1]Strictly speaking what we are about to define should be called a hermitian bundle gerbe but the extra terminology is overly burdensome.

the set of all trivialisations of a hermitian line bundle $L \to M$ is acted on by the set of all maps $M \to U(1)$.

One can think of bundle gerbes as one stage in a hierarchy of objects with each type of object having a characteristic class in $H^p(M, \mathbb{Z})$. For example if $p = 1$ we have maps from M to $U(1)$, the characteristic class is the pull-back of dz. When $p = 2$ we have hermitian line bundles on M with characteristic class the Chern class. When $p = 3$ we have bundle gerbes and they have a characteristic class $d(L) = d(L, Y) \in H^3(M, \mathbb{Z})$, the Dixmier-Douady class of (L, Y). The Dixmier-Douady class is the obstruction to the gerbe being trivial. It is shown in [27] that

Theorem 2.1 ([27])**.** *A bundle gerbe (L, Y) has zero Dixmier-Douady class precisely when it is trivial.*

Strictly speaking the theorem in [27] dealt with bundle gerbes defined using line bundles or $\mathbb{C}^\times$ principal bundles not hermitian line bundles. To see that it generalises we need to know that if $L = \delta(J)$ for $J \to Y$ a line bundle then we can choose an inner product on J so that $\delta(J)$ has an isomorphic inner product to that on L. Notice that if V is a one dimensional hermitian inner product then the set of vectors of unit length is an orbit under $U(1)$. It follows that any two hermitian inner products differ by multiplication by e^λ for some real number λ. So if we choose any hermitian inner product on the fibres of J the induced inner product on $\delta(J)$ differs from that on L by a function e^g where $g\colon Y^{[2]} \to (0, \infty)$. Because these inner products are compatible with the bundle gerbe product we will have that $\delta(g)(y_1, y_2, y_3) = g(y_2, y_3) - g(y_1, y_3) + g(y_1, y_2) = 0$. If we change the inner product on J then g is altered by addition of $\delta(h)(y_1, y_2) = h(y_2) - h(y_1)$ where $h\colon Y \to \mathbb{R}$. So we need to solve $\delta(g) = h$ and this can be done using the exact sequence in Section 8 of [27].

Notice that the same is true of the other objects in our hierarchy, line bundles are trivial if and only if their chern class vanishes and maps into $U(1)$ are trivial (i.e. homotopic to the constant map 1) if and only if the pull-back of dz vanishes in cohomology.

The construction of the Dixmier-Douady class is natural in the sense that if $Z \to N$ is another locally split map and $\hat{\phi}\colon Z \to Y$ is a fibre map covering $\phi\colon N \to M$ then it is straightforward to check from the definition that

$$d(\phi^{-1}(L), Z) = \phi^*(d(L, Y)). \tag{2.1}$$

In particular if $M = N$ and ϕ is the identity then

$$d(\phi^{-1}(L)) = d(L). \tag{2.2}$$

From [27] we also have

Theorem 2.2 ([27])**.** *If L and J are bundle gerbes over M then*

1. *$d(L^*) = -d(L)$ and*
2. *$d(L \otimes J) = d(L) + d(J)$.*

2.2. **Lifting bundle gerbes.** We will need one example of a bundle gerbe in a number of places. Consider a central extension of groups

$$U(1) \to \hat{G} \to G.$$

If $Y \to M$ is a principal G bundle then it is well known that the obstruction to lifting Y to a $\hat{G}$ bundle is a class in $H^3(M, \mathbb{Z})$. It was shown in [27] that a bundle gerbe can be constructed from Y, the so-called lifting bundle gerbe, whose Dixmier-Douady class is the obstruction to lifting Y to a $\hat{G}$ bundle. The construction of the lifting bundle gerbe is quite simple. As Y is a principal bundle there is a map $g\colon Y^{[2]} \to G$ defined by $y_1 g(y_1, y_2) = y_2$. We use this to pull back the $U(1)$ bundle $\hat{G} \to G$ and form the associated hermitian line bundle $L \to Y^{[2]}$. The bundle gerbe product is induced by the group structure of $\hat{G}$.

We will be interested in the lifting bundle gerbes for

$$U(1) \to U(n) \to PU(n)$$

and

$$U(1) \to U(\mathcal{H}) \to PU(\mathcal{H})$$

for $\mathcal{H}$ an infinite dimensional, separable, Hilbert space.

3. Stable isomorphism of bundle gerbes

Equation (2.2) shows that there are many bundle gerbes which have the same Dixmier-Douady class but which are not isomorphic. For bundle gerbes there is a notion called *stable isomorphism* which corresponds exactly to two bundle gerbes having the same Dixmier-Douady class. To motivate this consider the case of two hermitian line bundles $L \to M$ and $J \to M$ they are isomorphic if there is a bijective map $L \to J$ preserving all structure, i.e. the projections to M and the $U(1)$ action on the fibres. Such isomorphisms are exactly the same thing as trivialisations of $L^* \otimes J$. For the case of bundle gerbes the latter is the correct notion and we have

Definition 3.1. A stable isomorphism between bundle gerbes (L, Y) and (J, Z) is a trivialisation of $L^* \otimes J$.

We have [28]

Proposition 3.2. *A stable isomorphism exists from* (L, Y) *to* (J, Z) *if and only if* $d(L) = d(J)$.

If a stable isomorphism exists from (L, Y) to (J, Z) we say that (L, Y) and (J, Z) are stably isomorphic.

It follows easily that stable isomorphism is an equivalence relation. It was shown in [27] that every class in $H^3(M, \mathbb{Z})$ is the Dixmier-Douady class of some bundle gerbe. Hence we can deduce from Proposition 3.2 that

Theorem 3.3. *The Dixmier-Douady class defines a bijection between stable isomorphism classes of bundle gerbes and* $H^3(M, \mathbb{Z})$.

It is shown in [28] that a morphism from (L, Y) to (J, Z) induces a stable isomorphism but the converse is not true.

Assume that we have a stable isomorphism α from (L, Y) to (J, Z) and another stable isomorphism β from (J, Z) to (K, X) then it is shown in [37] that there is a stable isomorphism $\beta \circ \alpha$ from (L, Y) to (K, X) called the composition of α and β. To define this we note that $d(L, Y) = d(J, Z) = d(K, X)$ so that there exists a stable isomorphism γ from (L, Y) to (K, X). By definition α is trivialisation of $L^* \otimes J$ and β is a trivialisation of $J^* \otimes K$. It is straightforward to show [28] that $J^* \otimes J$ has a canonical trivialisation say ϵ. Trivialisations can be multiplied so we have two trivialisations $\alpha \otimes \beta$ and $\gamma \otimes \epsilon$ of $L^* \otimes J \otimes J^* \otimes K$. It follows that there is a hermitian line bundle S over M such that $\alpha \otimes \beta = \gamma \otimes \epsilon \otimes \pi^{-1}(S)$. We define

$\beta \circ \alpha = \gamma \otimes \pi^{-1}(S)$ or equivalently we define $\beta \circ \alpha$ so that $\alpha \otimes \beta = (\beta \circ \alpha) \otimes \epsilon$. The composition of stable isomorphisms is not quite associative, see [37] for details.

Notice that this construction also applies to line bundles. If L, J and K are line bundles over M and α is a section of $L^* \otimes J$ and β is a section of $J^* \otimes K$ then $\beta \circ \alpha \colon L \to K$ is a section of $L^* \otimes K$ satisfying $\alpha \otimes \beta = (\beta \circ \alpha) \otimes \epsilon$ where ϵ is the canonical section of $J^* \otimes J$.

4. Bundle gerbe modules

Let (L, Y) be a bundle gerbe over a manifold M and let $E \to Y$ be a finite rank, hermitian vector bundle. Assume that there is a hermitian bundle isomorphism

$$\phi \colon L \otimes \pi_1^{-1} E \xrightarrow{\sim} \pi_2^{-1} E \tag{4.1}$$

which is compatible with the bundle gerbe multiplication in the sense that the two maps

$$L_{(y_1,y_2)} \otimes (L_{(y_2,y_3)} \otimes E_{y_3}) \to L_{(y_1,y_2)} \otimes E_{y_2} \to E_{y_1}$$

and

$$(L_{(y_1,y_2)} \otimes L_{(y_2,y_3)}) \otimes E_{y_3} \to L_{(y_1,y_3)} \otimes E_{y_3} \to E_{y_1}$$

are the same. In such a case we call E a bundle gerbe module and say that the bundle gerbe acts on E. Bundle gerbe modules have also been considered for the case that Y is a disjoint union of open sets in [19] and in [20] where they are called twisted bundles.

We define two bundle gerbe modules to be isomorphic if they are isomorphic as vector bundles and the isomorphism preserves the action of the bundle gerbe. Denote by $\mathrm{Mod}(L)$ the set of all isomorphism classes of bundle gerbe modules for L. If (L, Y) acts on E and also on F then it acts on $E \oplus F$ in the obvious diagonal manner. The set $\mathrm{Mod}(L)$ is therefore a semi-group.

Notice that if E has rank one then it is a trivialisation of L. Moreover if E has rank r then L^r acts on $\wedge^r(E)$ and we deduce

Proposition 4.1. *If (L, Y) has a bundle gerbe module $Y \to E$ of rank r then its Dixmier-Douady class $d(L)$ satisfies $rd(L) = 0$.*

Recall that if $E \to Y$ is a bundle then *descent data* [6] for E is a collection of hermitian isomorphisms $\chi(y_1, y_2) \colon E_{y_2} \to E_{y_1}$ such that $\chi(y_1, y_2) \circ \chi(y_2, y_3) = \chi(y_1, y_3)$. The existence of descent data is equivalent to the existence of a bundle $F \to M$ and an isomorphism $E \to \pi^{-1}(F)$.

If L is a trivial bundle gerbe then $L_{(y_1,y_2)} = K_{y_2} \otimes K^*_{y_1}$ so if E is an (L, Y) module we have isomorphisms $K_{y_2} \otimes E_{y_2} \simeq K_{y_1} \otimes E_{y_1}$ which are descent data and hence $K \otimes E$ is the pull back of a bundle on M. Conversely if F is a bundle on M then L acts on $K \otimes \pi^{-1}(F)$. Denote by $\mathrm{Bun}(M)$ the semi-group of all isomorphism classes of vector bundles on M. Then we have we have

Proposition 4.2. *A trivialisation of (L, Y) defines a semi-group isomorphism from $\mathrm{Mod}(L)$ to $\mathrm{Bun}(M)$.*

Notice that this isomorphism is not canonical but depends on the choice of the trivialisation. If we change the trivialisation by tensoring with the pull-back of a line bundle J on M then the isomorphism changes by composition with the endomorphism of $\mathrm{Bun}(M)$ defined by tensoring with the J.

Recall that a stable isomorphism from a bundle gerbe (L, Y) to a bundle gerbe (J, X) is a trivialisation of $L^* \otimes J$. This means there is a bundle $K \to Y \times_f X$ and an isomorphism $L^* \otimes J \to \delta(K)$ or, in other words for every (y_1, y_2) and (x_1, x_2) we have an isomorphism

$$L^*_{(y_1,y_2)} \otimes J_{(x_1,x_2)} \to K_{(y_2,x_2)} \otimes K^*_{(y_1,x_1)}.$$

Let $E \to Y$ be an L module and define $\hat{F}_{(y,x)} = K^*_{(y,x)} \otimes E_y$ a bundle on $Y \times_f X$. We have isomorphisms

$$\begin{aligned}\hat{F}_{(y_2,x)} &= K^*_{(y_2,x)} \otimes E_{y_2} \\ &= K^*_{(y_1,x)} \otimes L_{(y_1,y_2)} \otimes E_{y_2} \\ &= K^*_{(y_1,x)} \otimes E_{y_1} \\ &= \hat{F}_{(y_1,x)}.\end{aligned}$$

These define a descent map for $\hat{F}$ for the map $Y \times_\pi X \to X$ and hence define a bundle F on X. Note that as the inner products are everywhere preserved F is also a hermitian bundle.

We also have

$$\begin{aligned}J_{(x_1,x_2)} \otimes F_{x_2} &= J_{(x_1,x_2)} \otimes K^*_{(y,x_2)} \otimes E_y \\ &= K^*_{(y,x_1)} \otimes E_y \\ &= F_{x_1}\end{aligned}$$

and this makes F a (J, X) module.

So the choice of stable isomorphism has defined a map

$$\mathrm{Mod}(L) \to \mathrm{Mod}(J).$$

In a similar fashion we can define a map

$$\mathrm{Mod}(J) \to \mathrm{Mod}(L)$$

which is an inverse. Hence we have

Proposition 4.3. *A stable isomorphism from* (L, Y) *to* (J, Y) *induces an isomorphism of semi-groups between* $\mathrm{Mod}(L)$ *and* $\mathrm{Mod}(J)$*.*

Note that, as in the trivial case, this isomorphism is not canonical but depends on the choice of stable isomorphism. Changing the stable isomorphism by tensoring with the pull-back of a line bundle J over M changes the isomorphism in Prop. 4.3 by composition with the endomorphism of $\mathrm{Mod}(J)$ induced by tensoring with the pull-back of J.

There is a close relationship between bundle gerbe modules and bundles of projectives spaces. Recall that a bundle of projective spaces $\mathcal{P} \to M$ is a fibration whose fibres are isomorphic to $P(V)$ for V a Hilbert space, either finite or infinite dimensional, and whose structure group is $PU(V)$. This means that there is a $PU(V)$ bundle $X \to M$ and $\mathcal{P} = X \times_{PU(V)} P(V)$. Associated to X is a lifting bundle gerbe $J \to X^{[2]}$ and a Dixmier-Douady class. This Dixmier-Douady class is the obstruction to $\mathcal{P}$ being the projectivisation of a vector bundle. The lifting bundle gerbe acts naturally on the bundle gerbe module $E = X \times H$ because each $J_{(x_1,x_2)} \subset U(V)$ by construction 2.2.

Let (L, Y) be a bundle gerbe and $E \to Y$ a bundle gerbe module. Then the projectivisation of E descends to a projective bundle $\mathcal{P}_E \to M$ because of the bundle gerbe action. It is straightforward to check that the class of this projective bundle is $d(L)$. Conversely if $\mathcal{P} \to M$ is a projective bundle with class $d(L)$ the associated lifting bundle gerbe has class $d(L)$ and hence is stably isomorphic to (L, Y). So the module on which the lifting bundle gerbe acts defines a module on which (L, Y) acts. From the discussion before 4.3 one can see that if two modules are related by a stable isomorphism they give rise to the same projective bundle on M. We also have that $E \to Y$ and $F \to Y$ give rise to isomorphic projective bundles on M if and only if there is a line bundle $K \to M$ with $E = \pi^{-1}(K) \otimes F$. Denote by $\mathrm{Lin}(M)$ the group of all isomorphism classes of line bundles on M. Then this acts on $\mathrm{Mod}(L)$ by $E \mapsto \pi^{-1}(K) \otimes E$ for any line bundle $K \in \mathrm{Lin}(M)$. If $[H] \in H^3(M, \mathbb{Z})$ denote by $\mathrm{Pro}(M, [H])$ the set of all isomorphism classes of projective bundles with class $[H]$. Then we have

Proposition 4.4. *If (L, Y) is a bundle gerbe then the map which associates to any element of $\mathrm{Mod}(L)$ a projective bundle on M whose Dixmier-Douady class is equal to $d(L)$ induces a bijection*

$$\frac{\mathrm{Mod}(L)}{\mathrm{Lin}(M)} \to \mathrm{Pro}(M, d(L)).$$

5. K-theory for torsion bundle gerbes

Given a bundle gerbe (L, Y) with torsion Dixmier-Douady class we denote by $K(L)$ the Grothendieck group of the semi-group $\mathrm{Mod}(L)$ and call this the K group of the bundle gerbe. We immediately have from Prop. 4.3:

Proposition 5.1. *A choice of stable isomorphism from L to J defines a canonical isomorphism $K(L) \simeq K(J)$.*

Notice that the group $K(L)$ depends only on the class $d(L) \in H^3(M, \mathbb{Z})$ and for any class $[H]$ in $H^3(M, \mathbb{Z})$ we can define a bundle gerbe L with $d(L) = [H]$ and hence a group $K(L)$. When we want to emphasise the dependence on $[H]$ we denote this by $K_{bg}(M, [H])$.

It is easy to deduce from the theory of bundle gerbes various properties of this K-theory:

Proposition 5.2. *Bundle gerbe K theory satisfies the following properties:*

(1) If (L, Y) is a trivial bundle gerbe then $K_{bg}(L) = K(M)$.
(2) $K_{bg}(L)$ is a module over $K(M)$.
(3) If $[H]$ and $[H']$ are classes in $H^3(M, \mathbb{Z})$ there is a homomorphism

$$K_{bg}(M, [H]) \otimes K_{bg}(M, [H']) \to K_{bg}(M, [H] + [H']).$$

(4) If $[H]$ is a class in $H^3(M, \mathbb{Z})$ and $f\colon N \to M$ is a map there is a homomorphism

$$K_{bg}(M, [H]) \to K_{bg}(N, f^*([H])).$$

Proof. (1) This follows from applying Prop. 4.2 which shows that $\mathrm{Mod}(L)$ is isomorphic to the semi-group of all vector bundles on M

(2) If we pull a bundle back from M to Y and tensor it with a bundle gerbe module the result is still a bundle gerbe module.

(3) If $E \to Y$ is a bundle gerbe module for (L, Y) and $F \to X$ is a bundle gerbe module for (J, X) it is straightforward to see that $E \otimes F$ defines a bundle over the fibre product of Y and X which is a bundle gerbe module for $L \otimes J$.

(4) This follows easily by pull-back.

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There is another construction that associates to any class $[H]$ in $H^3(M,\mathbb{Z})$ a group $K(M,[H])$ or the *twisted* K group. Twisted K-theory shares the same properties as those in Prop. 5.2. In the next section we discuss twisted cohomology and show that, in the torsion case, bundle gerbe cohomology and twisted cohomology coincide.

6. Twisted K-theory and bundle gerbe modules

6.1. **Twisted K-theory.** We recall the definition of twisted cohomology [34]. In this discussion the class $[H] \in H^3(M,\mathbb{Z})$ is not restricted to be torsion.

Given a class $[H] \in H^3(M,\mathbb{Z})$ choose a $PU(\mathcal{H})$ bundle Y whose class is $[H]$. We can form an associated bundle

$$Y(\mathrm{Fred}) = Y \times_{PU(\mathcal{H})} \mathrm{Fred}$$

where Fred is the space of Fredholm operators on $\mathcal{H}$ acted on by conjugation. Let $[M, Y(\mathrm{Fred})]$ denote the space of all homotopy classes of sections of $Y(\mathrm{Fred})$ then we have [34]

Definition 6.1 ([34])**.** If $[H] \in H^3(M,\mathbb{Z})$ the twisted K theory of M is defined by

$$K(M,[H]) = [M, Y(\mathrm{Fred})].$$

It is a standard result that sections of $Y(\mathrm{Fred})$ are equivalent to $PU(\mathcal{H})$ equivariant maps from $Y \to \mathrm{Fred}$ so we have

$$K(M,[H]) = [M, Y(\mathrm{Fred})]_{PU(\mathcal{H})}$$

where $[M, Y(\mathrm{Fred})]_{PU(\mathcal{H})}$ is the space of all homotopy classes of equivariant maps with the homotopies being by equivariant maps.

6.2. **Bundle gerbe K-theory and twisted K-theory in the torsion case.** In the case when the Dixmier-Douady class $[H]$ is torsion, we will prove that bundle gerbe K theory and twisted K-theory are the same and indicate their relationship with equivariant K-theory.

The Serre-Grothendieck theorem cf. [8] says that, given a torsion class, there is a $PU(n)$ bundle $X \to M$, with Dixmier-Douady invariant equal to $[H]$. We can define an action $U(n)$ on $\mathbb{C}^n \otimes \mathcal{H} = \mathcal{H}^n$ letting g act as $g \otimes 1$. This gives a representation $\rho_n \colon U(n) \to U(\mathcal{H}^n)$ and induces a $PU(\mathcal{H}^n)$ bundle with Dixmier-Douady class $[H]$. As $\mathcal{H}^N \simeq \mathcal{H}$ and all $PU(\mathcal{H})$ bundles are determined by their Dixmier-Douady class we can assume that this bundle is Y and contains X as a $U(n)$ reduction. Then we have

$$(Y \times \mathrm{Fred})/PU(\mathcal{H}) \cong (X \times \mathrm{Fred})/PU(n),$$

so that

$$K(M,[H]) = [Y, \mathrm{Fred}]^{PU(\mathcal{H})} \cong [X, \mathrm{Fred}]^{PU(n)}.$$

The lifting bundle gerbe for $Y \to M$ pulls-back to become the lifting bundle gerbe L for $X \to M$. We will now prove that $K_{bg}(M,L) = K(M,[H])$. Notice that this will prove the result also for *any* bundle gerbe with torsion Dixmier-Douady class as we already know that bundle gerbe K-theory depends only on the Dixmier-Douady class.

In the case where there is no twist Atiyah showed that $K(M) = [M, \mathrm{Fred}]$ and we will follow his proof indicating just what needs to be modified to cover this equivariant case.

First we have have the following

Lemma 6.2. *If W is a finite dimensional subspace of $\mathbb{C}^n \otimes \mathcal{H}$ there is a finite co-dimensional subspace V of $\mathcal{H}$ such that $\mathbb{C}^n \otimes V \cap W = 0$.*

Proof. Let U be the image of V under the map $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H} \to \mathcal{H}$ were we contract the two copies of $\mathbb{C}^n$ with the inner product. Then $V \subset \mathbb{C}^n \otimes U$. So take $W = U^{\perp}$. □

Using the compactness of X and the methods in Atiyah we can show that if $f\colon X \to \mathrm{Fred}(\mathbb{C}^n \otimes \mathcal{H})$ then there is a subspace $V \subset \mathcal{H}$, of finite co-dimension, such that $\ker(f(x)) \cap \mathbb{C}^n \otimes V = 0$. Then $\mathcal{H}/V$ and $\mathcal{H}/f(V)$ will be vector bundles on X and moreover they will by acted on by $U(n)$ in such a way as to make them bundle gerbe modules. So we define

$$\mathrm{ind}\colon [X, \mathrm{Fred}(\mathbb{C}^n \otimes \mathcal{H})]_{U(n)} \to K_{bg}(M, L)$$

by $\mathrm{ind}(f) = \mathcal{H}/V - \mathcal{H}/f(V)$. Again the methods of [2] will show that this index map is well-defined and a homomorphism.

As in [2] we can identify the kernel of ind as $[X, U(\mathbb{C}^n \otimes \mathcal{H})]_{U(n)}$ and use the result of Segal [36] which shows that $U(\mathbb{C}^n \otimes \mathcal{H})$ is contractible so ind is injective.

Finally we consider surjectivity. First we need from [35] the following

Proposition 6.3. *If $E \to X$ is a bundle gerbe module for L then there is a representation $\mu\colon U(n) \to U(N)$ such that E is a sub-bundle gerbe module of $\mathbb{C}^N \otimes X$.*

If $E \to X$ is a bundle gerbe module then Proposition 6.3 enables us to find a $U(n)$ equivariant map $\tilde{f}\colon X \to \mathrm{Fred}(\mathbb{C}^N \otimes X)$ whose index is E. The action of $U(n)$ used here on $\mathbb{C}^N \otimes X$ is that induced from the representation μ. To prove surjectivity of the index map it suffice to find a map $f\colon X \to \mathrm{Fred}(\mathbb{C}^n \otimes X)$ whose index is E. Then if $E - F$ is a class in $K_{bg}(M, L)$ we can apply a similar technique to obtain a map whose index is $-F$ and combine these to get a map whose index is $E - F$ and we are done.

To construct f we proceed as follows. We have a representation $\rho_n\colon U(n) \to \mathbb{C}^n \otimes \mathcal{H}$ and a representation $\rho_N\colon U(n) \to \mathbb{C}^N \otimes \mathcal{H}$. These can be used to induce a $PU(\mathbb{C}^n \otimes \mathcal{H})$ bundle and a $PU(\mathbb{C}^N \otimes \mathcal{H})$ bundle, both with Dixmier-Douady class $[H]$. So they must be isomorphic. We need the precise form of this isomorphism. Choose an isomorphism $\phi\colon \mathbb{C}^n \otimes \mathcal{H} \to \mathbb{C}^N \otimes \mathcal{H}$. This induces an isomorphism $U(\mathbb{C}^n \otimes \mathcal{H}) \to U(\mathbb{C}^N \otimes \mathcal{H})$ given by $u \mapsto \phi u \phi^{-1}$ which we will denote by $\phi[u]$ for convenience. There is a similar identification $\mathrm{Fred}(\mathbb{C}^n \otimes \mathcal{H}) \to \mathrm{Fred}(\mathbb{C}^N \otimes \mathcal{H})$. The two PU bundles are given by $X \times_{\rho_n} U(\mathbb{C}^n \otimes \mathcal{H})$ and $X \times_{\rho_N} U(\mathbb{C}^N \otimes \mathcal{H})$ and consist of cosets $[x, u] = [xg, \rho_n^{-1}(g)u]$ and $[x, u] = [xg, \rho_N^{-1}(g)u]$ respectively. The action of $U(\mathbb{C}^n \otimes \mathcal{H})$ is $[x, u]v = [x, uv]$ and similarly for $U(\mathbb{C}^N \otimes \mathcal{H})$. Because these are isomorphic bundles there must be a bundle map

$$\phi\colon X \times_{\rho_n} U(\mathbb{C}^n \otimes \mathcal{H}) \to \times_{\rho_N} U(\mathbb{C}^N \otimes \mathcal{H})$$

satisfying $\phi([x, u]v) = \phi([x, u])\phi[v]$ and hence $\phi([x, u]) = \phi([x, 1])\phi[u]$. Define $\alpha\colon X \to U(\mathbb{C}^N \otimes \mathcal{H})$ by requiring that $\phi([x, 1]) = [x, \alpha(x)]$. Then if $g \in U(n)$

we have

$$\begin{aligned}[xg, \alpha(xg)] &= \phi([xg, 1]) \\ &= \phi([x, \rho_n(g)]) \\ &= \phi([x, 1])\phi[\rho_n(g)] \\ &= [x, \alpha(x)]\phi[\rho_n(g)] \\ &= [x, \alpha(x)\phi[\rho_n(g)]] \\ &= [xg, \rho_N(g)^{-1}\alpha(x)\phi[\rho_n(g)]]\end{aligned}$$

so that

$$\alpha(xg) = \rho_N(g)^{-1}\alpha(x)\phi[(\rho_n(g)]. \tag{6.1}$$

We can now define $f\colon X \to \mathrm{Fred}(\mathbb{C}^n \otimes X)$ by

$$f(x) = \alpha(x)\phi^{-1}[\tilde{f}(x)]$$

and it is straightforward to see that this is $U(n)$ equivariant by applying the equation (6.1). It is clear that $\alpha(x)$ and ϕ can be used to establish an isomorphism between $\ker(f)$ and $\ker(\tilde{f})$ and hence between $\ker(f)$ and E.

This proves

Proposition 6.4. *If L is a bundle gerbe over M with Dixmier-Douady class $[H]$ which is torsion then $K_{bg}(M, L) = K(M, [H])$.*

The lifting bundle gerbe for $Y \to M$ pulls-back to become the lifting bundle gerbe for $X \to M$. A bundle gerbe module for this is a bundle $E \to X$ with a $U(n)$ action covering the action of $PU(n)$ on X. This $U(n)$ action has to have the property that the center $U(1) \subset U(n)$ acts on the fibres of E by scalar multiplication. Considered from this perspective we see that we are in the context of equivariant K theory [35]. Notice that by projecting to $PU(n)$ we can make $U(n)$ act on X. Of course this action is not free, the center $U(1)$ is the isotropy subgroup at every point. The equivariant K theory $K_{U(n)}(Q)$ is the K theory formed from vector bundles on Q which have an $U(n)$ action covering the action on X. We need a subset of such bundles with a particular action. To understand this note that because the center $U(1) \subset U(n)$ is the isotropy subgroup for the $U(n)$ action on X it must act on the fibres of E and hence define a representation of $U(1)$ on $\mathbb{C}^r$ if the bundle E has rank r. This defines an element of $R(U(1))$, the representation ring of $U(1)$, and to be a bundle gerbe module this representation must be scalar multiplication on $\mathbb{C}^r$. In terms of equivariant K theory we can consider the map which is restriction to a fibre of $X \to M$ and then we have

$$K_{U(n)}(X) \to K_{U(n)}(U(n)/U(1)) = R(U(1)).$$

$K_{bg}(M, L)$ is the pre-image under this map of the representation of $U(1)$ on $\mathbb{C}^n$ by scalar multiplication.

6.3. **Characteristic classes of bundle gerbe modules.** In this section we discuss the Chern character of a twisted bundle gerbe module. Suppose (L, Y) is a bundle gerbe on M and that $E \to Y$ is a bundle gerbe module. Recall (4.1) that this means that there is an isomorphism:

$$\phi\colon L \otimes \pi_1^{-1}E \ \xrightarrow{\sim}\ \pi_2^{-1}E$$

which is compatible with the bundle gerbe product on L. Recall from [27] that a *bundle gerbe connection* on L is a connection ∇_L on L which is compatible in the obvious sense with the bundle gerbe product on L. Furthermore, one can show (see [27]), that the curvature F_L of ∇_L satisfies $F_L = \delta(f) = \pi_1^* f - \pi_2^* f$ for some 2-form f on Y. f is unique up to 2-forms pulled back from M. We call a choice of such an f a *curving* for the connection ∇_L. In [27] it is shown that there is a closed, integral 3-form ω on M such that $df = \pi^*\omega$. ω is called the 3-curvature of the connection ∇_L and curving f. It is the image in real cohomology of the Dixmier-Douady class of L. In our case, since L has torsion Dixmier-Douady class, one can choose a bundle gerbe connection ∇_L for L and a curving f for ∇_L such that $df = 0$. We want a connection D on E so that ϕ is a connection preserving isomorphism of vector bundles, where $L \otimes \pi_1^{-1} E$ is given the tensor product connection $\nabla_L \otimes I + \pi_1^{-1} D$.

Take an open cover $\{U_i\}_{i\in I}$ of M such that there exist local sections over U_i of $\pi\colon Y \to M$ and such that there exists a partition of unity $\{\rho_i\}_{i\in I}$ of M subordinate to U_i. Then L is trivialised over U_i — say $L = \delta(K_i)$ over U_i. The connection ∇_L on L induces a connection ∇_i on K_i. The bundle $E \otimes K_i$ on $Y_i = Y|_{U_i}$ descends to a bundle F_i on U_i. Choose any connection ∇_E on E, and a connection ∇_{F_i} on F_i. Then the pullback connection $\pi^{-1}\nabla_{F_i}$ on $\pi^{-1}F_i$ differs from the connection $\nabla_E + \nabla_i \otimes I$ on $E \otimes K_i$ by an $\mathrm{End}(E \otimes L_i) = \mathrm{End}(E)$ valued 1-form B_i on Y_i. Give $E|_{Y_i}$ the connection $D_i = \nabla_E - B_i$. Then, over $Y_i^{[2]}$, ϕ induces an isomorphism of vector bundles with connection

$$L|_{Y_i^{[2]}} \otimes \pi_1^{-1} E|_{Y_i^{[2]}} \xrightarrow{\sim} \pi_2^{-1} E|_{Y_i^{[2]}}.$$

Using the partition of unity $\{\rho_i\}_{i\in I}$ pulled back to $\{Y_i\}_{i\in I}$ we can patch together the local connections D_i on $E|_{Y_i}$ to get a connection D on E which is compatible with ∇_L under ϕ in the above sense. Calculating curvatures we get the following equality of $\mathrm{End}(\pi_1^{-1}E \otimes L) = \mathrm{End}(\pi_2^{-1}E)$ valued 2-forms on $Y^{[2]}$:

$$F_{\pi_1^{-1}D} + F_L I = \phi^{-1} \circ F_{\pi_2^{-1}D} \circ \phi. \tag{6.2}$$

Writing $F_L = \pi_1^* f - \pi_2^* f$ we get

$$\pi_1^*(F_D + fI) = \phi^{-1} \circ \pi_2^*(F_D + fI) \circ \phi.$$

If P is an invariant polynomial in Lie algebra valued variables then this equation shows that

$$\pi_1^*(P(F_D + fI, \dots, F_D + fI)) = \pi_2^*(P(F_D + fI, \dots, F_D + fI))$$

and hence the Chern-Weil $2k$-forms on Y descend to M. Moreover

$$\begin{aligned} & dP(F_D + fI, \dots, F_D + fI) \\ = & \sum P(F_D + fI, \dots, dF_D + df I, \dots, F_D + fI) \\ = & \sum P(F_D + fI, \dots, [F_D, A], \dots, F_D + fI). \end{aligned}$$

Using the standard trick of writing $g_t = \exp(tA)$ and using the invariance of P we get

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} P(g_t^{-1} F_D g_t + fI, \dots, g_t^{-1} F_D g_t + fI) \\ &= \sum P(F_D + fI, \dots, [F_D, A], \dots, F_D + fI). \end{aligned}$$

So the $2k$-forms $P(F_D + fI, \dots, F_D + fI)$ on M are all closed.

The usual definition of the chern character can be applied to define $\mathrm{Ch}(E) \in H^*(M, \mathbb{Q})$ for any bundle gerbe module. This satisfies $\mathrm{Ch}(E+F) = \mathrm{Ch}(E) + \mathrm{Ch}(F)$ and hence defines a chern character:

$$\mathrm{Ch}\colon K_{bg}(M) \to H^*(M, \mathbb{Q}).$$

7. Twisted K-theory in the non-torsion case.

We have seen that approaching twisted K-theory via finite rank bundle gerbes is not possible if the class $[H]$ is not torsion as there are then no finite rank bundle gerbe modules. A possible generalisation would be to allow bundle gerbe modules which are infinite Hilbert bundles. In that case the induced projective bundle on M is a $PU(\mathcal{H})$ bundle for $\mathcal{H}$ an infinite dimensional Hilbert bundle and it is well known that there is only one such bundle for a given Dixmier-Douady class and hence Proposition 4.4 implies that

Proposition 7.1. *Every bundle gerbe admits exactly one bundle gerbe module which is a bundle of infinite dimensional Hilbert spaces with structure group $U(\mathcal{H})$.*

In particular if E and F are Hilbert bundle gerbe modules then $E = F$ so that the class $E = F$ in the induced K group is zero. So the K group is zero.

In the remainder of this section we discuss another approach to twisted cohomology where the structure group of the bundle gerbe module is the group $U_{\mathcal{K}}$, the subgroup of $U(\mathcal{H})$ of unitaries which differ from the identity by a compact operator (here $\mathcal{K}$ denotes the compact operators on $\mathcal{H}$). To see how this arises notice that in Rosenberg's definition 6.1 we can replace Fred by a homotopy equivalent space. For our purposes we choose $BU_{\mathcal{K}} \times \mathbb{Z}$. This can be done in a $PU(\mathcal{H})$ equivariant fashion as follows. For $BU_{\mathcal{K}}$ we could choose the connected component of the identity of the invertibles in the Calkin algebra $B(\mathcal{H})/\mathcal{K}$ which is homotopy equivalent in a $PU(\mathcal{H})$ equivariant way to the Fredholms of index zero under the quotient map $\pi : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}$. Note however that the identity component of the invertibles in the Calkin algebra is just $GL(\mathcal{H})/GL_{\mathcal{K}}$ where $GL(\mathcal{H})$ denotes the invertible operators on $\mathcal{H}$ and $GL_{\mathcal{K}}$ are the invertibles differing from the identity by a compact. Thus we can take $BU_{\mathcal{K}}$ to be $GL(\mathcal{H})/GL_{\mathcal{K}}$ and this choice is $PU(\mathcal{H})$ equivariant. We could equally well take $BU_{\mathcal{K}} = U(\mathcal{H})/U_{\mathcal{K}}$.

As $U(\mathcal{H})$ acts on $U_{\mathcal{K}}$ by conjugation there is a semi-direct product

$$U_{\mathcal{K}} \to U_{\mathcal{K}} \rtimes PU(\mathcal{H}) \to PU(\mathcal{H}).$$

Note that this means that any $U_{\mathcal{K}} \rtimes PU(\mathcal{H})$ bundle over M induces a $PU(\mathcal{H})$ bundle and hence a class in $H^3(M, \mathbb{Z})$. If $R \to Y$ is a $U_{\mathcal{K}}$ bundle we call it $PU(\mathcal{H})$ covariant if there is an action of $PU(\mathcal{H})$ on the right of R covering the action on Y such that $(rg)[u] = r[u]u^{-1}gu$ for any $r \in R$, $[u] \in PU(\mathcal{H})$ and $g \in U_{\mathcal{K}}$. Here $[u]$ is the projective class of some $u \in U(\mathcal{H})$.

Because $BU_{\mathcal{K}}$ is homotopy equivalent to only the connected component of index 0 of Fred it is convenient to work with reduced twisted K theory, $\widetilde{K}(M, [H])$, defined by

$$\widetilde{K}(M, [H]) = [M, Y(BU_{\mathcal{K}})].$$

We have

Proposition 7.2. *Given a $PU(\mathcal{H})$ bundle $Y \to M$ with class $[H] \in H^3(M, \mathbb{Z})$ the following are equivalent spaces*

1. $\widetilde{K}(M, [H])$

2. *space of homotopy classes of sections of* $Y \times_{PU(\mathcal{H})} BU_{\mathcal{K}}$
3. *space of homotopy classes of* $PU(\mathcal{H})$ *equivariant maps from* Y *to* $BU_{\mathcal{K}}$
4. *space of isomorphism classes of* $PU(\mathcal{H})$ *covariant* $U_{\mathcal{K}}$ *bundles on* Y*, and*
5. *space of isomorphism classes of* $U_{\mathcal{K}} \rtimes PU(\mathcal{H})$ *bundles on* M *whose projection to a* $PU(\mathcal{H})$ *bundle has class* $[H]$.

Proof. (1) $\iff$ (2) This is just the reduced version of Rosenberg's definition of twisted K-theory 6.1.

(2) $\iff$ (3) This is a standard construction.

(3) $\implies$ (4) Notice that the $U_{\mathcal{K}}$ bundle $EU_{\mathcal{K}} \to BU_{\mathcal{K}}$ is $PU(\mathcal{H})$ covariant. It follows that if we pull it back to Y by a $PU(\mathcal{H})$ equivariant map $Y \to BU_{\mathcal{K}}$ that we must get a $PU(\mathcal{H})$ covariant bundle on Y.

(4) $\iff$ (5) Let $R \to Y$ be a $PU(\mathcal{H})$ covariant $U_{\mathcal{K}}$ bundle. By composing the projections $R \to Y \to M$ we think of R as a bundle on M. Both groups $U_{\mathcal{K}}$ and $PU(\mathcal{H})$ act on R and the combined action is an action of the semi-direct product and realises R as a bundle over M for this semi-direct product. Conversely consider a bundle $R \to M$ for the semi-direct product for which the induced $PU(\mathcal{H})$ bundle is isomorphic (as a $PU(\mathcal{H})$ bundle) to Y. Identify this bundle with Y and hence R is a bundle over Y and, in fact, a $PU(\mathcal{H})$ covariant $U_{\mathcal{K}}$ bundle.

(5) $\implies$ (3) A $U_{\mathcal{K}} \rtimes PU(\mathcal{H})$ bundle over M is determined by a classifying map $\phi\colon M \to B(U_{\mathcal{K}} \rtimes PU(\mathcal{H}))$. A little thought shows that we can realise this latter space as $EPU(\mathcal{H}) \times_{PU(\mathcal{H})} BU_{\mathcal{K}}$ which fibres over $BPU(\mathcal{H})$. The composition $\tilde{\phi}\colon M \to BPU(\mathcal{H})$ of ϕ with the projection to $BPU(\mathcal{H})$ is the classifying map of the induced $PU(\mathcal{H})$ bundle which is Y. This means that we can find a $PU(\mathcal{H})$ equivariant map $\hat{\phi}\colon Y \to EPU(\mathcal{H})$ covering $\tilde{\phi}$. Using this we define $\rho\colon Y \to BU_{\mathcal{K}}$ by $\phi(\pi(y)) = [\hat{\phi}(y), \rho(y)]_{PU(\mathcal{H})}$. This is well-defined. Moreover if $g \in PU(\mathcal{H})$ then $\pi(yg) = \pi(y)$ so that $[\hat{\phi}(y), \rho(y)]_{PU(\mathcal{H})} = [\hat{\phi}(yg), \rho(yg)]_{PU(\mathcal{H})} = [\hat{\phi}(y)g, \rho(yg)]_{PU(\mathcal{H})} = [\hat{\phi}(yg), g\rho(yg)]_{PU(\mathcal{H})}$ and hence $\rho(yg) = g^{-1}\rho(y)$ proving equivariance.

□

Note 7.1. Notice that if we worked with Fred instead of $BU_{\mathcal{K}}$ then it has connected components Fred_n consisting of operators of index n. We can then consider sections of $Y \times_{PU(\mathcal{H})} \mathrm{Fred}_n$ for every n, not just zero. Such a section will pull back a K class and if we take the determinant of this K class we will obtain a line bundle on Y on which the gerbe L^n acts. Hence we will have $n[H] = nd(L) = 0$ as in Prop. 4.1 and so we deduce the result noted in [3] that if $[H]$ is not torsion then there are no sections of $Y \times_{PU(\mathcal{H})} \mathrm{Fred}_n$ except when $n = 0$ so $\widetilde{K}(M, [H]) = K(M, [H])$.

7.1. $U_{\mathcal{K}}$ **bundle gerbe modules.** Given a $PU(\mathcal{H})$ covariant $U_{\mathcal{K}}$ bundle R over Y we can define the associated bundle

$$E = R \times_{U_{\mathcal{K}}} \mathcal{H} \to Y. \tag{7.1}$$

We claim that this is a bundle gerbe module for the lifting bundle gerbe P. Let $[r, v] \in E_{y_1}$ be a $U_{\mathcal{K}}$ equivalence class where $r \in R_{y_1}$, the fibre of R over $y_1 \in Y$ and $v \in \mathcal{H}$. Let $u \in L_{y_1 y_2}$ be an element of the lifting bundle gerbe. Then, by definition, $u \in U(\mathcal{H})$ and $y_1[u] = y_2$. We define the action of u by $[r, v]u = [r[u], u^{-1}v]$. It is straightforward to check that this is well defined. Hence we have associated to any $PU(\mathcal{H})$ covariant $U_{\mathcal{K}}$ bundle R on Y a module for the lifting bundle gerbe.

The inverse construction is also possible if the bundle gerbe module is a $U_{\mathcal{K}}$ bundle gerbe module which we now define. Let $E \to Y$ be a Hilbert bundle with structure group $U_{\mathcal{K}}$. We recall what it means for a Hilbert bundle to have structure group $U_{\mathcal{K}}$. To any Hilbert bundle there is associated a $U(\mathcal{H})$ bundle $U(E)$ whose fibre, $U(E)_y$, at y is all unitary isomorphisms $f\colon \mathcal{H} \to E_y$. If $u \in U(\mathcal{H})$ it acts on $U(\mathcal{H})_y$ by $fu = f \circ u$ and hence $U(E)$ is a principal $U(\mathcal{H})$ bundle. For E to have structure group $U_{\mathcal{K}}$ means that we have a reduction of $U(E)$ to a $U_{\mathcal{K}}$ bundle $R \subset U(E)$. Each $R_y \subset U(E)_y$ is an orbit under $U_{\mathcal{K}}$, that is R is a principal $U_{\mathcal{K}}$ bundle.

For E to be a $U_{\mathcal{K}}$ bundle gerbe module we need to define an action of the bundle gerbe on it. By comparing with the action on the bundle E defined in (7.1) we see that we need to make the following definition. If $u \in U(\mathcal{H})$ such that $y_1[u] = y_2$ then $u \in L_{(y_1,y_2)}$ where $L \to Y^{[2]}$ is the lifting bundle gerbe so if $f \in R_{y_1}$ then $ufu^{-1} \in U(E)_{y_2}$. We require that $ufu^{-1} \in R_{y_2}$. So a lifting bundle gerbe module which is a $U_{\mathcal{K}}$ Hilbert bundle and satisfies this condition we call a $U_{\mathcal{K}}$ bundle gerbe module. By construction we have that the associated R is a $U_{\mathcal{K}}$ bundle over Y on which $PU(\mathcal{H})$ acts. Let us denote by $\mathrm{Mod}_{U_{\mathcal{K}}}(M,[H])$ the semi-group of all $U_{\mathcal{K}}$ bundle gerbe modules for the lifting bundle gerbe of the $PU(\mathcal{H})$ bundle with three class $[H]$. As any two $PU(\mathcal{H})$ bundles with the same three class are isomorphic we see that $\mathrm{Mod}_{U_{\mathcal{K}}}(M,[H])$ depends only on $[H]$.

We have now proved

Proposition 7.3. *If (L,Y) is the lifting bundle gerbe for a $PU(\mathcal{H})$ bundle with Dixmier-Douady class $[H]$*

$$\widetilde{K}(M,[H]) = \mathrm{Mod}_{U_{\mathcal{K}}}(M,[H]).$$

If L_1 and L_2 are two $PU(\mathcal{H})$ covariant $U_{\mathcal{K}}$ bundles on Y note that $L_1 \times L_2$ is a $U_{\mathcal{K}} \times U_{\mathcal{K}}$ bundle. Choose an isomorphism $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ which induces an isomorphism $U_{\mathcal{K}} \times U_{\mathcal{K}} \to U_{\mathcal{K}}$ and hence defines a new $PU(\mathcal{H})$ covariant bundle $L_1 \otimes L_2$. It is straightfoward to check that

$$(L_1 \otimes L_2)(\mathcal{H}) \simeq L_1(\mathcal{H}) \times L_2(\mathcal{H}).$$

This makes $\mathrm{Mod}_{U_{\mathcal{K}}}(M,[H])$ a semi-group and the map $\widetilde{K}(M,[H]) = \mathrm{Mod}_{U_{\mathcal{K}}}(M,[H])$ is a semi-group isomorphism. Note that with our definition $BU_{\mathcal{K}}$ is a group. Moreover the space of all equivariant maps $Y \to BU_{\mathcal{K}}$ is a group as well. To see this notice that if f and g are equivariant maps and we multiply pointwise then for $y \in Y$ and $[u] \in PU(\mathcal{H})$ we have

$$\begin{aligned}(fg)(y[u]) &= f(y[u])g(y[u])\\ &= (u^{-1}f(y)u)(u^{-1}g(y)u)\\ &= u^{-1}(fg)(y)u\end{aligned}$$

and if f^{-1} is the pointwise inverse then $f^{-1}(y[u]) = \left(u^{-1}f(y)u\right)^{-1} = u^{-1}f^{-1}(y)u$. This induces a group structure on $\mathrm{Mod}_{U_{\mathcal{K}}}(M,[H])$. We have already noted that it is a semi-group but this implies more, for every $U_{\mathcal{K}}$ bundle gerbe module E there is a $U_{\mathcal{K}}$ bundle gerbe module E^{-1} such that $E \oplus E^{-1}$ is the trivial $U_{\mathcal{K}}$ bundle gerbe. Hence we have

Proposition 7.4. *If (L, Y) is the lifting bundle gerbe for a $PU(\mathcal{H})$ bundle with Dixmier-Douady class $[H]$ then*

$$K_{U_{\mathcal{K}}}(M, [H]) = \mathrm{Mod}_{U_{\mathcal{K}}}(M, [H]) = \widetilde{K}(M, [H])$$

Note 7.2. (1) The group $U_{\mathcal{K}}$ used here could be replaced by any other group to which it is homotopy equivalent by a homotopy equivalence preserving the $PU(\mathcal{H})$ action. In particular we could consider U_1, the subgroup of $U(\mathcal{H})$ consisting of unitary operators which differ from the identity by a trace class operator. In Section 9 we show that the computation in Section 6 of bundle gerbe characteristic classes generalizes, with some modifications, to U_1 bundle gerbe modules.

(2) In [12] it is argued that $U_{\mathcal{K}}$ is the appropriate gauge group for non-commutative gauge theory.

7.2. **Local description of $U_{\mathcal{K}}$ bundle gerbe modules.** Let $\{U_i\}_{i\in I}$ be a good cover of M and let $U_{ij\ldots k} = U_i \cap U_j \cap \cdots \cap U_k$. The trivial bundle has a sections s_i which are related by

$$s_i = s_j[u_{ji}]$$

where $[u_{ji}] \colon U_{ij} \to PU(\mathcal{H})$ for some $u_{ji} \colon U_{ij} \to U(\mathcal{H})$ where $u_{ij}u_{jk}u_{ki} = g_{ijk}1$ where 1 is the identity operator and the g_{ijk} are non-zero scalars.

Over each of the $s_i(U_i)$ are sections σ_i of the $U_{\mathcal{K}}$ bundle R. We can compare σ_i and $\sigma_j[u_{ji}]$ so that

$$\sigma_i = \sigma_j[u_{ji}]g_{ji}$$

where $g_{ij} \colon U_{ij} \to U_{\mathcal{K}}$. These satisfy

$$g_{ki} = ([u_{ji}^{-1}]g_{kj}[u_{ji}])g_{ji}. \tag{7.2}$$

If $Y_i = \pi^{-1}(U_i)$ you can define a section of R over all of Y_i by $\hat{\sigma}_i(s_i[u]) = \sigma_i[u]$. The transition functions for these are $\hat{g}_{ij}$ where $\hat{g}_{ij}(s_j[u]) = [u^{-1}]g_{ji}[u]$ and the identity (7.2) is equivalent to $\hat{g}_{ki} = \hat{g}_{kj}\hat{g}_{ji}$.

8. Examples

This section contains calculations of twisted K-theory, mainly for 3 dimensional manifolds. In the ensuing computations, we sometimes make use of the following observation in defining the connecting homomorphisms. Dixmier-Douady classes can be regarded canonically as elements in K^1-theory in 3 dimensions. This is because when X is a 3 dimensional manifold, then it is a standard fact, since $SU(2) \cong S^3$, that $H^3(X, \mathbb{Z}) = [X, SU(2)]$, where [,] means homotopy classes. But $SU(2)$ includes canonically as a subgroup of $U(\infty) = \varinjlim_n U(n)$, so that any map $f : X \to SU(2)$ can be regarded canonically as an element in $K^1(X) = [X, U(\infty)]$. It follows that the map $\phi : H^3(X, \mathbb{Z}) \to K^1(X)$ is a homomorphism of groups, for 3 dimensional manifolds X. There is also a homomorphism $\mathrm{Ch}_3 : K^1(X) \to H^3(X, \mathbb{Z})$ that is derived from the Chern character and is given by the formula $\mathrm{Ch}_3(t) = \frac{1}{12\pi}\mathrm{Tr}((t^{-1}dt)^3)$. Noting that $H^3(X, \mathbb{Z})$ is torsion-free, a calculation shows that the composition $\mathrm{Ch}_3 \circ \phi$ is the identity, since $\frac{1}{12\pi}\mathrm{Tr}((g^{-1}dg)^3)$ is the volume form of $SU(2)$ and so the differential form representative of a map $f : X \to SU(2)$ is just $\mathrm{Ch}_3(f)$. Therefore ϕ is injective, which allows us to identify Dixmier-Douady classes with elements in $K^1(X)$.

8.1. **The three-sphere.** We first discuss a few methods to compute the K-theory $K^\bullet(S^3)$ in the untwisted case, and then generalize to the twisted case.

8.1.1. *Mayer-Vietoris.* Suppose $X = U_1 \cup U_2$, where $U_i, i = 1, 2$, are closed subsets of a locally compact space X. Then we have the short exact sequence of C^*-algebras

$$0 \longrightarrow C_0(X) \xrightarrow{\imath} C_0(U_1) \oplus C_0(U_2) \xrightarrow{\pi} C_0(U_1 \cap U_2) \longrightarrow 0 \quad (8.1)$$

and the associated six-term exact (Mayer-Vietoris) sequence on K-theory [18, Th. 4.18] [2]

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{\imath_*} & K^0(U_1) \oplus K^0(U_2) & \xrightarrow{\pi_*} & K^0(U_1 \cap U_2) \\ \uparrow & & & & \downarrow \\ K^1(U_1 \cap U_2) & \xleftarrow{\pi_*} & K^1(U_1) \oplus K^1(U_2) & \xleftarrow{\imath_*} & K^1(X) \end{array} \quad (8.2)$$

Now consider $X = S^3$. Take for the U_i the upper and lower (closed) hemispheres $D_\pm$, respectively. Then, since $D_\pm$ is contractible, we have $K^0(D_\pm) = \mathbb{Z}$, $K^1(D_\pm) = 0$, while $D_+ \cap D_- \sim_h S^2$. Hence (8.2) reduces to

$$\begin{array}{ccccc} K^0(S^3) & \xrightarrow{\imath_*} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\pi_*} & K^0(S^2) \\ \uparrow & & & & \downarrow \\ K^1(S^2) & \xleftarrow{\pi_*} & 0 & \xleftarrow{\imath_*} & K^1(S^3) \end{array} \quad (8.3)$$

If we use the fact that $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$ and $K^0(S^2) = 0$, then we have a short exact sequence

$$0 \longrightarrow K^0(S^3) \xrightarrow{\imath_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow K^1(S^3) \longrightarrow 0 \quad (8.4)$$

where the map π_* is easily seen to be given by $\pi_*(m, n) = (m - n, 0)$. We conclude $K^0(S^3) = \mathbb{Z}$, $K^1(S^3) = \mathbb{Z}$.

We can use the same procedure to compute the twisted K-theory of S^3 (or, more generally, when the $PU(\mathcal{H})$ bundle $\mathcal{E}_{[H]}$ is trivial over U_i). In that case, $C_0(X, \mathcal{E}_{[H]})$ is given by pasting $C_0(U_1) \otimes \mathcal{K}$ and $C_0(U_2) \otimes \mathcal{K}$ over $U_1 \cap U_2$ via a map $L_{[H]} : U_1 \cap U_2 \to PU(\mathcal{H})$, i.e.

$$C_0(X, \mathcal{E}_{[H]}) = \{(f_1, f_2) \mid f_i \in C_0(U_i) \otimes \mathcal{K}\,,\ f_{1|U_1 \cap U_2} = L_{[H]}(f_{2|U_1 \cap U_2})\}\,,$$

and we have a short exact sequence

$$0 \longrightarrow C_0(X, \mathcal{E}_{[H]}) \xrightarrow{\imath} \bigoplus_i C_0(U_i) \otimes \mathcal{K} \xrightarrow{\pi} C_0(U_1 \cap U_2) \otimes \mathcal{K} \longrightarrow 0$$

where

$$\imath(f_1, f_2) = f_1 \oplus f_2\,, \qquad \pi(f_1 \oplus f_2) = f_{1|U_1 \cap U_2} - L_{[H]}(f_{2|U_1 \cap U_2})\,.$$

The associated six-term exact sequence in twisted K-theory is given by [33]

$$\begin{array}{ccccc} K^0(X, [H]) & \xrightarrow{\imath_*} & K^0(U_1) \oplus K^0(U_2) & \xrightarrow{\pi_*} & K^0(U_1 \cap U_2) \\ \uparrow & & & & \downarrow \\ K^1(U_1 \cap U_2) & \xleftarrow{\pi_*} & K^1(U_1) \oplus K^1(U_2) & \xleftarrow{\imath_*} & K^1(X, [H]) \end{array} \quad (8.5)$$

[2] There exists an analogous sequence if the U_i's are open subsets [18, Th. 4.19].

and in the case of S^3 collapses to

$$0 \longrightarrow K^0(S^3,[H]) \xrightarrow{\imath_*} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z}\oplus\mathbb{Z} \longrightarrow K^1(S^3,[H]) \longrightarrow 0 \tag{8.6}$$

where now the map π_* is given by $\pi_*(m,n) = (m-n,-nN)$ if $[H] = N[H_0]$ where $[H_0]$ is the generator of $H^3(S^3,\mathbb{Z}) = \mathbb{Z}$. We conclude $K^0(S^3,[H]) = 0$, $K^1(S^3,[H]) = \mathbb{Z}/N\mathbb{Z}$. This computation was initially performed in [33] and was recently reviewed in the context of D-branes in [23].

8.1.2. *The three-sphere at infinity.* We can think of S^3 as being the boundary of the closed four-ball B^4. This leads to a short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^4) \xrightarrow{\imath} C(B^4) \longrightarrow C(S^3) \longrightarrow 0 \tag{8.7}$$

and associated six-terms sequence in K-theory

$$\begin{array}{ccccc} K^0(\mathbb{R}^4) & \xrightarrow{\imath_*} & K^0(B^4) & \longrightarrow & K^0(S^3) \\ \uparrow & & & & \downarrow \\ K^1(S^3) & \longleftarrow & K^1(B^4) & \longleftarrow & K^1(\mathbb{R}^4) \end{array} \tag{8.8}$$

Using Bott-periodicity, $K^0(\mathbb{R}^4) = \mathbb{Z}$, $K^1(\mathbb{R}^4) = 0$, while contractibility of B^4 implies $K^0(B^4) = \mathbb{Z}$, $K^1(B^4) = 0$. Using furthermore that the map $\imath_* : K^0(\mathbb{R}^4) \to K^0(B^4)$ is trivial in this case, we again have $K^0(S^3) = K^1(S^3) = \mathbb{Z}$. However, since in general a $PU(\mathcal{H})$-bundle $\mathcal{E}_{[H]}$ over S^3 does not extend to B^4 we can not generalize (8.7) to the twisted case.

8.1.3. *Excision of a point.* Excising a point x_0 from S^3 we have

$$0 \longrightarrow C_0(\mathbb{R}^3) \longrightarrow C(S^3) \longrightarrow C(\{x_0\}) \longrightarrow 0 \tag{8.9}$$

and, accordingly,

$$\begin{array}{ccccc} K^0(\mathbb{R}^3) & \longrightarrow & K^0(S^3) & \longrightarrow & K^0(\{x_0\}) \\ \uparrow & & & & \downarrow \Delta \\ K^1(\{x_0\}) & \longleftarrow & K^1(S^3) & \longleftarrow & K^1(\mathbb{R}^3) \end{array} \tag{8.10}$$

Again, using Bott-periodicity, $K^0(\mathbb{R}^3) = 0$, $K^1(\mathbb{R}^3) = \mathbb{Z}$, while $K^0(\{x_0\}) = \mathbb{Z}$, $K^1(\{x_0\}) = 0$. In this case the connecting map $\Delta : K^0(\{x_0\}) \to K^1(\mathbb{R}^3)$ is trivial. And thus, we conclude $K^0(S^3) = K^1(S^3) = \mathbb{Z}$. In this case we can generalize (8.9) to the twisted case, namely

$$0 \longrightarrow C_0(\mathbb{R}^3)\otimes\mathcal{K} \longrightarrow C(S^3,\mathcal{E}_{[H]}) \longrightarrow C(\{x_0\})\otimes\mathcal{K} \longrightarrow 0 \tag{8.11}$$

The associated six-term sequence in twisted K-theory is similar to (8.10), except that $\Delta : K^0(\{x_0\}) = \mathbb{Z} \to K^1(\mathbb{R}^3) = \mathbb{Z}$, is now given by $\Delta(m) = mN$. We conclude again $K^0(S^3,[H]) = 0$, $K^1(S^3,[H]) = \mathbb{Z}/N\mathbb{Z}$.

8.2. **Product of one- and two-sphere.** The case $X = S^1 \times S^2$ is interesting since an explicit realization of the principle $PU(\mathcal{H})$-bundles over X, for $[H] \in H^3(S^1 \times S^2, \mathbb{Z}) = \mathbb{Z}$, is known [6]. We can compute $K^\bullet(S^1 \times S^2, [H])$ using the Mayer-Vietoris sequence as in Section 8.1.1, but the analogue of the procedure in section 8.1.3 is more convenient. Take a point $x_0 \in S^1$. We have

$$0 \longrightarrow C_0(\mathbb{R} \times S^2) \longrightarrow C(S^1 \times S^2) \longrightarrow C(\{x_0\} \times S^2) \longrightarrow 0 \tag{8.12}$$

Using $K^n(\mathbb{R} \times S^2) = K^{n+1}(S^2)$ and $K^\bullet(\{x_0\} \times S^2) = K^\bullet(S^2)$, we have for the twisted analogue of (8.12)

$$\begin{array}{ccccc} K^1(S^2) & \longrightarrow & K^0(S^1 \times S^2, [H]) & \longrightarrow & K^0(S^2) \\ \uparrow & & & & \downarrow \Delta \\ K^1(S^2) & \longleftarrow & K^1(S^1 \times S^2, [H]) & \longleftarrow & K^0(S^2) \end{array} \tag{8.13}$$

Now, $K^0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$ and $K^1(S^2) = 0$. The connecting map $\Delta : K^0(S^2) \to K^0(S^2)$ corresponds to taking the cup-product with $[H]$. Say, if $[\omega]$ is the generating line-bundle of $K^0(S^2)$ then $\Delta(m \cdot 1 + n[\omega]) = (m \cdot 1 + n[\omega]) \cup [H]$. We conclude $\Delta : (m, n) = (0, mN)$. Hence, $K^0(S^1 \times S^2, [H]) = \mathbb{Z}$ and $K^1(S^1 \times S^2, [H]) = \mathbb{Z} \oplus (\mathbb{Z}/N\mathbb{Z})$ for $N \neq 0$, while for $N = 0$ we have $K^0(S^1 \times S^2, [H]) = K^1(S^1 \times S^2, [H]) = \mathbb{Z} \oplus \mathbb{Z}$, as it should. The same conclusion is reached from the (twisted) Atiyah-Hirzebruch spectral sequence, cf. [33].

8.3. **The real projective three-space.** The K-theory for the real projective spaces $\mathbb{R}\mathrm{P}^n$ is given in [2, Prop. 2.7.7].

$$\begin{aligned} \widetilde{K}^0(\mathbb{R}\mathrm{P}^{2n+1}) = \widetilde{K}^0(\mathbb{R}\mathrm{P}^{2n}) = \mathbb{Z}_{2^n}\,, \\ K^1(\mathbb{R}\mathrm{P}^{2n+1}) = \mathbb{Z}\,, \qquad\qquad K^1(\mathbb{R}\mathrm{P}^{2n}) = 0\,. \end{aligned} \tag{8.14}$$

For $\mathbb{R}\mathrm{P}^3$ part of this result is derived by looking at the six-term sequence related to the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^3) \longrightarrow C(\mathbb{R}\mathrm{P}^3) \longrightarrow C(\mathbb{R}\mathrm{P}^2) \longrightarrow 0 \tag{8.15}$$

i.e., the exact sequence corresponding to the pair $(\mathbb{R}\mathrm{P}^3, \mathbb{R}\mathrm{P}^2)$. The associated six-term exact sequence is

$$\begin{array}{ccccc} K^0(\mathbb{R}^3) & \longrightarrow & K^0(\mathbb{R}\mathrm{P}^3) & \longrightarrow & K^0(\mathbb{R}\mathrm{P}^2) \\ \uparrow & & & & \downarrow \Delta \\ K^1(\mathbb{R}\mathrm{P}^2) & \longleftarrow & K^1(\mathbb{R}\mathrm{P}^3) & \longleftarrow & K^1(\mathbb{R}^3) \end{array} \tag{8.16}$$

Now, $K^0(\mathbb{R}^3) = 0$ and $K^1(\mathbb{R}^3) = \mathbb{Z}$. So, the result of [2], $K^0(\mathbb{R}\mathrm{P}^3) = K^0(\mathbb{R}\mathrm{P}^2) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $K^1(\mathbb{R}\mathrm{P}^2) = 0$, $K^1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}$ is perfectly consistent with (8.15) provided the connecting map $\Delta : K^0(\mathbb{R}\mathrm{P}^2) \to K^1(\mathbb{R}^3)$ vanishes in this case (which is not too hard to check independently, i.e. as in (8.13)). In the twisted version of (8.15) and (8.16) the connecting map Δ is given by $\Delta(m, n) = mN$, if $[H]$ is N times the generator of $H^3(\mathbb{R}\mathrm{P}^3, \mathbb{Z}) = \mathbb{Z}$. Thus $K^0(\mathbb{R}\mathrm{P}^3, [H]) = \mathbb{Z}_2$ and $K^1(\mathbb{R}\mathrm{P}^3, [H]) = \mathbb{Z}/N\mathbb{Z}$.

Alternatively, we may use $\mathbb{R}\mathrm{P}^3 = S^3/\mathbb{Z}_2$ and compute $K^\bullet(\mathbb{R}\mathrm{P}^3)$ through the $\mathbb{Z}_2$-equivariant K-theory of S^3. In that case, however, none of the sequences for S^3 discussed in Section 1 are appropriate since we need $\mathbb{Z}_2$ to act on the subspace. We can however slightly modify (8.9) by cutting out two points $\{x_0, x_1\} \in S^3$ related by the $\mathbb{Z}_2$ action. I.e.

$$0 \longrightarrow C_0(\mathbb{R}\times S^2) \longrightarrow C(S^3) \longrightarrow C(\{x_0,x_1\}) \longrightarrow 0 \qquad (8.17)$$

The associated six-term sequence in $\mathbb{Z}_2$-equivariant K-theory is

$$\begin{array}{ccccc} K^0_{\mathbb{Z}_2}(\mathbb{R}\times S^2) & \longrightarrow & K^0(\mathbb{R}\mathrm{P}^3) & \longrightarrow & K^0(\{x_0\}) \\ \uparrow & & & & \downarrow \Delta \\ K^1(\{x_0\}) & \longleftarrow & K^1(\mathbb{R}\mathrm{P}^3) & \longleftarrow & K^1_{\mathbb{Z}_2}(\mathbb{R}\times S^2) \end{array} \qquad (8.18)$$

where we have used $K^\bullet_{\mathbb{Z}_2}(\{x_0,x_1\}) = K^\bullet(\{x_0\})$ since $\mathbb{Z}_2$ acts freely on $\{x_0, x_1\}$. Note however that while $\mathbb{Z}_2$ does act freely on $\mathbb{R}\times S^2$ it does not act freely on $\mathbb{R}$ separately. Hence it would be wrong to conclude that $K^0_{\mathbb{Z}_2}(\mathbb{R}\times S^2)$ is equal to $K^0(\mathbb{R}\times\mathbb{R}\mathrm{P}^2) = K^1(\mathbb{R}\mathrm{P}^2) = 0$. In fact, the connecting map Δ vanishes in this case. Using our result for $K^\bullet(\mathbb{R}\mathrm{P}^3)$ then yields $K^0_{\mathbb{Z}_2}(\mathbb{R}\times S^2) = \mathbb{Z}_2$, $K^1_{\mathbb{Z}_2}(\mathbb{R}\times S^2) = \mathbb{Z}$ (this also follows from [18, Prop. 2.4]). In the twisted case $\Delta(m) = mN$ and again we find $K^0(\mathbb{R}\mathrm{P}^3, [H]) = \mathbb{Z}_2$ and $K^1(\mathbb{R}\mathrm{P}^3, [H]) = \mathbb{Z}/N\mathbb{Z}$.

8.4. **Lens spaces.** In the case of a Lens space $L_p = S^3/\mathbb{Z}_p$, for p a prime, we can compute $K^\bullet(L_p)$ as in [2] with the result

$$K^0(L_p) = \mathbb{Z}\oplus\mathbb{Z}_p\,, \qquad K^1(L_p) = \mathbb{Z}\,. \qquad (8.19)$$

Generalizing the equivariant computation of section 3 immediately gives

$$K^0(L_p,[H]) = \mathbb{Z}_p\,, \qquad K^1(L_p,[H]) = \mathbb{Z}/N\mathbb{Z}\,. \qquad (8.20)$$

D-branes on Lens spaces were recently considered in [21].

8.5. **Group manifolds.** In the case of $G = SU(n)$ we have
$H^\bullet(G,\mathbb{Q}) = \bigwedge\{c_3, c_5, \ldots, c_{2n-1}\}$ where $c_n \in H^n(G,\mathbb{Q})$. The K-groups are given similarly in terms of certain appropriately normalized linear combinations of the c_n. Since the generator of $H^3(G,\mathbb{Z}) = \mathbb{Z}$ corresponds to c_3 (appropriately normalized), the third differential d_3 in the (twisted) Atiyah-Hirzebruch spectral sequence corresponds to taking the wedge product with Nc_3 if $[H] = Nc_3$. Thus

$$\mathrm{Ker}\ d_3 = \mathbb{Z}c_3 \wedge \left(\bigwedge\{c_5,\ldots,c_{2n-1}\}\right) \qquad (8.21)$$

while

$$\mathrm{Im}\ d_3 = N\mathbb{Z}c_3 \wedge \left(\bigwedge\{c_5,\ldots,c_{2n-1}\}\right)\,. \qquad (8.22)$$

I.e., at the 3rd term of the spectral sequence, we have

$$E_3^\bullet(SU(n),[H]) = (\mathbb{Z}/N\mathbb{Z})\ c_3 \wedge \left(\bigwedge\{c_5,\ldots,c_{2n-1}\}\right)\,. \qquad (8.23)$$

The spectral sequence collapses at the 3rd term for $G = SU(2)$, e.g.

$$K^0(SU(2),[H]) = 0\,, \qquad K^1(SU(2),[H]) = \mathbb{Z}/N\mathbb{Z}\,, \qquad (8.24)$$

(in agreement with the results of Section 8.1.1), but for $G = SU(n), n > 2$, the higher order differentials are nonzero [14] (see also [22]). D-branes on group manifolds were studied in, e.g., [1, 9, 10].

9. The Chern character in the non-torsion case

9.1. $U_{\mathcal{K}}$ **bundle gerbe modules.** Recall that in the finite dimensional case we were able to consider connections ∇ on a finite rank bundle gerbe module E which were compatible with a bundle gerbe connection on the bundle gerbe L, so-called 'bundle gerbe module connections'. If F_∇ denoted the curvature 2-form of ∇, we were able to show that the 2-form $F_\nabla + fI$ with values in the bundle $\Omega^2(\mathrm{End}(E))$ was compatible with the descent isomorphism for $\mathrm{End}(E)$, where f was the 'curving' for the bundle gerbe connection on L. It followed that the forms $\mathrm{tr}(F_\nabla + fI)^k$ descend to M.

In the case of infinite rank bundle gerbe modules, to make sense of the trace, we need to restrict our attention to bundle gerbe modules with a reduction of the structure group to U_1, the group of unitaries differing from the identity by a trace class operator. As remarked earlier, in Note 7.2, in the definition of twisted K-theory we can replace Fred by any homotopy equivalent space, as long as that space is $PU(\mathcal{H})$ equivariant. The notion of a bundle gerbe module connection continues to make sense in this setting however it is not possible to find module connections so that the bundle-valued 2-forms $F_\nabla + fI$ take trace class values, ie lie in the adjoint bundle $\Omega^2(\mathrm{ad}(P))$ associated to the $PU(\mathcal{H})$ covariant principal U_1 bundle P via the adjoint action of U_1 on its Lie algebra $\mathcal{L}_1$, the ideal of trace class operators on $\mathcal{H}$. Instead, given a pair of $PU(\mathcal{H})$ covariant principal U_1 bundles P and Q, defining a class in twisted K-theory, we can consider differences of bundle-valued 2-forms $(\mathcal{F}_P+fI)-(\mathcal{F}_Q+fI)$ coming from module connections on the Hilbert vector bundles associated to P and Q. We will show that it is still possible to make sense of the trace in this setting and that we can define forms on M representing classes in the twisted cohomology group $H^\bullet(M;H)$. We propose that these forms on M define the Chern character for reduced twisted K-theory $\tilde{K}^0(M;[H])$. Recall that the Chern character for (reduced) twisted K-theory is a homomorphism $ch_{[H]}\colon \tilde{K}^0(M;[H]) \to H^\bullet(M;[H])$. $ch_{[H]}$ is uniquely characterised by requiring that it is functorial with respect to pullbacks, respects the $\tilde{K}^0(M)$-module structure of $\tilde{K}^0(M;[H])$ and reduces to the ordinary Chern character in the untwisted case when $[H]=0$.

9.2. **Remarks on the projective unitary group.** Dixmier and Douady's 1963 work on continuous fields of C^*-algebras exploited the fact that there is a natural bijection between $H^3(M;\mathbb{Z})$ and isomorphism classes of principal PU bundles on M. They used the strong operator topology on $U(\mathcal{H})$, the group of unitary operators on an infinite dimensional separable Hilbert space $\mathcal{H}$. Neither $U(\mathcal{H})$ with the strong operator topology nor $PU(\mathcal{H})$ with the induced topology are Lie groups. In 1965 Kuiper proved that $U(\mathcal{H})$ equipped with the norm topology is contractible. $U(\mathcal{H})$, equipped with the norm topology, is a Lie group (see for instance [24]) and one can show that $PU(\mathcal{H})$ equipped with the topology induced by the norm topology on $U(\mathcal{H})$ is a Lie group modelled locally on the quotient $Lie(U(\mathcal{H}))/i\mathbb{R}$ [38].

9.3. **Twisted cohomology.** There are several definitions of twisted cohomology that are well known among experts and which are all probably equivalent. One such definition is given by Atiyah in [3]. We give another definition here. If H is a closed, differential 3-form on M then we can use H to introduce a 'twist' on the usual cohomology of M and consider the twisted cohomology group $H^\bullet(M;H)$. $H^\bullet(M;H)$ is constructed from the algebra $\Omega^\bullet(M)$ of differential forms on M by introducing a twisted differential δ on $\Omega^\bullet(M)$ given by $\delta = d - H$, where d is the usual exterior derivative of differential forms on M. It is easy to see that $\delta^2 = 0$ using the fact that H is of degree three and hence $H^2 = 0$. We then set

$$H^\bullet(M;H) = \ker\{\delta\colon \Omega^\bullet(M) \to \Omega^\bullet(M)\}/\mathrm{im}\{\delta\colon \Omega^\bullet(M) \to \Omega^\bullet(M)\}.$$

$H^\bullet(M;H)$ is then a group under addition which satisfies the obvious functorial property that a smooth map $f\colon N \to M$ induces a homomorphism $f^*\colon H^\bullet(M;H) \to H^\bullet(N;f^*H)$. Note that although there is no algebra structure on $H^\bullet(M;H)$, there exist homomorphisms $H^\bullet(M;H) \otimes H^\bullet(M;H') \to H^\bullet(M;H+H')$. When $H = H' = 0$ this is just the usual wedge product of forms. If $H' = 0$ only then this map defines an action of the ordinary cohomology algebra $H^\bullet(M)$ on $H^\bullet(M;H)$ making $H^\bullet(M;H)$ into a $H^\bullet(M)$-module. Note that if λ is a 2-form on M then we can define a map $H^\bullet(M;H) \to H^\bullet(M;H+d\lambda)$ by sending a representative ω of a class $[\omega]$ in $H^\bullet(M;H)$ to the class in $H^\bullet(M;H+d\lambda)$ represented by $\exp(\lambda)\omega$. This is well defined, since if $\omega' = \omega + d\mu - H\mu$ then

$$\begin{aligned}\exp(\lambda)\omega' &= \exp(\lambda)\omega + \exp(\lambda)d\mu - H\exp(\lambda)\mu \\ &= \exp(\lambda)\omega + d(\exp(\lambda)\mu) - (H+d\lambda)\exp(\lambda)\mu.\end{aligned}$$

Suppose that the closed 3-form H is the representative for the image, in real cohomology, of the Dixmier-Douady class of a principal $PU(\mathcal{H})$ bundle Y on M. We interpret H as the 3-curvature of a bundle gerbe connection ∇_L and curving f for the lifting bundle gerbe $L \to Y^{[2]}$ associated to Y. Recall that a curving f for ∇_L satisfies $\delta(f) = F_{\nabla_L}$ and $df = \pi^*H$ where F_{∇_L} is the curvature of the connection ∇_L on the line bundle L and $\pi\colon Y \to M$ is the projection. Any two curvings for ∇_L differ by the pullback of a 2-form λ on M. We define $H^\bullet(M;[H])$ to be the set of equivalence classes of quadruples $([\omega], L, \nabla_L, f)$, where L is the lifting bundle gerbe for a principal $PU(\mathcal{H})$ bundle Y with Dixmier-Douady class $[H]$, f is a curving for the bundle gerbe connection ∇_L with 3-curvature H (so that $df = \pi^*H$) and $[\omega] \in H^\bullet(M;H)$. If J is the lifting bundle gerbe for another principal $PU(\mathcal{H})$ bundle X on M whose Dixmier-Douady class is also equal to $[H]$, then Y and X are isomorphic. This extends to an isomorphism between the lifting bundle gerbes L and J. We declare two quadruples $([\omega], L, \nabla_L, f_L)$ and $([\omega'], J, \nabla_J, f_J)$ to be equivalent if, under the ismorphism $L = J$, we have $\nabla_J = \nabla_L + \delta(\rho)$ for some complex valued 1-form ρ on X, $f_J = f_L + d\rho + \pi^*\lambda$ for some 2-form λ on M and $[\omega'] = [\exp(\lambda)\omega]$, where $[\omega] \in H^\bullet(M;H)$ and $[\omega'] \in H^\bullet(M;H+d\lambda)$. Here we are identifying $[\omega']$ with the image of $[\omega]$ under the isomorphism of complexes $H^\bullet(M;H) \to H^\bullet(M;H+d\lambda)$ defined above. For a curving f for ∇_L we will also define $H^\bullet(M;L,\nabla_L,f)$ to be $H^\bullet(M;H)$ where H is the 3-curvature of the pair (∇_L, f). Then $H^\bullet(M;[H])$ is equal to the quotient of the union of the $H^\bullet(M;L,\nabla_L,f)$ over all bundle gerbe connections ∇_L and curvings f on the lifting bundle gerbes L, under the equivalence relation defined above. The twisted cohomology groups satisfy the following properties. If $f\colon N \to M$ is a smooth map then there is an induced map $f^*\colon H^\bullet(M;[H]) \to H^\bullet(N;f^*[H])$. $H^\bullet(M;[H])$ is a module over $H^\bullet(M)$. This in turn follows from the property that if $[H]$ and $[H']$ are classes in $H^3(M;\mathbb{Z})$, then there is a homomorphism $H^\bullet(M;[H]) \otimes H^\bullet(M;[H']) \to H^\bullet(M;[H]+[H'])$. These properties are analogous to those for twisted K-theory.

9.4. **Defining the Chern Character.** Suppose $L \to Y^{[2]}$ is a bundle gerbe with bundle gerbe connection ∇_L. Recall that a module connection ∇_E on a bundle gerbe module E for L is a connection on the vector bundle E which is compatible with the bundle gerbe connection ∇_L, ie under the isomorphism $\pi_1^{-1}E \otimes L \to \pi_2^{-1}E$ the tensor product connection $\pi_1^{-1}\nabla_E \otimes \nabla_L$ on $\pi_1^{-1}E \otimes L$ is mapped into the connection $\pi_2^{-1}\nabla_E$ on $\pi_2^{-1}E$. Suppose now that $L \to Y^{[2]}$ is the lifting bundle gerbe for the principal $PU(\mathcal{H})$ bundle $Y \to M$ and that ∇_L is a bundle gerbe connection on L with curving f such that the associated 3-curvature (which represents the image, in real cohomology, of the Dixmier-Douady class of L) is equal to the closed, integral 3-form H. If E is a U_1 bundle gerbe module for E then we can consider module connections ∇_E on E; however, as remarked above, the algebra valued 2-form $F_E + fI$ cannot take trace class values (here F_E denotes the curvature of the connection ∇_E). If F is another U_1 bundle gerbe module for L, so that the difference $E - F$ represents a class in $\tilde{K}^0(M; [H])$ under the isomorphism of Proposition 7.3, we can consider module connections ∇_E and ∇_F on E and F respectively such that the difference of connections $\nabla_E - \nabla_F$ is trace class. By this we mean that in local trivialisations of E and F such that the connections ∇_E and ∇_F are given by $d + A_E$ and $d + A_F$ respectively, the difference $A_E - A_F$ is trace class. It follows that the difference of curvatures $F_E - F_F$ in local trivialisations of E and F respectively is trace class. One can show that it is always possible to find such module connections.

It follows that the differences $(F_E + fI) - (F_F + fI)$ and hence $(F_E + fI)^k - (F_F + fI)^k$ take trace class values (considered in local trivialisations of E and F respectively). Therefore the $2k$-forms $\operatorname{tr}((F_E + fI)^k - (F_F + fI)^k)$ on Y are well defined. To see this, note that in a local trivialisation of E, F_E is given by operator valued 2-forms F_E^i which are related on overlaps by $F_E^j = g_{ij}^{-1} F_E^i g_{ij}$, where g_{ij} are the U_1 valued transition functions for E. Similarly, in local trivialisations of F defined over the same open cover of Y as the local trivialisations of E, the curvature 2-form F_F is given locally by the operator valued 2-forms F_F^i which are related on overlaps by $F_F^j = h_{ij}^{-1} F_F^i h_{ij}$ where h_{ij} are the U_1 valued transition functions for F. Therefore the local $2k$-forms $\operatorname{tr}((F_E^i + fI)^k - (F_F^i + fI)^k)$ define global forms, since

$$\begin{aligned}
& \operatorname{tr}((F_E^j + fI)^k - (F_F^j + fI)^k) \\
= \; & \operatorname{tr}(g_{ij}^{-1}(F_E^i + fI)^k g_{ij} - h_{ij}^{-1}(F_F^i + fI)^k h_{ij}) \\
= \; & \operatorname{tr}(g_{ij}^{-1}(F_E^i + fI)^k g_{ij} - (F_E^i + fI)^k) + \operatorname{tr}((F_E^i + fI)^k - (F_F^i + fI)^k) \\
& + \operatorname{tr}((F_F^i + fI)^k - h_{ij}^{-1}(F_F^i + fI)^k h_{ij}) \\
= \; & \operatorname{tr}((F_E^i + fI)^k - (F_F^i + fI)^k).
\end{aligned}$$

We want to know that the forms we have defined live on M. This follows from the fact that the F_E and F_F are curvatures of module connections, and therefore satisfy equation 6.2. More precisely, suppose that $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Sigma}$ is an open cover of M such that there exist local sections $s_\alpha \colon U_\alpha \to Y$ of the $PU(\mathcal{H})$ bundle $Y \to M$. Suppose that Y has transition functions $g_{\alpha\beta}$ relative to this open covering. If E is a U_1 bundle gerbe module for L then we can use the sections s_α to pullback E to form Hilbert vector bundles E_α on U_α whose structure group reduces to U_1. The transition functions $g_{\alpha\beta}$ for Y then provide maps $E_\alpha \stackrel{g_{\alpha\beta}}{\to} E_\beta$. If F is the curvature of a module connection on E, then the pullbacks $F_\alpha + f_\alpha I$ are related by $F_\beta + f_\beta I = \hat{g}_{\alpha\beta}^{-1}(F_\alpha + f_\alpha I)\hat{g}_{\alpha\beta}$. On taking traces of differences of powers as above we see that these forms are globally defined. Note that

$$\operatorname{tr}(\exp(F_E + fI) - \exp(F_F + fI)) = \exp(f) \operatorname{tr}(\exp(F_E) - \exp(F_F)). \tag{9.1}$$

We have the following Proposition.

Proposition 9.1. *Suppose that E and F are U_1 bundle gerbe modules for the lifting bundle gerbe $L \to Y^{[2]}$ equipped with a bundle gerbe connection ∇_L and curving f, such that the associated 3-curvature is H. Suppose that ∇_E and ∇_F are module connections on E and F respectively such that the difference $\nabla_E - \nabla_F$ is trace class, considered in local trivialisations of E and F. Let $ch_H(\nabla_E, \nabla_F) \in \Omega^\bullet(M)$ denote the differential form on M whose lift to Y is given by $\exp(f)\mathrm{tr}(\exp(F_E) - \exp(F_F))$. Then $ch_H(\nabla_E, \nabla_F)$ is closed with respect to the twisted differential $d - H$ on $\Omega^\bullet(M)$ and hence represents a class in $H^\bullet(M; H)$. The class $[ch_H(\nabla_E, \nabla_F)]$ is independent of the choice of module connections ∇_E and ∇_F on E and F.*

Remark 9.1. Some care needs to be taken when working with connections on infinite dimensional vector bundles, as it is not always clear that the difference of two connections on E is a section of $\Omega^1(\mathrm{End}(E))$, where $\mathrm{End}(E)$ denotes the bundle on Y whose fibre at y is the space of all bounded linear operators $E_y \to E_y$. We will avoid this problem by fixing a module connection on E and then only consider module connections which differ from this fixed connection by a section of $\Omega^1(\mathrm{End}(E))$.

To show that $ch_H(\nabla_E, \nabla_F)$ is closed under $d - H$ it is sufficient to show that $\mathrm{tr}(\exp(F_E) - \exp(F_F))$ is closed. We have

$$\begin{aligned}
& d\,\mathrm{tr}((F_E^i)^k - (F_F^i)^k) \\
= \; & \mathrm{tr}(\sum F_E^i \cdots dF_E^i \cdots F_E^i - \sum F_F^i \cdots dF_F^i \cdots F_F^i) \\
= \; & \mathrm{tr}(\sum F_E^i \cdots [F_E^i, A_E^i] \cdots F_E^i - \sum F_F^i \cdots [F_F^i, A_E^i] \cdots F_F^i) \\
= \; & \mathrm{tr}(\sum F_E^i \cdots [F_E^i, A_E^i] \cdots F_E^i - \sum F_F^i \cdots [F_F^i, A_E^i] \cdots F_F^i \\
& + \sum F_F^i \cdots [F_F^i, A_E^i - A_F^i] \cdots F_F^i) \\
= \; & \mathrm{tr}(\sum F_E^i \cdots [F_E^i, A_E^i] \cdots F_E^i - \sum F_F^i \cdots [F_F^i, A_E^i] \cdots F_F^i) \\
& + \mathrm{tr}(\sum F_F^i \cdots [F_F^i, A_E^i - A_F^i] \cdots F_F^i)
\end{aligned}$$

Note that the second term makes sense as the difference of the two module connections $A_E^i - A_F^i$ is trace class. The first term vanishes by the usual argument for Chern-Weil theory, the trace is invariant: $\mathrm{tr}(g^{-1}Ag) = \mathrm{tr}(A)$ for any $g \in U(\mathcal{H})$ so long as A is trace class. The second term also vanishes: we could write it as

$$k\,\mathrm{tr}(F_F^i \cdots F_F^i [F_F^i, A_E^i - A_F^i]),$$

since the F_F are all even forms we can shuffle them around in the trace to show that this is zero.

We now show that the forms $ch_H(\nabla_E, \nabla_F)$ are independent of the choice of module connections ∇_E and ∇_F on the U_1 bundle gerbe modules E and F. So suppose that ∇'_E and ∇'_F is another pair of module connections on the U_1 bundle gerbe modules E and F respectively such that the difference $\nabla'_E - \nabla'_F$ is trace class when considered in local trivialisations of E and F. Form the families of module connections $\nabla_E(t)$ and $\nabla_F(t)$ on E and F respectively given by $\nabla_E(t) = t\nabla_E + (1-t)\nabla'_E$ and $\nabla_F(t) = t\nabla_F + (1-t)\nabla'_F$. It is clear that the difference

$\nabla_E(t) - \nabla_F(t)$ is trace class. Consider the $2k-1$ forms Ψ_k on Y given locally by

$$\Psi_k = \mathrm{tr}(\frac{d}{dt}(A_E(t))(F_E(t)+fI)^{k-1} - \frac{d}{dt}(A_F(t))(F_F(t)+fI)^{k-1}). \quad (9.2)$$

It is easy to see that Ψ_k is in fact a global $2k-1$-form on Y and that moreover Ψ_k descends to a form on M. We calculate the exterior derivative of Ψ_k:

$$\begin{aligned} & d\Psi_k \\ = \; & \mathrm{tr}(\frac{d}{dt}(dA_E(t))(F_E(t)+fI)^{k-1} - \frac{d}{dt}(dA_F(t))(F_F(t)+fI)^{k-1} \\ & + \frac{d}{dt}(A_F(t))\sum (F_F(t)+fI)\cdots([F_F(t),A_F(t)]+HI)\cdots(F_F(t)+fI) \\ & - \frac{d}{dt}(A_E(t))\sum (F_E(t)+fI)\cdots([F_E(t),A_E(t)]+HI)\cdots(F_E(t)+fI)) \\ = \; & \mathrm{tr}(\frac{d}{dt}(dA_E(t))(F_E(t)+fI)^{k-1} - \frac{d}{dt}(dA_F(t))(F_F(t)+fI)^{k-1} \\ & + \frac{d}{dt}(A_F(t))\sum (F_F(t)+fI)\cdots[F_F(t),A_F(t)]\cdots(F_F(t)+fI) \\ & - \frac{d}{dt}(A_E(t))\sum (F_E(t)+fI)\cdots[F_E(t),A_E(t)]\cdots(F_E(t)+fI)) \\ & + (k-1)H\Psi_{k-1}. \end{aligned}$$

We now examine d/dt of $\mathrm{tr}((F_E(t)+fI)^k - (F_F(t)+fI)^k)$. Using $\dot{F}_t = d\dot{\theta}_t + \dot{\theta}_t\theta_t + \theta_t\dot{\theta}_t$ we get

$$\begin{aligned} & \frac{d}{dt}\,\mathrm{tr}((F_E(t)+fI)^k - (F_F(t)+fI)^k) \\ = \; & \mathrm{tr}(\sum (F_E(t)+fI)\cdots(\frac{d}{dt}(dA_E(t)) + \frac{d}{dt}(A_E(t))A_E(t) \\ & + A_E(t)\frac{d}{dt}(A_E(t)))\cdots(F_E(t)+fI) - \sum (F_F(t)+fI)\cdots \\ & (\frac{d}{dt}dA_F(t) + \frac{d}{dt}(A_F(t))A_F(t) + A_F(t)\frac{d}{dt}(A_F(t)))\cdots(F_F(t)+fI)) \\ = \; & k\,\mathrm{tr}(\frac{d}{dt}(dA_E(t))(F_E(t)+fI)^{k-1} - \frac{d}{dt}(dA_F(t))(F_F(t)+fI)^{k-1}) \\ & - k\,\mathrm{tr}(\frac{d}{dt}(A_E(t))\sum (F_E(t)+fI)\cdots[F_E(t),A_E(t)]\cdots(F_E(t)+fI) \\ & + \frac{d}{dt}(A_F(t))\sum (F_F(t)+fI)\cdots[F_F(t),A_F(t)]\cdots(F_F(t)+fI)). \end{aligned}$$

Hence we see that $kd\Psi_k - k(k-1)H\Psi_{k-1} = d/dt\,\mathrm{tr}((F_E(t)+fI)^k - (F_F(t)+fI)^k)$. Integrating from $t=0$ to $t=1$ shows that the $(1/(k-1)!)\int_0^1 \Psi_k\, dt$ relate the two character forms $ch_H(\nabla_E,\nabla_F)$ and $ch_H(\nabla'_E,\nabla'_F)$.

Recall that $\mathrm{Mod}_{U_1}(L)$ denotes the semi-group of all U_1 bundle gerbe modules for the lifting bundle gerbe $L \to Y^{[2]}$ associated to a principal $PU(\mathcal{H})$ bundle $Y \to M$ with Dixmier-Douady class equal to $[H]$ (previously we were interested in $U_{\mathcal{K}}$ bundle gerbe modules but, as we have already mentioned, the extension of the theory to U_1 presents no difficulties). For a fixed bundle gerbe connection ∇_L on L and a curving f for ∇_L, the character forms $ch_H(\nabla_E,\nabla_F)$ define a map $ch_H(L,\nabla_L,f)\colon \mathrm{Mod}_{U_1}(L) \to H^\bullet(M;L,\nabla_L,f)$. Specifically, $ch_H(L,\nabla_L,f)(E-F) = [ch_H(\nabla_E,\nabla_F)] \in H^\bullet(M;L,\nabla_L,f)$. We have shown that this is independent of the choice of module connections ∇_E and ∇_F for E and F. We need to investigate the effect that changing the bundle gerbe connection ∇_L on L and the curving f has on the character forms $ch_H(\nabla_E, \nabla_F)$. It is only possible to change the bundle gerbe connection ∇_L on L by $\delta(a)$ for some complex valued 1-form a on Y. Then ∇_E and ∇_F no longer define module connections on E and F, instead $\nabla_E - aI$ and $\nabla_F - aI$ define module connections on E and F for the new bundle gerbe connection $\nabla_L + \delta(a)$. It is easy to check that $[ch_H(\nabla_E - aI, \nabla_F - aI)] = [ch_H(\nabla_E, \nabla_F)]$. Changing the curving f by the pullback of a 2-form λ on M to Y changes the character forms by the exponential factor $\exp(\lambda)$; the maps $ch_H(L, \nabla_L, f)\colon \mathrm{Mod}_{U_1}(L) \to H^\bullet(M; L, \nabla_L, f)$ and $ch_H(L, \nabla_L, f + \pi^*\lambda)\colon \mathrm{Mod}_{U_1}(L) \to H^\bullet(M; L, \nabla_L, f + \pi^*\lambda)$ are related by $ch_H(L, \nabla_L, f + \pi^*\lambda) = \exp(\lambda) ch_H(L, \nabla_L, f)$. Any two principal $PU(\mathcal{H})$ bundles Y and X with Dixmier-Douady class $[H]$ are isomorphic, and this isomorphism extends to an isomorphism of the lifting bundle gerbes $L \to Y^{[2]}$ and $J \to X^{[2]}$ associated to Y and X respectively. There exist isomorphisms $\mathrm{Mod}_{U_1}(L) = \mathrm{Mod}_{U_1}(J)$ and we write $\mathrm{Mod}_{U_1}(M, [H])$ for this isomorphism class of semi-groups. It is clear that the maps $ch_H(L, \nabla_L, f_L)$ and $ch_H(J, \nabla_J, f_J)$ are compatible under the isomorphisms $\mathrm{Mod}_{U_1}(L) = \mathrm{Mod}_{U_1}(J)$ and hence descend to define a map $ch_{[H]}\colon \mathrm{Mod}_{U_1}(M, [H]) \to H^\bullet(M; [H])$. Under the isomorphism $\mathrm{Mod}_{U_1}(M, [H]) = \tilde{K}^0(M; [H])$ of Proposition 7.3 we get a map

$$ch_{[H]}\colon \tilde{K}^0(M; [H]) \to H^\bullet(M; [H]).$$

We propose that this map defines the Chern character for (reduced) twisted K-theory. It can be shown that the Chern character for twisted K-theory is uniquely characterised by requiring that it is a functorial homomorphism which is compatible with the $\tilde{K}^0(M)$-module structure on $\tilde{K}^0(M; [H])$ and reduces to the ordinary Chern character when $[H] = 0$. It is easy to check that the map $ch_{[H]}\colon \tilde{K}^0(M; [H]) \to H^\bullet(M; [H])$ is functorial with respect to smooth maps $f\colon N \to M$. To show that $ch_{[H]}$ is a homomorphism, it is sufficient to show that the various maps $ch_H(L, \nabla_L, f)\colon \mathrm{Mod}_{U_1}(L) \to H^\bullet(M; L, \nabla_L, f)$ are homomorphisms. To see this, recall that the semi-group structure of $\mathrm{Mod}_{U_1}(L)$ is defined via the direct sum of U_1 bundle gerbe modules (the direct sum $E \oplus F$ acquires a U_1 reduction rather than a $U_1 \times U_1$ reduction from a fixed isomorphism $\mathcal{H} \oplus \mathcal{H} = \mathcal{H}$). ¿From here it is easy to see that $ch_H(\nabla_{E_1} \oplus \nabla_{E_2}, \nabla_{F_1} \oplus \nabla_{F_2}) = ch_H(\nabla_{E_1}, \nabla_{F_1}) + ch_H(\nabla_{E_2}, \nabla_{F_2})$. We do not show here that $ch_{[H]}$ is compatible with the $\tilde{K}^0(M)$-module structure of $\tilde{K}^0(M; [H])$ (this can easily be shown to be true when $[H]$ is torsion, it is more difficult to prove this when $[H]$ is not torsion).

10. Conclusion

Let us conclude with a final remark about C^* algebras and bundle gerbes. There is a well-known construction of a continuous trace C^* algebra from a groupoid [31]. This can be used to construct a C^* algebra from some bundle gerbes as follows. If the fibres of $Y \to M$ have an appropriate measure on them then we can define a product on two sections $f, g\colon Y^{[2]} \to P$ by

$$(fg)(y_1, y_2) = \int f(y_1, y)g(y, y_2)dy$$

where in the integrand we use the bundle gerbe product so that $f(y_1, y)g(y, y_2) \in L_{(y_1, y_2)}$. Appropriately closing this space of sections gives a C^* algebra with spectrum M and Dixmier-Douady class the Dixmier-Douady class of (L, Y). Some constructions in the theory of C^* algebras become easy from this perspective. For example if A is an algebra with spectrum X and $f\colon Y \to X$ is a continuous map there is an algebra $f^{-1}(A)$ with spectrum Y. This is just the pullback of bundle gerbes. It is tempting to define the K-theory of a bundle gerbe to be the K-theory of the associated C^* algebra. However a result from [27] is an obstruction to this. If a bundle gerbe has non-torsion Dixmier-Douady class then the fibres of $Y \to M$ are either infinite-dimensional (in which case there is no measure) or disconnected. The simplest example of the disconnected case is the one originally used by Raeburn and Taylor [30], in their proof that every three class is the Dixmier-Douady class of some C^* algebra, which is to take Y the disjoint union of an open cover so the fibres are discrete and counting measure suffices. It follows from general theory however that when the C^* algebra can be defined, because it has the same Dixmier-Douady class as the bundle gerbe, its K-theory is the K-theory of the bundle gerbe.

Errata to [5]: Bouwknegt and Mathai would like to correct the following errors.
• page 5 in [5], last paragraph. The $\mathcal{K}$-bundles with torsion Dixmier-Douady class $[H]$ described there are those that are pulled back from the classifying space of the fundamental group, even though it is not explicitly mentioned there. In general there are $\mathcal{K}$-bundles with torsion Dixmier-Douady class that can not be described in this manner.
• page 7 in [5], in section 3. Since $\mathcal{K}$ has no unit, one has to add a unit in defining the K-theory of the algebra of sections of the bundle of compact operators $\mathcal{E}_{[H]}$. The definition given in page 7 works only in the case when the Dixmier-Douady class $[H]$ is torsion, since in this case it can be shown that that the relevant algebra of sections has an approximate identity of idempotents, cf. [4] §5.5.4. This affects the discussion in the remaining part of the section starting from the last paragraph on page 8, in the sense that it is valid only when the Dixmier-Douady class $[H]$ is torsion. In particular, the finite dimensional description of elements in twisted K-theory is valid only in the torsion case. In the general case, sections of the twisted Fredholm operators as in equation (3.2) in [5] define elements in twisted K-theory.
• page 8, equation (3.3) in [5] should read

$$K^1(X, [H]) = [Y, U_{\mathcal{K}}]^{Aut(\mathcal{K})}$$

where $U_{\mathcal{K}}$ is the group of unitaries on a Hilbert space $\mathcal{H}$ of the form, identity operator + compact operator.

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