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TWISTOR SPACES WITH HERMITIAN RICCI TENSOR

JOHANN DAVIDOV AND OLEG MUŠKAROV

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ABSTRACT. The twistor space Z of an oriented Riemannian 4-manifold M admits a natural 1-parameter family of Riemannian metrics h_t compatible with the almost-complex structures J_1 and J_2 introduced, respectively, by Atiyah, Hitchin and Singer, and Eells and Salamon. In the present note we describe the (real-analytic) manifolds M for which the Ricci tensor of (Z, h_t) is J_n-Hermitian, $n = 1$ or 2. This is used to supply examples giving a negative answer to the Blair and Ianus question of whether a compact almost-Kähler manifold with Hermitian Ricci tensor is Kählerian.

1. INTRODUCTION

Given a compact symplectic manifold M, one can consider the integrals $\int_M s\, dV_g$ and $\int_M (s-s^*)\, dV_g$, s(resp. s^*) being the scalar (resp. *-scalar) curvature, as functionals on the set of metrics associated with the symplectic structure. D. E. Blair and S. Ianus [3] proved that the critical points of these functionals are the associated almost-Kähler metrics for which the Ricci tensor is Hermitian with respect to the corresponding almost-complex structure. Since the Kähler metrics satisfy this condition, Blair and Ianus raised the question of whether a compact almost-Kähler manifold with Hermitian Ricci tensor is Kählerian. A purpose of this note is to show that the twistor space of a compact oriented Riemannian 4-manifold which is Einstein, self-dual, and with negative scalar curvature supplies a negative answer to the question above.

The twistor space Z of an oriented Riemannian 4-manifold admits a natural 1-parameter family of Riemannian metrics h_t (cf., e.g. [8, 9, 13]) compatible with the almost-complex structures J_1 and J_2 on Z introduced, respectively, by Aityah, Hitchin and Singer [1], and Eells and Salamon [7]. Motivated by the Blair and Ianus result, we consider the problem when the Ricci tensor of (Z, h_t) is J_n-Hermitian, $n = 1$ or 2, and prove the following theorem:

Theorem. *Let M be a connected oriented real-analytic Riemannian 4-manifold. If the Ricci tensor of the twistor space (Z, h_t) is J_n-Hermitian, then either*

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(i) *M is Einstein and self-dual*

or

(ii) *M is self-dual with constant scalar curvature $s = 12/t$ and, for each point of M, at least three eigenvalues of its Ricci operator coincide.*

Conversely, if M is a (smooth) Riemannian 4-manifold satisfying (i) *or* (ii), *then the Ricci tensor of (Z, h_t) is J_n-Hermitian.*

The proof is based on an explicit formula for the Ricci tensor of (Z, h_t) in terms of the curvature of M [4].

Remarks. (1) Let M be an oriented Riemannian 4-manifold. If M is Einstein, self-dual, and with negative scalar curvature s, then (h_t, J_2) for $t = -12/s$ is an almost-Kähler structure on the twistor space Z [12]. This structure is not Kählerian since the almost-complex structure J_2 is never integrable [7]. By the theorem, the Ricci tensor of (Z, h_t) is J_2-Hermitian, so, if M is compact, (Z, h_t, J_2) gives a negative answer to the Blair and Ianus question. Note that the only known examples of such manifolds M are compact quotients of the unit ball in $\mathbf{C}^2$ with the metric of constant negative curvature or the Bergman metric (cf., e.g. [13]).

On the other hand the classification of compact connected self-dual Einstein 4-manifolds M with nonnegative scalar curvature s is well known: If $s > 0$, M is the unit sphere S^4 or the complex projective space $\mathbf{CP}^2$ with their standard metrics [9, 11]. If $s = 0$, the universal covering of M is a $K3$-surface with a Ricci flat Kähler metric or M is flat [10].

(2) Let S^1 and S^3 be the unit spheres of dimensions one and three. Then $M = S^1 \times S^3$ with the product-metric is a non-Einstein manifold satisfying the conditions (ii) of the theorem. Other examples of such manifolds M can be obtained as warped-products of S^1 and S^3 (cf. [5]).

2. Preliminaries

Let M be an oriented Riemannian 4-manifold with metric g. Then g induces a metric on the bundle $\bigwedge^2 TM$ of 2-vectors by $g(A_1 \wedge A_2, A_3 \wedge A_4) = 1/2 \cdot \det(g(A_i, A_j))$. Let ∇ be the Riemannian connection of (M, g). For the curvature tensor R of ∇, we adopt the following definition $R(A, B) = \nabla_{[A,B]} - [\nabla_A, \nabla_B]$. The curvature operator $\mathscr{R}$ is the self-adjoint endomorphism of $\bigwedge^2 TM$ defined by $g(\mathscr{R}(A \wedge B), C \wedge D) = g(R(A, B)C, D)$. The Hodge star operator defines an endomorphism $*$ of $\bigwedge^2 TM$ with $*^2 = Id$. Hence $\bigwedge^2 TM = \bigwedge^2_+ TM \oplus \bigwedge^2_- TM$ where $\bigwedge^2_\pm TM$ are the subbundles of $\bigwedge^2 TM$ corresponding to the (± 1)-eigenvalues of $*$. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$
\begin{aligned}
&s_1 = E_1 \wedge E_2 - E_3 \wedge E_4, \qquad && \bar{s}_1 = E_1 \wedge E_2 + E_3 \wedge E_4, \\
&s_2 = E_1 \wedge E_3 - E_4 \wedge E_2, \qquad && \bar{s}_2 = E_1 \wedge E_3 + E_4 \wedge E_2, \qquad (2.1)\\
&s_3 = E_1 \wedge E_4 - E_2 \wedge E_3, \qquad && \bar{s}_3 = E_1 \wedge E_4 + E_2 \wedge E_3.
\end{aligned}
$$

Then (s_1, s_2, s_3) (resp., $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$) is a local oriented orthonormal frame of $\bigwedge^2_- TM$ (resp., $\bigwedge^2_+ TM$). The block-decomposition of $\mathscr{R}$ with respect to the above splitting of $\bigwedge^2 TM$ is

$$\mathscr{R} = \begin{bmatrix} s/6 \cdot Id + \mathscr{W}_+ & \mathscr{B} \\ t_B & s/6 \cdot Id + \mathscr{W}_- \end{bmatrix},$$

where s is the scalar curvature of M; $s/6 \cdot Id + \mathscr{B}$ and $\mathscr{W} = \mathscr{W}_+ + \mathscr{W}_-$ represent the Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is said to be self-dual (anti-self-dual) if $\mathscr{W}_- = 0$ $(\mathscr{W}_+ = 0)$. It is Einstein exactly when $\mathscr{B} = 0$.

The twistor space of M is the 2-sphere bundle $\pi: Z \to M$ consisting of the unit vectors of $\bigwedge^2_- TM$. The Riemannian connection of M gives rise to a splitting $TZ = \mathscr{H} \oplus \mathscr{V}$ of the tangent bundle of Z into horizontal and vertical components. We further consider the vertical space $\mathscr{V}_\sigma$ at $\sigma \in Z$ as the orthogonal complement of σ in $\bigwedge^2_- T_pM$, $p = \pi(\sigma)$.

Each point $\sigma \in Z$ defines a complex structure K_σ on T_pM, $p = \pi(\sigma)$, by

$$g(K_\sigma A, B) = 2g(\sigma, A \wedge B), \qquad A, B \in T_pM. \tag{2.2}$$

This structure is compatible with the metric g and the opposite orientation of M at p.

Denote by $\times$ the usual vector product in the oriented three-dimensional vector space $\bigwedge^2_- T_pM$. Following [1] and [7], define two almost-complex structures J_1 and J_2 on Z by

$$J_n V = (-1)^n \sigma \times V \quad \text{for } V \in \mathscr{V}_\sigma,$$
$$\pi_*(J_n X) = K_\sigma(\pi_* X) \quad \text{for } X \in \mathscr{H}_\sigma.$$

It is well known [1] that J_1 is integrable (i.e. comes from a complex constructure on Z) iff M is self-dual. Unlike J_1, the almost-complex structure J_2 is never integrable [7].

As in [9], define a pseudo-Riemannian metric h_t on Z by $h_t = \pi^* g + t g^v$, where $t \neq 0$ and g^v is the restriction of the metric of $\bigwedge^2 TM$ on the vertical distribution $\mathscr{V}$. Then h_t is compatible with the almost-complex structures J_1 and J_2.

3. Proof of the theorem

Lemma. *The Ricci tensor c_Z of (Z, h_t) is Hermitian with respect to J_n iff for each point $\sigma \in Z$ one has:*

$$(12 - ts(p) + 6tg(\mathscr{W}_-(\sigma), \sigma))\mathscr{B}(\sigma) = 0, \tag{3.1}$$

where $p = \pi(\sigma)$ and s is the scalar curvature of M.

$$\|\mathscr{R}(\cdot)\| = \text{const} \textit{ on the fibre } Z_p \textit{ through } \sigma. \tag{3.2}$$

$$g((\delta\mathscr{R})(X), \sigma \times V) = (-1)^{n+1} g((\delta\mathscr{R})(K_\sigma X), V) \tag{3.3}$$

for every $X \in T_pM$ and $V \in \mathscr{V}_\sigma$. Here $\delta\mathscr{R}$ is the codifferential of $\mathscr{R}$ and K_σ is the complex structure on T_pM determined by σ via (2.2).

Proof. If $E \in TZ$, $X = \pi_* E$ and V is the vertical component of E, then [4]

$$(3.4)\quad c_Z(E, E) = c(X, X) + tg((\delta\mathscr{R})(X), \sigma \times V) + (t^2/4)\|\mathscr{R}(\sigma \times V)\|^2 - (t/2)\|i_X \circ \mathscr{R}_-\|^2 + (t/2)\|(i_X \circ \mathscr{R})(\sigma)\|^2 + \|V\|^2,$$

where c is the Ricci tensor of M, $i_X\colon \bigwedge^2 TM \to TM$ is the interior product and $\mathscr{R}_-$ is the restriction of $\mathscr{R}$ on $\bigwedge^2_- TM$.

We first show that c_Z is J_n-Hermitian on horizontal vectors iff (3.1) holds for every $\sigma \in Z$. In fact, it follows from (3.4) that c_Z is J_n-Hermitian on the horizontal space $\mathscr{H}_\sigma$ iff

$$(3.5)\quad 2c(X, X) - t\|R(\tau)X\|^2 - t\|R(\sigma \times \tau)X\|^2 = 2c(K_\sigma X, K_\sigma X) - t\|R(\tau)K_\sigma X\|^2 - t\|R(\sigma \times \tau)K_\sigma X\|^2$$

for $X \in T_pM$ and $\tau \in Z_p$, $\tau \perp \sigma$. Fix $\tau \in Z_p$, $\tau \perp \sigma$ and $E \in T_pM$, $\|E\| = 1$. Since $K_\sigma \circ K_\tau = -K_{\sigma\times\tau}$, $(E_1, E_2, E_3, E_4) = (E, K_\sigma E, K_\tau E, K_{\sigma\times\tau} E)$ is an oriented orthonormal basis of T_pM such that $\sigma = s_1$, $\tau = s_2$, and $\sigma \times \tau = s_3$, where s_1, s_2, s_3 are defined by (2.1). For $X \in T_pM$, denote

$$V_i = X \wedge E_i - K_\sigma X \wedge K_\sigma E_i, \qquad \overline{V}_i = X \wedge E_i + K_\sigma X \wedge K_\sigma E_i, \qquad i = 1, \dots, 4.$$

Then

$$(3.6)\quad \begin{gathered} c(X, X) - c(K_\sigma X, K_\sigma X) = \sum_{i=1}^{4} g(\mathscr{R}(V_i), \overline{V}_i) \\ \|R(\tau)X\|^2 - \|R(\tau)K_\sigma X\|^2 = \sum_{i=1}^{4} g(\mathscr{R}(\tau), V_i) g(\mathscr{R}(\tau), \overline{V}_i). \end{gathered}$$

If $X = \sum_{i=1}^4 \lambda_i E_i$, then

$$(3.7)\quad \begin{aligned} V_1 &= -\lambda_3 s_2 - \lambda_4 s_3, & \overline{V}_1 &= -\lambda_2(s_1 + \bar{s}_1) - \lambda_3 \bar{s}_2 - \lambda_4 \bar{s}_3, \\ V_2 &= \lambda_3 s_3 - \lambda_4 s_2, & \overline{V}_2 &= \lambda_1(s_1 + \bar{s}_1) - \lambda_3 \bar{s}_3 + \lambda_4 \bar{s}_2, \\ V_3 &= \lambda_1 s_2 - \lambda_2 s_3, & \overline{V}_3 &= -\lambda_4(\bar{s}_1 - s_1) + \lambda_1 \bar{s}_2 + \lambda_2 \bar{s}_3, \\ V_4 &= \lambda_1 s_3 + \lambda_2 s_2, & \overline{V}_4 &= \lambda_3(\bar{s}_1 - s_1) - \lambda_2 \bar{s}_2 + \lambda_1 \bar{s}_3. \end{aligned}$$

Substituting (3.6) and (3.7) into (3.5) and then varying $(\lambda_1, \dots, \lambda_4)$, one sees that the identity (3.5) holds iff

$$(3.8)\quad (2 - tg(\mathscr{R}(\sigma), \sigma)\mathscr{B}(\sigma)) - tg(\mathscr{R}(\sigma), \tau)\mathscr{B}(\tau) = 0$$

for all $\tau \in Z_p$, $\tau \perp \sigma$. Taking a point $\tau \in Z_p$ such that $\tau \perp \sigma$ and $g(\mathscr{R}(\sigma), \tau) = 0$, one obtains (3.1). Conversely, assume that the identity (3.1) holds for every $\sigma \in Z$. Fix a point $p \in M$. Then either $\mathscr{B}_p = 0$ or $12 - ts(p) + 6tg(\mathscr{W}_-(\sigma), \sigma) = 0$ for all $\sigma \in Z_p$. In the second case, $12 - ts(p) = 0$

since trace $\mathscr{W}_- = 0$ and therefore $(\mathscr{W}_-)_p = 0$. So $g(\mathscr{R}(\sigma), \tau) = 0$ for every $\sigma, \tau \in Z_p$, $\sigma \perp \tau$. In both cases (3.8) is fulfilled and c_Z is J_n-Hermitian on horizontal vectors.

It is obvious from (3.4) that c_Z is J_n-Hermitian on vertical vectors iff $\|\mathscr{R}(\sigma)\| = \|\mathscr{R}(\tau)\|$ for every σ, $\tau \in Z$ with $\pi(\sigma) = \pi(\tau)$ and $\sigma \perp \tau$, which is equivalent to (3.2). Formula (3.4) also shows that $c_Z(J_n E, J_n V) = c_Z(E, V)$ for all $E \in \mathscr{H}_\sigma$, $V \in \mathscr{V}_\sigma$ iff (3.3) holds. Thus the lemma is proved.

To prove the theorem, first assume that the Ricci tensor c_Z of (Z, h_t) is J_n-Hermitian. Then the identity (3.1) of the lemma and the principle of analytic continuation imply that either $\mathscr{B} \equiv 0$ or

$$12 - t(s \circ \pi)(\sigma) + 6tg(\mathscr{W}_-(\sigma), \sigma) \equiv 0 \quad \text{on } Z. \tag{3.9}$$

We shall show that in the first case M is self-dual. Consider $\mathscr{W}_-$ as a self-adjoint endomorphism of $\bigwedge^2_- T_pM$, $p \in M$, and denote by μ_1, μ_2, μ_3 its eigenvalues. Since $\mathscr{B} = 0$, $\mathscr{R}(\sigma) = (s/6)\sigma + \mathscr{W}_-(\sigma)$ for $\sigma \in \bigwedge^2_- T_pM$, and the condition (3.2) of the lemma gives $|\mu_1 + s/6| = |\mu_2 + s/6| = |\mu_3 + s/6|$. Moreover, $\mu_1 + \mu_2 + \mu_3 = \text{trace } \mathscr{W}_- = 0$. Hence either $\mu_1 = \mu_2 = \mu_3 = 0$ or $\{\mu_1, \mu_2, \mu_3\} = \{s/3, s/3, -2s/3\}$. It follows that either $\|\mathscr{W}_-\| \equiv 0$ or $\|\mathscr{W}_-\| \equiv 2s^2/3$. So we have to consider only the case when $\|\mathscr{W}_-\| \equiv 2s^2/3$. Since M is Einstein, $\delta\mathscr{W}_- = 0$ (cf., e.g. [2, §16.5]) and Proposition 5, (iii) of [6] gives $\nabla\mathscr{W}_- = 0$. For every oriented Riemannian 4-manifold with $\delta\mathscr{W}_- = 0$, one has [2, §16.73]

$$\Delta\|\mathscr{W}_-\|^2 = -s\|\mathscr{W}_-\|^2 + 18 \det \mathscr{W}_- - 2\|\nabla\mathscr{W}_-\|^2,$$

which implies in our case $s = 0$. Hence $\mathscr{W}_- = 0$.

Now assume that the identity (3.9) is satisfied. Then $s = 12/t$ since trace $\mathscr{W}_- = 0$. Therefore $g(\mathscr{W}_-(\sigma), \sigma) \equiv 0$ which shows that $\mathscr{W}_- = 0$. Thus $\mathscr{R}(\sigma) = (2/t)\sigma + \mathscr{B}(\sigma)$ for $\sigma \in Z$, and (3.2) of the lemma is equivalent to $\|\mathscr{B}(\cdot)\|$ being constant on the fibre through each point $\sigma \in Z$. Let $C\colon T_pM \to T_pM$, $p \in M$, be the Ricci operator and (E_1, E_2, E_3, E_4) an oriented orthonormal basis of T_pM consisting of eigenvectors of C. Denote by λ_i, $i = 1, \dots, 4$, the corresponding eigenvalues. Let $(\bar{s}_i, s_i)$ be the basis of $\bigwedge^2 T_pM$ defined by (2.1). Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = s$ and $\mathscr{B}(X \wedge Y) = C(X) \wedge Y + X \wedge C(Y) - (s/2) X \wedge Y$, one has $\mathscr{B}(s_1) = (\lambda_1 + \lambda_2 - s/2)\bar{s}_1$, $\mathscr{B}(s_2) = (\lambda_1 + \lambda_3 - s/2)\bar{s}_2$, $\mathscr{B}(s_3) = (\lambda_1 + \lambda_4 - s/2)\bar{s}_3$. Therefore $\|\mathscr{B}(\cdot)\| = \text{const}$ on Z_p iff $|\lambda_1 + \lambda_2 - s/2| = |\lambda_1 + \lambda_3 - s/2| = |\lambda_1 + \lambda_4 - s/2|$, i.e. iff at least three eigenvalues of C coincide.

Conversely, let M be a (smooth) Einstein self-dual 4-manifold. Then $\mathscr{R}(\sigma) = (s/6)\sigma$, $\sigma \in Z$, $\delta\mathscr{R} = 0$ (cf., e.g. [2, §16.3]), and the three conditions of the lemma obviously hold. Now, assume that M satisfies the condition (ii) of the theorem. Then (3.1) is obvious and (3.2) follows from the arguments above. Since $s = 12/t$ and $\mathscr{W}_- = 0$, one has $\delta\mathscr{R} = 2\delta\mathscr{W} = 2\delta\mathscr{W}_+$ ([2, §16.5]), so $(\delta\mathscr{R})(X) \in \bigwedge^2_+ T_pM$. Hence, (3.3) holds and the theorem is proved.

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Institute of Mathematics, Bulgarian Academy of Sciences, ul. "Acad. G. Bontchev", bl.8, 1090-Sofia, Bulgaria