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Two alternative derivations of Bridgman's theorem

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One of the fundamental results of Dimensional Analysis is the so-called Bridgman's theorem. This theorem states that the only functions that may have dimensional arguments are products of powers of the base quantities of a given system of units. In this work, Bridgman's theorem is discussed and rederived in two different ways, one not involving calculus, and a second one based on a Taylor series expansion analysis.

There is nothing absolute about dimensions, but that they may be anything consistent with a set of definitions which agree with the experimental facts.

P.W. Bridgman [2, p. 78].

1. Introduction

Dimensional Analysis is a basic and important subject in physics [2], and consequently, in physical chemistry [4,6]. One of its parts constitutes the so-called quantity calculus, which is the theory and algebra of physical quantities [2,4,6]. The remaining part of Dimensional Analysis is similarity theory (or theory of models), where the effect of scale on physical systems is studied.

A *physical quantity* **Q** is any property of a physical system that can in principle be measured, i.e., evaluated against a standard, **U** [2].[1] Its value, Q, is given by $Q = m(Q \mid U)U$, where $m(Q \mid U)$, a pure number, is the measure of **Q**, given the unit U. The form of this equation ensures the invariance of Q. In fact, being a property of the system, it must be the same, irrespective of the standard or system of units used.

The fact that physical quantities are (usually) dimensional implies some differences between their algebra and that of pure numbers. For instance, if the equality $A = B$ holds, then $A^2 = B^2$ and adding the two one gets $A + A^2 = B + B^2$, a relation

[1] In most cases this is not really done, **Q** being instead calculated from other directly measured quantities.

that is correct in the algebra of pure numbers, but nonsense in physics, because it violates the *principle of dimensional homogeneity*. It will be correct in physics only if all quantities are previously divided by the unit of A and B, U, $(A/U)+(A/U)^2 = (B/U)+(B/U)^2$, that is, if the equation is written in a dimensionless form.[2] Dimensional forms are also possible, e.g., $A \cdot U + A^2 = B \cdot U + B^2$.

Considering again that $A = B$ holds, and supposing that $A > 0$, it also appears obvious that $\ln A = \ln B$. Again, this is correct only for pure numbers. For dimensional quantities one must write $\ln(A/B) = 0$, or divide both members by the unit of A and B, U, $\ln(A/U) = \ln(B/U)$.

Why is this so, and what are the functions whose arguments can be dimensional?

2. Base and derived quantities

The type and number of base quantities, **B**, and respective units, U_B, of a given *system of units* are to some extent arbitrary, and defined by convention. They are mainly chosen for practical reasons. All base quantities are said to correspond to a different dimension in that system of units, and are considered to be dimensionally independent. For each, a unit is defined. In the SI system of units, there are seven base physical quantities [1]: the *length*, **L** (unit: meter, $U_L = \mathrm{m}$), the *mass*, **M** (unit: kilogram, $U_M = \mathrm{kg}$), the *time*, **T** (unit: second, $U_T = \mathrm{s}$), the *electric current*, **I** (unit: ampere, $U_I = \mathrm{A}$), the *thermodynamic temperature*, Θ (unit: kelvin, $U_\Theta = \mathrm{K}$), the *amount of substance* **N** (unit: mole, $U_N = \mathrm{mol}$), and the *luminous intensity*, $\mathbf{I}_v$ (unit: candela, $U_{I_v} = \mathrm{cd}$). All other physical quantities are called *derived quantities*, as they are defined in terms of the base quantities by means of *defining equations*.

It may be remarked that of the seven SI base quantities, only the first four correspond to fundamental physical properties. Of these, two are attributes of the space–time (length, time), and the other two are independent attributes of matter (mass and electric charge). The thermodynamic temperature, the amount of substance and the luminous intensity could in principle be excluded from a minimum set of fundamental base quantities, and this indeed happened in the past (recall the old CGS and MKSA systems).

But a system of units may still be consistent and not even imply the use of all four mentioned quantities. A single base quantity suffices [3].

[2] One of the less known achievements of René Descartes (1596–1650) was a major break with Greek mathematical tradition: instead of considering x^2 and x^3, for example, as an area and a volume, he interpreted them also as lines. This permitted him to abandon the principle of homogeneity, at least, explicitly, and yet retain geometric meaning. Descartes could write an expression such as $a^2b^2 - b$, for, as he expressed it, one "must consider the quantity a^2b^2 divided once by unity (that is, be u the unit line segment, then, $[a^2b^2/u]$ = [volume]) and the quantity b multiplied twice by unity (that is, $[a \cdot u \cdot u]$ = [volume])." (English translation given in [5].) With this Descartes made his geometric algebra more flexible, so flexible indeed that today we read $x \cdot x$ as "x squared $\equiv x^2$" without seeing a square in our mind. What for Descartes's time was a difficult move towards abstraction, is now so natural that the dimensional viewpoint is often overlooked.

Consider now the following two questions:

(i) Is there any restriction to the type of function that can be used to define a derived quantity from the base quantities?

(ii) Is there any restriction to the type of function that can be used to define a physical quantity from general (dimensionally compounded) quantities?

The second question encompasses the first question, but it is convenient to treat the two separately.

3. Bridgman's theorem

The answer to the first question is given by one of the fundamental results of Dimensional Analysis, known as *Bridgman's theorem* [2], that states that the only allowed functions are of the type

$$\mathbf{Q} = \alpha \mathbf{B}_1^{\beta_1} \mathbf{B}_2^{\beta_2} \cdots, \tag{1}$$

$$Q = m(Q \mid U)U = m(Q \mid U_1, U_2, \ldots) U_1^{\beta_1} U_2^{\beta_2} \cdots, \tag{2}$$

where α and β_1, β_2, etc. are numerical constants. Any other type of function does not warrant the form of Q as the product of a pure number by a dimensional part (such is the case of exponential and logarithmic dependences). This is in fact a familiar result, namely that all derived quantities have units that are obtained from the base ones by multiplication and division. Note, however, that most of these quantities are not usually directly defined with respect to the base ones, but with respect to others, already dimensionally compounded. The final dimensions of a given quantity are therefore usually the result of several multiplications and divisions of powers of the same units, that may, for this reason, be even absent in the final result.

4. Two alternative proofs of Bridgman's theorem

The original proof of Bridgman's theorem, making use of differentiation, is given in [2, pp. 17–25], and reproduced in the appendix. We give here two alternative proofs, one not involving calculus, and a second one based on a Taylor series expansion analysis.

4.1. *First proof*

Consider a given quantity Q', defined by

$$Q' = f(Q), \tag{3}$$

where Q is another physical quantity. We have for both

$$Q' = m\left(Q' \,\middle|\, U'\right)U', \tag{4}$$

$$Q = m(Q \mid U)U. \tag{5}$$

One then obtains from equations (3)–(5) that

$$m\bigl(Q' \mid U'\bigr)U' = f\bigl[m(Q \mid U)U\bigr]. \tag{6}$$

It is also true that equation (6) holds for the pure numerical part of the quantities,

$$m\bigl(Q' \mid U'\bigr) = f\bigl[m(Q \mid U)\bigr]. \tag{7}$$

Insertion of equation (7) into equation (6) yields

$$f\bigl[m(Q \mid U)\bigr]U' = f\bigl[m(Q \mid U)U\bigr]. \tag{8}$$

For the particular case $Q = U$ one has $m(Q \mid U) = 1$, and from equation (8),

$$U' = \frac{f(U)}{f(1)}, \tag{9}$$

a result that holds for any value of Q. Insertion of equation (9) into equation (8) finally gives

$$f\bigl[m(Q \mid U)\bigr]\frac{f(U)}{f(1)} = f\bigl[m(Q \mid U)U\bigr]. \tag{10}$$

If Q is dimensionless ($U \equiv 1$), equation (10) is always obeyed, regardless of the form of function f.

However, if Q has dimensions, not all functions satisfy equation (10). This equation can be rewritten as

$$f(xy) = \frac{f(x)f(y)}{f(1)}, \tag{11}$$

whose general solution is seen to be

$$f(x) = \alpha x^{\beta}, \tag{12}$$

where α is a dimensional or dimensionless constant and β is a pure number. This is the one-variable form of Bridgman's theorem.

4.2. *Second proof*

Here we apply the principle of dimensional homogeneity to a Taylor series expansion of a general function $f(x)$,

$$f(x) = f(a) + (x-a)\frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_a + \frac{(x-a)^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\bigg|_a + \cdots. \tag{13}$$

According to the principle of dimensional homogeneity, both members of equation (13) must have the same dimensions, the same being true of every term. This will happen if

$$[f] = [x]\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right], \tag{14}$$

$$[f] = \left[x^2\right]\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = [x]^2\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right], \tag{15}$$

$$\vdots$$

where the square brackets stand, as usual, for "dimensions of". Equalities (14)–(15) will hold if

$$\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \frac{[f]}{[x]}, \tag{16}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right] = \frac{[f]}{[x]^2}, \tag{17}$$

$$\vdots$$

But equation (16) can be rewritten as

$$\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \frac{[f]}{[x]} = \left[\frac{f}{x}\right]. \tag{18}$$

Since these relations must be valid for any x, they imply that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \beta\frac{f}{x}, \tag{19}$$

where β is a dimensionless constant. The solution of equation (19) is

$$f(x) = \alpha x^{\beta}, \tag{20}$$

where α is again a dimensional or dimensionless constant. Equation (20) also satisfies the higher terms conditions (equation (17), etc.).

5. General results

The answer to the second (and more general) question is that the *final* form of the derived quantity must again be like equation (1), and therefore only products of powers of the quantities can occur, unless some of the independent quantities are algebraically combined to yield dimensionless groups, whose functions are grouped together in the factor α of equation (1). Dimensionless groups can also be formed with the participation of dimensional constants. These constants may have unit value, and occur associated with standard states (e.g., in thermodynamics) or definitions (e.g., the definition of pH), or may have definite values, and be called physical constants, like the Boltzmann and Planck constants, and the speed of light in the vacuum.

For all dimensionless groups, no restrictions on the (dimensionless) functional form exist, that may be "transcendental to the worst degree" [2, p. 44]. For instance, exponentials and logarithms of such groups are common, but much more complex forms are possible and do indeed exist, as a perusal of any Handbook of Physics shows. Only their number can be obtained from Dimensional Analysis (Buckingham's Π theorem [2]).

6. Discussion and conclusions

We now consider two examples taken from the literature where functions other than powers have dimensional arguments.

The first example is the Arrhenius equation for the rate of a chemical reaction, $k = A\exp(-E_\mathrm{a}/RT)$, where k is the rate constant, A the frequency factor, E_a the energy of activation, R is the perfect gas constant, and T is the temperature. For the purpose of evaluating A and E_a from experimental data, the equation is frequently linearized in the form $\ln k = \ln A - E_\mathrm{a}/RT$. None of the two logarithms is correct *per se*, as discussed. Therefore, if a meaning is to be attributed to individual terms in the expression above, the arguments of the logarithms must be made dimensionless, and the equation written as $\ln(k/k_0) = \ln(A/k_0) - E_\mathrm{a}/RT$, k_0 being the unit rate constant in the chosen units. This is in fact what is done when the logarithms of the experimental quantities are calculated: only the number is used, not the associated units. On the other hand, if the quantities (pure number times unit) are used directly in $\ln k = \ln A - E_\mathrm{a}/RT$, logarithms of the units appear on both sides. No meaning can be attributed to these. Nevertheless, a cancellation of terms occurs, the logarithms of the units disappear, and the final result is the same. All reduces in this case to a matter of formal rigor.

The second example is the chemical potential of a perfect gas, sometimes written as $\mu = \mu^0 + RT\ln p$, where $\mu^0 = \mu(p^0, T)$ is the chemical potential of the standard state, p^0 being the pressure of the standard state, R is the perfect gas constant, T is the temperature and p is the pressure. The fault is in this case that pressure is implicitly assumed to be given in a certain unit, usually the bar. The equation should in fact be $\mu = \mu^0 + RT\ln(p/p^0)$. In this second example, the situation is more serious, and the equation is indeed dimensionally incorrect.

In conclusion, problems that occur with dimensional arguments result either from a lack of formal rigor or from dimensionally incorrect expressions, that can only be correctly understood if accompanied by a sentence stating which units are to be used. This amounts to divide the respective quantities by such units, thus rendering them dimensionless. No harm usually results, but the practice is confusing and unnecessary. Computational errors may also be the outcome of such a procedure.

In this work, Bridgman's theorem was rederived in two new ways, and its significance was discussed.

Appendix: Original proof of Bridgman's theorem

We give here the one-variable proof. The generalization to n variables is straighforward [2]. Let

$$Q' = f(Q). \tag{A1}$$

Let the unit of **Q** to be U. Take two different numerical values of **Q**, m_1 and m_2. If the unit of **Q**, U, is now changed to U/x, then the numerical values of **Q** become xm_1 and xm_2. The following relation must clearly hold:

$$\frac{f(m_1)}{f(m_2)} = \frac{f(xm_1)}{f(xm_2)}. \tag{A2}$$

This equation can be rewritten as

$$f(xm_1) = \frac{f(m_1)}{f(m_2)} f(xm_2). \tag{A3}$$

Differentiation with respect to x yields

$$m_1 \dot{f}(xm_1) = \frac{f(m_1)}{f(m_2)} m_2 \dot{f}(xm_2), \tag{A4}$$

where the dot represents the derivative of the function with respect to the argument. Making now $x = 1$, one gets

$$m_1 \frac{\dot{f}(m_1)}{f(m_1)} = m_2 \frac{\dot{f}(m_2)}{f(m_2)}. \tag{A5}$$

Because this equation must apply to any m_1 and m_2, it implies the differential equation

$$m \frac{\dot{f}(m)}{f(m)} = \beta, \tag{A6}$$

where β is some pure number. The solution of equation (A6) is

$$f(m) = \alpha m^{\beta}, \tag{A7}$$

where α is a dimensional or dimensionless constant and β is a pure number.

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