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# TWO APPLICATIONS OF A POISSON APPROXIMATION FOR DEPENDENT EVENTS[1]

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Recent results have estimated the error when sums of dependent non-negative integer-valued random variables are approximated in distribution by a Poisson variable. Two problems are considered where these results can be used to provide simple solutions. The first problem studies the asymptotic behavior, as $\alpha \to 0$, of the number of independent random arcs of length $\alpha$ needed to cover a circle of unit circumference at least $m$ times ($m \geqq 1$). The second problem deals with urn schemes.

**1. Introduction.** Freedman [2] has studied the problem of approximating in distribution a sum of dependent random variables by a Poisson variable. We first introduce some notation. For $\theta > 0$, let $N(\theta)$ denote a Poisson variable with parameter $\theta$. If $X_1$ and $X_2$ are any two nonnegative integer-valued random variables, then define

$$d(X_1, X_2) = \sup_A |P(X_1 \in A) - P(X_2 \in A)|$$

where $A$ ranges over all subsets of nonnegative integers. Also for any set $B$, $I_B$ is its indicator function.

Theorem A (Freedman [2]). *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space. Let* $A_1, A_2, \cdots$ *be events*, i.e., *elements of* $\mathscr{F}$. *Let* $\{\mathscr{F}_i\}$ *be a collection of sub-$\sigma$-algebras of* $\mathscr{F}$ *and assume that*

$$A_i \in \mathscr{F}_i, \quad i \geqq 1, \qquad \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots.$$

*Let* $p_i = P(A_i \mid \mathscr{F}_{i-1})$. *Let $N$ equal the number of $A_i$ that occur with* $i \leqq n$:

$$N = \sum_{i=1}^n I_{A_i}.$$

*Let* $a \leqq b$ *be nonnegative real numbers. Let* $\varepsilon$, $\delta$ *be nonnegative real numbers less than* $\frac{1}{2}$. *Suppose*

$$P(a \leqq \sum_{i=1}^n p_i \leqq b,\ \sum_{i=1}^n p_i^2 \leqq \varepsilon) \geqq 1 - \delta. \tag{1}$$

*Then there is a positive constant $C$ such that*

$$d(N, N(a)) \leqq C\varepsilon + (b - a) + 2\delta.$$

The difficulties in using Theorem A are to compute the $\{p_i\}$ and to establish (1). The next result considers a simple situation, where sufficient conditions are given which imply (1).

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COROLLARY 1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $\{A_{ij}\}$ be a collection of events and $\{\mathscr{F}_{ij}\}$ a collection of sub-$\sigma$-algebras of $\mathscr{F}$. Assume that*

$$A_{ij} \in \mathscr{F}_{ij} \quad \text{and} \quad \mathscr{F}_{0j} \subset \mathscr{F}_{1j} \subset \cdots \quad \text{for all} \quad i, j .$$

*Let $p_{ij} = P(A_{ij} \mid \mathscr{F}_{ij})$ and $\{n_j\}$ a sequence of integers tending to $\infty$. Let*

$$N_j = \sum_{i=1}^{n_j} I_{A_{ij}} .$$

*Suppose that as $j \to \infty$*

$$\sum_{i=1}^{n_j} E(p_{ij}) \to \theta \quad \text{a positive finite constant,} \tag{2}$$

$$E(|\sum_{i=1}^{n_j} (p_{ij} - E(p_{ij}))|) = o(1) , \tag{3}$$

$$\sum_{i=1}^{n_j} E(p_{ij}^2) = o(1) . \tag{4}$$

*Then*

$$d(N_j, N(\theta)) = o(1) .$$

PROOF. Let $0 \leqq \varepsilon \leqq \frac{1}{2}$, $0 \leqq \delta \leqq \frac{1}{2}$. In view of (3) and (4), there exists a $J_0$ such that $j > J_0$ implies

$$P(|\sum_{i=1}^{n_j} (p_{ij} - E(p_{ij}))| < \varepsilon, \sum_{i=1}^{n_j} p_{ij}^2 < \varepsilon) > 1 - \delta .$$

Hence by Theorem A, for all $j > J_0$,

$$d(N_j, N(\sum_{i=1}^{n_j} E(p_{ij}) - \varepsilon)) \leqq \varepsilon(2 + C) + 2\delta . \tag{5}$$

The result now follows from (2) and (5) and the observation ([2], Lemma 8) that

$$d(N(\sum_{i=1}^{n_j} E(p_{ij}) - \varepsilon), N(\theta)) \leqq |\sum_{i=1}^{n_j} E(p_{ij}) - \varepsilon - \theta| .$$

The purpose of this note is to give two applications where the computation of the $\{p_{ij}\}$ is possible and the conditions of Corollary 1 are verifiable.

The first application is to a problem studied by Flatto [1]. Let $\mathscr{C}$ be a circle with unit circumference and suppose that arcs of length $\alpha$ are thrown independently and uniformly on $\mathscr{C}$. Set $N_\alpha(m)$ equal to the number of arcs necessary to cover $\mathscr{C}$ at least $m$ times ($m \geqq 1$). We then have

THEOREM 1 (Flatto [1]). *Let $-\infty < x < \infty$, and $m \geqq 1$. Then,*

$$\lim_{\alpha \to 0} P\left(N_\alpha(m) \leqq \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + m \log\log\frac{1}{\alpha} + x\right)\right) = e^{-e^{-x}/(m-1)!} .$$

Using Corollary 1 and some properties of the Poisson process, we will give a simple proof of Theorem 1. The details are carried out in Section 2.

Our second problem was considered by Sevast'yanov [4]. Suppose we perform a sequence of experiments, where, for the $l$th experiment, we independently toss $n_l$ balls into $N_l$ urns according to some distribution $\{b_l(k)\}_{k=1}^{N_l}$. For ease of notation we drop the subscript $l$. We assume that on the set of integers $\{1, 2, \cdots, N\}$, there is a metric $\rho(i, j)$ possibly depending on $l$. We write $Z_i$ for the number of the urn picked on the $i$th throw and let $Z$ denote the number of pairs of tosses $(i, j)$, $i < j$, with outcomes $Z_i$ and $Z_j$ such that $\rho(Z_i, Z_j) \leqq a$, where $a \geqq 0$ is a certain fixed number. Let $\pi_k = \sum_{\rho(k,j)\leqq a} b(j)$ be the probability of picking an urn whose number is in the sphere of radius $a$ with center at $k$ for one trial. We then have:

THEOREM 2. *Assume*

$$n^3 \sum_{k=1}^{N} b(k)\pi_k^2 = o(1) \tag{6}$$

*and*

$$\binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k \to \lambda < \infty . \tag{7}$$

*Then,*

$$d(Z, N(\lambda)) = o(1) .$$

(It should be noted that all the above limits are taken as $l \to \infty$.)

A proof of Theorem 2 is given in Section 3.

**2. Proof of Theorem 1.** For the remainder of this section we assume $x$ and $m$ are fixed and set

$$n = n(\alpha, m, x) = \left[\frac{1}{\alpha}\left(\log\frac{1}{\alpha} + m \log\log\frac{1}{\alpha} + x\right)\right],$$

where for any positive $y$, $[y]$ is the greatest integer in $y$. Our starting point is a result of Flatto [1].

LEMMA 1. *Let $n - 1$ ($n > m$) points be chosen uniformly and independently on $[0, 1]$. Let $(Z_{(1)}, Z_{(2)}, \cdots, Z_{(n-1)})$ denote the order statistics. For convenience set $Z_{(0)} \equiv 0$ and $Z_{(n)} \equiv 1$. Let*

$$L_i = Z_{(i)} - Z_{(i-1)} \qquad 1 \leqq i \leqq n$$

*and define $L_i$ for all $i$ by putting $L_i = L_{i+n}$. Set*

$$S_i = \sum_{j=0}^{m-1} L_{i-j} \qquad m \leqq i \leqq n + m - 1 .$$

*Then,*

$$P(N_\alpha(m) \leqq n) = P(S_i \leqq \alpha, m \leqq i \leqq n + m - 1) . \tag{8}$$

Lemma 1 is just a restatement of Theorem 2.1 of [1].

The problem is to determine the asymptotic behavior of the right-hand side of (8). Since the $\{L_i\}$ are exchangeable random variables ([3], Chapter 8), all the $\{S_i\}$ have the same distribution. Furthermore, Flatto [1] showed that $P(S_m > \alpha) = o(1)$. Thus there is no loss of generality in studying

$$A_\alpha(n) = P(S_i \leqq \alpha, m \leqq i \leqq n - 1) .$$

The analysis depends on Corollary 1 and some properties of the Poisson process. Let $\{X(t)\}_{0\leqq t\leqq 1}$ be a Poisson process with parameter equal to $n$. Denote the interarrival times by $\{T_i\}_{i\geqq 1}$, and recall that they are i.i.d., having a negative exponential distribution with parameter $n$. Let

$$W_i = \sum_{j=i-m+1}^{i} T_j \qquad i \geqq m .$$

We then have

$$(9)\qquad P(W_i \leqq \alpha,\, m \leqq i \leqq X(1);\, X(1) \geqq m) = \sum_{k=m}^{\infty} \Lambda(k)\, \frac{e^{-n}n^k}{k!}$$

where

$$\Lambda(k) = P(W_i \leqq \alpha;\, m \leqq i \leqq k \mid X(1) = k)\,.$$

It is well known ([3], Chapter 8) that conditioned on $X(1) = k$, $\{\sum_{j=1}^{i} T_j\}_{1\leqq i\leqq k}$ have the same distribution as a set of order statistics of size $k$ taken from a uniform distribution on [0, 1]. Thus $\Lambda(n-1) = A_n(\alpha)$.

The idea now is to show, using Corollary 1, that the left-hand side of (9) is asymptotic to $\exp(-e^{-x}/(m-1)!)$, and that the asymptotic difference between the right-hand side of (9) and $A_n(\alpha)$ is negligible; to do this we use the central limit theorem (CLT).

We first consider the left-hand side of (9). Let $\varepsilon > 0$. It follows from the law of large numbers that

$$\begin{aligned} &P(W_i \leqq \alpha;\, m \leqq i \leqq X(1);\, X(1) > m) \\ &\qquad = P\left(W_i \leqq \alpha;\, m \leqq i \leqq X(1);\, \left|\frac{X(1)}{n} - 1\right| < \varepsilon\right) + o(1)\,. \end{aligned}$$

Let $n_0 = n(1-\varepsilon)$ and $n_1 = n(1+\varepsilon)$. Using monotonicity we then have

$$(10)\qquad \begin{aligned} &P(W_i \leqq \alpha;\, m \leqq i \leqq n_1) + o(1) \\ &\qquad \leqq P\left(W_i \leqq \alpha;\, m \leqq i \leqq X(1);\, \left|\frac{X(1)}{n} - 1\right| < \varepsilon\right) \\ &\qquad \leqq P(W_i \leqq \alpha;\, m \leqq i \leqq n_0) + o(1)\,. \end{aligned}$$

Consider the right-hand side of (10). The argument for the left-hand side is analogous. Define

$$A_i = (W_i > \alpha) \qquad\qquad m \leqq i \leqq n_0\,.$$

Then,

$$P(W_i \leqq \alpha;\, m \leqq i \leqq n_0) = P\left(\sum_{i=m}^{n_0} I_{(W_i > \alpha)} = 0\right)\,.$$

We now verify the conditions of Corollary 1. First we compute $\theta = \sum_{i=m}^{n_0} P(A_i)$. Since all the $\{A_i\}$ have the same probability

$$\theta = (n_0 - m)P(A_m)\,.$$

But

$$\begin{aligned} P(A_m) &= P\left(\sum_{i=1}^{m} T_i > \alpha\right) \\ &= \int_{\alpha}^{\infty} \frac{n}{(m-1)!}\,(nx)^{m-1}e^{-nx}\,dx \\ &= \frac{(n\alpha)^{m-1}}{(m-1)!}\,e^{-n\alpha} + o\left(\frac{1}{n}\right) \\ &= \frac{\alpha e^{-x}}{(m-1)!\,\log 1/\alpha} + o\left(\frac{1}{n}\right)\,. \end{aligned}$$

(The last equality follows from the definition of $n$.)

Hence

$$\theta = \frac{(1-\varepsilon)e^{-x}}{(m-1)!} + o(1)\,.$$

This proves (2).

Let $\mathscr{F}_i = \sigma(T_1, \cdots, T_i)$, $i \geqq m$. Then

$$\begin{aligned} p_i &= P(A_j \mid \mathscr{F}_{i-1}) \\ &= P(T_i > \alpha - \textstyle\sum_{j=i-m+1}^{i-1} T_j \mid \mathscr{F}_{i-1}) \\ &= I_{(\sum_{j=i-m+1}^{i-1} T_j > \alpha)} + \exp[-n(\alpha - \textstyle\sum_{j=i-m+1}^{i-1} T_j)] I_{(\sum_{j=i-m+1}^{i-1} T_j \leqq \alpha)} \\ &= W_{1i} + W_{2i}\,. \end{aligned}$$

To prove (3) it suffices to verify

$$B_j = E(|\textstyle\sum_{i=m}^{n_0} (W_{ji} - E(W_{ji}))|) = o(1) \qquad j = 1, 2\,. \tag{11}$$

A straightforward computation shows that

$$\textstyle\sum_{i=m}^{n_0} E(W_{1i}) = o(1) \tag{12}$$

which in turn implies that $B_1 = o(1)$. To handle $B_2$ we argue as follows. Observe

$$\begin{aligned} &E([\textstyle\sum_{i=m}^{n_0} (W_{2i} - E(W_{2i}))]^2) \\ &\qquad \leqq \textstyle\sum_{i=m}^{n_0} E(W_{2i}^2) + \sum_{i=m}^{n_0} [\sum_{j \neq i} E((W_{2i} - EW_{2i})(W_{2j} - EW_{2j}))]\,. \end{aligned}$$

By the independence of the $\{T_i\}$, the sum in the brackets has at most $2m$ nonzero terms and each of these, by Schwarz's inequality, is bounded by $E(W_{2i}^2)$. Thus

$$\begin{aligned} E([\textstyle\sum_{i=m}^{n_0} (W_{2i} - E(W_{2i}))]^2) &\leqq 3m \textstyle\sum_{i=m}^{n_0} E(W_{2i}^2) \\ &= 3mn_0 E(W_{2m}^2) \quad \text{by exchangeability.} \end{aligned}$$

But

$$\begin{aligned} E(W_{2m}^2) &= \int_0^\alpha e^{-2n(\alpha - z)} \frac{n}{(m-2)!} (nz)^{m-2} e^{-nz}\, dz \\ &= \frac{e^{-2n\alpha}}{(m-2)!} \int_0^{n\alpha} z^{m-2} e^z\, dz \\ &= \frac{e^{-2n\alpha}}{(m-2)!} (n\alpha)^{m-2} e^{n\alpha} [1 + o(1)] \\ &= o\left(\frac{1}{n}\right)\,. \end{aligned} \tag{13}$$

This completes the proof of (3). The proof of (4) follows from (12), (13) and the observation that

$$\textstyle\sum_{i=m}^{n_0} E(p_i^2) \leqq 2 \sum_{i=m}^{n_0} (E(W_{1i}) + E(W_{2i}^2))\,.$$

We now conclude from Corollary 1 that as $\alpha \to 0$

$$|P(W_i \leqq \alpha;\, m \leqq i \leqq n_0) - e^{-(1-\varepsilon)e^{-x}/(m-1)!}| = o(1)\,. \tag{14}$$

Similarly

$$|P(W_i \leqq \alpha;\, m \leqq i \leqq n_1) - e^{-(1+\varepsilon)e^{-x}/(m-1)!}| = o(1)\,. \tag{15}$$

Since $\varepsilon$ is arbitrary we obtain from (10), (14), (15) that

$$|P(W_i \leqq \alpha; m \leqq i \leqq X(1); X(1) \geqq m) - e^{-e^{-x}/(m-1)!}| = o(1) .$$

We now turn to the right-hand side of (9). It follows from the CLT that for any $\varepsilon > 0$, there exists a $D > 0$ such that for all $n$ sufficiently large (i.e., for all $\alpha$ sufficiently small),

$$P\left(\left|\frac{X(1) - n}{n^{\frac{1}{2}}}\right| > D\right) < \varepsilon .$$

It therefore suffices to show for any $D > 0$,

$$\eta = \sup_{|k-n| \leqq Dn^{\frac{1}{2}}} |\Lambda(k) - \Lambda(n-1)| = o(1) . \tag{16}$$

To prove (16) it is convenient to modify the definition of $\Lambda(k)$. Let

$$\tilde{\Lambda}(k) = P(W_i \leqq \alpha; m \leqq i \leqq X(1); 1 - \textstyle\sum_{j=1}^{X(1)-m+1} T_j \leqq \alpha \,|\, X(1) = k) .$$

Then for $|k - n| \leqq Dn^{\frac{1}{2}}$

$$\begin{aligned} |\Lambda(k) - \tilde{\Lambda}(k)| &\leqq P(1 - \textstyle\sum_{j=1}^{X(1)-m+1} T_j > \alpha \,|\, X(1) = k) \\ &= O\left(\frac{1}{n}\right) \quad \text{(uniform in } k) . \end{aligned} \tag{17}$$

The last equality is a straightforward computation which is omitted.

We next estimate

$$\beta_k = |\tilde{\Lambda}(k) - \tilde{\Lambda}(k-1)| , \qquad |k - n| \leqq Dn^{\frac{1}{2}} .$$

Let $Z_{(0)}, Z_{(1)}, \cdots, Z_{(k)}, Z_{(k+1)}$ be a set of order statistics of size $k$ taken from the uniform distribution on $[0, 1]$, and set $S_i = Z_{(i)} - Z_{(i-m)}$ ($Z_{(0)} \equiv 0$, $Z_{(k+1)} = 1$). Let $X$ be one more observation from uniform $[0, 1]$, $\{Z'_{(0)}, Z'_{(1)}, \cdots, Z'_{(k+1)}, Z'_{(k+2)}\}$ the new set of order statistics, and $S_i' = Z'_{(i)} - Z'_{(i-m)}$. Then,

$$\beta_k = |\tilde{\Lambda}(k) - \tilde{\Lambda}(k-1)| \leqq \textstyle\sum_{i=m}^{k} P(S_i > \alpha; S'_{(j)} \leqq \alpha; m \leqq j \leqq k+1) .$$

But

$$\begin{aligned} P(S_i > \alpha, S_j' \leqq \alpha; m \leqq j \leqq k + 1) & \\ &\leqq P(\alpha < S_i \leqq m\alpha; Z_{(i-m)} < X < Z_{(i)}) \\ &\leqq m\alpha P(\alpha < S_i \leqq m\alpha) \\ &= m\alpha P(\alpha < S_m \leqq m\alpha) \quad \text{(by exchangeability)} \\ &= m\alpha \frac{(k-1)!}{(m-1)!\,(k-1-m)!} \int_\alpha^{m\alpha} x^{m-1}(1-x)^{k-1-m}\,dx \\ &= O\left(\frac{\alpha^2}{(\log 1/\alpha)^m}\right) \quad \text{uniform in } k . \end{aligned} \tag{18}$$

Hence

$$\beta_k = O(\alpha) \qquad \text{uniform in } k , \tag{19}$$

and so using (17) and (19) we obtain

$$\eta \leqq \textstyle\sum_{|k-n|<Dn^{\frac{1}{2}}} \beta_k + 2Dn^{\frac{1}{2}} \sup_{|k-n| \leqq Dn^{\frac{1}{2}}} |\Lambda(k) - \tilde{\Lambda}(k)| = o(1) .$$

This establishes (16) and completes the proof of Theorem 1.

REMARK. Flatto's method for proving Theorem 1 was to estimate $P(\bigcup_{i=m}^{n+m-1}(S_i > \alpha))$. Our use of Corollary 1 allows us to avoid many of the analytical difficulties which he encountered.

**3. Proof of Theorem 2.** Define

$$\zeta_{ij} = 1 \quad \text{if} \quad \rho(Z_i, Z_j) \leqq a$$
$$= 0 \quad \text{otherwise} \qquad 1 \leqq i < j \leqq n\,,$$

and

$$X_j = \sum_{i<j} \zeta_{ij} \qquad 2 \leqq j \leqq n\,.$$

Note that

$$Z = \sum_{j=2}^{n} X_j\,.$$

Since $E(\zeta_{ij}) = \sum_{i=1}^{N} b(i)\pi_i$, for all $i$, $j$,

(20) $$E(Z) = [\sum_{j=2}^{n}(j-1)] \sum_{i=1}^{N} b(i)\pi(i)\,.$$

It follows then from (7) that $E(Z) \to \lambda$. Let $A_j = \{X_j = 1\}$, $\tilde{N} = \sum_{j=2}^{n} I_{(X_j=1)}$, and $p_j = P(A_j \mid Z_1, Z_2, \cdots, Z_{j-1})$, $j \geqq 2$.

Using the independence of the $\{Z_i\}$, it is not difficult to show that

(21) $$P(X_j = 1) = E(P(X_j = 1 \mid Z_j)) = (j-1) \sum_{i=1}^{N} b(i)\pi_i(1-\pi_i)^{j-2}\,.$$

Thus

(22) $$E(|Z - \tilde{N}|) \leqq \sum_{j=2}^{n} E(|X_j - I_{(X_j=1)}|) = \sum_{j=2}^{n}(j-1)[\sum_{i=1}^{N} b(i)\pi_i(1-(1-\pi_i)^{j-2})]\,.$$

The last equality follows from (21) and the fact that $X_j \geqq I_{(X_j=1)}$ with probability 1. Thus

(23) $$E(|Z - \tilde{N}|) \leqq \{\sum_{j=2}^{n}(j-1)(j-2)\}\{\sum_{i=1}^{N} b(i)\pi_i^2\} = o(1)\,.$$

The last equality follows from (6) and the observation that $\sum_{j=2}^{n}(j-1)(j-2) = O(n^3)$. To complete the proof of the theorem, we therefore need only show

(24) $$d(\tilde{N}, N(\lambda)) = o(1)\,.$$

To do this we use Corollary 1, and so we need to verify its conditions. (2) follows from (23) and the already noted fact that $E(Z) \to \lambda$. We next verify (3). Adding and subtracting the appropriate quantities leads to:

$$\begin{aligned} E(|\textstyle\sum_{j=2}^{n}(p_j - E(p_j))|) &\leqq E(|\textstyle\sum_{j=2}^{n} p_j - \sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1})|) \\ &\quad + E(|\textstyle\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \sum_{j=2}^{n} E(X_j)|) \\ &\quad + E(|\textstyle\sum_{j=2}^{n} E(X_j) - \sum_{j=2}^{n} E(p_j)|) \\ &= I_1 + I_2 + I_3\,. \end{aligned}$$

Since for all $i$, $p_i \leqq E(X_i \mid \mathscr{F}_{i-1})$ with probability 1,

$$I_1 = \sum_{j=2}^{n} E(X_j) - \sum_{j=2}^{n} E(p_j) = I_3\,.$$

Thus to establish (3) it suffices to prove $I_2 = o(1)$ and $I_3 = o(1)$. It follows from (22) and (7) that $I_3 = o(1)$. To show $I_2 = o(1)$, we argue as follows. In view of (21) it is enough to prove

$$E(|\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k|) = o(1) .$$

However,

$$\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k = \sum_{j=1}^{n-1} (n-j)(\pi_{Z_j} - E(\pi_{Z_j})) .$$

Squaring, taking expectations and using independence yields

$$\begin{aligned} E\{(\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k)^2\} &\leqq \{\sum_{j=1}^{n-1} (n-j)^2\} E(\pi_{Z_1}^2) \\ &\leqq Cn^3 \sum_{k=1}^{N} b(k)\pi_k^2 \\ &= o(1) . \end{aligned}$$

Thus $I_3 = o(1)$ and the proof of (3) is complete.

Finally we need to establish (4). Observe

$$p_j \leqq E(X_j \mid Z_1, \cdots, Z_{j-1}) = \sum_{i=1}^{j-1} \pi_{Z_i} .$$

Thus

$$\begin{aligned} E(\sum_{j=2}^{n} p_j^2) &\leqq E(\sum_{j=2}^{n} (\sum_{i=1}^{j-1} \pi_{Z_i})^2) \\ &= \sum_{j=2}^{n} E(\sum_{i=1}^{j-1} \pi_{Z_i}^2 + \sum_{\substack{k=1 \\ i \neq k}}^{j-1} \sum_{i=1}^{j-1} \pi_{Z_i}\pi_{Z_k}) \\ &\leqq \sum_{j=2}^{n} ((j-1) + 2\tbinom{j-1}{2})(\sum_{i=1}^{N} b(i)\pi_i^2) \\ &\leqq Cn^3 \sum_{i=1}^{N} b(i)\pi_i^2 \\ &= o(1) . \end{aligned}$$

This proves (4). Hence we can apply Corollary 1 and conclude that (24) is true, completing the proof of the theorem.

REMARK. Theorem 2 is slightly stronger than Sevast'yanov's result since he needed to assume that $\max_{1 \leqq k \leqq N} \pi_k = o(1/n)$.

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