Two approximation problems in function spaces

Lars Inge Hedberg[1]

0. Introduction

The first problem we shall treat is an approximation problem in the Sobolev space $W_m^q(\mathbf{R}^d)$. This space is defined as the Banach space of functions (distributions) f whose partial derivatives $D^\alpha f$ of order $|\alpha|\leqq m$ all belong to $L^q(\mathbf{R}^d)$. Let K be a closed set in $\mathbf{R}^d$. The problem is to determine the closure in $W_m^q(\mathbf{R}^d)$ of $C_0^\infty(\complement K)$ the set of smooth functions which vanish on some neighborhood of K.

The second problem is closely related to the first one by duality. It concerns approximation in L^p, $\frac{1}{p}+\frac{1}{q}=1$, on compact sets by solutions of elliptic partial differential equations of order m.

After some necessary (and well-known) preliminaries it is easy to give a condition that f has to satisfy in order to be approximable as above. We recall that $W_m^q(\mathbf{R}^d)$ is continuously imbedded in $C(\mathbf{R}^d)$ if $mq>d$, but not if $mg\leqq d$. (We assume throughout that $1<q<\infty$.) In the case $mq\leqq d$ the deviation from continuity is measured by an (m, q)-capacity which is naturally associated to the space. For a compact K this capacity is defined by

$$C_{m,q}(K) = \inf_{\varphi} \|\varphi\|_{m,q}^q,$$

where the infimum is taken over all C^∞ functions φ such that $\varphi\geqq 1$ on K, and $\|\cdot\|_{m,q}$ denotes a norm on $W_m^q(\mathbf{R}^d)$. The definition is extended to arbitrary sets E by setting

$$C_{m,q}(E) = \sup_{K\subset E} C_{m,q}(K), \quad K \text{ compact}.$$

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If a statement is true except on a set $E \subset \mathbf{R}^d$ with $C_{m,q}(E)=0$ we say that it is true (m, q)-a.e.

Now let $f \in W_m^q(\mathbf{R}^d)$ and let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of test functions such that $\lim_{n\to\infty} \|f-\varphi_n\|_{m,q}=0$. Then it is well known that there is a subsequence $\{\varphi_{n_i}\}_{i=1}^\infty$ such that $\{\varphi_{n_i}(x)\}_{i=1}^\infty$ converges (m, q)-a.e., and uniformly outside an open set with arbitrarily small (m, q)-capacity. This makes it possible to define $f(x)$ (m, q)-a.e. as $\lim_{n_i\to\infty} \varphi_{n_i}(x)$. We then say that f is strictly defined. In what follows we shall always assume that Sobolev functions are strictly defined. In particular the (distribution) partial derivatives $D^\alpha f$ of order $|\alpha|$, which belong to $W_{m-|\alpha|}^q(\mathbf{R}^d)$, are strictly defined in that space.

The following necessary condition for approximation is now obvious.

Theorem 0.1. Let $K \subset \mathbf{R}^d$ *be closed and suppose that* $f \in W_m^q(\mathbf{R}^d)$, $1<q<\infty$, *can be approximated arbitrarily closely by functions in* $C_0^\infty(\complement K)$. *Then* $f(x)=0$ *for* (m, q)*-a.e.* $x \in K$, *and* $D^\alpha f(x)=0$ *for* $(m-|\alpha|, q)$*-a.e.* $x \in K$ *for all multiindices* α *with* $|\alpha|=1, 2, \dots, m-1$.

Our problem, therefore, is to decide whether for all closed K this necessary condition for approximation is also sufficient. When this is the case we say that *K has the approximation property for* $W_m^q(\mathbf{R}^d)$.

It is possible that all closed sets have this property, but we can only prove this for $q > \max\left(\frac{d}{2}, 2-\frac{1}{d}\right)$ (Corollary 5.3). In the general case we need a weak condition on K. The precise results are formulated in Theorems 3.1, 4.1, and 5.1. These results go considerably further than earlier results in this direction due to J. C. Polking [37] and the author [23].

The problem has also been treated earlier for more general function spaces (Bessel potential spaces, Besov spaces etc.) but to the author's knowledge only when K is a $(d-1)$-dimensional smooth manifold. See J. L. Lions and E. Magenes [24], [25], [26], and H. Triebel [41].

It must also be said that our results are new only when $m \neq 1$. The case $m=1$ is much simpler because of the fact that truncations (and other contractions) operate on W_1^q. The difficulty in the general case comes from the presence of higher derivatives. It is, in fact, known that all closed K have the approximation property for $W_1^q(\mathbf{R}^d)$, $1<q<\infty$. For $q=2$ this is (in dual formulation) a spectral synthesis result of A. Beurling and J. Deny [9] (see also J. Deny [16]). For $2 \leqq q < \infty$ the result is due to V. P. Havin [19], and in the general case to T. Bagby [6]. See also the author [21; Lemma 4] for a simpler proof. A similar result for Cauchy transforms of bounded functions was proved by L. Bers [8]. Our method of proof in the present paper goes back to that paper.

Dually, the approximation problem can be stated in the following way. Let T be a distribution in $W^p_{-m}(\mathbf{R}^d)$, $1<p<\infty$, $pq=p+q$, with support in K. Can T be approximated in the Banach space $W^p_{-m}(\mathbf{R}^d)$ by measures supported in K and their derivatives?

In this formulation the problem leads directly to our second approximation problem. Let $X\subset\mathbf{R}^d$ be compact, and let $P(x, D)$ be a linar elliptic partial differential operator of order m with coefficients that are in C^∞ in a neighborhood of X. We say that $u\in\mathscr{H}(X)$ if u satisfies $P(x, D)u=0$ in some neighborhood of X, and we denote by $\mathscr{H}^p(X)$ the subspace of $L^p(X)$ consisting of functions u such that $P(x, D)u=0$ in the interior X^0. The problem is whether $\mathscr{H}(X)$ is dense in $\mathscr{H}^p(X)$. This problem is dealt with in the last section of the paper. Using the results from the earlier sections we improve the earlier results of Polking [37] and the author [23].

The case $m=1$, i.e. the Cauchy—Riemann operator, is again special, and has been treated earlier by S. O. Sinanjan [38], L. Bers [8], V. P. Havin [19], T. Bagby [6] and the author [21]. See also the survey article of M. S. Mel'nikov and S. O. Sinanjan [33].

In the next section we shall give some facts about (m, q)-capacities and the related potentials, which although known may not be well-known. Some new results about non-linear potentials are found in Section 4.

The proofs of our main results depend on an estimate given in Section 2 (Lemma 2.1), which generalizes an estimate of V. G. Maz'ja [28], and may be of some interest in itself.

1. Preliminaries

We use the abbreviated notation $\nabla^k f=\{D^\alpha f;\ |\alpha|=k\}$, and $|\nabla^k f|=\sum_{|\alpha|=k}|D^\alpha f|$. Thus the space $W^q_m(\mathbf{R}^d)$ is normed by $\|f\|_{m,q}=\sum_{k=0}^m\|\nabla^k f\|_q$.

We shall use the Bessel potential spaces $\mathscr{L}^q_s(\mathbf{R}^d)=\{J_s(f);\ f\in L^q(\mathbf{R}^d)\}$, $s\in\mathbf{R}$, where the operator $J_s=(I-\Delta)^{-s/2}$ is defined as convolution with the inverse Fourier transform G_s of $\hat{G}_s(\xi)=(1+4\pi^2|\xi|^2)^{-s/2}$. For $0<s<d$ the "Bessel kernel" G_s is a positive function which satisfies

$$(1.1)\qquad A_1|x|^{s-d}\leqq G_s(x)\leqq A_2|x|^{s-d}\quad\text{for}\quad |x|\leqq 1,$$

and tends to zero exponentially at infinity.

We write $J_{-s}(f)=f^{(s)}$, i.e. if $f\in\mathscr{L}^q_s$ we have $f=J_s(f^{(s)})=G_s*f^{(s)}$, $f^{(s)}\in L^q$. We norm $\mathscr{L}^q_s$ by $\|f\|_{s,q}=\|f^{(s)}\|_q$. When s is an integer and $1<q<\infty$ this norm is equivalent to the Sobolev space norm. For this reason we shall not distinguish between the norms of W^q_m and $\mathscr{L}^q_m$ for integral m, and by $\|\cdot\|_{m,q}$ we shall mean whichever norm that is most convenient for the moment. For the above (and other) properties of Bessel kernels and Bessel potentials we refer to A. P. Calderón [11] and N. Aronszajn and K. T. Smith [5].

We now define an (s, q)-capacity for arbitrary $s>0$ and arbitrary sets $E\subset \mathbf{R}^d$ by setting $C_{s,q}(E)=\inf_f \|f\|_{s,q}^q$, where the infimum is taken over all $f\in \mathscr{L}_s^q(\mathbf{R}^d)$ such that $f^{(s)}\geqq 0$ and $f(x)\geqq 1$ for all $x\in E$. The definition makes sense since $f(x)=\int_{\mathbf{R}^d} G_s(x-y)f^{(s)}(y)\,dy$ is defined everywhere.

When s is an integer and K is compact this definition clearly gives a capacity which is equivalent to the capacity we defined before. That this equivalence extends to all Borel (and Suslin) sets is a deeper fact which was proved by B. Fuglede [18] and N. G. Meyers [34] using Choquet's theory of capacities. In fact, for any Suslin set E we have $C_{s,q}(E)=\sup_K C_{s,q}(K)$ for compact $K\subset E$. Because of this equivalence we shall not distinguish the differently defined capacities by different letters.

Practically by the very definition of (s, q)-capacity the functions in $\mathscr{L}_s^q$ are defined (s, q)-a.e. The values of these functions agree (s, q)-a.e. with the values of the strictly defined functions defined before, according to a generalization of a theorem of H. Wallin [42] due to V. G. Maz'ja and V. P. Havin [31, Lemma 5.8]. (See also T. Sjödin [39], where Wallin's proof is generalized.) Therefore we shall not distinguish between functions in W_m^q and $\mathscr{L}_m^q$.

We also note the following Lebesgue property. If $f\in \mathscr{L}_s^q$ then

$$\lim_{\delta\to 0} |B(x, \delta)|^{-1}\int_{B(x,\delta)} |f(y)-f(x)|^q\,dy = 0$$

for (s, q)-a.e. x. ($B(x, \delta)$ denotes the ball $\{y;\ |y-x|\leqq\delta\}$ and $|B(x, \delta)|$ its volume.) Thus also $\lim_{\delta\to 0} |B(x, \delta)|^{-1}\int_{B(x,\delta)} f(y)\,dy=f(x)$ for (s, q)-a.e. x. This and other results are found in T. Bagby and W. P. Ziemer [7]. (See also Remark 2 in Section 2.)

Fuglede and Meyers also proved that $C_{s,q}$ can be given a dual definition. In fact, for all Suslin sets E

(1.2) $C_{s,q}(E)^{1/q}=\sup \mu(E)$, where the supremum is taken over all positive measures μ with support in E such that $\|J_s(\mu)\|_p\leqq 1$, $\dfrac{1}{p}+\dfrac{1}{q}=1$.

These dual extremal problems are connected in the following way: There exists a positive measure ν supported by the closure $\bar{E}$ of E such that

$$f(x) = V_{s,q}^{\nu}(x) = J_s\big((J_s(\nu))^{p-1}\big)(x) \geqq 1 \quad (s, q)\text{-a.e. on } E \tag{1.3}$$

and

$$\|f^{(s)}\|_q^q = \|J_s(\nu)\|_p^p = C_{s,q}(E). \tag{1.4}$$

For the theory of such "non-linear potentials" we refer to the papers by N. G. Meyers [34], V. G. Maz'ja and V. P. Havin [31], [32], D. R. Adams and N. G. Meyers [2], [3], other papers by these authors, and Hedberg [21].

We shall need the fact that there is a constant A independent of E such that the capacitary potential satisfies

$$V^{\nu}_{s,q}(x) \leqq A \quad \text{for all} \quad x. \tag{1.5}$$

This "boundedness principle" is due to Maz'ja and Havin [31, Theorem 3.1] and Adams and Meyers [3, Theorem 2.3].

Throughout the paper we shall use the letter A to denote various positive constants that may take different values even in the same string of estimates.

If $d-sq<0$, then $C_{s,q}(\{x\})>0$. Thus only the empty set has non-zero capacity. If $d-sq>0$, then

$$A^{-1}\delta^{d-sq} \leqq C_{s,q}\big(B(x,\delta)\big) \leqq A\delta^{d-sq}, \quad 0<\delta\leqq 1, \tag{1.6}$$

and if $d-sq=0$, then

$$A^{-1}(\log 2/\delta)^{1-q} \leqq C_{s,q}\big(B(x,\delta)\big) \leqq A(\log 2/\delta)^{1-q}, \quad 0<\delta\leqq 1. \tag{1.7}$$

For any set $E\subset\mathbf{R}^d$ we define the Hausdorff measure $\Lambda_\alpha(E)$, $\alpha>0$, by

$$\Lambda_\alpha(E) = \lim_{\varrho\to 0}\Lambda_\alpha^{(\varrho)}(E), \quad \text{where} \quad \Lambda_\alpha^{(\varrho)}(E) = \inf\Big\{\sum_i r_i^\alpha;\ E\subset\bigcup_{i=1}^\infty B(x_i,r_i),\ r_i\leqq\varrho\Big\}.$$

Then, if E is Suslin and $d-sq>0$

$$C_{s,q}(E) \leqq A\Lambda_{d-sq}^{(\infty)}(E), \tag{1.8}$$

and

$$\Lambda_{d-sq}(E)<\infty \Rightarrow C_{s,q}(E)=0. \tag{1.9}$$

See Meyers [34], and Maz'ja and Havin [31].

Let $E\subset B(x,\delta)$. In the case $d-sq=0$ we shall sometimes use the capacity $C_{s,q}\big(E;B(x,2\delta)\big)$ defined by

$$C_{s,q}\big(E;B(x,2\delta)\big)^{1/q} = \sup\{\mu(E);\ \|J_s(\mu)\|_{L^p(B(x,2\delta))}\leqq 1,\ \operatorname{supp}\mu\subset E\}. \tag{1.10}$$

It is then easily seen that

$$A^{-1} \leqq C_{s,q}\big(B(x,\delta);B(x,2\delta)\big) \leqq A, \quad 0<\delta\leqq 1. \tag{1.11}$$

For any set $E\subset\mathbf{R}^d$ we set

$$c_{s,q}(E,x,\delta) = \begin{cases} C_{s,q}\big(E\cap B(x,\delta)\big)\delta^{sq-d}, & \text{if} \quad d-sq\geqq 0\\ 1, \quad \text{if} \quad d-sq<0. \end{cases} \tag{1.12}$$

For $d=sq$ we write

$$c_{s,q}(E,x,\delta;2\delta) = C_{s,q}\big(E\cap B(x,\delta);B(x,2\delta)\big) \tag{1.13}$$

Following Meyers [36] we say that E *is* (s,q)*-thin at* x *if*

$$\int_0 c_{s,q}(E,x,\delta)^{p-1}\delta^{-1}\,d\delta<\infty. \tag{1.14}$$

Otherwise E is (s,q)-fat at x. (See also Adams and Meyers [2] and the author [21], where other definitions of (s,q)-thinness are given.) Thus, if $sq>d$, every E is (s,q)-fat at all of its points.

We shall need the following generalization of Kellogg's lemma. See [21; Theorem 6 and Corollaries].

Theorem 1.1. *If* $q>2-\frac{s}{d}$ *the subset of* $E\subset\mathbf{R}^d$ *where* E *is* (s, q)*-thin has* (s, q)*-capacity zero. In particular* $C_{s,q}(E)=0$ *if* E *is* (s, q)*-thin at all of its points.*

Whether this theorem is true for all $q>1$ is unknown to the author. The following is known, however ([21, Theorem 8]).

We say that E is uniformly (s, q)-thin on F if there is an increasing function h such that $\int_0 h(\delta)^{p-1}\delta^{-1}\,d\delta<\infty$ and $\limsup_{\delta\to 0} c_{s,q}(E, x, \delta)/h(\delta)<\infty$ for all $x\in F$.

Theorem 1.2. *Let* $1<q<\infty, s>0$. *Then any subset* F *of* $E\subset\mathbf{R}^d$ *where* E *is uniformly* (s, q)*-thin has* $C_{s,q}(F)=0$.

The following continuity property will be used in Section 6. See the author [21, Theorem 5], and Meyers [36; Theorem 3.1].

Theorem 1.3. *Let* $f\in\mathcal{L}_s^q$, $1<q<\infty$, $s>0$. *For* (s, q)*-a.e.* x_0 *the set* $\{x;\ |f(x)-f(x_0)|\geqq\varepsilon\}$ *is* (s, q)*-thin at* x_0 *for all* $\varepsilon>0$.

In Section 4 it will be convenient for us to use Riesz potentials $I_s(g)$,

$$I_s(g)(x) = \int_{\mathbf{R}^d} |x-y|^{s-d} g(y)\, dy, \quad 0 < s < d,$$

instead of the Bessel potentials $J_s(g)$.

Any function f in $W_m^q(\mathbf{R}^d)$ or $\mathcal{L}_s^q(\mathbf{R}^d)$ can be represented as a Riesz potential, $f=I_s(f^{(s)})$, where $f^{(s)}\in L^q(\mathbf{R}^d)$, but the converse is not true in general. (We have used $f^{(s)}$ to denote two different functions, but this should not create confusion.) If $g\in L^q(\mathbf{R}^d)$, then $I_s(g)\in L^{q'}(\mathbf{R}^d)$, $\frac{1}{q'}=\frac{1}{q}-\frac{s}{d}$, by Sobolev's inequality. Thus $I_s(g)$ belongs to L^q_{loc}, but not necessarily to L^q.

(s, q)-capacities, say $C'_{s,q}(\cdot)$, can be defined using Riesz potentials in exactly the same way as for Bessel potentials, if $0<sq<d$. Then

$$C'_{s,q}(E) \leqq AC_{s,q}(E)$$

for all sets E, and

$$C_{s,q}(E) \leqq AC'_{s,q}(E)$$

for all sets E contained in a fixed ball.

If $sq=d$ this definition would make the (s, q)-capacity equal to zero for all bounded sets. In this case we modify the definition by only considering sets contained in a fixed ball, and by taking norms with respect to a ball of twice the radius. With this modification

$$A^{-1}C_{s,q}(E) \leqq C'_{s,q}(E) \leqq AC_{s,q}(E).$$

In what follows we shall only use capacities in situations where $C_{s,q}$ and $C'_{s,q}$ are equivalent. Therefore we shall not hereafter take the trouble to distinguish them by different notation.

The maximal function will be denoted $M(f)$, i.e.

$$M(f)(x) = \sup_{\delta>0} |B(x,\delta)|^{-1} \int_{B(x,\delta)} |f(y)|\, dy.$$

Then, by the Hardy—Littlewood—Wiener maximal theorem.

$$\|M(f)\|_q \leqq A\|f\|_q, \quad 1<q<\infty. \tag{1.15}$$

The following elementary lemma will be used repeatedly.

Lemma 1.4. (a) *Let f be measurable. If $0<s<d$, then for all $x\in\mathbf{R}^d$ and all $\delta>0$*

$$\int_{B(x,\delta)} |x-y|^{s-d}|f(y)|\, dy \leqq A\delta^s M(f)(x).$$

(b) *If $s>0$, then for all $x\in\mathbf{R}^d$ and all $\delta>0$*

$$\int_{|y-x|\geqq\delta} |x-y|^{-s-d}|f(y)|\, dy \leqq A\delta^{-s} M(f)(x).$$

The following is a simple consequence. See Hedberg [22; Theorem 3].

Lemma 1.5. *If $f\geqq 0$ is measurable on $\mathbf{R}^d$, $0<s<d$, and $0<\theta<1$, then*

$$I_{s\theta}(f)(x) \leqq AM(f)(x)^{1-\theta} I_s(f)(x)^\theta.$$

Corollary 1.6. *Let $f\in W^q_m(\mathbf{R}^d)$, and let $1\leqq j\leqq k\leqq m$. Set $|f^{(m)}|=g$. Then (j,q)-a.e.*

$$|\nabla^{m-j} f| \leqq AI_j(g) \leqq AM(g)^{1-j/k} I_k(g)^{j/k}.$$

2. An estimate

In this section we shall give an estimate, which will be crucial for what follows, for $f(x)$ near a set where f and a certain number of its derivatives vanish.

Lemma 2.1. *Let $f\in W^q_m(\mathbf{R}^d)$, $1<q<\infty$, $m\in Z^+$, let k be an integer, $1\leqq k\leqq m$, and suppose that $\nabla^j f(x)=0 \cap (k,q)$-a.e. on a set K for all j, $0\leqq j\leqq m-k$. Then, for all balls $B(x_0,\delta)$,*

$$\int_{B(x_0,\delta)} |f(y)|^q\, dy \leqq A\frac{\delta^{(m-k+1)q}}{c_{k,q}(K,x_0,\delta)} \sum_{i=1}^k \delta^{(i-1)q} \int_{B(x_0,2\delta)} |\nabla^{m-k+i} f(y)|^q\, dy.$$

If $kq=d$, the inequality is still true if $c_{k,d/k}(K,x_0,\delta)$ is replaced by $c_{k,d/k}(K,x_0,\delta;2\delta)$. (See (1.12) and (1.13).)

Remark 1. In the case $k=m$ (i.e. $j=0$) the lemma is due to V. G. Maz'ja [28, Lemma 1]. He also showed that the estimate is sharp in a certain sense. (See also Maz'ja [29], and [30].) Maz'ja's lemma was later rediscovered by J. C. Polking [37; Lemma 2.10], and used in a context similar to the present one. Our proof follows that of Polking.

Remark 2. T. Bagby and W. P. Ziemer [7] have proved the following related result: Let $f\in W_m^q(\mathbf{R}^d)$, and let k be an integer, $1\leqq k\leqq m$. Then, for (k,q)-a.e. x there is a polynomial $P_x^{(m-k)}$ of degree $\leqq m-k$ such that as $\delta\to 0$

$$\delta^{-d}\int_{B(x,\delta)}|f(y)-P_x^{(m-k)}(y)|^q\,dy = o(\delta^{(m-k)q}).$$

For a full statement of their theorem we refer to [7]. See also Meyers [35], and C. P. Calderón, E. B. Fabes, and N. M. Rivière [13].

Remark 3. Meyers [36; Theorem 2.1] has proved that if $g\in L^q$, then

$$\int_0^\infty\Big\{\delta^{sq-d}\int_{B(x,\delta)}|g(y)|^q\,dy\Big\}^{p-1}\delta^{-1}\,d\delta<\infty$$

for (s,q)-a.e. x.
In the case $k=1$ Lemma 2.1 gives that

$$\delta^{-d-(m-1)q}\int_{B(x,\delta)}|f(y)|^q\,dy\leqq A\delta^{q-d}\int_{B(x,2\delta)}|\nabla^m f(y)|^q\,dy\cdot\frac{1}{c_{1,q}(K,x,\delta)}.$$

It follows from Meyers' theorem and the definition (1.14) of $(1,q)$-thinness that for all x such that the set K in Lemma 2.1 is $(1,q)$-fat at x we have

$$\liminf_{\delta\to 0}\delta^{-d-(m-1)q}\int_{B(x,\delta)}|f(y)|^q\,dy=0.$$

Thus the polynomial $P_x^{(m-1)}\equiv 0$ for $(1,q)$-a.e. $x\in K$ if $q>2-\frac{1}{d}$, according to Theorem 1.1.

Proof of Lemma 2.1. We prove the lemma for $kq\leqq d$, the case $kq>d$ being easier. We first let f be an arbitrary C^∞ function. Then, for all x and y in $\mathbf{R}^d$ we have by Taylor's formula

$$f(x)=P_y^{(m-k)}(x)+R_y^{(m-k)}(x),$$

where

$$P_y^{(m-k)}(x)=\sum_{j=0}^{m-k}\frac{1}{j!}\big((x-y)\cdot\nabla\big)^j f(y),$$

and

$$R_y^{(m-k)}=\frac{1}{(m-k)!}\int_0^t(t-\tau)^{m-k}(\sigma\cdot\nabla)^{m-k+1}f(y+\tau\sigma)\,d\tau.$$

Here

$$t=|x-y|,\quad\text{and}\quad\sigma=(x-y)/t.$$

Without loss of generality we set $x_0=0$. Let φ be a C^∞ function such that $\varphi(y)=1$

on $B(0,\delta,)$ $\varphi(y)=0$ off $B(0,2\delta)$, and $|\nabla^j\varphi(y)|\leqq A\delta^{-j}$ for $j\leqq m$. Let μ be a positive measure with support on $K\cap B(0,\delta)$ such that $\|J_k(\mu)\|_p=1$. Let $x\in B(0,\delta)$. We have

$$f(x)\|\mu\| = f(x)\int\varphi(y)\,d\mu(y) = \int\varphi(y)P_y^{(m-k)}(x)\,d\mu(y)+\int\varphi(y)R_y^{(m-k)}(x)\,d\mu(y)$$
$$= I_1(x)+I_2(x).$$

Here $|I_2(x)|\leqq A\|\varphi R_y^{(m-k)}(x)\|_{k,q}\|J_k(\mu)\|_p$. In order to estimate $I_2(x)$ it is sufficient to estimate $\left\|\nabla_y^k(\varphi(y)R_y^{(m-k)}(x))\right\|_q$. By Leibniz' formula and the assumption on φ this reduces to estimating $\sum_{i=0}^k\delta^{-i}\|\nabla_y^{k-i}R_y^{(m-k)}(x)\|_{q,B(0,2\delta)}$.

We first let $i=k$. We have

$$|R_y^{(m-k)}(x)| \leqq At^{m-k}\int_0^t|\nabla^{m-k+1}f(y+\tau\sigma)|\,d\tau$$
$$\leqq At^{m-k+1/p}\left\{\int_0^t|\nabla^{m-k+1}f(y+\tau\sigma)|^q\,d\tau\right\}^{1/q}.$$

Thus, using polar coordinates centered at x,

$$\int_{B(0,2\delta)}|R_y^{(m-k)}(x)|^q\,dy \leqq A\delta^{(m-k)q+q-1}\int_{B(0,2\delta)}dy\int_0^t|\nabla^{m-k+1}f(y+\tau\sigma)|^q\,d\tau$$
$$\leqq A\delta^{(m-k)q+q-1+d-1}\int_{|\sigma|=1}d\sigma\int_0^{t(\sigma)}dt\int_0^t\left|\nabla^{m-k+1}f(x-(t-\tau)\sigma)\right|^q\,d\tau$$
$$\leqq A\delta^{(m-k+1)q+d-1}\int_{B(0,2\delta)}|\nabla^{m-k+1}f(\xi)|^q|\xi-x|^{1-d}\,d\xi.$$

Integrating over $|x|<\delta$ we obtain

$$\delta^{-kq}\int_{B(0,\delta)}\|R_y^{(m-k)}(x)\|_{q,B(0,2\delta)}^q\,dx \leqq A\delta^{(m-2k+1)q+d}\int_{B(0,2\delta)}|\nabla^{m-k+1}f(\xi)|^q\,d\xi.$$

Now let $i\leqq k-1$. We have.

$$\nabla_yR_y^{(m-k)}(x) = \nabla_y\Big(f(x)-\sum_{j=0}^{m-k}\frac{1}{j!}((x-y)\cdot\nabla)^jf(y)\Big)$$
$$= -\sum_{j=0}^{m-k}\frac{1}{j!}((x-y)\cdot\nabla)^j\nabla f(y)+\sum_{j=0}^{m-k}\frac{1}{j!}\nabla_x((x-y)\cdot\nabla)^jf(y)$$
$$= -\sum_{j=0}^{m-k}\frac{1}{j!}((x-y)\cdot\nabla)^j\nabla f(y)+\sum_{j=1}^{m-k}\frac{1}{(j-1)!}((x-y)\cdot\nabla)^{j-1}\nabla f(y)$$
$$= -\frac{1}{(m-k)!}((x-y)\cdot\nabla)^{m-k}\nabla f(y).$$

It follows from Leibniz' formula that for $0\leqq i\leqq k-1$

$$|\nabla_y^{k-i}R_y^{m-k}(x)| \leqq A\sum_{j=0}^{k-i-1}|x-y|^{m-2k+1+i+j}|\nabla^{m-k+1+j}f(y)|$$
$$\leqq A\sum_{j=0}^{k-i-1}\delta^{m-2k+1+i+j}|\nabla^{m-k+1+j}f(y)|.$$

Thus

$$\sum_{i=0}^{k-1}\delta^{-iq}\|\nabla_y^{k-i}R_y^{(m-k)}(x)\|_{q,B(0,2\delta)}^q$$
$$\leqq A\delta^{(m-2k+1)q}\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\delta^{jq}\|\nabla^{m-k+1+j}f\|_{q,B(0,2\delta)}^q$$
$$\leqq A\delta^{(m-2k+1)q}\sum_{j=0}^{k-1}\delta^{jq}\|\nabla^{m-k+1+j}f\|_{q,B(0,2\delta)}^q.$$

Integrating over $|x|<\delta$ and combining with the estimate for $i=k$ we finally obtain

$$\int_{B(0,\delta)} |I_2(x)|^q\,dx \leqq A\delta^{(m-2k+1)q+d} \sum_{j=0}^{k-1} \delta^{jq} \int_{B(0,2\delta)} |\nabla^{m-k+1+j} f(x)|^q\,dx.$$

Now let $f\in W_m^q$, and suppose $\nabla^j f(x)=0$ (k, q)-a.e. on K for all j, $0\leqq j\leqq m-k$. Then there exists a sequence $\{f_n\}_1^\infty$ of C^∞ functions such that $\lim_{n\to\infty} \|f-f_n\|_{m,q}=0$, and such that $|\nabla^j f(x)-\nabla^j f_n(x)|\to 0$ uniformly for $0\leqq j\leqq m-k$, except on a set G with, say, $C_{k,q}(G)<\frac{1}{2}C_{k,q}(K\cap B(0,\delta))$. Our measure μ is now chosen with support in $(K\cap B(0,\delta))\setminus G$, with $\|\mu\|\geqq\frac{1}{2}C_{k,q}(K\cap B(0,\delta))$, and $\|J_k(\mu)\|_p=1$.

If the above Taylor expansion is applied to f_n for arbitrarily large n, we obtain that $I_1(x)$ is arbitrarily small, and the lemma follows by letting n tend to infinity.

The modification for $kq=d$ is proved in the same way since it is easily seen that what is really needed is only that $\|J_k(\mu)\|_{L^p(B(0,2\delta))}\leqq 1$.

3. The approximation property for everywhere fat sets

This section is devoted to proving the following theorem.

Theorem 3.1. *Suppose that K is compact and $(1, q)$-fat at each of its points. Then K has the approximation property for all W_m^q, $m=1, 2, \ldots$.*

Proof. Let $f\in W_m^q$, and suppose that $\nabla^k f(x)=0$ $(1, q)$-a.e. on K for $0\leqq k\leqq m-1$. (It follows that $\nabla^m f(x)=0$ Lebesgue a.e. on K). Suppose that K is as in the theorem.

We want to construct a C^∞ function ω such that $\omega(x)=1$ in a neighborhood of K and $\|f\omega\|_{m,q}$ is small. Then a suitable regularization of $f(1-\omega)$ is a C^∞ function that vanishes on a neighborhood of K and approximates f.

We decompose $\mathbf{R}^d$ into a mesh of unit cubes, whose interiors are disjoint, and we denote this mesh by $\mathcal{M}_0$. By successively decomposing each cube into 2^d equal cubes, we obtain meshes $\mathcal{M}_1, \mathcal{M}_2, \ldots$, so that $\mathcal{M}_n$ is a mesh of cubes with side 2^{-n}. The cubes in $\mathcal{M}_n$ are enumerated in an arbitrary way and denoted by Q_{ni}, $i=0, 1, 2, \ldots$. By rQ_{ni}, $r>0$, we mean the concentric cube with side $r2^{-n}$.

The definition of $(1, q)$-fatness can be formulated equivalently as

$$\sum_{n=0}^{\infty} \{C_{1,q}(K\cap B(x, 2^{-n}))2^{n(d-q)}\}^{p-1}=\infty,\quad x\in K. \tag{3.1}$$

We set $\{C_{1,q}(K\cap 5Q_{ni})2^{n(d-q)}\}^{p-1}=\lambda_{ni}$, and observe that if Q_{n0} intersects K, and Q_{ni} is adjacent to Q_{n0} (i.e. $Q_{ni}\subset 3Q_{n0}$), then for some $x_0\in K$ we have $B(x_0, 2^{-n})\subset 3Q_{n0}\subset 5Q_{ni}$, so that

$$\lambda_{ni}\geqq \{C_{1,q}(K\cap B(x_0, 2^{-n}))2^{n(d-q)}\}^{p-1}. \tag{3.2}$$

Lemma 2.1 applied to $\nabla^{m-k}f$ (the components of which belong to W^q_k) gives that for each Q_{ni}

$$\int_{Q_{ni}} |\nabla^{m-k}f|^q\,dx \leqq A\lambda_{ni}^{1-q}2^{-nkq}\int_{7Q_{ni}} |\nabla^m f|^q\,dx. \tag{3.3}$$

Using (3.1) and (3.2) we shall construct the function ω in such a way that its derivatives match the factor λ_{ni}^{1-q} in (3.3). The idea of such a weight function goes back to a construction of Ahlfors (see L. Bers [8], and also the author's papers [20] and [23]), but in the present case the construction is complicated a great deal by the fact that we have assumed no uniformity of the fatness of K. An easier construction would also be possible if we only wanted to control the first derivatives of ω. The construction of ω is the object of the following lemma.

Lemma 3.2. *Under the above assumptions there exists a C^∞ function ω with the following properties:*

(a) $\omega(x)=0$ *outside an arbitrarily prescribed neighborhood V of K;*

(b) $\omega(x)=1$ *on a neighborhood of K;*

(c) $0\leqq\omega(x)\leqq 1$;

(d) *For all x there is a Q_{ni} containing x such that*

$$|\nabla^k\omega(x)| \leqq A\lambda_{ni}2^{nk}, \quad k=1,2,\ldots; \tag{3.4}$$

(A is allowed to depend on k.)

(e) *There is a constant A, only depending on d, such that for all x*

$$\textstyle\sum_{n=0}^{\infty}\sum_i \lambda_{ni}\chi(x;\,7Q_{ni}) \leqq A, \tag{3.5}$$

where the sum is extended over only those indices i for which $\nabla\omega$ is not identically zero on Q_{ni}. ($\chi(\cdot, E)$ denotes the characteristic function of E.)

We assume the lemma for the moment, and proceed with the proof of the theorem.

$\int_{\mathbf{R}^d}|\omega f|^q dx\leqq\int_V |f|^q dx$ is clearly arbitrarily small, so it is enough to estimate $\int_{\mathbf{R}^d}|\nabla^m(\omega f)|^q dx$. Thus, by the Leibniz formula, it is enough to estimate

$$\int_{\mathbf{R}^d}|\nabla^k\omega|^q|\nabla^{m-k}f|^q\,dx \quad\text{for}\quad k=0,1,2,\ldots,m.$$

We decompose $\mathbf{R}^d$ as a disjoint union $\bigcup_{(n,i)\in I}Q'_{ni}$, where Q'_{ni} is a subset of Q_{ni} such that (3.4) holds for all $x\in Q'_{ni}$. Then, for $k=1,2,\ldots,m$, by (3.3) and (3.4)

$$\int_{\mathbf{R}^d}|\nabla^k\omega|^q|\nabla^{m-k}f|^q\,dx = \textstyle\sum_{(n,i)\in I}\displaystyle\int_{Q'_{ni}}|\nabla^k\omega|^q|\nabla^{m-k}f|^q\,dx$$
$$\leqq A\textstyle\sum'_{(n,i)\in I}\lambda_{ni}^q2^{nkq}\displaystyle\int_{Q'_{ni}}|\nabla^{m-k}f|^q\,dx \leqq A\textstyle\sum'_{(n,i)\in I}\lambda_{ni}\displaystyle\int_{7Q_{ni}}|\nabla^m f|^q\,dx.$$

Here $\sum'$ indicates that we sum over only those Q_{ni} where $\nabla\omega$ is not identically zero. Thus, the sum is finite, although K is covered by infinitely many cubes $7Q_{ni}$ with $(n, i)\in I$.

By (3.5) we obtain

$$\sum'_{(n,i)\in I} \lambda_{ni}\int_{7Q_{ni}} |\nabla^m f|^q\,dx = \int_{V'} \big(\sum'_{(n,i)\in I} \lambda_{ni}\chi(x; 7Q_{ni})\big)|\nabla^m f|^q\,dx$$
$$\leqq A\int_{V'} |\nabla^m f|^q\,dx,$$

where $V'\supset V$ is small if V is small.

For $k=0$ we have

$$\int_{\mathbf{R}^d} |\omega\nabla^m f|^q\,dx \leqq \int_V |\nabla^m f|^q\,dx.$$

Since $\nabla^m f(x)=0$ a.e. on K the right hand side in these inequalities is arbitrarily small, and the theorem follows.

Proof of Lemma 3.2: Before constructing the function ω we make some preliminary observations.

Let $x_0\in K$, and let $\{Q_{n0}\}_{n=0}^\infty$, $Q_{n0}\in\mathcal{M}_n$, be a sequence of closed cubes that contain x_0. There is some arbitrariness in the choice only if x_0 belongs to the boundary of some of the cubes. Consider the sequence $\{3Q_{n0}\}_0^\infty$.

Set $\underline{\lambda}_n=\min\{\lambda_{ni};\, Q_{ni}\subset 3Q_{n0}\}$. It follows from (3.2) and (3.1) that $\sum\underline{\lambda}_n=\infty$. Set $\bar{\lambda}_n=\max\{\lambda_{ni};\, Q_{ni}\subset 3Q_{n0}\}$. If $Q_{ni}\subset 3Q_{n0}$ we have

$$\lambda_{ni} = \{C_{1,q}(K\cap 5Q_{ni})2^{n(d-q)}\}^{p-1} \geqq \{C_{1,q}(K\cap 5Q_{n+1,j})2^{(n+1)(d-q)}\}^{p-1}2^{-(d-q)(p-1)}$$

for all $Q_{n+1,j}\subset 3Q_{n+1,0}$. Thus,

$$\underline{\lambda}_n \geqq M^{-1}\bar{\lambda}_{n+1}, \tag{3.6}$$

where $M=2^{(d-q)(p-1)}$.

Now to the actual construction. For each Q_{ni} we define λ^*_{ni} by

$$\lambda^*_{ni}2^n = \max_{m\leqq n}\{\lambda_{mj}2^m;\, Q_{mj}\supset Q_{ni}\}.$$

We set

$$\varrho_n(x) = \min_i\left\{\lambda_{ni}2^n;\, x\in\frac{3}{2}Q_{ni}\right\}.$$

Thus $\varrho_n(x)\leqq\lambda_{ni}2^n$ for $x\in\frac{3}{2}Q_{ni}$. It follows that if $\varphi\geqq 0$ has support in $B(0, 2^{-n-2})$ and $\int\varphi\,dx=1$, then

$$(\varrho_n*\varphi)(x) \leqq \lambda_{ni}2^n \quad\text{for}\quad x\in Q_{ni}. \tag{3.7}$$

We denote by G_n the union of Q_{ni} such that

$$\lambda_{ni} > \frac{1}{2}M^{-1}\lambda^*_{ni}, \quad (\lambda_{0i}>0 \text{ for } n=0) \tag{3.8}$$

and we set

(3.9) $$G_n' = \{x\in G_n;\ \mathrm{dist}(x, \partial G_n) \geqq 2^{-n-2}\}.$$

We define a function ω_0 by setting

$$\omega_0(x) = 0 \quad \text{for} \quad x\notin G_0',$$

$$\omega_0(x) = \min\left\{1, \inf\int_{\gamma(x)} \varrho_0(t)\,|dt|\right\} \quad \text{for} \quad x\in G_0',$$

where the infimum is taken over all paths $\gamma(x)$ that join $\partial G_0'$ to x. ω_0 is clearly Lipschitz, and $|\nabla\omega_0(x)|\leqq\varrho_0(x)$.

Let $\varphi\geqq 0$ be a C^∞ function with support in the unit ball such that $\int\varphi(x)\,dx=1$. Set $\varphi_n(x)=2^{nd}\varphi(2^n x)$, $n=1, 2, \ldots$. We observe that the convolution $\varphi_n*\varphi_{n+1}*\ldots*\varphi_{n+m}$ has its support in $B(0, 2^{-n+1})$ for all m.

We regularize ω_0 by setting $\tilde\omega_0=\omega_0*\varphi_3$. It follows from (3.7) that

$$|\nabla\tilde\omega_0*\varphi_4*\ldots*\varphi_l(x)| \leqq \lambda_{0i} \quad \text{for} \quad x\in Q_{0i} \quad \text{and for all } l,$$

and that for all k

$$|\nabla^k(\tilde\omega_0*\varphi_4*\ldots*\varphi_l)(x)| \leqq |\nabla\omega_0*\nabla^{k-1}\varphi_3*\varphi_4*\ldots*\varphi_l(x)| \leqq A\lambda_{0i}$$

for $x\in Q_{0i}$ and all l. Here A is allowed to depend on k.

We now assume that ω_m and $\tilde\omega_m=\omega_m*\varphi_{m+3}$ have been defined for $m=1, 2, \ldots, n-1$. We define ω_n by setting

$$\omega_n(x) = \tilde\omega_{n-1}(x) \quad \text{for} \quad x\notin G_n',$$

and

$$\omega_n(x) = \min\left\{1, \inf(\tilde\omega_{n-1}(y))+\int_{\gamma(y,x)} \max_{m\leqq n} \varrho_m(t)\,|dt|\right\}, \quad \text{for} \quad x\in G_n',$$

where the infimum is taken over all $y\in\partial G_n'$ and over all paths $\gamma(y, x)$ joining y to x. We then set $\tilde\omega_n=\omega_n*\varphi_{n+3}$.

We assume that $\tilde\omega_{n-1}$ has the following property: Suppose $m\leqq n-1$ and let $Q_{mi}\subset G_m$. Then for all $x\in Q_{mi}\setminus\left(\bigcup_{m+1}^{n-1} G_j\right)$

(3.10) $$|\nabla\tilde\omega_{n-1}*\varphi_{n+3}*\ldots*\varphi_{n+l}(x)| \leqq \lambda_{mi}^* 2^m \quad \text{for all } l,$$

and

(3.11) $$|\nabla^k\tilde\omega_{n-1}*\varphi_{n+3}*\ldots*\varphi_{n+l}(x)| \leqq A\lambda_{mi}^*\, 2^{mk} \quad \text{for all } k \text{ and } l,$$

where A is allowed to depend on k.

We claim that $\tilde\omega_n$ has the same property. Let $Q_{ni}\subset G_n$. On G_n' we have $\nabla\omega_n(x)\leqq \max_{m\leqq n}\varrho_m(x)$, and outside G_n' we have $\nabla\omega_n(x)=\nabla\tilde\omega_{n-1}(x)$. It follows easily from (3.7) and (3.10) that

$$|\nabla\tilde\omega_n*\varphi_{n+4}*\ldots*\varphi_{n+l}(x)| \leqq \lambda_{ni}^*\, 2^n,$$

and

$$|\nabla^k(\tilde{\omega}_n * \varphi_{n+4} * \dots * \varphi_{n+l})(x)| \leqq |\nabla\omega_n * \nabla^{k-1}\varphi_{n+3} * \varphi_{n+4} * \dots * \varphi_{n+l}(x)|$$
$$\leqq A\lambda^*_{ni}2^{nk}, \quad \text{for} \quad x \in Q_{ni}.$$

For $x \notin G_n$ we have $\operatorname{dist}(x, G'_n) \geqq 2^{-n-2}$. Thus $\tilde{\omega}_n(x) = \tilde{\omega}_{n-1} * \varphi_{n+3}(x)$, and $\nabla^k\tilde{\omega}_n = \nabla^k\tilde{\omega}_{n-1} * \varphi_{n+3}$. The claim follows from (3.10) and (3.11).

Let $Q_{mi} \subset G_m$, $x \in Q_{mi} \setminus \bigcup_{m+1}^{\infty} G_j$. By (3.8) we have

$$(3.12) \qquad |\nabla^k\tilde{\omega}_n(x)| \leqq AM\lambda_{mi}2^{mk} \quad \text{for all} \quad n \geqq m.$$

We claim that $\tilde{\omega}_n(x) = 1$ in a neighborhood of K if n is sufficiently large. Consider again $x_0 \in K$ and the sequence $\{Q_{n0}\}_{n=0}^{\infty}$ of cubes containing x_0.

Let $\{\bar{\lambda}_{n_\nu}2^{n_\nu}\}_{\nu=0}^{\infty}$ be the sequence of succesive maxima of $\{\bar{\lambda}_n 2^n\}_0^{\infty}$, i.e. $\bar{\lambda}_n 2^n < \bar{\lambda}_{n_\nu}2^{n_\nu}$ for $n < n_\nu$, $\bar{\lambda}_n 2^n \leqq \bar{\lambda}_{n_\nu}2^{n_\nu}$ for $n_\nu \leqq n < n_{\nu+1}$, $\bar{\lambda}_{n_\nu}2^{n_\nu} < \bar{\lambda}_{n_{\nu+1}}2^{n_{\nu+1}}$.

Then $\sum_{n_\nu+1}^{n_{\nu+1}-1}\bar{\lambda}_n \leqq \bar{\lambda}_{n_\nu}2^{n_\nu}\sum_{n_\nu+1}^{\infty}2^{-n} = \bar{\lambda}_{n_\nu}$, so that $\sum_{n=0}^{\infty}\bar{\lambda}_n \leqq 2\sum_{\nu=0}^{\infty}\bar{\lambda}_{n_\nu}$, which implies that the last series diverges. It follows from (3.6) that also $\sum_{\nu=0}^{\infty}\lambda_{n_\nu-1} = \infty$.

Moreover, (3.6) implies that $3Q_{n_\nu-1,0} \subset G_{n_\nu-1}$. In fact $\underline{\lambda}_{n_\nu-1} \geqq M^{-1}\bar{\lambda}_{n_\nu}$ by (3.6) and $\bar{\lambda}_{n_\nu}2^{n_\nu} > \lambda^*_{n_\nu-1,i}2^{n_\nu-1}$ for all i such that $Q_{n_\nu-1,i} \subset 3Q_{n_\nu-1,0}$. Thus $\lambda_{n_\nu-1,i} > \frac{1}{2}M^{-1}\lambda^*_{n_\nu-1,i}$ for these i, which is (3.8).

Thus $\frac{5}{2}Q_{n_\nu-1,0} \subset G'_{n_\nu-1}$. Since $3Q_{n,0} \subset 2Q_{n-1,0}$, it follows that the distance from $3Q_{n_\nu,0}$ to $\partial G'_{n_\nu-1}$ is at least $2^{-n_\nu-1}$. Thus, if $x \in 3Q_{n_\nu,0}$ and $y \in \partial G'_{n_\nu-1}$, we have $\int_{\gamma(y,x)}\max_{m \leqq n_\nu-1}\varrho_m(t)|dt| \geqq 2^{-n_\nu-1}\underline{\lambda}_{n_\nu-1}2^{n_\nu-1} = \frac{1}{4}\underline{\lambda}_{n_\nu-1}$. If $x \in Q_{n_\nu,0}$ the integral is $\geqq \frac{3}{4}\underline{\lambda}_{n_\nu-1}$. Consequently, if $\tilde{\omega}_{n_\nu-2}(x) \geqq L$ on $3Q_{n_\nu-1,0}$, it follows that $\omega_{n_\nu-1}(x) \geqq L + \frac{1}{4}\underline{\lambda}_{n_\nu-1}$ on $3Q_{n_\nu,0}$, and that the convolutions $\omega_{n_\nu-1} * \varphi_{n_\nu+2} * \dots * \varphi_{n_\nu+l}$ satisfy the same inequality, as long as $L + \frac{3}{4}\underline{\lambda}_{n_\nu-1} \leqq 1$. In any case, the divergence of $\sum_{\nu=0}^{\infty}\underline{\lambda}_{n_\nu-1}$ implies by induction that $\tilde{\omega}_n(x) = 1$ in a neighborhood of x_0 for sufficiently large n. It follows from the compactness of K that ultimately $\tilde{\omega}_n(x) = 1$ in a neighborhood of K.

We set $\omega = \tilde{\omega}_n$ for some sufficiently large n. It is clear that by starting the construction from $\mathcal{M}_{n_0}$ for some large n_0 instead of from $\mathcal{M}_0$ we can construct ω with support in an arbitrary neighborhood of K.

All that remains to prove now is (e). Let x be arbitrary and let $N(x) = N$ be the largest index n that appears in the sum in (3.5). Let x_0 be the point in K that is nearest to x, and let $x_0 \in Q_{n0}$, $n = 0, 1, \dots,$ as before.

Suppose $x \in 7Q_{ni}$. For each n there are only A_d such cubes, where A_d only depends on d, so that $\sum_i \chi(x, 7Q_{ni}) \leqq A_d$. Moreover, if $\lambda_{ni} > 0$ the cube $5Q_{ni}$ intersects K, so that $5Q_{ni} \subset AQ_{n0}$ for some A. It follows that $\lambda_{ni} \leqq A\bar{\lambda}_{n-n_0}$ for some A and n_0, and hence that $\sum_i \lambda_{ni}\chi(x, 7Q_{ni}) \leqq A\bar{\lambda}_{n-n_0}$.

On the other hand $\sum_{n=0}^{N}\bar{\lambda}_n \leqq A\omega(x_0) = A$ by the construction above. Since λ_{ni} is always bounded by a fixed constant (3.5) follows.

4. The approximation property for sets with zero capacity

Theorem 4.1. *Suppose that K is compact, and that $C_{k-1,q}(K)=0$ for some integer k, $2 \leqq k \leqq m$. Then K has the approximation property for $W_m^q(\mathbf{R}^d)$ if $\liminf_{\delta\to 0} c_{k,q}(K, x, \delta)>0$ for all $x \in K$ (thus in particular if $kq>d$). In the case $kq=d$ the result is true with $c_{k,q}(K, x, \delta)$ replaced by $c_{k,q}(K, x, \delta; 2\delta)$.*

The plan of the proof is the following: We assume that $f \in W_m^q$, and that $f(x)=\nabla f(x)=\ldots=\nabla^{m-k}f(x)=0$ (k, q)-a.e. on K. (Note that the higher derivatives, $\nabla^{m-k+i}f(x)$, $i=1, 2, \ldots$, automatically vanish $(k-i, q)$-a.e. on K, since $C_{k-i,q}(K)=0$.) Again we shall estimate $\|f\omega\|_{m,q}$ where the function ω equals 1 in a neighborhood of K and this time is such that $\|\omega\|_{k-1,q}$ is small.

ω will be constructed by modifying a non-linear potential, and the additional information we need about such potentials will be given in a series of lemmas.

The information we need about f is contained in Lemma 2.1, and in the following lemma.

Lemma 4.2. a) *Let $f \in \mathscr{L}_s^q(\mathbf{R}^d)$, where $1<q<\infty$, $s>0$, and $sq \leqq d$. Let E_ε denote the set of points x where*

$$M_q(f)(x) = \sup_{r>0} \left\{r^{-d} \int_{B(x,r)} |f(y)|^q \, dy\right\}^{1/q} > 1/\varepsilon.$$

Then $C_{s,q}(E_\varepsilon) \leqq A\varepsilon^q \|f\|_{s,q}^q$.

b) *Let $f \in \mathscr{L}_{s-t}^q(\mathbf{R}^d)$, where $1<q<\infty$, $0<t<s$, and $sq \leqq d$. Let E_ε denote the set of points x where*

$$M_{t,q}(f)(x) = \sup_{r>0} r^t \left\{r^{-d} \int_{B(x,r)} |f(y)|^q \, dy\right\}^{1/q} > 1/\varepsilon.$$

Then $C_{s,q}(E_\varepsilon) \leqq A\Lambda_{d-sq}^{(\infty)}(E_\varepsilon) \leqq A\varepsilon^q \|f\|_{s-t,q}^q$.

The lemma is contained (somewhat implicitly) in the papers of A. P. Calderón and A. Zygmund [12; Theorem 4, p. 175, and 195—197] for $t>0$, and T. Bagby and W. P. Ziemer [7; Theorem 3.1 (c), p. 136] for $t=0$. For the reader's convenience we prove the lemma here.

Proof of a). We have $f=J_s(f^{(s)})$, $f^{(s)} \in L^q$. It is no loss of generality to assume that $f^{(s)} \geqq 0$.

Suppose that $r^{-d} \int_{B(x,r)} f(y)^q \, dy > \varepsilon^{-q}$.

Then, either $r^{-d} \int_{B(x,r)} dy \left\{\int_{B(x,2r)} G_s(z-y) f^{(s)}(z) \, dz\right\}^q \geqq A^{-1}\varepsilon^{-q}$, or else $\int_{\mathbf{R}^d} G_s(z-y) f^{(s)}(z) \, dz \geqq A^{-1}\varepsilon^{-1}$ for all $y \in B(x, r)$.

In fact, for any $y_0 \in B(x, r)$ we have

$$\int_{|z-x|\geq 2r} G_s(z-y_0) f^{(s)}(z)\,dz \leq A \inf_{y\in B(x,r)} \int_{|z-x|\geq 2r} G_s(z-y) f^{(s)}(z)\,dz.$$

But for any $y \in B(x, r)$ we have by Lemma 1.4 and (1.1)

$$\int_{B(x,2r)} G_s(z-y) f^{(s)}(z)\,dz \leq AM(f^{(s)})(y) r^s.$$

Thus, either

$$r^{sq-d} \int_{B(x,r)} M(f^{(s)})^q\,dy \geq A^{-1}\varepsilon^{-q},$$

or $J_s(f^{(s)})(y) \geq A^{-1}\varepsilon^{-1}$ on $B(x, r)$.

By definition a union U_1 of balls where the second alternative holds has $C_{s,q}(U_1) \leq A\varepsilon^q \|f\|_{s,q}^q$. If $d > sq$ any union U_2 of disjoint balls such that the first alternative holds has $C_{s,q}(U_2) \leq A\Lambda_{d-sq}^{(\infty)}(U_2) \leq A\varepsilon^q \int M(f^{(s)})^q\,dy \leq A\varepsilon^q \int (f^{(s)})^q\,dy = A\varepsilon^q \|f\|_{s,q}^q$, by (1.8) and (1.15). If $d = sq$ the first alternative is impossible if ε is small enough. An application of a well-known covering lemma finishes the proof. (See e.g. Stein [40; Lemma I. 1.6], see also Bagby and Ziemer [7; Lemma 3.2].)

Part b of Lemma 4.2 is a consequence of the following lemma. (Notation as in Lemma 4.2).

Lemma 4.3. *Let* $f \in \mathcal{L}_{s-t}^q(\mathbf{R}^d)$, $1 < q < \infty$, $0 < t < s$, $sq \leq d$. *Then* $M_{t,q}(f)(x) \leq AM_{s,q}(f^{(s-t)})(x)$.

Proof. We set $x = 0$ and assume that $f^{(s-t)} \geq 0$. For $|z| \leq r$ we obtain

$$\int_{|y|\geq 2r} G_{s-t}(y-z) f^{(s-t)}(y)\,dy \leq A \int_{|y|\geq 2r} |y-z|^{s-t-d} f^{(s-t)}(y)\,dy$$

$$\leq A \int_{|y|\geq r} |y|^{s-t-d} f^{(s-t)}(y)\,dy = A \sum_{n=1}^{\infty} \int_{r2^{n-1}\leq |y| < r2^n} |y|^{s-t-d} f^{(s-t)}(y)\,dy$$

$$\leq A \sum_{n=1}^{\infty} (r2^{n-1})^{s-t-d} \left\{\int_{|y|\leq r2^n} (f^{(s-t)})^q\,dy\right\}^{1/q} (r2^n)^{d/p}$$

$$\leq AM_{s,q}(f^{(s-t)})(0) r^{-t} \sum_{n=1}^{\infty} 2^{-nt} = AM_{s,q}(f^{(s-t)})(0) r^{-t}.$$

On the other hand, for the same values of z we have by Lemma 1.4

$$\int_{|y|\leq 2r} G_{s-t}(y-z) f^{(s-t)}(y)\,dy \leq A \int_{|y|\leq 2r} |y-z|^{s-t-d} f^{(s-t)}(y)\,dy \leq AM(f^{(s-t)})(z) r^{s-t},$$

where $M(f^{(s-t)})$ here denotes the maximal function of the restriction of $f^{(s-t)}$ to the ball $B(0, 2r)$.

Thus, for all $r > 0$,

$$r^{tq-d} \int_{|z|\leq r} (G_{s-t} * f^{(s-t)}(z))^q\,dz$$

$$\leq AM_{s,q}(f^{(s-t)})(0)^q + Ar^{sq-d} \int_{|z|\leq r} M(f^{(s-t)})^q\,dz$$

$$\leq AM_{s,q}(f^{(s-t)})(0)^q + Ar^{sq-d} \int_{|z|\leq 2r} (f^{(s-t)})^q\,dz \leq AM_{s,q}(f^{(s-t)})(0)^q.$$

Here the second inequality follows from (1.15).

Lemma 4.2 now follows, because it is easily seen that the set G_ε where $M_{s,q}(g)>1/\varepsilon$, $g\in L^q$, has $\Lambda^{(\infty)}_{d-sq}(G_\varepsilon)<A\varepsilon^q\|g\|_q^q$.

We now turn to the function ω. In section 3 we defined meshes $\mathcal{M}_n$ of cubes Q with side 2^{-n}. According to a well-known lemma of H. Whitney (see e.g. Stein [40; Theorem 1.3]) the complement $\complement K$ is a union of cubes Q with disjoint interiors, such that each Q belongs to some $\mathcal{M}_n$, and such that for each Q

$$\operatorname{diam} Q \leqq \operatorname{dist}(Q, K) \leqq 4 \operatorname{diam} Q.$$

We choose such a covering of $\complement K$. In what follows the cubes in this covering will be called Whitney cubes with respect to K.

For technical reasons it will be more convenient to prove the following lemmas for Riesz potentials than for Bessel potentials.

Lemma 4.4. *Let $V^\nu_{s,q}=I_s(g)$, $g=(I_s(\nu))^{p-1}$, where ν is a positive measure with compact support, $0<s<d$, and $1<q<\infty$. Let Q be a Whitney cube with respect to supp ν with side 2^{-n}. Then $V^\nu_{s,q}$ has the following properties.*

a) *For $0\leqq j<s$ and $x\in Q$*

$$|\nabla^j V^\nu_{s,q}(x)| \leqq A I_{s-j}(g)(x).$$

b) *For any x and y in Q*

$$A^{-1}I_{s-j}(g)(y) \leqq I_{s-j}(g)(x) \leqq A I_{s-j}(g)(y)$$

(the Harnack property).

c) *For all integers j and for all $x\in Q$*

$$|\nabla^j V^\nu_{s,q}(x)| \leqq A 2^{jn} V^\nu_{s,q}(x).$$

d) *There is a function $h\geqq 0$ with*

$$\|h\|_q \leqq A\|g\|_q,$$

such that for all $j\geqq s$ and $x\in Q$

$$|\nabla^j V^\nu_{s,q}(x)| \leqq A 2^{(j-s)n} h(x),$$

and for all x and y in Q

$$A^{-1}h(y) \leqq h(x) \leqq A h(y).$$

Proof. (a) follows immediatelly from the fact that $\left|\nabla^j |x|^{s-d}\right| \leqq A|x|^{s-j-d}$.

We prove (b) by proving that for any α and β, $0<\alpha$, $\beta<d$, $V(x)=\int |x-y|^{\beta-d}\{\int |y-z|^{\alpha-d}\,d\nu(z)\}^{p-1}dy$ has the Harnack property, $A^{-1}V(y)\leqq V(x)\leqq AV(y)$ for x and y in a Whitney cube Q. Essentially the same result was proved by Adams and Meyers [2; Theorem 6.1] and the author [21; p. 305], but we include a proof here for the sake of completeness.

Let $x=0$, and suppose $\operatorname{dist}(0, \operatorname{supp} \nu)=\delta>0$. It is enough to prove that $V(y) \leqq AV(0)$ for $|y| \leqq \frac{1}{4}\delta$. Set $(I_\alpha(\nu))^{p-1}=g$. Then

$$V(y)=\int_{|t| \leqq (3/8)\delta}+\int_{|t| \geqq (3/8)\delta} |y-t|^{\beta-d} g(t)\, dt.$$

For $|t| \geqq \frac{3}{8}\delta$ we have $|y| \leqq \frac{1}{4}\delta \leqq \frac{2}{3}|t|$, $|y-t| \geqq |t|-|y| \geqq \frac{1}{3}|t|$. Thus $\int_{|t| \geqq (3/8)\delta} |y-t|^{\beta-d} g(t)\, dt \leqq A \int_{|t| \geqq (3/8)\delta} |t|^{\beta-d} g(t)\, dt \leqq AV(0)$. On the other hand,

$$\int_{|t| \leqq (3/8)\delta} |y-t|^{\beta-d} g(t)\, dt=\int_{|y-\tau| \leqq (3/8)\delta} |\tau|^{\beta-d} g(y-\tau)\, d\tau$$

$$\leqq \int_{|\tau| \leqq (5/8)\delta} |\tau|^{\beta-d}\left\{\int_{|z| \geqq \delta} |y-\tau-z|^{\alpha-d}\, d\nu(z)\right\}^{p-1} d\tau.$$

For $|\tau| \leqq \frac{5}{8}\delta$ we have $|z+\tau| \geqq |z|-|\tau| \geqq |z|-\frac{5}{8}|z|=\frac{3}{8}|z|$, $|y| \leqq \frac{1}{4}|z| \leqq \frac{2}{3}|z+\tau|$, and thus $|y-\tau-z| \geqq |z+\tau|-|y| \geqq \frac{1}{3}|z+\tau|$.

Thus

$$\int_{|\tau| \leqq (5/8)\delta} |\tau|^{\beta-d}\left\{\int_{|z| \geqq \delta} |y-\tau-z|^{\alpha-d}\, d\nu(z)\right\}^{p-1} d\tau$$

$$\leqq A \int_{|\tau| \leqq (5/8)\delta} |\tau|^{\beta-d}\left\{\int_{|z| \geqq \delta} |\tau+z|^{\alpha-d}\, d\nu(z)\right\}^{p-1} d\tau$$

$$\leqq A \int_{|\tau| \leqq (5/8)\delta} |\tau|^{\beta-d} g(\tau)\, d\tau \leqq AV(0), \quad \text{which proves (b).}$$

Now let j be arbitrary, let $|y| \leqq \frac{1}{8}\delta$, and consider $\nabla^j I_s(g)(y)$. We split the kernel $|x|^{s-d}$ by setting $|x|^{s-d}=R_1(x)+R_2(x)$, where $R_2 \in C^\infty$, and

$$R_1(x)=|x|^{s-d} \quad \text{for} \quad |x| \leqq \frac{1}{2}\delta;$$

$$R_1(x)=0 \quad \text{for} \quad |x| \geqq \frac{3}{4}\delta;$$

$$|\nabla^j R_1(x)| \leqq A\delta^{s-j-d} \quad \text{for} \quad \frac{1}{2}\delta \leqq |x| \leqq \frac{3}{4}\delta.$$

We have $\nabla^j(R_1 * g)(y)=(R_1 * \nabla^j g)(y)$

$$=\int R_1(\tau) \nabla^j g(y-\tau)\, d\tau=\int_{|\tau| \leqq 3\delta/4} R_1(\tau) \nabla_y^j\left\{\int_{|z| \geqq \delta} |y-\tau-z|^{s-d}\, d\nu(z)\right\}^{p-1} d\tau.$$

Now $|y-\tau-z| \geqq |z|-|y|-|\tau| \geqq \delta-\frac{1}{8}\delta-\frac{3}{4}\delta=\frac{1}{8}\delta$, and thus $\left|\nabla_y^j |y-\tau-z|^{s-d}\right| \leqq A\delta^{-j}|y-\tau-z|^{s-d}$ for all j. Thus

$$|\nabla^j I_s(\nu)(y-\tau)|=\left|\nabla^j \int_{|z| \geqq \delta} |y-\tau-z|^{s-d}\, d\nu(z)\right| \leqq A\delta^{-j} I_s(\nu)(y-\tau).$$

By Leibniz' formula and induction we obtain $|\nabla^j g(y-\tau)|=\left|\nabla^j\left(I_s(\nu)(y-\tau)\right)^{p-1}\right| \leqq A\delta^{-j} g(y-\tau)$, and hence

$$|\nabla^j(R_1 * g)(y)| \leqq A\delta^{-j}(R_1 * g)(y). \tag{4.1}$$

Moreover, we have $g(y-\tau)\leqq Ag(0)$, so

$$(4.2)\qquad |\nabla^j(R_1*g)(y)| \leqq A\delta^{-j}g(0)\int R_1(\tau)\,d\tau \leqq Ag(0)\delta^{s-j}.$$

On the other hand, $|\nabla^j(R_2*g)(y)|=\left|\int \nabla^j R_2(y-t)g(t)\,dt\right|\leqq A\delta^{-j}\int R_2(y-t)g(t)\,dt$. Together with (4.1) this proves (c).

But for $j>s$ we also have

$$\left|\int \nabla^j R_2(y-t)\,g(t)\,dt\right| \leqq A\int_{|y-t|\geqq(1/2)\delta}|y-t|^{s-j-d}g(t)\,dt \leqq A\delta^{s-j}M(g)(y),$$

by Lemma 1.4.

Since $\|M(g)\|_q\leqq A\|g\|_q$, and since it is easily seen that $M(g)(y)\leqq AM(g)(0)$, this proves (d) $\big($with $h=M(g)\big)$ for $j>s$.

The case $j=s$ has to be treated separately. It is easy to see that

$$|\nabla^j(R_2*g)(y)-\nabla^j(R_2*g)(0)| \leqq \int|\nabla^j R_2(y-t)-\nabla^j R_2(-t)|\,g(t)\,dt$$

$$\leqq \int_{|t|\geqq(1/2)\delta}|y||t|^{-d-1}g(t)\,dt \leqq A|y|\,\delta^{-1}M(g)(0) \leqq AM(g)(0)$$

by Lemma 1.4. According to (4.2) we have $|\nabla^j(R_1*g)(y)|\leqq Ag(0)$.

Thus $\left|\nabla^j(I_j(g))(y)\right|\leqq A\left|\nabla^j(I_j(g))(0)\right|+AM(g)(0)$. The lemma follows since $\left\|\nabla^j(I_j(g))\right\|_q\leqq A\|g\|_q$ by the theory of singular integrals.

Now let $V^{\nu}_{s,q}$, $sq\leqq d$, be the capacitary potential for a compact set F, so that $V^{\nu}_{s,q}(x)\leqq 1$ on $\operatorname{supp}\nu\subset F$. Then $V^{\nu}_{s,q}(x)\leqq A$ for all x by the boundedness principle (1.5). Let $\Phi(r)$, $r\geqq 0$, be a non-decreasing C^∞ function such that $\Phi(0)=0$, and $\Phi(r)=1$ for $r\geqq 1$. Set $\omega=\Phi\circ V^{\nu}_{s,q}$.

Lemma 4.5. *There is a function $h\geqq 0$ and constants A such that for any Whitney cube Q with respect to F with side 2^{-n}*

$$\text{(a)}\qquad \int_{\mathbf{R}^d} h(x)^q\,dx \leqq AC_{s,q}(F).$$

(If $sq=d$ the integral is taken over a fixed ball containing F.)

(b) $A^{-1}h(y)\leqq h(x)\leqq Ah(y)$ *for x and y in Q*

(c) $|\nabla^j\omega(x)|\leqq Ah(x)^{j/s}$ *for $j\leqq s$ and $x\notin F$*

(d) $|\nabla^j\omega(x)\leqq Ah(x)2^{n(j-s)}$ *for $j>s$ and $x\in Q$.*

Proof. Cf. Littman [27], and Adams and Polking [4]. Set $\psi=V^{\nu}_{s,q}=I_s(g)$, $g=I_s(\nu)^{p-1}$.

Then $\nabla\omega=\Phi'\cdot\nabla\psi$, $|\nabla^2\omega|\leqq|\Phi''|\,|\nabla\psi|^2+|\Phi'|\,|\nabla^2\psi|$, etc., $|\nabla^j\omega|\leqq A\sum_{i=1}^j|\Phi^{(i)}|\sum\prod_{l=1}^i|\nabla^{\alpha_l}\psi|$, where the last sum is taken over all i-tiples $(\alpha_1,\dots,\alpha_l)$ such that $\sum_{l=1}^i\alpha_l=j$, and all $\alpha_l\geqq 1$.

If $\alpha_l < s$ we have by Lemmas 4.4 (a) and 1.5

$$|\nabla^{\alpha_l}\psi| \leqq AI_{s-\alpha_l}(g) \leqq AM(g)^{\theta_l}\psi^{1-\theta_l}, \quad \text{where} \quad \theta_l = \frac{\alpha_l}{s}. \tag{4.4}$$

By Lemma 4.4 (c) we also have

$$|\nabla^{\alpha_l}\psi| \leqq A2^{n\alpha_l}\psi \quad \text{in} \quad Q. \tag{4.5}$$

For $\alpha_l \geqq s$ Lemma 4.4 (d) gives

$$|\nabla^{\alpha_l}\psi| \leqq A2^{n(\alpha_l - s)}h, \tag{4.6}$$

where h has the Harnack property.

Thus, for $j<s$, we find by (4.4) and (1.5)

$$|\nabla^j \omega| \leqq A \sum_{i=1}^{j} \sum_{\theta_1+\ldots+\theta_i = j/s} \prod_{l=1}^{i} M(g)^{\theta_l}\psi^{1-\theta_l} \leqq AM(g)^{j/s}\psi^{j-j/s} \leqq AM(g)^{j/s},$$

and similarly by using (4.5) and (4.6) $|\nabla^s\omega| \leqq A(M(g)+h)$ (if s is an integer), and for $j>s$ $|\nabla^j\omega| \leqq A(M(g)+h)2^{n(j-s)}$. Since both $M(g)$ and h have the Harnack property the lemma follows.

For technical reasons we shall need the following lemma.

Lemma 4.6. *Let F be compact, and let ν be a positive measure such that $V_{s,q}^{\nu}(x) = I_s(I_s(\nu)^{p-1})(x) \geqq 1$ (s, q)-a.e. on F, and $V_{s,q}^{\nu}(x) \leqq M$ everywhere. Suppose that F contains a cube Q. Then there is a constant $c>0$, independent of F and Q, such that $V_{s,q}^{\nu}(x) \geqq c$ for $x \in 2Q$.*

The lemma follows immediately from the following somewhat more general lemma.

Lemma 4.7. *Let F be compact, and let ν be a positive measure such that $V_{s,q}^{\nu}(x) = I_s(I_s(\nu)^{p-1})(x) \geqq 1$ (s, q)-a.e. on F, and $V_{s,q}^{\nu}(x) \leqq M$ everywhere. Suppose that $C_{s,q}(F \cap B(x_0, \delta))\delta^{sq-d} \geqq c > 0$ $(C_{s,q}(F\cap B(x_0,\delta); B(x_0, 2\delta)) \geqq c$ if $sq=d)$ for some $\delta>0$. Then $V_{s,q}^{\nu}(x_0) \geqq Ac^{p-1}$, where A is independent of ν, F, x_0, δ, and c.*

Proof. The proof is basically the same as that of the Wiener Criterion (Theorem 2) in [21].

Set $x_0=0$. Let σ_δ be a unit measure on $F\cap B(0,\delta) = F_\delta$, such that $\|I_s(\sigma_\delta)\|_p \leqq 2C_{s,q}(F_\delta)^{-1/q}$ (such that $\{\int_{|y|\leqq(3/2)\delta} I_s(\sigma_\delta)^p\,dx\}^{1/p} \leqq 2C_{s,q}(F_\delta; B(0,2\delta))^{-1/q}$ if $sq=d$). Such a measure exists by the dual definition of $C_{s,q}$. Then $1 \leqq \int V_{s,q}^{\nu}\,d\sigma_\delta = \int_{\mathbf{R}^d} I_s(\sigma_\delta) I_s(\nu)^{p-1}\,dy$. We denote $V_{s,q}^{\nu}(0)$ by V and assume that $V<1$. If $|y| \geqq \frac{3}{2}\delta$ we have $I_s(\sigma_\delta)(y) \leqq A|y|^{s-d}$, and thus

$$V = \int |y|^{s-d} I_s(\nu)^{p-1}\,dy \geqq A^{-1}\int_{|y|\geqq(3/2)\delta} I_s(\sigma_\delta)\, I_s(\nu)^{p-1}\,dy$$
$$\geqq A^{-1}\Big(1 - \int_{|y|\leqq(3/2)\delta} I_s(\sigma_\delta)\, I_s(\nu)^{p-1}\,dy\Big).$$

We denote the restriction of ν to $B(0, 4\delta)$ by $\nu_{4\delta}$. Using the definition of σ_δ and the boundedness of $V^\nu_{s,q}$, Hölder's inequality gives

$$\int_{|y|\leq(3/2)\delta} I_s(\sigma_\delta) I_s(\nu_{4\delta})^{p-1}\,dy \leq 2C_{s,q}(F_\delta)^{-1/q}\|I_s(\nu_{4\delta})\|_p^{p-1}$$
$$\leq 2C_{s,q}(F_\delta)^{-1/q}M^{1/q}\nu\big(B(0, 4\delta)\big)^{1/q}.$$

We want to estimate $\int_{|y|\leq(3/2)\delta} I_s(\sigma_\delta) I_s(\nu-\nu_{4\delta})^{p-1}\,dy=$ $\int d\sigma_\delta(x)\int_{|y|\leq(3/2)\delta}|x-y|^{s-d}\{\int_{|t|\geq 4\delta}|y-t|^{s-d}d\nu(t)\}^{p-1}dy$. For these x, y, and t we have $|y-t|\geq\frac{1}{3}|t-(y-x)|$, and thus $|y-t|^{s-d}\leq A|t-(y-x)|^{s-d}$. It follows that

$$\int_{|y|\leq(3/2)\delta}|x-y|^{s-d}\Big\{\int_{|t|\geq 4\delta}|y-t|^{s-d}\,d\nu(t)\Big\}^{p-1}dy$$
$$\leq A\int_{|y|\leq(3/2)\delta}|x-y|^{s-d}\Big\{\int_{\mathbf{R}^d}|t-(y-x)|^{s-d}\,d\nu(t)\Big\}^{p-1}dy$$
$$\leq A\int_{|z|\leq(5/2)\delta}|z|^{s-d}I_s(\nu)^{p-1}(z)\,dz \leq AV.$$

Thus $\int_{|y|\leq(3/2)\delta} I_s(\sigma_\delta) I_s(\nu)^{p-1}\,dy\leq AC_{s,q}(F_\delta)^{-1/q}\nu\big(B(0,4\delta)\big)^{1/q}+AV$.

But according to [21; (4), p. 303] we have for $sq\leq d$

$$V\geq A\int_0^{5\delta}\big(\nu(B(0,r))r^{sq-d}\big)^{p-1}r^{-1}\,dr \geq A\big(\nu(B(0,4\delta))\delta^{sq-d}\big)^{p-1}.$$

By assumption $C_{s,q}(F_\delta)\geq c\delta^{d-sq}$. Thus $C_{s,q}(F_\delta)^{-1/q}\nu\big(B(0,4\delta)\big)^{1/q}\leq Ac^{-1/q}V^{1/p}$, and thus

$$\int_{|y|\leq(3/2)\delta} I_s(\sigma_\delta) I_s(\nu)^{p-1}\,dy \leq A(c^{-1/q}V^{1/p}+V) \leq Ac^{-1/q}V^{1/p}.$$

Hence, either $Ac^{-1/q}V^{1/p}\geq\frac{1}{2}V^{1/p}\geq\frac{1}{2}A^{-1}c^{1/q}$, or else $AV\geq 1-Ac^{-1/q}V^{1/p}\geq\frac{1}{2}$. But since $V\leq V^{1/p}$, the last inequality gives $Ac^{-1/q}V^{1/p}\geq 1$. The lemma follows.

Proof of Theorem 4.1. K is the given compact set, $C_{k-1,q}(K)=0$ for some integer k, $2\leq k\leq m$. Let $\{Q\}$ be a Whitney covering of $\complement K$.

Let $f\in W^q_m(\mathbf{R}^d)$, and suppose that $f(x)=\nabla f(x)=\ldots=\nabla^{m-k}f(x)=0$ (k, q)-a.e. on K.

Lemma 2.1, applied to f and to $\nabla^{m-j}f$, $j=k, k+1, \ldots, m-1$, gives for a Whitney cube Q with side 2^{-n} and center x_Q

(4.7)
$$\int_Q|\nabla^{m-j}f|^q \leq Ac_{k,q}(K, x_Q, L_1 2^{-n})^{-1}2^{-(j-k+1)nq}\sum_{i=1}^k 2^{-(i-1)nq}\int_{L_2 Q}|\nabla^{m-k+i}f|^q\,dy.$$

Here L_1 and L_2 are suitable constants, only depending on d, chosen so that $L_1 2^{-n}\geq 2\,\mathrm{dist}\,(x_Q, K)$, and $L_2 Q\supset B(x_Q, L_1 2^{-n})$.

Let $\varepsilon>0$ and denote by $G'_\varepsilon=\bigcup_{n,l}Q_{nl}$ the union of all Whitney cubes Q_{nl} such that

(4.8) $$\sum_{i=1}^k 2^{nd-(i-1)nq}\int_{L_2 Q_{nl}}|\nabla^{m-k+i}f|^q\,dy > \varepsilon^{-q},$$

or

(4.9) $$2^{nd}\int_{Q_{nl}} I_{k-1}(f^{(m)})^q\,dy > \varepsilon^{-q}.$$

By Lemma 4.2 we have $C_{k-1,q}(G'_\varepsilon) < A\varepsilon^q\|f\|^q_{m,q}$. Therefore we can choose a neighborhood G_ε of K such that $G'_\varepsilon \subset G_\varepsilon$, and such that $C_{k-1,q}(G_\varepsilon) < A\varepsilon^q\|f\|^q_{m,q}$. We can also assume that $\bar{G}_\varepsilon \setminus K$ is a union of Whitney cubes.

Let ν be the $(k-1, q)$-capacitary measure for G_ε, so that $V^\nu_{k-1,q}(x) \geqq 1$ on G_ε. Let $U_\varepsilon = \bigcup(9Q_{nl})$, the union being taken over all Whitney cubes $Q_{nl} \subset G_\varepsilon$. Then $V^\nu_{k-1,q}(x) \geqq c > 0$ on U_ε by Lemma 4.6.

Now set $\omega = \Phi\circ(c^{-1}V^\nu_{k-1,q})$, where $\Phi(r)$, $r \geqq 0$, is a non-decreasing C^∞ function such that $\Phi(r) = 0$ for $0 \leqq r \leqq \frac{1}{2}$ and $\Phi(r) = 1$ for $r \geqq 1$. Thus ω has compact support and $\omega(x) = 1$ on U_ε.

Consider a Whitney cube Q contained in $\bar{G}_\varepsilon$. Then $\omega(x) = 1$ on $9Q$. Since any Whitney cube adjacent to Q has at most 4 times the side of Q, it follows that $\omega(x) = 1$ on any such cube. Thus, for a Whitney cube Q with side 2^{-n} such that $\nabla\omega(x) \not\equiv 0$ on Q, we have $\operatorname{dist}(Q, \partial G_\varepsilon) \geqq A \operatorname{dist}(Q, K) \geqq A2^{-n}$. Therefore Lemma 4.5 applies to ω and the Whitney covering of $\complement K$, although ν is supported by $\bar{G}_\varepsilon$.

We now assume for the moment that $c_{k,q}(K, x, \delta) \geqq \eta > 0$ for all $x \in K$ as soon as $\delta \leqq \delta_0$.

We have to estimate $\int_{\mathbf{R}^d} |\nabla^j\omega|^q |\nabla^{m-j}f|^q\,dx$ for all j, $0 \leqq j \leqq m$. Let Q be a Whitney cube where $\nabla\omega$ does not vanish identically.

First we consider the case $k \leqq j \leqq m$, i.e. $0 \leqq m-j \leqq m-k$. For large enough n we have by Lemma 4.5, (4.7), and (4.8)

$$\int_Q |\nabla^j\omega|^q |\nabla^{m-j}f|^q\,dx$$
$$\leqq Ah(x_Q)^q 2^{(j-k+1)nq}\eta^{-1}2^{-(j-k+1)nq}\sum_{i=1}^k 2^{-(i-1)nq}\int_{L_2Q} |\nabla^{m-k+i}f|^q\,dx$$
$$\leqq A\eta^{-1}h(x_Q)^q 2^{-nd}\varepsilon^{-q} \leqq A\eta^{-1}\varepsilon^{-q}\int_Q h(x)^q\,dx.$$

Thus

$$\int_{\mathbf{R}^d} |\nabla^j\omega|^q |\nabla^{m-j}f|^q\,dx \leqq \sum_Q \int_Q |\nabla^j\omega|^q |\nabla^{m-j}f|^q\,dx$$
$$\leqq A\eta^{-1}\varepsilon^{-q}\int_{\mathbf{R}^d} h(x)^q\,dx \leqq A\eta^{-1}\|f\|^q_{m,q}.$$

Now let $1 \leqq j \leqq k-1$. Set $j/(k-1) = \theta$. We can assume that $f^{(m)} \geqq 0$. By Lemma 4.5, Corollary 1.6 and (4.9) we have

$$\int_Q |\nabla^j\omega|^q |\nabla^{m-j}f|^q\,dx \leqq Ah(x_Q)^{q\theta}\int_Q |\nabla^{m-j}f|^q\,dx$$
$$\leqq Ah(x_Q)^{q\theta}\int_Q M(f^{(m)})^{(1-\theta)q} I_{k-1}(f^{(m)})^{\theta q}\,dx$$
$$\leqq A\big(h(x_Q)^q 2^{-nd}\big)^\theta \Big\{\int_Q M(f^{(m)})^q\,dx\Big\}^{1-\theta}\Big\{2^{nd}\int_Q I_{k-1}(f^{(m)})^q\,dx\Big\}^\theta$$
$$\leqq A\Big\{\int_Q h(x)^q\,dx\Big\}^\theta\Big\{\int_Q M(f^{(m)})^q\,dx\Big\}^{1-\theta}\varepsilon^{-q\theta}.$$

By Hölder's inequality for sums

$$\int_{\mathbf{R}^d} |\nabla^j \omega|^q |\nabla^{m-j} f|^q \, dx \leqq A \left\{ \int_{\mathbf{R}^d} h(x)^q \, dx \right\}^\theta \left\{ \int_{\mathbf{R}^d} M(f^{(m)})^q \, dx \right\}^{1-\theta} \varepsilon^{-q\theta}$$
$$\leqq A C_{k-1,q}(G_\varepsilon)^\theta \|f\|_{m,q}^{q(1-\theta)} \varepsilon^{-q\theta} \leqq A \|f\|_{m,q}^q.$$

Finally $\int_{\mathbf{R}^d} |\omega \nabla^m f|^q \, dx \leqq \int_{\operatorname{supp} \omega} |\nabla^m f|^q \, dx$ is arbitrarily small, since $\operatorname{mes} K=0$. Thus by the Leibniz formula $\int_{\mathbf{R}^d} |\nabla^m (\omega f)|^q \, dx$ is uniformly bounded, independently of ε. On the other hand, $\omega(x) f(x) \to 0$ pointwise on $\complement K$ as $\varepsilon \to 0$. By weak compactness there is a sequence $\{\omega_n\}$ such that $\{\omega_n f\}$ converges weak* in $W_m^q(\mathbf{R}^d)$. By the Banach—Saks theorem there exists a sequence of averages ω_n' such that $\{\omega_n' f\}$ converges strongly in $W_m^q(\mathbf{R}^d)$, which finishes the proof under the restriction made on K.

(Instead of using the weak compactness argument we could also use a strong type estimate of D. R. Adams [1]. His estimate implies in fact that $\liminf_{\varepsilon \to 0} \varepsilon^{-q} C_{k-1,q}(G_\varepsilon')=0$, which is all we need.)

Now assume that K satisfies only the hypothesis in the theorem. We can write $K=\bigcup_1^\infty K_n$ where $K_n=\{x \in K;\ c_{k,q}(K, x, \delta) \geqq 2^{-n} \text{ for } \delta \leqq 2^{-n}\}$. Then it is easily seen that the closure $\bar{K}_n \subset K_{n+1}$. By the above proof f can be approximated arbitrarily closely by a function that vanishes on a neighborhood of $\bar{K}_n$ for each n. By the compactness of K one of these neighborhoods is a neighborhood of K, which proves the theorem.

5. The approximation property for general sets

Putting the results from Sections 3 and 4 together we obtain the following theorem.

Theorem 5.1. *Let $K \subset \mathbf{R}^d$ be a closed set. Then K has the approximation property for W_m^q if the following conditions are satisfied.*

(a) *The subset $E_1 \subset K$ where K is $(1, q)$-this has $C_{1,q}(E_1)=0$.*

(b) *For $2 \leqq k \leqq m$ the subset $E_k \subset E_{k-1}$ where $\liminf_{\delta \to 0} c_{k,q}(K, x, \delta)=0$ $(c_{k,q}(K, x, \delta; 2\delta)$ in case $kq=d)$ has $C_{k,q}(E_k)=0$.*

Lemma 5.2. *Let $f \in W_m^q(\mathbf{R}^d)$, and let $F \subset \mathbf{R}^d$ with $C_{m,q}(F)=0$. Then for any $\varepsilon>0$ there exists a function $\omega \in W_m^q$ such that $\omega=1$ in a neighborhood of F, $f(1-\omega) \in W_m^q \cap L^\infty$, and $\|f\omega\|_{m,q}<\varepsilon$.*

Proof. We assume, without loss of generality, that f can be written $f=I_m(f^{(m)})$, $f^{(m)} \geqq 0$. Let $G_\lambda=\{x;\ f(x)>\lambda^{-1}\}$. Then G_λ is open and $C_{m,q}(G_\lambda)<A\lambda^q \|f^{(m)}\|_q^q$. There is a function ω such that $\omega(x)=1$ on G_λ, $0 \leqq \omega(x) \leqq 1$, and $\|\omega\|_{m,q}^q \leqq A C_{m,q}(G_\lambda)$.

We want to estimate $\|f\omega\|_{m,q}$. It is enough to estimate $\int |\nabla^j \omega|^q |\nabla^{m-j} f|^q \, dx$ for $0 \leqq j \leqq m$. The term for $j=0$ is easily seen to be arbitrarily small. For $0<j<m$ we use Lemma 4.5. Thus $|\nabla^j \omega(x)| \leqq Ah(x)^{j/m}$, $\|h\|_q^q \leqq AC_{m,q}(G_\lambda)$. By Corollary 1.6 we also have

$$|\nabla^{m-j} f| \leqq AI_j(f^{(m)}) \leqq AM(f^{(m)})^{1-\theta} I_m(f^{(m)})^\theta, \quad \theta = \frac{j}{m}.$$

Since $\nabla^j \omega(x)=0$ wherever $f(x)>\lambda^{-1}$ we obtain

$$\int_{\mathbf{R}^d} |\nabla^j \omega|^q |\nabla^{m-j} f|^q \, dx \leqq A\lambda^{-qj/m} \int_{\mathbf{R}^d} \left(h^{j/m} M(f^{(m)})^{1-j/m}\right)^q dx$$

$$\leqq A\lambda^{-qj/m} \left\{\int_{\mathbf{R}^d} h^q \, dx\right\}^{j/m} \left\{\int_{\mathbf{R}^d} M(f^{(m)})^q \, dx\right\}^{1-j/m}$$

$$\leqq A \|f^{(m)}\|_q^{qj/m} \|M(f^{(m)})\|_q^{q(1-j/m)} \leqq A \|f\|_{m,q}^q.$$

Again an application of weak compactness and the Banach—Saks theorem or of D. R. Adams' estimate [1] finishes the proof.

Proof of Theorem 5.1. Suppose that K satisfies the above conditions, and that $f \in W_m^q$ and $\nabla^{m-j} f(x)=0$ (j, q)-a.e. on K for $j=1, \ldots, m$. Since we can always assume that f has compact support, it is no restriction to assume that K is compact. It is clear from the proof of Theorem 3.1 that $K \setminus E_1$ is a countable union of compact sets each of which has the approximation property, and similarly it is clear from the proof of Theorem 4.1 that each of the sets $E_1 \setminus E_2, \ldots, E_{m-1} \setminus E_m$, is also a countable union of compact sets with the approximation property. Now by Lemma 5.2 f can be approximated by a function f_1 that vanishes in a neighborhood of E_m, and still satisfies the hypothesis of the theorem. Then, by Theorem 4.1, f_1 and thus f can be approximated by a function f_2 that also satisfies the hypothesis and vanishes on a neighborhood of a part of E_{m-1}, etc. By Theorem 3.1 f can be approximated by f_{m+1} that vanishes in a neighborhood of a compact part of K. The theorem now follows from the compactness of K.

The following corollary follows immediately from Theorems 5.1 and 1.1.

Corollary 5.3. *Every closed $K \subset \mathbf{R}^d$ has the approximation property for W_m^q for all m if $q > \max\left(\frac{d}{2}, 2-\frac{1}{d}\right)$.*

Remark. That the approximation property holds for $q>d$ was known before. See. J. C. Polking [37], and V. I. Burenkov [10].

Remark. If we could weaken the hypothesis (b) to requiring only that the set $E_k \subset E_{k-1}$ where K is (k, q)-thin has $C_{k,q}(E_k)=0$, it would follow that the approximation property holds for $q>2-\frac{1}{d}$ for all K. If in addition Theorem 1.1 could be extended to $1<q<\infty$ the approximation property would follow for all K and W_m^q, $1<q<\infty$.

We give another corollary that can be formulated without using capacities.

Corollary 5.4. *Let $K\subset \mathbf{R}^d$ be a closed set, and suppose that every compact subset of K has finite k-dimensional Hausdorff measure for some integer k, $1\leqq k\leqq d$. Suppose furthermore that K is sufficiently regular so that for (m, q)-a.e. $x\in K$ there exists a truncated cone $V_x\subset K$ with vertex at x such that $\Lambda_k(V_x)>0$. Then K has the approximation property for $W^q_m(\mathbf{R}^d)$, $1<q<\infty$.*

Proof. The assumption that $\Lambda_k(K\cap B(0, R))<\infty$ implies that $C_{j,q}(K)=0$ for $jq\leqq d-k$, by (1.9). Let j_0 denote the integer part of $(d-k)/q$. Then $(j_0+1)q> d-k$, $k>d-(j_0+1)q$, and it follows that $C_{j_0+1,q}(V_x)>0$. (Maz'ja and Havin [31; Theorem 7.1]). Then it is easy to prove by a homogeneity argument that $C_{j_0+1,q}(K\cap B(x,\delta))\geqq C_{j_0+1,q}(V_x\cap B(x,\delta))\geqq A\delta^{d-(j_0+1)q}C_{j_0+1,q}(V_x)$, if $d>(j_0+1)q$, for δ small enough, and that $C_{j_0+1,q}(K\cap B(x,\delta); B(x, 2\delta))\geqq$ $C_{j_0+1,q}(V_x\cap B(x,\delta); B(x, 2\delta))\geqq AC_{j_0+1,q}(V_x)$ if $d=(j_0+1)q$.

6. Approximation in L^p by solutions of elliptic partial differential equations

We first state as a theorem the dual formulation of the approximation property given in the introduction.

Theorem 6.1. *A closed set $K\subset\mathbf{R}^d$ has the approximation property for W^q_m if and only if (signed) measures with support in K and their partial derivatives are dense in $W^p_{-m}(K)$, the distributions in $W^p_{-m}(\mathbf{R}^d)$ with support on K.*

Proof. A distribution T in $W^p_{-m}(\mathbf{R}^d)$, i.e. a bounded linear functional on $W^q_m(\mathbf{R}^d)$, belongs to $W^p_{-m}(K)$ if and only if $(T,\varphi)=0$ for all C^∞ functions φ with support off K.

Denote by $L(K)$ the linear span of all distributions in $W^p_{-m}(K)$ that are measures or derivatives of measures. Suppose $f\in W^q_m(\mathbf{R}^d)$. It is easily seen that $(T,f)=0$ for all $T\in L(K)$ if and only if $\nabla^k f(x)=0$ $(m-k, q)$-a.e. on K for $k=0, 1, \ldots, m-1$.

Thus $L(K)$ and $W^p_{-m}(K)$ have the same annihilators if and only if K has the approximation property for $W^q_m(\mathbf{R}^d)$, which proves the theorem.

Now let $P(x, D)=\sum_{|\alpha|\leqq m} a_\alpha(x)D^\alpha$ be a linear elliptic partial differential operator of order m with C^∞ coefficients defined in an open set $\Omega\subset\mathbf{R}^d$. If F is relatively compact in Ω we denote by $\mathscr{H}(F)$ the set of all functions u that satisfy $P(x, D)u=0$ in some neighborhood of F. We let $1<p<\infty$, $pq=p+q$, and we set $\mathscr{H}^p(F)=\mathscr{H}(F^0)\cap L^p(F)$, i.e. the subspace of $L^p(F)$ that consists of functions u such that $P(x, D)u(x)=0$ in the interior of F.

Following Polking [37] we assume that $P(x, D)$ has a bi-regular fundamental solution $E(x, y)$ on Ω. I.e. $E(x, y)\in L^1_{\text{loc}}(\Omega\times\Omega)$, is infinitely differentiable off the diagonal in $\Omega\times\Omega$, and satisfies the equations $P(x, D)E(x, y)=\delta_x$, and ${}^tP(y, D)E(x, y)=\delta_y$.

It follows moreover that for each compact $F\subset\Omega$, and each multiindex α,

$$|D^\alpha E(x, y)| \leqq A|x-y|^{m-|\alpha|-d}, \qquad x, y\in F, \quad \text{if} \quad |\alpha|+d > m,$$

and

$$|D^\alpha E(x, y)| \leqq A_1+A_2\left|\log|x-y|\right|, \quad x, y\in F, \quad \text{if} \quad |\alpha|+d = m.$$

(See also Fernström and Polking [17] for more details.)

Let $G\subset\Omega$ be open and relatively compact. It follows from the above that if μ is a measure with compact support in $\Omega\setminus G$, such that $J_{m-k}(\mu)\in L^p(\mathbf{R}^d)$ for some $k=0, 1, \ldots, m-1$, and $1<p<\infty$, then $u(x)=\int D_y^\alpha E(x, y)\,d\mu(y)\in\mathscr{H}^p(G)$ for $|\alpha|\leqq k$. The following is an immediate consequence of Theorem 6.1.

Theorem 6.2. *$\mathscr{H}^p(G)$ is spanned by solutions of the form $u(x)=\int D_y^\alpha E(x, y)\,d\mu(y)$,* $\operatorname{supp}\mu\subset\Omega\setminus G$, *if and only if $\complement G$ has the approximation property for $W_m^q(\mathbf{R}^d)$.*

We now assume that G is the interior of a compact set $X\subset\Omega$. We ask if the measures in Theorem 6.2 can be replaced by point masses in $\Omega\setminus X$, in other words if $\mathscr{H}(X)$ is dense in $\mathscr{H}^p(X)$. That this is the case if $\complement X$ is not too fat on too big a part of ∂X is the content of the following theorems, which improve on earlier results of Polking [37] and the author [23], to which papers we refer for more information concerning the problem. In particular necessary and sufficient conditions are given in the case when X has no interior, so that $\mathscr{H}^p(X)=L^p(X)$. A related problem is solved by Fernström and Polking in [17].

Theorem 6.3. *$\mathscr{H}(X)$ is dense in $\mathscr{H}^p(X^0)$ if $\complement X^0$ has the approximation property for $W_m^q(\mathbf{R}^d)$, and if furthermore $\complement X$ is (k, q)-fat (k, q)-a.e. on ∂X for $k=1, 2, \ldots, m$.*

Theorem 6.4. *$\mathscr{H}(X)$ is dense in $\mathscr{H}^p(X^0)$ if $\complement X^0$ has the approximation property for $W_m^q(\mathbf{R}^d)$ and if furthermore there is an $\eta>0$ such that $C_{k,d}(U\setminus X)\geqq\eta C_{k,d}(U\setminus X^0)$ for $k=1, 2, \ldots, m$ and all open sets U.*

Proof. Suppose that $g\in L^q(X)$ and that $\hat g(y)=\int g(x)E(x, y)\,dx=0$ for all $y\in\Omega\setminus X$. Thus $\hat g\in W_m^q$ and $\hat g(y)$ vanishes on $\complement X$. If X satisfies either of the assumptions, it follows that $\hat g(y)$ and $\nabla^k\hat g(y)$ vanish $(m-k, q)$-a.e. on ∂X for $k=0, 1, \ldots, m-1$. In the case of Theorem 6.3 this is a consequence of Theorem 1.3, and in the case of Theorem 6.4 the result is found in [21; Theorem 11].

By the approximation property $\hat g$ can be approximated in $W_m^q(\mathbf{R}^d)$ by C^∞ functions φ with support in X^0. But if $u\in L^p(X)$ (we set $u=0$ on $\complement X$) and $P(x, D)u(x)=0$ on X^0 we have $(g, u)=({}^tP(y, D)\hat g, u)=(\hat g, P(x, D)u)=\lim_{\varphi\to\hat g}(\varphi, P(x, D)u)=0.$

It follows that u can be approximated in $L^p(X)$ by linear combinations $\sum_1^N a_i E(\cdot, y_i)$, $y_i \in \Omega \setminus X$, which proves the theorems.

Finally we apply Theorem 5.1 to obtain a result where the approximation property does not enter explicitly in the assumptions.

Theorem 6.5. *$\mathscr{H}(X)$ is dense in $\mathscr{H}^p(X)$ if $\complement X$ is $(1, q)$-fat $(1, q)$-a.e. on ∂X, and if* $\liminf_{\delta\to 0} c_{k,d}(\complement X, x, \delta) > 0$ *(k, q)-a.e. on ∂X for $k=2, \dots, m$.*
($\liminf_{\delta\to 0} c_{k,d}(\complement X, x, \delta; 2\delta) > 0$, *if $kq=d$.*)

Proof. By Theorem 5.1 the conditions imply that $\complement X$ has the approximation property. The theorem follows as before.

The question of the necessity of the above conditions is somewhat mysterious. The condition $C_{m,q}(U\setminus X) = C_{m,q}(U\setminus X^0)$ for all open U is necessary (Polking [37: Theorem 2.7]). In the case when X^0 is empty this condition is both necessary and sufficient (Polking [37; Theorem 2.6]), in particular $\mathscr{H}(X)$ is always dense in $L^p(X)$ if $mq>d$. It might be tempting to believe that $\mathscr{H}(X)$ is always dense in $\mathscr{H}^p(X)$ if $mq>d$, even if X has interior. This would be analogous to the fact that for holomorphic functions in the plane $(m=1)$ one always has density in $\mathscr{H}^p(X)$ if $p<2$ $(q>2)$, but not if $p\geqq 2$, whether or not X has an interior. However, the following example shows that the presence of an interior really complicates the situation, and that $\mathscr{H}(X)$ is dense in $\mathscr{H}^p(X)$ for all X only if $q>d$. (I am grateful to A. A. Gončar for prompting me to construct such an example.)

Example 6.6. Let $q=d$, and let $m\geqq 1$. Then there is a compact set $X\subset \mathbf{R}^d$ such that $\mathscr{H}(X)$ is not dense in $\mathscr{H}^p(X)$ for any $P(x, D)$ of order m satisfying the above conditions.

Proof. It is enough to construct a set X and a function $\varphi \in W_m^d(\mathbf{R}^d)$ such that $\operatorname{supp}\varphi \subset X$, and $\nabla^{m-1}\varphi(x) \neq 0$ on a subset of ∂X with positive $(1, d)$-capacity.

Denote the unit ball in $\mathbf{R}^d$ by B_0 and the $(d-1)$-dimensional ball $\{x\in \mathbf{R}^d;\ |x|\leqq \frac{1}{2}, x_d=0\}$ by D. We shall choose suitable disjoint balls B_k, $k=1, 2, \dots$, $B_k = \{x;\ |x-x_k|<r_k\}$, $x_k\in D$, and set $X = B_0 \setminus \left(\bigcup_{k=1}^\infty B_k\right)$.

Let $R_k > r_k$, and let $\chi_k \in C^\infty(0, \infty)$ be such that $\chi_k(r)=1$ for $0\leqq r\leqq r_k$, $\chi_k(r)=0$ for $r\geqq R_k$, $0\leqq \chi_k \leqq 1$, and $|D^j \chi_k(r)| \leqq A r^{-j} (\log R_k/r_k)^{-1}$, $1\leqq j\leqq m$. Set $\psi_k(x) = \chi_k(|x-x_k|)$, and choose a function $\varphi_0 \in C_0^\infty(B_0)$ such that $\varphi_0(x) = x_d^{m-1}$ in a neighborhood of D.

It is easily verified that $\int |\nabla^m(\varphi_0\psi_k)|^d\,dx \leqq A(\log R_k/r_k)^{1-d}$, if R_k is small enough. Now choose R_k so that $\sum_1^\infty R_k^{d-1} < 2^{1-d}$, and x_k so that the balls $\{x;\ |x-x_k|\leqq R_k\}$ are disjoint. Finally choose r_k so that $\sum_{k=1}^\infty (\log R_k/r_k)^{1-d} < \infty$, and set $\varphi = \varphi_0\left(1-\sum_1^\infty \psi_k\right)$. Clearly $\varphi \in W_m^d$, and $\operatorname{supp}\varphi \subset X$. But every $x\in D$ that is not contained in one of the balls $\{x;\ |x-x_k|\leqq R_k\}$ is a boundary point of X. On the line perpendicular to D through such a point we have $\varphi = \varphi_0$, and thus

$\partial_d^{m-1}\varphi(x)=(m-1)!$. Since the set of such points has positive $(d-1)$-dimensional measure, φ has the desired properties.

An easy modification gives the following example.

Example 6.7. Let $d=q+1$, and let $m\geqq 1$. Then there is a compact set $X\subset \mathbf{R}^d$ with connected complement which has the properties of Example 6.6.

Proof. Let $X_0\subset \mathbf{R}^{d-1}$ be the set constructed in Example 6.6, and set $X=X_0\times[0, 1]$. Let $\varphi\in W_m^{d-1}(\mathbf{R}^{d-1})$ be the function constructed in Example 6.6, and set $\Phi=\varphi\psi$, where $\psi\in C_0^\infty[0, 1]$. Then Φ has the desired properties.

Remarks added in November 1977: After this paper had already been accepted for publication I became aware of some earlier related work that deserves comment.

The problem of approximation in L^2 by solutions of elliptic equations was raised in 1961 by I. Babuška [43; Section VI] in connection with a study of the stability of the Dirichlet problem for the polyharmonic equation $\Delta^m u=0$. It is easily seen that Babuška's definition of Δ^m-stability can be formulated in the following way (See [43; Def. 5.1], and also the recent monograph by B.-W. Shulze and G. Wildenhain [44; Def. IX. 5.6].):

Let G be a bounded domain which is equal to the interior of its closure. Then *G is Δ^m-stable if every function f in $W_m^2(\mathbf{R}^d)$ that vanishes off $\bar{G}$ can be approximated in $W_m^2(\mathbf{R}^d)$ by functions in $C_0^\infty(G)$.*

Thus, as Babuška observed [43; Theorem 6.3 and Remarks] (See also Polking [37; Theorem 1.1].), *approximation in $L^2(G)$ by solutions of an elliptic equation of order m is equivalent to the Δ^m-stability of G,* and our Theorems 6.3—6.5 give sufficient conditions for Δ^m-stability. Babuška gave some geometric sufficient conditions for Δ^m-stability, and he also gave examples of a domain in $\mathbf{R}^2$ which is Δ-unstable, and a domain in $\mathbf{R}^5$ which is Δ^2-unstable. Our Example 6.6 gives a *domain in $\mathbf{R}^2$ which is Δ^m-unstable for all $m\geqq 1$.*

A necessary and sufficient condition for Δ^m-stability, expressed in terms of a different capacity, was given by È. M. Saak [45]. Let the capacity $N_{m,q}$ be defined for compact F by $N_{m,q}(F)=\inf\{\|\omega\|_{m,q}^q;\ \omega\in C_0^\infty,\ \omega(x)=1$ in a neighborhood of $F\}$, and for arbitrary E by $N_{m,q}(E)=\sup\{N_{m,q}(F);\ F\subset E,\ F$ compact$\}$. (Then it is known that $N_{m,q}(F)$ and $C_{m,q}(F)$ are equivalent in the sense that they have bounded ratios. See [32, § 5], or [4].) Then Saak's necessary and sufficient condition can be formulated as follows: *G is Δ^m-stable if and only if $N_{m,2}(B\setminus\bar{G})=N_{m,2}(B\setminus G)$ for all open balls B.* (In order to facilitate comparison we have modified his statements somewhat. Also, Saak assumes $2m<d$.

The approximation property for W_m^q studied in this paper (in its dual formulation as given in Theorem 6.1) was introduced by B. Fuglede in 1968 in the case

$q=2$ (unpublished, see [44; IX. § 5.1]). Fuglede called this property the *2m-spectral synthesis property.* He noticed that *the fine Dirichlet problem for the polyharmonic equation $\Delta^m u=0$ in a domain G has a unique solution if and only if $\complement G$ satisfies 2m-spectral synthesis.* In other words, $2m$-spectral synthesis is true for $\complement G$ if and only if every u in $W^2_m(\mathbf{R}^d)$ which satisfies $\Delta^m u=0$ in G and vanishes on $\complement G$ together with its derivatives of order up to $m-1$ (i.e. $\nabla^k u=0$ $(m-|k|, 2)$-a.e. on $\complement G$ for $|k|=0, 1, 2, \dots, m-1$), has to vanish identically.

It is proved in [44; Satz IX. 5.4] that the fine Dirichlet problem for Δ^m is uniquely solvable in G if G is Δ^m-stable, and a weaker result was given by Babuška [43; Theorem 7.3]. This is an immediate consequence of Theorem 6.1 above. Moreover, our Corollary 5.3 shows that *the fine Dirichlet problem for Δ^m is uniquely solvable in all G in $\mathbf{R}^2$ and $\mathbf{R}^3$.*

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Lars Inge Hedberg
Department of Mathematics
University of Stockholm
Box 6701
S—113 85 Stockholm