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Two-Arc Transitive Graphs and Twisted Wreath Products

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Abstract. The paper addresses a part of the problem of classifying all 2-arc transitive graphs: namely, that of finding all groups acting 2-arc transitively on finite connected graphs such that there exists a minimal normal subgroup that is nonabelian and regular on vertices. A construction is given for such groups, together with the associated graphs, in terms of the following ingredients: a nonabelian simple group T, a permutation group P acting 2-transitively on a set Ω, and a map $F : \Omega \to T$ such that $x = x^{-1}$ for all $x \in F(\Omega)$ and such that T is generated by $F(\Omega)$. Conversely we show that all such groups and graphs arise in this way. Necessary and sufficient conditions are found for the construction to yield groups that are permutation equivalent in their action on the vertices of the associated graphs (which are consequently isomorphic). The different types of groups arising are discussed and various examples given.

Keywords: graph, group, two-arc transitive graph, twisted wreath product

1. Introduction

An *s-arc*, $s \geq 1$, in a graph Γ is a sequence $(\alpha_0, \ldots, \alpha_s)$ of $s + 1$ vertices of Γ such that for all $1 \leq i \leq s$ the vertex α_{i-1} is adjacent to α_i, and for all $1 \leq i < s$ the vertices α_{i-1} and α_{i+1} are distinct. A graph Γ is said to be *s-arc transitive* if its automorphism group $Aut\Gamma$ is transitive on the set of s-arcs of Γ; a subgroup G of $Aut\Gamma$ is said to be *s-arc transitive on* Γ if G is transitive on the set of s-arcs of Γ.

Weiss [17] has shown that the only finite 8-arc transitive graphs are disjoint unions of cycles. Of fundamental importance is the easy observation that, in a group acting s-arc transitively on a graph for $s \geq 2$, the vertex stabilizer is 2-transitive on the neighbors of that vertex. As this condition is sufficient for the group to be 2-arc transitive (given also that it is vertex-transitive), the problem of classifying all finite 2-arc transitive graphs is highly attractive from the group theoretic viewpoint.

A program which attempts to do exactly this has been initiated by Praeger (see [11, 13, 15]). This starts by reducing to connected graphs and then treating the two cases, bipartite and nonbipartite, separately. In each case one proceeds via reduction theorems which state that such graphs may be obtained as covers of other 2-arc transitive graphs that satisfy certain extra conditions. (A graph Γ

is a *cover* of a graph Σ if there is a surjection θ from the vertex set of Γ to that of Σ which preserves adjacency and is such that, for each vertex α of Γ, the sets $\Gamma(\alpha)$ and $\Sigma(\theta(\alpha))$ of neighbors of α and $\theta(\alpha)$ in Γ and Σ respectively have the same size.) Indeed in the nonbipartite case we have the following.

THEOREM 1.1 **(Praeger [13, 4.2]).** *Let Γ be a finite connected graph which is not bipartite, and suppose that Γ has a group of automorphisms G which is s-arc transitive on Γ with $s \geq 2$. Then Γ is a cover of a graph Σ such that the group of automorphisms of Σ induced by G is s-arc transitive and quasiprimitive on Σ.*

(A permutation group is said to be *quasiprimitive* if each of its nontrivial normal subgroups is transitive.) The natural way to proceed then is to start by classifying those graphs Γ that admit a group of automorphisms that is quasiprimitive on vertices and transitive on 2-arcs, and then to find all 2-arc transitive covers of these. The starting point here is Theorem 1 of [13] due to Praeger that is an analog for quasiprimitive permutation groups of the O'Nan-Scott theorem for primitive permutation groups: the theorem states that a quasiprimitive permutation group is permutation equivalent to a group from one of several well-defined classes of quasiprimitive permutation groups. One of these classes is known as the quasiprimitive permutation groups of *twisted wreath type* and consists precisely of those quasiprimitive permutation groups G whose socle is regular and nonabelian, but not simple; note that the socle $\mathrm{Soc}\, G$ must then be a minimal normal subgroup of G since quasiprimitivity implies that any nontrivial normal subgroup of G is transitive and so has size at least $|\mathrm{Soc}\, G|$. In this paper we study 2-arc transitive graphs that admit such groups of automorphisms; however it will be convenient to include also other closely related groups and so we make the following definitions.

Definition 1.1. We say that (Γ, G) is a *pair* if Γ is a graph admitting G as a group of automorphisms. The pair (Γ, G) is *nice* if Γ is connected and G is transitive on the 2-arcs of Γ; the pair is *special* if, in addition, G has a minimal normal subgroup M that is both nonabelian and regular on the vertices of Γ; the pair is *QP-nice*, respectively *QP-special*, if, in addition, G is quasiprimitive on vertices.

Our intention then is to find all special pairs (Γ, G). However, as the ideal end result of Praeger's program would be to obtain a list of 2-arc transitive graphs with a single representative for every isomorphism type, we also wish to consider the situation where (Γ, G) and (Γ_1, G_1) are nice pairs with Γ and Γ_1 isomorphic as graphs. We will consider this in the context of the following definitions.

Definition 1.2. The pairs (Γ, G), (Γ_1, G_1) are *equivalent* if there exists an edge-preserving permutation equivalence between the actions of G and G_1 on the vertex sets of Γ and Γ_1 respectively. The pair (Γ, G) is *contained in* the pair (Γ_1, G_1) if (Γ, G) is equivalent to the pair (Γ_1, H) for some subgroup H of G_1.

Thus, we would like to classify all special pairs up to equivalence, and also to find all containments between special pairs. So suppose that (Γ, G) is a special pair. As the minimal normal subgroup M is nonabelian and characteristically simple, it is the direct product of isomorphic nonabelian simple groups and so defines a nonabelian simple group T up to isomorphism. Also, for a vertex α of Γ, the action of G_α on the neighbors of α gives rise to a 2-transitive permutation group P on a set Ω which is well-defined up to permutation equivalence. We say that T and P on Ω are the *invariants* of the pair (Γ, G), and it seems sensible, given the classification of the finite nonabelian simple groups and that of the 2-transitive permutation groups, to regard T and P on Ω as being fixed and then to find all special pairs with such invariants. It turns out that the following concept is the key to such special pairs.

Definition 1.3. A map $F : \Omega \to T$ is *admissible* if $F(\Omega)$ is a set of involutions in T (and we include the identity as an involution) that together generate T.

We then have the following theorem.

THEOREM 1.2. *Given a nonabelian simple group T, a 2-transitive permutation group P on Ω and an admissible map $F : \Omega \to T$, there exists (and the proof is constructive) a special pair with T and P on Ω as invariants. Conversely, all special pairs arise in this way.*

The proof of this theorem takes place in §4 and consists of Construction 4.1, which constructs a pair given the appropriate ingredients, and Theorem 4.1, which says the constructed pair is special, and that all special pairs are equivalent to one so obtained. Prior to this, in §2 and §3, we present some preliminaries on twisted wreath products and graphs.

In §5 we consider the different types of special pairs that may occur and give a criterion for determining when a special pair is QP-special. We also give an example of a special pair that is not QP-special.

In §6 we consider equivalences and containments between special pairs. Here we obtain precise criteria for equivalence in terms of a natural action of $\mathit{Aut}T \times N_{\mathrm{Sym}\Omega}(P)$ on the set of admissible maps: namely, two special pairs are equivalent if and only if their associated admissible maps lie in the same orbit in this action. We also obtain necessary and sufficient conditions on containment of special pairs.

Notation

All groups, sets, graphs, etc. are assumed to be finite; all graphs are undirected and without loops. We say that the permutation groups P on Ω and $\widehat{P}$ on $\widehat{\Omega}$ are permutation equivalent if and only if there exists a bijection $\theta : \Omega \to \widehat{\Omega}$ and an isomorphism $\chi : P \to \hat{P}$ such that

$$\theta(\omega p) = \theta(\omega)\chi(p) \qquad \text{for all } \omega \in \Omega \text{ and } p \in P.$$

Occasionally we use notation such as $\omega * p$ to denote the image of $\omega \in \Omega$ under the action of $p \in P$. The identity element of a group G is denoted id_G. We hope that the meaning of all other unexplained notation is self-evident.

2. Preliminaries on twisted wreath products

The concept of a twisted wreath product was originally due to B.H. Neumann [10]; here we generalize slightly the treatment of [16]. The ingredients for this construction are:

a group T, a group P, a subgroup Q of P,
and a homomorphism $\phi : Q \to AutT$.

Define the *complete base group* $\mathcal{B}$ to be the set of maps $f : P \to T$ with multiplication defined pointwise. There is an action of P on $\mathcal{B}$ given by

$$f^p(x) = f(px) \qquad \text{for all } x \in P,\ f \in \mathcal{B};$$

let $\mathcal{X}$ be the semidirect product $\mathcal{B} \times P$ with respect to this action. Define the *ϕ-base group* B_ϕ by

$$B_\phi = \{f : P \to T : f(pq) = f(p)^{\phi(q)} \qquad \text{for all } p \in P,\ q \in Q\}$$

and observe that B_ϕ is a subgroup of $\mathcal{B}$ normalized by P. The subgroup $X_\phi = B_\phi P$ of $\mathcal{X}$ is called the *twisted wreath product* of T by P with respect to ϕ, and we write

$$X = T twr_\phi P.$$

We refer to P as the *top group* of X_ϕ, to ϕ as the *twisting homomorphism*, to B_ϕ as simply the *base group* of X_ϕ, and to Q, the domain of ϕ, as the *twisting subgroup*. Note that $\mathcal{X}$ is itself the twisted wreath product with respect to the trivial map $\{\mathrm{id}_P\} \to AutT$, and that $B_\phi \cong T^k$ where $k = |P : Q|$.

Twisted wreath products arise quite naturally as the following result shows.

LEMMA 2.1 **(Bercov [3] and Lafuente [8]).** *Let G be a group with a normal subgroup M complemented by a subgroup P. Suppose that T is a subgroup of M such that for some $p_1 (= \mathrm{id}_P), p_2, \ldots, p_k \in P$ we can write*

$$M = T^{p_1} \times \cdots \times T^{p_k}$$

where conjugation by P permutes the T^{p_i} amongst themselves, i.e., $\{T^{p_1}, \ldots, T^{p_k}\}$ is the set of P-conjugates of T. Set $Q = N_P(T)$ and let $\phi : Q \to AutT$ be the map induced by the conjugation action of Q on T. Then there exists an isomorphism $G \to Ttwr_\phi P$ which maps M to the base group and P to the top group of the twisted wreath product.

The following two results are implicit in the proof of either [2, 3.1] or [6, 1.1].

LEMMA 2.2. *Suppose that ϕ, η are twisting homomorphisms with twisting subgroups Q_ϕ, Q_η, respectively. Then B_ϕ is a subgroup of B_η if and only if $Q_\phi \geq Q_\eta$ and the restriction of ϕ to Q_η is equal to η.*

LEMMA 2.3. *Suppose that T is a nonabelian simple group. Let H be a subgroup of the complete base group $\mathcal{B}$ that is normalized by P, and is such that the subgroup*

$$H(\mathrm{id}_P) = \{h(\mathrm{id}_P) : h \in H\}$$

of T is equal to T. Then H is a ϕ-base group for some ϕ.

We now consider twisted wreath products as quasiprimitive permutation groups. Define the *base group action* of $X_\phi = Ttwr_\phi P$ on B_ϕ by letting B_ϕ act by right multiplication and letting P act by conjugation. The following is relevant to this situation.

LEMMA 2.4 (**[2, 2.7(3)]**). *Let $X_\phi = Ttwr_\phi P$ and suppose that the center of T is trivial. Then there exists an isomorphism $C_{X_\phi}(B_\phi) \to \mathrm{Core}_P\phi^{-1}(InnT)$ which restricts to give an isomorphism $C_P(B_\phi) \to \mathrm{Core}_P\ker\phi$.*

(For a subgroup H of G, the *core* $\mathrm{Core}_G H$ of H in G is defined as the largest normal subgroup of G that is contained in H; if this is trivial then H is said to be *core-free in* G.) Hence, if T is a nonabelian simple group and if $\phi^{-1}(InnT)$ is a core-free subgroup of P, then we deduce that X_ϕ is faithful on B_ϕ, that B_ϕ is the unique minimal normal subgroup of X_ϕ (and is quite obviously transitive), and so X on B_ϕ is a quasiprimitive permutation group. Conversely, we have the following definition due to Praeger [13].

Definition 2.1. Let P be a group with a proper subgroup Q with a homomorphism $\phi : Q \to AutT$ such that $\mathrm{Core}_P\phi^{-1}(InnT)$ is trivial, where T is a nonabelian simple group. Then $X_\phi = Ttwr_\phi P$ in its base group action on B_ϕ is said to be a *quasiprimitive permutation group of twisted wreath type*. Also any permutation group that is permutation equivalent to such an X_ϕ is a quasiprimitive permutation group of twisted wreath type.

We remark that of the quasiprimitive permutation groups we constructed in the paragraph immediately preceeding Definition 2.1, the only ones that are not of twisted wreath type are those for which P itself is the twisting subgroup. In such a case we have that X_ϕ is almost simple with socle T and $P \lesssim OutT$.

3. Preliminaries on graphs

We start by giving the following result, which is elementary, well known, and fundamental to the study of 2-arc transitive graphs.

LEMMA 3.1. *Suppose that the graph Γ admits G as a vertex-transitive group of automorphisms; let α be a vertex of Γ. Then G is transitive on the 2-arcs of Γ if and only if the vertex stabilizer G_α is 2-transitive on the neighbors $\Gamma(\alpha)$ of α.*

Proof. We consider the action on 2-arcs of the form $(\alpha_0, \alpha, \alpha_2)$. □

We now assume that (Γ, G) is a special pair. Thus, G has a minimal normal subgroup M that is both nonabelian and regular. As M is characteristically simple we may write

$$M = T_1 \times \cdots \times T_k$$

where each T_i is isomorphic to the nonabelian simple group $T = T_1$. Fix a vertex α of Γ and set $P = G_\alpha$. Then P is a complement in G to M, since M is regular, and P permutes the T_i on conjugation; by Lemma 2.1 there exists a twisting homomorphism ϕ and an isomorphism $G \to X_\phi = Ttwr_\phi P$ that maps M to the base group B_ϕ and P to the top group. Such an isomorphism induces a permutation equivalence between the action of G on the vertices of Γ and the base group action of X_ϕ on B_ϕ; we assume from now on that $G = X_\phi$ and that the vertices of Γ are identified with B_ϕ. Let Y be the set of vertices that are adjacent to id_{B_ϕ} and suppose that x, y are any two adjacent vertices. As B_ϕ acting by right multiplication preserves edges we see that the vertices $\mathrm{id}_{B_\phi} = xx^{-1}$, yx^{-1} are also adjacent and so $yx^{-1} \in Y$. Conversely, it follows that if $x, y \in B_\phi$ are such that $yx^{-1} \in Y$, then x, y are adjacent vertices. Hence, Γ is the Cayley graph $\mathrm{Cay}(B_\phi, Y)$ of B_ϕ with respect to Y. (Note that a Cayley graph may have loops or directed edges.)

LEMMA 3.2. *Let T be a nonabelian simple group; let $X_\phi = Ttwr_\phi P$ be the twisted wreath product of T by some group P and with respect to some homomorphism $\phi : Q \to AutT$ with $Q \leq P$; let Y be a subset of B_ϕ, the base group of X_ϕ; let Γ be the Cayley graph* $\mathrm{Cay}(B_\phi, Y)$. *Then (Γ, X_ϕ) is a special pair in which the action of X_ϕ on the vertices of Γ is given by the base group action of X_ϕ, if and only if Y is a set of nontrivial involutions that generates B_ϕ, and that is acted on 2-transitively and faithfully on conjugation by P. Furthermore, if (Γ, X_ϕ) is a special pair, then its invariants are the nonabelian simple group T and the 2-transitive permutation group P on $\Omega = Y$ induced by the conjugation action of P on Y.*

Proof. We assume that (Γ, X_ϕ) is a special pair with X_ϕ acting on Γ as given. Then P acts on the vertex set B_ϕ of Γ by conjugation, and B_ϕ acts by right multiplication. We see that Y is invariant under P as P fixes id_{B_ϕ} and so also fixes the set of its neighbors. Moreover, P is 2-transitive on Y by Lemma 3.1. Also $\langle Y \rangle = B_\phi$ as Γ is connected. To see that P is faithful on Y, observe that the kernel of the action is $C_P(Y)$: but as Y generates B_ϕ this is equal to $C_P(B_\phi)$, which is the kernel of the action of P on the vertices of the graph – the latter is a faithful action and so $C_P(Y)$ is trivial. Now let $y \in Y$; then id_{B_ϕ}, y are adjacent, whence so are $y^{-1}, \mathrm{id}_{B_\phi}$, as right multiplication by elements of B_ϕ preserves edges, and so $y^{-1} \in Y$. (Equivalently, $y^{-1} \in Y$ as Γ is an undirected graph.) We now use an argument due to Praeger to show that Y is a set of nontrivial involutions (see [13, 6.2]). For $y \in Y$, the set $\{y, y^{-1}\}$ is a block of imprimitivity for the action of P on Y; but P acts 2-transitively on Y, whence we have either that $Y = \{y, y^{-1}\}$, or that $y = y^{-1}$ as required. The former is impossible as $B_\phi = \langle Y \rangle$ is not cyclic. Finally we deduce that $\mathrm{id}_{B_\phi} \notin Y$ as Γ has no loops.

The proofs of the converse to the above and of the final assertion of the lemma are straightforward and we leave them to the reader. □

Remark 3.1. In the context of Praeger's program to classify all 2-arc transitive graphs, it is interesting to note that if (Γ, X_ϕ) is a special pair then Γ is neither a cyclic nor a bipartite graph: it is noncyclic as $|Y| \geq 3$ since $B_\phi = \langle Y \rangle$ is neither a cyclic nor a dihedral group, and it is not bipartite as B_ϕ has no subgroup of index 2.

So we see that the problem of finding all special pairs with invariants T and P on Ω reduces to the group theoretic problem of finding all twisting homomorphisms ϕ and subsets Y of nontrivial involutions in B_ϕ such that Y generates B_ϕ, is invariant under conjugation by P and is such that P on Y is permutation equivalent to P on Ω.

4. A characterization in terms of admissible maps

Throughout this section T is a fixed nonabelian simple group and P on Ω a fixed 2-transitive permutation group. Recall that the map $F : \Omega \to T$ is admissible if $F(\Omega)$ is a set of involutions (possibly including id_T) that generates T. We start by showing that such maps do in fact arise quite naturally in our situation. Suppose that we have a special pair $(\mathrm{Cay}(B_\phi, Y), X_\phi = T twr_\phi P)$ with T and P on Ω as invariants. Now the action of P on Y is permutation equivalent to that of P on Ω and so we may write Y as

$$Y = \{f_\omega : \omega \in \Omega\} \tag{1}$$

with the conjugation action of $p \in P$ given by $f_\omega^p = f_{\omega p}$ for all $\omega \in \Omega$. We claim that the map $F : \Omega \to T$ defined by

$$F(\omega) = f_\omega(\mathrm{id}_P) \qquad \text{for all } \omega \in \Omega \tag{2}$$

is admissible: to see this we note that the fact that Y is a set of nontrivial involutions generating B_ϕ implies that $Y(\mathrm{id}_P) = \{f(\mathrm{id}_P) : f \in Y\}$ is a set of involutions generating $B_\phi(\mathrm{id}_P) = T$. Also we observe that for $x \in P$ and $\omega \in \Omega$

$$f_\omega(x) = f_\omega^x(\mathrm{id}_P) = f_{\omega x}(\mathrm{id}_P) = F(\omega x).$$

Conversely, if we start with an admissible map $F : \Omega \to T$ then we may define the elements $f_\omega \in \mathcal{B}$ by

$$f_\omega(p) = F(\omega p) \qquad \text{for all } \omega \in \Omega,\, p \in P.$$

Then $Y_F = \{f_\omega : \omega \in \Omega\}$ is a set of nontrivial involutions in $\mathcal{B}$ that is invariant under conjugation by P and is such that the action of P on Y_F is permutation equivalent to that of P on Ω (these remarks will be justified in the proof of Theorem 4.1). Set $H = \langle Y_F \rangle$, whence H is a subgroup of $\mathcal{B}$ normalized by P and such that $H(\mathrm{id}_P) = \langle Y_F(\mathrm{id}_P) \rangle = T$; by Lemma 2.3 we see that $H = B_{\phi_F}$ for some twisting homomorphism ϕ_F. It follows from Lemma 3.2 that, for such Y_F and ϕ_F (which are uniquely determined given F), the pair $(\mathrm{Cay}(B_{\phi_F}, Y_F), X_{\phi_F})$ is special. Thus, any special pair is determined by its invariants T and P on Ω together with an admissible map $F : \Omega \to T$. We have now essentially justified Theorem 1.2 except that these remarks hardly constitute a construction for the twisting homomorphism ϕ_F (and hence the special pair $(\mathrm{Cay}(B_{\phi_F}, Y_F), X_{\phi_F})$) in terms of the admissible map F; this is now achieved in what follows.

Construction 4.1. We suppose that $F : \Omega \to T$ is admissible, and construct various objects in terms of F. Let Σ_F be the partition of Ω given by

$$\Sigma_F = \{F^{-1}(t) : t \in F(\Omega)\}.$$

We denote by $P_{(\Sigma_F)}$ and P_{Σ_F} the pointwise and setwise stabilizers in P of Σ_F, respectively; thus,

$$P_{(\Sigma_F)} = \{x \in P : \Delta x = \Delta \text{ for all } \Delta \in \Sigma_F\}$$
and
$$P_{\Sigma_F} = \{x \in P : \Delta x \in \Sigma_F \text{ for all } \Delta \in \Sigma_F\}.$$

The quotient $P_{\Sigma_F}/P_{(\Sigma_F)}$ is denoted by $P^{F(\Omega)}$, and is viewed as a permutation group on $F(\Omega)$ so that for $x \in P_{\Sigma_F}$ the effect of $P_{(\Sigma_F)}x \in P^{F(\Omega)}$ on $t \in F(\Omega)$ is given by

$$t(P_{(\Sigma_F)}x) = F(\omega x)$$

where $\omega \in \Omega$ is such that $F(\omega) = t$. Let $(AutT)_{F(\Omega)}$ be the automorphisms of T that fix $F(\Omega)$ setwise; as $F(\Omega)$ generates T the action of $(AutT)_{F(\Omega)}$ on $F(\Omega)$ is faithful and we may view $(AutT)_{F(\Omega)}$ as a subgroup of the group $\mathrm{Sym}F(\Omega)$ of all permutations on $F(\Omega)$. We define the subgroup Q_F of P to be the inverse image in P_{Σ_F} of $P^{F(\Omega)} \cap (AutT)_{F(\Omega)}$ under the quotient map $P_{\Sigma_F} \to P^{F(\Omega)} = P_{\Sigma_F}/P_{(\Sigma_F)}$. Observe that this quotient map restricts to give a map $Q_F \to (AutT)_{F(\Omega)}$ which may be composed with an inclusion map to give

$$\phi_F : Q_F \to AutT.$$

(Essentially $x \in Q_F$ if x induces a permutation on $F(\Omega)$ which may be extended to an automorphism of T, and $\phi_F(x)$ is precisely this automorphism, which is well-defined as T is generated by $F(\Omega)$.) Let X_F be the twisted wreath product of T by P with respect to the twisting homomorphism ϕ_F. Finally set $Y_F = \{f_\omega : \omega \in \Omega\}$ where for $\omega \in \Omega$ the element f_ω is given by

$$f_\omega(x) = F(\omega x) \qquad \text{for all } x \in P,$$

and denote the graph $\mathrm{Cay}(B_{\phi_F}, Y_F)$ by Γ_F.

THEOREM 4.1. *Suppose that T is a nonabelian simple group, that P on Ω is a 2-transitive permutation group, and that $F : \Omega \to T$ is an admissible map; then (Γ_F, X_F) is a special pair. Conversely, suppose that (Γ, G) is a special pair with invariants T and P on Ω; then there exists an admissible map $F : \Omega \to P$ such that (Γ, G) and (Γ_F, X_F) are equivalent.*

Proof. It follows, from the remarks made just prior to Construction 4.1, that the first assertion of the theorem holds provided only that Y_F is a set of nontrivial involutions in B_{ϕ_F}, that Y_F is invariant under conjugation by P, that the action of P on Y_F is permutation equivalent to that of P on Ω, and that Y_F generates B_{ϕ_F}.

We recall that $Y_F = \{f_\omega : \omega \in \Omega\}$ where for $\omega \in \Omega$ we have

$$f_\omega(x) = F(\omega x) \qquad \text{for all } x \in P.$$

The f_ω are certainly involutions as $F(\Omega)$ is a set of involutions in T; also each f_ω is nontrivial as $f_\omega(x)$ must be nontrivial for some $x \in P$. We consider the conjugation action of P on Y_F: for $x, p \in P$ and $\omega \in \Omega$ we have

$$f_\omega^p(x) = f_\omega(px) = F(\omega px) = f_{\omega p}(x)$$

and Y_F is indeed invariant under P. The above also shows that the action of P on Y_F by conjugation is equivalent to that of P on some system of imprimitivity for P on Ω. However P on Ω is 2-transitive, hence primitive, and we have either that P on Ω is permutation equivalent to P on Y_F or that Y_F consists of a single element. The latter is however a contradiction $-F(\Omega)$ consists of at least 2 distinct elements (in fact at least 3) as $T = \langle F(\Omega) \rangle$ is a nonabelian simple group, and so there exist $\omega, \gamma \in \Omega$ with $F(\omega) \neq F(\gamma)$, whence $f_\gamma \neq f_\omega$.

It remains then to show that $\langle Y_F \rangle = B_{\phi_F}$. We have

$$\langle Y_F \rangle (\mathrm{id}_P) = \langle f_\omega(\mathrm{id}_P) : \omega \in \Omega \rangle = \langle F(\Omega) \rangle = T$$

and so by Lemma 2.3 we see that $\langle Y_F \rangle = B_\eta$ for some twisting homomorphism η (as pointed out at the beginning of this section). Moreover for $\omega \in \Omega$, $x \in P$ and $q \in Q_F$ we have

$$f_\omega(xq) = F(\omega xq) = F(\omega x)^{\phi_F(q)}, \quad \text{by definition of } \phi_F,$$
$$= f_\omega(x)^{\phi_F(q)}$$

whence $f_\omega \in B_{\phi_F}$ and $\langle Y_F \rangle = B_\eta \leq B_{\phi_F}$.

To demonstrate the reverse containment we note that by Lemma 2.2 it is enough to show that the domain Q_η of η is contained in Q_F and that η equals the restriction of ϕ_F to Q_η. So suppose that $q \in Q_\eta$; for $\omega \in \Omega$ we have that

$$F(\omega q) = f_\omega(q) = f_\omega(\mathrm{id}_P)^{\eta(q)} = F(\omega)^{\eta(q)},$$

and thus we see that not only is $q \in P_{\Sigma_F}$, but also that q induces a permutation on $F(\Omega)$ which is extendible to the automorphism $\eta(q)$ of T. This forces $q \in Q_F$ and $\phi_F(q) = \eta(q)$ by the definition of ϕ_F.

Conversely suppose that (Γ, G) is a special pair with invariants T and P on Ω. From Lemma 3.2 and the remarks preceeding it we deduce that there exist a twisting homomorphism $\phi : Q \to AutT$ with $Q \leq P$, and a subset Y of the base group B_ϕ of $X_\phi = Ttwr_\phi P$ with $B_\phi = \langle Y \rangle$, which is invariant under conjugation by P and is such that P on Y is permutation equivalent to P on Ω, such that the special pairs (Γ, G) and $(\mathrm{Cay}(B_\phi, Y), X_\phi)$ are equivalent. Let $F : \Omega \to T$ be as defined by (1) and (2), and let Y_F and ϕ_F be as given by Construction 4.1. Then it is clear that $Y = Y_F$, whence

$$B_\phi = \langle Y \rangle = \langle Y_F \rangle = B_{\phi_F}.$$

By applying Lemma 2.2 twice we deduce that $\phi = \phi_F$ and so the special pair $(\mathrm{Cay}(B_\phi, Y), X_\phi)$ and the special pair obtained via Construction 4.1 with input T, P on Ω and F are not just equivalent, but are in fact identical. □

5. Criteria for quasiprimitivity

In this section we address the following questions:

- If (Γ, G) is QP-special then of what type is the quasiprimitive permutation group G on the vertices of Γ?
- Which special pairs are QP-special?
- How easy is it to construct examples?

We start by looking at the first of these. Recall that we are using the description of quasiprimitive permutation groups that is due to Praeger, as given in [13]. So suppose that (Γ, G) is QP-special, whence G on the vertices of Γ is a quasiprimitive permutation group with a regular nonabelian normal subgroup. By inspecting Praeger's description we see that one of the following holds:

(1) G is almost simple (i.e., type II in the notation of [13]);
(2) G is of simple diagonal type with two minimal normal subgroups (i.e., type III(a)(ii));
(3) G is of product type derived from a group of simple diagonal type with 2 minimal normal subgroups (i.e., G is of type III(b)(ii) with two minimal normal subgroups);
(4) G is of twisted wreath type (i.e., type III(c)).

However in [13] Praeger shows that quasiprimitive permutation groups of types III(a)(ii) or III(b)(ii) cannot act 2-arc transitively on a connected graph (see [13, 5.2 and 5.3(a)]) and so we are left with G almost simple or of twisted wreath type depending on whether the socle of G is simple or not. (We remark that the proof of [13, 5.2] uses inspection of the list of almost simple 2-transitive permutation groups, and so depends on the classification of finite simple groups.)

We now turn to the questions of which special pairs are QP-special. Our answer is contained in the next result.

LEMMA 5.1. *Suppose that (Γ, G) is a special pair and that M is a minimal normal subgroup of G that is both nonabelian and regular on Γ. Then (Γ, G) is QP-special if and only if $C_G(M)$ is trivial.*

Proof. We identify the vertices of Γ with the elements of M so that M acts by right multiplication; also we set $P = G_{\mathrm{id}_M}$ and $Y = \Gamma(\mathrm{id}_M)$ so that $\Gamma = \mathrm{Cay}(M, Y)$, $G = M \rtimes P$, and P acts on Γ by conjugation.

Suppose now that G is not quasiprimitive on the vertices of Γ, so that (Γ, G) is not QP-special. Then there exists a normal subgroup N of G that is not transitive on Γ; so $N \not\geq M$, whence by the minimality of M we see that the intersection of M and N is trivial. This implies that $C_G(M)$ contains N and is nontrivial.

Conversely, if (Γ, G) is QP-special then G on the vertices of Γ is a quasiprimitive permutation group of either almost simple type or twisted wreath type as pointed out above. Observe in both cases that G has a unique minimal normal subgroup, whence $C_G(M)$ is indeed trivial. □

COROLLARY 5.1. *Suppose that T is a nonabelian simple group, that P on Ω is a 2-transitive permutation group, and that $F : \Omega \to T$ is an admissible map; let (Γ_F, X_F) be the special pair constructed via Construction 4.1. Then (Γ_F, X_F) is QP-special if and only if $\phi_F^{-1}(\mathrm{Inn}T)$ is a core-free subgroup of P.*

Proof. We translate into the notation of §4 and use Lemma 2.4. □

This brings us to the question of how easy it is to construct examples. Certainly Construction 4.1 gives us a nice method of finding examples of special pairs and let us start by looking at one.

Example 5.1. Let $T = A_5$ and $P = AGL_3(2)$, with Ω a vector space of three dimensions over the field $\mathbb{F}_2$ of two elements. We write elements of T as permutations on $\{1, \ldots, 5\}$, elements of Ω as 3-tuples with entries in $\mathbb{F}_2$, and elements of P as xA where $x \in \Omega$ and A is a 3×3 invertible matrix with entries in $\mathbb{F}_2$; multiplication inside P and the action of P on Ω is as usual. Define a map $F : \Omega \to T$ by

$$F(0, 0, 0) = F(0, 0, 1) = (23)(45)$$

$$F(1, 0, 0) = F(1, 0, 1) = (14)(35)$$

$$F(0, 1, 0) = F(0, 1, 1) = (14)(25)$$

$$F(1, 1, 0) = F(1, 1, 1) = (23)(15).$$

It is straightforward to verify that F is an admissible map (as there are no proper subgroups of A_5 that contain $F(\Omega)$) and so we may consider the special pair described by Construction 4.1. Observe that the elements $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1) \in P$ give rise to permutations on $F(\Omega)$ that are equivalent to conjugation by $(12)(34)$, $(13)(24)$, $\mathrm{id}_T \in T$, respectively. Thus, the socle of P is contained in $\phi_F^{-1}(\mathit{Inn}T)$ and by Corollary 5.1 the constructed special pair is not QP-special.

The key to the above example is the existence of the elementary abelian subgroup $\langle (12)(34), (13)(24) \rangle$ of T acting regularly by conjugation on a set of involutions in T that together generate T. This is borne out by the following.

PROPOSITION 5.1. *There exists a special pair that is not QP-special, with invariants T and P on Ω, where P is an affine 2-transitive permutation group with socle of size p^d for some prime p, if and only if there exists an elementary abelian p-subgroup H of T of size p^a with $a \leq d$ that acts regularly by conjugation on a set J of involutions which generates T.*

Proof. We assume that such an elementary abelian p-subgroup H of T exists and let P be an affine 2-transitive permutation group with socle of size p^d with $d \geq a$. Choose an epimorphism $\chi : \mathrm{Soc}P \to H$ and an involution $x \in J$. Identify Ω with $\mathrm{Soc}P$ and define $F : \Omega \to T$ by

$$F(\omega) = x^{\chi(\omega)} \qquad \text{for all } \omega \in \Omega.$$

It is clear that F is admissible and also that $\mathrm{Soc}P$ is contained in $\phi_F^{-1}(\mathit{Inn}T)$, whence by Corollary 5.1 the pair constructed by Construction 4.1 is not QP-special.

Conversely suppose that for a nonabelian simple group T and affine 2-transitive permutation group P on Ω with $|\mathrm{Soc}P| = p^d$ there exists an admissible map F

which gives rise to a pair that is not QP-special. Then we see that Soc$P$ is contained in $\phi_F^{-1}(InnT)$. Let H be the subgroup of T such that conjugation by elements of H corresponds to the inner automorphisms of T given by $\phi_F(\mathrm{Soc}P)$; also let $J = F(\Omega)$. Then H is a homomorphic image of Soc$P$ and so is an elementary abelian p-subgroup of T of size at most $|\mathrm{Soc}P|$. Also Soc$P$ is transitive on Ω which implies that the induced action of Soc$P$ on $F(\Omega)$, and likewise that of H on J, is transitive. As H is also faithful on J, it is regular on J. □

We give the analogous result for almost simple 2-transitive permutation groups.

PROPOSITION 5.2. *There exists a special pair, that is not QP-special, with invariants T and P on Ω, where P is an almost simple 2-transitive permutation group with socle P_0, if and only if there exists a subgroup H of T and a set J of involutions which generates T such that the permutation group H on J (acting by conjugation) is permutation equivalent to P_0 on Ω.*

Proof. The proof is analogous to the above result except that we also use the fact that P_0 is primitive on Ω in order to deduce that F is injective. (This fact follows by inspection of the list of almost simple 2-transitive permutation groups given in [4] and so depends upon the classification of finite simple groups.) □

The conditions of Proposition 5.2 seem more difficult to satisfy than those of Proposition 5.1, but we do have one example.

Example 5.2. In [5] Curtis shows that in the sporadic simple group M_{24} there exist seven involutions that:

(i) together generate M_{24};
(ii) are permuted faithfully and 2-transitively on conjugation by some maximal subgroup of M_{24} that is isomorphic to $L_2(7)$.

Thus Propositions 5.1 and 5.2 together describe under precisely what conditions a special pair may fail to be QP-special. These circumstances are somewhat restrictive and so there is some justification in the viewpoint that most special pairs are QP-special. Similarly the following gives credence to the intuitive feeling that nearly all QP-special pairs involve quasiprimitive permutation groups of twisted wreath, rather than almost simple, type.

PROPOSITION 5.3. *There exists a QP-special pair (Γ, G) in which G is an almost simple group with socle T, if and only if there exists a subgroup H of G that complements T and a set J of involutions with $\langle J\rangle = T$ such that H stabilizes, and is 2-transitive on, J by conjugation.*

Proof. We assume that such G, T, H, and J exist; let Γ be the Cayley graph

Cay(T, J) on T with respect to J. It follows that (Γ, G) is a QP-special pair with G almost simple.

Conversely suppose that (Γ, G) is a QP-special pair in which G is an almost simple group with socle T. Then T is regular on vertices and we may assume that Γ = Cay(T, J) for some set $J \subseteq T$ of involutions that generates T. Set $H = G_{\mathrm{id}_T}$, the stabilizer in G of the vertex id_T. Clearly H complements T in G; also H stabilizes and is 2-transitive on J by conjugation as J is the set of neighbors in Γ of id_T. □

The author is grateful to Marston Conder for the following example satisfying the conditions of Proposition 5.3.

Example 5.3. Let $T = L_3(4)$ and $\tau \in AutT$ be the 'inverse-transpose' automorphism; also let $\sigma \in AutT$ be the automorphism induced on conjugation by the following matrix in $GL_3(4)$:

$$\begin{pmatrix} 1 & \omega & \omega \\ \omega & 0 & 1 \\ \omega & 1 & 1 \end{pmatrix}$$

where ω generates $\mathbb{F}_4^*$. Identify T with $InnT$ and set $H = \langle \sigma, \tau \rangle$ and $G = T.H$; it is clear that H is isomorphic to S_3 and complements T in G. Let $\alpha \in T$ correspond to the following matrix in $GL_3(4)$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then α is an involution with $C_H(\alpha) = \langle \tau \rangle$; let J be the set of conjugates of α under H. To show that G, T, H, and J satisfy the conditions of Proposition 5.3 it remains to show that J generates T. This can be verified by hand by showing, using various elementary ad hoc arguments, that no proper subgroup of T contains J.

6. Equivalences and containments

We start by pointing out that the concepts of equivalence and containment for pairs, as defined in §1, are sensible. Indeed, the graphs Γ and Σ are isomorphic if and only if the pairs ($\Gamma, Aut\Gamma$) and ($\Sigma, Aut\Sigma$) are equivalent. Also, any pair (Γ, G) is contained in the pair ($\Gamma, Aut\Gamma$). Thus, the process of determining all equivalences and containments between special pairs is a first step toward finding the complete automorphism groups of the graphs involved.

So suppose that T_1, T_2 are nonabelian simple groups, that P_1 on Ω_1, P_2 on Ω_2 are 2-transitive permutation groups, and that $F_1 : \Omega_1 \to T_1$, $F_2 : \Omega_2 \to T_2$ are admissible maps; we assume that the special pairs (Γ_{F_1}, X_{F_1}), (Γ_{F_2}, X_{F_2}) obtained via Construction 4.1 are equivalent. (From now on we will write Γ_1 for Γ_{F_1}, B_1 for $B_{\phi_{F_1}}$ etc.) Hence, there exists an edge-preserving permutation equivalence between the actions of X_1 and X_2 on the vertices of Γ_1 and Γ_2, respectively. We wish to find a canonical form for this permutation equivalence. (The following work uses similar ideas to those found in [2, 6.2].)

We assume that the permutation equivalence is given by a bijection $\theta : B_1 \to B_2$ between vertex sets and an isomorphism $\chi : X_1 \to X_2$, both compatible with the group actions: in other words, for b, $f \in B_1$ and $p \in P_1$ we have

$$\theta(b * fp) = \theta(b) \star \chi(fp),$$

where $*$, $\star$ denote the actions in X_1 and X_2, respectively. (Obviously we know that $b * fp$ is equal to $(bf)^p$ but we must be very careful with the action in X_2; we do not necessarily have that $\chi(B_1) = B_2$.) As we have a permutation equivalence that preserves edges we also assume that

$$\theta(x)\theta(y)^{-1} \in Y_2 \iff xy^{-1} \in Y_1$$

for all x, $y \in B_1$.

Suppose that $\theta(\mathrm{id}_{B_1}) = b \neq \mathrm{id}_{B_2}$; define θ', χ' by

$$\theta'(f) = \theta(f)b^{-1}, \quad \chi'(x) = b\chi(x)b^{-1}$$

for all $f \in B_1$ and $x \in X_1$. It is straightforward to verify that θ', χ' preserve the base group actions of X_1 and X_2; also θ' preserves edges, and so we may assume that $\theta(\mathrm{id}_{B_1})$ does indeed equal the identity in B_2. From this it follows that χ restricts to give an isomorphism between the point stabilizers of id_{B_1} and id_{B_2}, namely P_1 and P_2 respectively.

We now consider $\chi(B_1)$. As B_1 is a regular, nonabelian, minimal normal subgroup of X_1, we have that $\chi(B_1)$ is regular, nonabelian and minimal normal in X_2.

LEMMA 6.1. *Suppose that* (Γ, G) *is a special pair. Then* G *has a unique regular, nonabelian, minimal normal subgroup.*

Proof. Certainly by the definition of a special pair G has such a subgroup; suppose that G has two such, namely M_1 and M_2. By minimality M_1, M_2 intersect trivially and centralize each other: thus

$$M_1 \times M_2 \leq \mathrm{Soc}G \leq M_1 \times C_G(M_1).$$

However, $C_G(M_1)$ is semiregular as M_1 is regular, whence by orders $C_G(M_1) = M_2$ and $\mathrm{Soc}G = M_1 \times M_2$. Hence G is quasiprimitive on vertices with two distinct minimal normal subgroups that centralize each other—this contradicts Lemma 5.1. ◻

So by the lemma we may assume that $\chi(B_1) = B_2$. This has the great advantage that we have

$$\theta((bf)^p) = \theta(b) \star \chi(fp) = (\theta(b)\chi(f))^{\chi(p)} \qquad \text{for all } b,\, f \in B_1,\, p \in P$$

which implies, setting $p = b = \mathrm{id}_{X_1}$, that $\theta(f) = \chi(f)$ for all $f \in B_1$.

To summarize what has happened so far, we have shown that if the special pairs (Γ_1, X_1), (Γ_2, X_2) are equivalent, then we may assume that there exists an isomorphism $\chi : X_1 \to X_2$ which maps base group B_1 to base group B_2, top group P_1 to top group P_2, and generating set Y_1 to generating set Y_2. At this stage it is clear that $T_1 \cong T_2$ as the two base groups are isomorphic via χ; also χ induces a permutation equivalence between P_1 on Y_1 and P_2 on Y_2. Hence, the special pairs (Γ_1, X_1), (Γ_2, X_2) have the same invariants and we will assume that $P = P_1 = P_2$, $\Omega = \Omega_1 = \Omega_2$, and $T = T_1 = T_2$. This identification means that the base groups B_1, B_2 are both subgroups of the complete base groups $\mathcal{B} = \{f : P \to T\}$, but not that they are the same. Indeed the base groups are determined by the admissible maps and we now attempt to find conditions on F_1 and F_2 that determine the equivalence of the special pairs.

We continue with our search for a canonical choice for χ. For $i = 1, 2$ and $t \in T$ define the elements $t^{(i)}$ of B_i by

$$t^{(i)}(p) = \begin{cases} \mathrm{id}_T & \text{if } p \notin Q_i \\ t^{\phi_i(p)} & \text{otherwise,} \end{cases}$$

for all $p \in P$ and where Q_i, ϕ_i are the twisting subgroup and twisting homomorphism respectively. Let $T^{(i)} = \{t^{(i)} : t \in T\}$ for $i = 1, 2$; observe that $t \mapsto t^{(i)}$ is an isomorphism $T \to T^{(i)}$, and that $T^{(i)}$ is a minimal normal subgroup of B_i. Thus $\chi\left(T^{(1)}\right) = \left(T^{(2)}\right)^x$ for some $x \in P$ and by conjugating by x we may assume that χ additionally maps $T^{(1)}$ to $T^{(2)}$. We define the automorphism $\sigma \in \mathit{AutT}$ by requiring that

$$\sigma(t)^{(2)} = \chi\left(t^{(1)}\right) \qquad \text{for all } t \in T.$$

Consider now the effect of χ on the top groups, which are both equal to P, and on the generating sets Y_1, Y_2. For $i = 1, 2$ we write $Y_i = \{f_\omega^{(i)} : \omega \in \Omega\}$ where

$$f_\omega^{(i)}(p) = F_i(\omega p) \qquad \text{for all } p \in P.$$

We have that χ maps Y_1 to Y_2 and so we may define the permutation $\pi \in \mathrm{Sym}\,\Omega$ by requiring that

$$\chi\left(f_\omega^{(1)}\right) = f_{\omega\pi}^{(2)} \quad \text{for all } \omega \in \Omega.$$

Then for $p \in P$ and $\omega \in \Omega$ we have

$$f^{(2)}_{\omega\chi(p)} = \left(f^{(2)}_\omega\right)^{\chi(p)} = \left(\chi\left(f^{(1)}_{\omega\pi^{-1}}\right)\right)^{\chi(p)} = \chi\left(\left(f^{(1)}_{\omega\pi^{-1}}\right)^p\right) = \chi\left(f^{(1)}_{\omega\pi^{-1}p}\right) = f^{(2)}_{\omega\pi^{-1}p\pi}.$$

This shows that the permutations $p^\pi \in \mathrm{Sym}\,\Omega$ and $\chi(p) \in P \leq \mathrm{Sym}\,\Omega$ agree on Ω and so are equal, whence $\pi \in N_{\mathrm{Sym}\,\Omega}(P)$.

We claim that $\sigma \in AutT$ and $\pi \in N_{\mathrm{Sym}\,\Omega}(P)$ completely determine χ. Certainly for $p \in P$ we have $\chi(p) = p^\pi$ and so it remains to consider $\chi(b)$ for some $b \in B_1$. Now B_1 is the direct product of the P-conjugates of $T^{(1)}$ and so there exist $p_1 (= \mathrm{id}_P), p_2, \ldots, p_k \in P$ and $t_1, \ldots, t_k \in T$ such that

$$b = \prod_{i=1}^{k} \left(t_i^{(1)}\right)^{p_i},$$

whence

$$\chi(b) = \prod_{i=1}^{k} \chi\left(t_i^{(1)}\right)^{p_i^\pi} = \prod_{i=1}^{k} \left(\sigma(t_i)^{(2)}\right)^{p_i^\pi}.$$

We are now able to describe the effect of χ on the admissible maps F_1 and F_2 purely in terms of $\sigma \in AutT$ and $\pi \in N_{\mathrm{Sym}\,\Omega}(P)$. Recall that for $\omega \in \Omega$ and $i = 1, 2$ we have $F_i(\omega) = f_\omega^{(i)}(\mathrm{id}_P)$. For $\omega \in \Omega$ choose $p_1 (= \mathrm{id}_P), p_2, \ldots, p_k \in P$ and $t_1, \ldots, t_k \in T$ so that $f_\omega^{(1)} = \prod (t_i^{(1)})^{p_i}$. As $Q_1 = N_P(T^{(1)})$ we see that $\{p_1, \ldots, p_k\}$ are representatives of the right cosets of Q_1 in P, and so $p_2, \ldots, p_k \notin Q_1$ whence

$$\left(t_i^{(1)}\right)^{p_i}(\mathrm{id}_P) = \mathrm{id}_T \qquad \text{for each } 2 \leq i \leq k$$

and so $f_\omega^{(1)}(\mathrm{id}_P) = t_1^{(1)}(\mathrm{id}_T) = t_1$; also as χ maps $T^{(1)}$ to $T^{(2)}$ we have that $\chi(Q_1) = Q_2$ and we deduce that $\chi\left(f_\omega^{(1)}\right)(\mathrm{id}_P) = \chi\left(t_1^{(1)}\right)(\mathrm{id}_P) = \sigma(t_1)$. Hence

$$F_2(\omega\pi) = f_{\omega\pi}^{(2)}(\mathrm{id}_P) = \chi(f_\omega^{(1)})(\mathrm{id}_P) = \sigma(t_1) = \sigma(f_\omega^{(1)}(\mathrm{id}_P)) = \sigma(F_1(\omega)).$$

THEOREM 6.1. *Two special pairs are equivalent only if they have the same invariants. Moreover if T is a nonabelian simple group, P on Ω is a 2-transitive permutation group, and $F_1, F_2 : \Omega \to T$ are admissible maps, then the special pairs constructed with these ingredients are equivalent if and only if there exists $\sigma \in AutT$ and $\pi \in N_{\mathrm{Sym}\,\Omega}(P)$ such that*

$$F_2(\omega\pi) = \sigma(F_1(\omega)) \qquad \textit{for all } \omega \in \Omega.$$

Proof. We have already proven the first statement and the first half of the second. To see the second half we note that we may construct an equivalence of pairs from σ and π by merely reversing the previous steps. □

It is now clear that the special pairs are, up to equivalence, in one-to-one correspondence with triples $(T, P \text{ on } \Omega, F)$ where we let T run through a set of representatives of isomorphism classes of nonabelian simple groups, P on Ω through a set of representatives of permutation equivalence classes of 2-transitive permutation groups, and F through a set of representatives of orbits for the action of $AutT \times N_{\mathrm{Sym}\Omega}(P)$ on the set of admissible maps given by

$$(F * (\sigma, \pi))(\omega) = \sigma^{-1}(F(\omega\pi^{-1}))$$

for all $\sigma \in AutT$, $\pi \in N_{\mathrm{Sym}\Omega}(P)$, and admissible maps F.

We now turn to the question of containments. Having given a complete analysis for equivalences we need only consider the following problem: *given a special pair* (Γ, G), *find all subgroups* H *of* G *such that* (Γ, H) *is also a special pair.*

LEMMA 6.2. *If* (Γ, G) *is a special pair with a regular nonabelian minimal normal subgroup* M *and a subgroup* H *such that* (Γ, H) *is also a special pair, then* M *is contained in* H *as a minimal normal subgroup of* H.

Proof. As (Γ, H) is a special pair, there exists a minimal normal subgroup $\widehat{M}$ of H that is both nonabelian and regular on vertices. We concentrate now on the G-action on the vertices of Γ, which we may assume have been identified with the elements of M. This action leads to an embedding of G in Sym$M$; in fact, as G normalizes M we have an embedding of G in the holomorph of M in Sym$M$ and deduce that the quotient $G/(M \times C_G(M))$ is isomorphic to a subgroup of $OutT \times S_k$ where M is isomorphic to the direct product of k copies of the nonabelian simple group T. Now the minimality of $\widehat{M}$ implies that either $(M \times C_G(M)) \cap \widehat{M} = \widehat{M}$, or $(M \times C_G(M)) \cap \widehat{M}$ is trivial. If the former, then by orders (noting that by the results of the previous section $|C_G(M)| < |M|$) we have that $M \cap \widehat{M}$ is nontrivial, whence again by minimality $M = \widehat{M} \leq H$ as required; if the latter then we deduce that $\widehat{M}$ is isomorphic to a subgroup of $OutT \times S_k$. But $\widehat{M}$ is characteristically simple and nonabelian, and so is isomorphic to the direct product of say $\hat{k}$ copies of some nonabelian simple group $\widehat{T}$, whence $|T|^k = |\widehat{T}|^{\hat{k}}$; by a result of Teague (see [4, Appendix] or [7]) we deduce that $|T| = |\widehat{T}|$ and $k = \hat{k}$. By the Schreier conjecture $OutT$ is soluble and it follows that $\widehat{M} \cong \widehat{T}^k$ is isomorphic to a subgroup of S_k; this gives a contradiction as if p is any prime dividing $|\widehat{T}|$ then p^k divides $k!$, which is impossible as the p-part of $k!$ is at most p^{k-1}. (Note that both Teague's result and the Schreier conjecture depend upon the classification of finite simple groups.) $\square$

Suppose then that T is a nonabelian simple group, that P on Ω is a 2-transitive permutation group, and that $F : \Omega \to T$ is an admissible map; let (Γ_F, X_F) be the special pair obtained via Construction 4.1. Suppose also that (Γ_F, H) is a special pair with $H \leq X_F$. By the above lemma we see that the base group

$B = B_{\phi_F}$ of X_F is a minimal normal subgroup of H, and so $H = BK$ where $K = H \cap P$ is the stabilizer in H of the vertex id_B. As K is 2-transitive on the neighbors of id_B we have that K is a proper subgroup of P that is also 2-transitive on Ω; also K is transitive on the simple direct factors of B as the latter is minimal normal in H.

Conversely, given a proper subgroup of P on Ω that is 2-transitive on Ω and transitive on the simple direct factors of B, then it is clear that (Γ_F, BK) will be a special pair contained in (Γ_F, X_F). Moreover, (Γ_F, BK) is equivalent to the special pair obtained via Construction 4.1 with T, K on Ω, and F as ingredients.

We now consider to what extent these two conditions on proper subgroups of P on Ω are necessary for the existence of containments. In fact it is quite easy to see that they are independent conditions, as the following two examples show.

Example 6.1. Recall Example 5.1 and set $K = GL_3(2)$. Then K certainly does not act 2-transitively on Ω – it is not even transitive – but it is transitive on the simple direct factors of the base group. This can be seen by noting that $KQ_F = P$ since Q_F contains the socle of P, to which K is a complement.

Example 6.2. Let T be a nonabelian simple group with no outer automorphisms. Let C be a conjugacy class of nontrivial involutions in T. Set $n = |C|$ and choose a bijection $F : \Omega = \{1, \dots, n\} \to C \subseteq T$. Note that both A_n and S_n acting on Ω in the natural way are 2-transitive permutation groups, and that $F : \Omega \to T$ is admissible. Let Γ, $\hat{\Gamma}$ be the graphs obtained via Construction 4.1 with respectively T, S_n on Ω, F and T, A_n on Ω, F as ingredients. We claim that Γ and $\hat{\Gamma}$ are not isomorphic – in fact they do not even have the same number of vertices. To see this recall that $|V\Gamma| = |T|^{|S_n:Q|}$ and $|V\hat{\Gamma}| = |T|^{|A_n:\hat{Q}|}$ where Q, $\hat{Q}$ are the appropriate twisting subgroups defined in terms of F, and note that $Q = \hat{Q}$ as all permutations of Ω that give rise to automorphisms of T are even.

Remark 6.1. We observe that, for a special pair (Γ, G), the above example shows that the number of vertices of Γ is not determined by its valency.

At this point it seems appropriate to give an actual example of a containment of special pairs.

Example 6.3. Recall Example 5.1 and set $K = L_2(7)$ which is a 2-transitive subgroup of $AGL_3(2)$ that complements the socle 2^3. As this socle is contained in the twisting subgroup Q_F we do indeed get a containment based on such a K.

This example is interesting as it produces a containment of a QP-special pair in a special pair that is not QP-special. Indeed such behavior seems to be quite rare. To see this, suppose that (Γ, G) is a QP-special pair such that there exists a subgroup X of $Aut\Gamma$ containing G that is not quasiprimitive on vertices: hence

(Γ, X) is a nice pair that contains (Γ, G) but is not QP-nice. So there exists a nontrivial normal subgroup of X that is intransitive on vertices. We let N be minimal subject to being nontrivial, normalized by G and intransitive on vertices; then (Γ, NG) is a nice pair that contains (Γ, G) and that is not QP-nice. Let M be the minimal normal subgroup of G that is both nonabelian and regular; as (Γ, G) is QP-special we see that G is quasiprimitive of almost simple or twisted wreath type, and that in either case $M = \mathrm{Soc}G$. We now consider two cases, depending on whether $C_G(N)$ is trivial or not. (We shall see that this is equivalent to asking whether (Γ, NG) is special or not.)

First we suppose that $C_G(N)$ is nontrivial whence it must contain M. (Observe that M is then a normal subgroup of NG whence (Γ, NG) is a special pair; conversely if (Γ, NG) is a special pair with a regular nonabelian minimal normal subgroup $\widehat{M}$, then $N \leq C_{NG}(\widehat{M})$ and so obviously $C_{NG}(N) \geq \widehat{M}$: by Lemma 6.2 $\widehat{M} = M \leq G$ and this case applies.) By identifying the vertices of Γ with elements of M (so that M acts by right multiplication and P, the stabilizer in G of id_M, acts by conjugation) we deduce that there exists a proper subgroup K of M and an isomorphism $K \to N$, $k \mapsto \tilde{k}$ such that the action of $\tilde{k}$ on the vertex $m \in M$ is given by

$$m * \tilde{k} = k^{-1}m.$$

By considering the action of $(\tilde{k})^p$ on $m \in M$ for $p \in P$ and $k \in K$, we see that $(\tilde{k})^p = (\widetilde{k^p})$ whence K is also a minimal nontrivial subgroup that is normalized by P. Let Y be the set of neighbors of id_M; then we may assume that Γ is the Cayley graph $\mathrm{Cay}(M, Y)$ on M with respect to Y and by considering the action of $N = \tilde{K}$ on edges we deduce that $K \leq N_G(Y)$. Recall that Y generates M; hence $C_G(Y) = C_G(M) = \{\mathrm{id}_G\}$, since $M = \mathrm{Soc}G$, and we have an embedding $KP \to \mathrm{Sym}Y$. Thus, KP on Y is a permutation group with a minimal normal subgroup K and a 2-transitive subgroup P, whence KP is itself 2-transitive. Given the classification of 2-transitive permutation groups this is an extremely strong situation. Note first that the socle of a 2-transitive permutation group is itself a minimal normal subgroup and is either a nonabelian simple group or is abelian. So suppose that K is a nonabelian simple group; then P is isomorphic to a subgroup of $\mathit{Out}K$ and by the Schreier conjecture is soluble. In the terminology of [12], (P, KP) is a (HA, AS)-inclusion of primitive permutation groups and by Proposition 5.1 of [12] is one of a limited number of possibilities. By inspection we see that no such examples exist. Hence K is abelian, the 2-transitive permutation group is of affine type, and (P, KP) is either a (AS, HA)-inclusion or a (HA, HA)-inclusion. As $\mathrm{Soc}KP = K \not\leq P$ whence $\mathrm{Soc}P \neq \mathrm{Soc}KP$, we deduce from Proposition 5.2 of [12] that $P = L_2(7)$ and $KP = AGL_3(2)$ as in our example. (Again note that we have again used the classification of finite simple groups as both the results of [12] and the Schreier conjecture depend upon it.)

Second suppose that $C_G(N)$ is trivial. Then we have an embedding of G

in the automorphism group of N. Recall that N is minimal subject to being normalized by G; hence N is characteristically simple and

$$AutN = \quad GL_d(p) \quad \text{or} \quad (AutE) \wr S_d,$$

for some prime p, nonabelian simple group E and integer d, depending on whether N is abelian or nonabelian. In the nonabelian case observe that as $|N| = |E|^d < |M|$ we must have

$$(AutE)^d \cap M = \{\mathrm{id}_M\},$$

whence we in fact have an embedding of G in S_d. Intuitively, one feels that $GL_d(p)$ and S_d are both too small to contain G – however, at present this intuition has refused to be incorporated into a proof. Certainly if N is isomorphic to a subgroup of M then we would be able to reach a contradiction in a similar fashion to Lemma 6.2. But all we know is the following: let Σ be the N-orbit on vertices that contains id_M; then $K = M_\Sigma$ is a subgroup of M that is regular on Σ and normalizes N. Unfortunately this is not enough information to determine the isomorphism type of K – for instance in $AGL_3(2)$ acting on its natural vector space V there exists a subgroup of order 8 in $L_2(7)$ that is regular on V yet not isomorphic to $\mathrm{Soc}AGL_3(2)$.

7. Final remarks

7.1. 3-Arc transitivity

In the introduction we noted that the 8-arc transitive graphs have been determined and then promptly jumped to the study of 2-arc transitive graphs: we would like to put forward some reasons justifying this jump. (In fact Weiss in [17] deduces a great deal assuming s-arc transitivity with $s \geq 4$ and so it is really only $s = 3$ that has been passed over.) It should first be realized that the analysis used in the $s \geq 4$ case is very different from that used in the present paper: [17] depends very much on knowledge of the detailed structure of the nonabelian simple groups, whereas in this paper the main results are notable for their independence of choice of nonabelian simple group with which to work. This suggests that if we were to try to specialize our results on special pairs to cases involving s-arc transitivity for $s \geq 3$, then at some point the mere simplicity of a nonabelian simple group T would cease to be enough and examples would also depend on the detailed structure of T. To see that $s = 3$ necessarily involves such problems, suppose that the nonabelian simple group T, 2-transitive permutation group P on Ω, and admissible map F give rise to a special pair (Γ, G) in which G acts 3-arc transitively on Γ. By considering the action of P on 3-arcs which start at id_G, we see that P is transitive on points that are at distance 3 from id_G. The argument given at the end of §3.4 of [14] may be generalized to show that

$$T = \{\mathrm{id}_T\} \cup F(\Omega) \cup F(\Omega)^2,$$

i.e., that every element of T may be written as a product of at most two involutions from $F(\Omega)$. This is obviously a very strong restriction on T.

7.2. *Primitive examples*

Here we consider the problem of determining those QP-special pairs (Γ, G) in which G is not only quasiprimitive, but in fact primitive on vertices. One approach is to start with our description of the QP-special pairs and then decide which satisfy the extra condition of primitivity. However, we feel that this is the wrong approach as not only is primitivity a very much stronger condition for twisted wreath products than that of acting 2-arc transitively on a graph but it is also very much more difficult to work with. Instead it seems better to first consider the primitivity of twisted wreath products $Ttwr_\phi P$ where P is a 2-transitive permutation group, to second decide which of these have a doubly transitive subconstituent, and then last look for graphs admitting such groups as 2-arc transitive groups of automorphisms. A start is made on this problem in [1] where we look at the case $P = S_n$. We do note however that primitive examples do exist.

Example 7.1. We start by constructing a primitive permutation group of twisted wreath type. Let $T = A_8$ and let σ be a fixed-point free involution in T. Note that $C_{S_8}(\sigma) \cong S_2 \wr S_4$ is a maximal subgroup of S_8. Set $N = |S_8 : C_{S_8}(\sigma)|$ and view S_8 as a subgroup of $P = S_N$ via the primitive permutation action of S_8 by right multiplication on the cosets of $C_{S_8}(\sigma)$ in S_8; set $Q = S_8 \leq P$. Let $\phi : Q \to AutT$ be induced by the conjugation action of Q and define $X = Ttwr_\phi P$, the twisted wreath product of T by P with respect to ϕ.

We claim that X in its base group action is a primitive permutation group of twisted wreath type. By Lemma 2.4 we see that both $C_X(B)$ and $C_P(B)$ are trivial, where B is the base group of X. This shows that the base group action is faithful and also that $B = \mathrm{Soc}X$. Thus X is indeed a primitive permutation group of twisted wreath type provided only that P is a maximal subgroup of X. Now by [9] we see that Q is a maximal subgroup of P. The maximality of P now follows from Lemma 3.1 of [2], or from Theorem 1.1 of [6].

We now construct a graph on which X acts 2-arc transitively via its base group action on vertices. Let Ω be a set of N points on which P acts naturally so that P on Ω is a 2-transitive permutation group. Choose $\omega \in \Omega$ so that $Q \cap P_\omega = C_Q(\sigma)$. Define a map $F : \Omega \to T$ by

$$F(\omega q) = \sigma^q \qquad \text{for all } q \in Q.$$

This is well-defined as Q is transitive on Ω and as $P_\omega \cap Q \leq C_Q(\sigma)$. Also F is admissible since σ is an involution and as $F(\Omega)$ is the entire conjugacy class in

T containing σ. Let (Γ_F, X_F) be the special pair constructed via Construction 4.1. By inspecting the construction we see that $X_F = X$ and we have our required example.

7.3. *Covers*

Recall that in §5 we gave various conditions for the existence of examples of special pairs that fail to be QP-special. By Theorem 1.1 we obtain, on quotienting such pairs, new pairs that are QP-nice. Such examples may have implications for QP-nice pairs of other types. For instance this process applied to Example 5.2 yields a connected graph with 1,457,280 vertices that admits the Mathieu group M_{24} as a 2-arc transitive group of automorphisms.

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