

Two-body Dirac equation with a scalar linear potential

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A semirelativistic equation which describes the relative motion of a Dirac particle and its antiparticle, interacting through a scalar potential, linearly dependent on the relative distance, is investigated. The simplest case, corresponding to 1S_0 and 3P_0 solutions, of total angular momentum $J=0$ and positive and negative parities, respectively, is easily worked out for massless constituents. The case of massive constituents, which gives rise to a different regime, is also examined together with other salient features of our approach. A treatment of the light- and heavy-meson $J=0$ sectors is briefly discussed.

Phenomenological potential models based on the Dirac equation with a scalar central potential, linearly rising with distance, have been proposed in the literature.¹ Such potentials give rise to confinement since the Dirac particle gets heavier and heavier when its separation from the fixed center increases indefinitely. Furthermore, the confinement by a scalar potential is also known to be relativistically consistent. By this we mean that it is free of the disease, consisting in the tunneling of the particle through the otherwise confining potential, known as the Klein paradox.²

We note, on the other hand, that relativistic potential models have been applied to investigate the hadronic structure under the assumption that the constituent quarks (and/or antiquarks) are spin- $\frac{1}{2}$ Dirac particles which, in a first approximation, move independently of each other inside the confining potential. An obvious feature of this "independent quark approach" is the inherent necessity of subtracting the spurious energy due to the center-of-mass motion of the system.³ In the present work the need of these center-of-mass corrections is circumvented *ab initio* if we restrict ourselves to mesons, described by a two-body Dirac equation formulated in the center-of-mass frame of the quark-antiquark system. As we shall see, such an approach, when implemented by the introduction of Lorentz-scalar linear potential terms for both a quark and an antiquark, seems to provide a convenient basis for a treatment of mesons in which relativistic effects are incorporated in a simple and direct way.

Having this in mind, let us consider a system consisting of a spin- $\frac{1}{2}$ Dirac particle (say 1) and its antiparticle (say 2), of common rest mass m and obeying, in the center-of-mass reference system, the equation ($\hbar=c=1$)

$$[-i\alpha_1 \cdot \nabla + \beta_1(m + \lambda r) + i\alpha_2 \cdot \nabla + \beta_2(m + \lambda r) - W]\psi(\mathbf{r}) = 0. \quad (1)$$

In Eq. (1), a linear scalar potential λr is introduced in both particle and antiparticle mass terms. The operator $-i\nabla$ appearing in it is the momentum conjugate to

$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $r = |\mathbf{r}|$. The wave function $\psi(\mathbf{r})$ is a 16-component spinor ψ_{ij} , the subscripts i and j ($i, j = 1, 2, 3, 4$) referring to particle (1) and antiparticle (2), respectively. We note that Eq. (1) may be considered as a special case of the Kemmer equation⁴ which was also applied by Fermi and Yang⁵ to pions, regarded as a bound state of a nucleon and an antinucleon.

We shall confine our attention here to solutions of Eq. (1) corresponding to $J=0$, J being the total angular momentum of the system. These solutions are labeled as 1S_0 and 3P_0 and have negative and positive parities, respectively.

Several formalisms⁶ exist in order to derive the radial equations relative to Eq. (1). For the present purposes it is reasonable to adopt the more compact treatment given by Moseley and Rosen⁷ and Nagasaki.⁸ From their work, we know that there are *five* linear combinations involving the 16 components ψ_{ij} , which are nonvanishing for the 1S_0 state. With respect to rotations, they are two scalars denoted by I and A_4 , and one vector $\mathbf{F} = (F_1, F_2, F_3)$. They obey the set of equations

$$\begin{aligned} WI + 2i\nabla \cdot \mathbf{F} + 4(m + \lambda r)A_4 &= 0, \\ W\mathbf{F} + 2i\nabla I &= 0, \quad WA_4 + 4(m + \lambda r)I = 0, \end{aligned} \quad (2)$$

where

$$I = f_1(r), \quad A_4 = f_2(r), \quad \mathbf{F} = f_3(r) \frac{\mathbf{r}}{r}. \quad (3)$$

It is easily seen that (2) yields the following system of radial equations:

$$\begin{aligned} Wf_1(r) + 2i \left[r \frac{d}{dr} + 3 \right] \frac{1}{r} f_3(r) + 2(m + \lambda r)f_2(r) &= 0, \\ Wf_2(r) + 2(m + \lambda r)f_1(r) &= 0, \\ Wf_3(r) + 2i \frac{d}{dr} f_1(r) &= 0. \end{aligned} \quad (4)$$

By eliminating the components f_2 and f_3 , one gets (with a prime denoting d/dr)

$$u'' + \frac{W^2}{4} \left[1 - 4 \frac{(m + \lambda r)^2}{W^2} \right] u(r) = 0, \quad (5)$$

where we set $u(r) = r f_1(r)$.

The energy eigenvalues W are obtained by means of the boundary conditions

$$u(0) = 0, \quad u(\infty) = 0. \quad (6)$$

For simplicity, we shall restrict our discussion to *massless* constituents ($m = 0$). In this case it is interesting to point out that, although we started from linear potentials in Eq. (1), the radial Eq. (5) corresponds to the radial Schrödinger equation for a three-dimensional isotropic harmonic oscillator, in the S wave. This feature is clearly a consequence of the relativistic kinematics in which Eq. (1) was formulated.

The eigenfunctions $u(r)$, satisfying conditions (6), are given in standard form as

$$u(r) = r \exp \left[-\frac{\lambda}{2} r^2 \right] {}_1F_1 \left(\frac{1}{2} \left(\frac{3}{2} + a \right), \frac{3}{2}; \lambda r^2 \right). \quad (7)$$

In Eq. (7),

$$-a = \frac{W^2}{8\lambda} = 2n_r + \frac{3}{2}, \quad (8)$$

where n_r is a non-negative integer ($n_r = 0, 1, 2, \dots$). The corresponding eigenvalues W are given by

$$\frac{W}{\sqrt{\lambda}} = 2(4n_r + 3)^{1/2}. \quad (9)$$

The 1S_0 states, Eq. (3), are normalized by means of the relation

$$4\pi \mathcal{N}^2 \int_0^\infty (|f_1|^2 + |f_2|^2 + |f_3|^2) r^2 dr = 1.$$

For the lowest-energy state ($n_r = 0$), one finds $\mathcal{N} = (\lambda^3/4\pi^3)^{1/4}$. We note from (9) that W goes to zero as the interaction is cut out adiabatically to zero, as expected.

The state 3P_0 , of positive parity, can be worked out in a similar way,^{6,7} in terms of *seven* linear combinations, denoted as $J = f_4(r)$, $\mathbf{G} = f_5(r)\mathbf{r}/r$, and $\mathbf{U} = f_6(r)\mathbf{r}/r$. We get, with $g(r) = r f_5(r)$, the radial equation

$$g'' - \frac{2}{r^2} g(r) + \frac{W^2}{4} \left[1 - 4 \frac{(m + \lambda r)^2}{W^2} \right] g(r) = 0, \quad (10)$$

with the conditions $g(0) = g(\infty) = 0$.

We note that Eq. (10) differs from Eq. (5) only by the presence of a P -wave centrifugal term. Again, for *massless* constituents, the radial wave function $g(r)$ is of the form

$$g(r) = r^2 \exp \left[-\frac{\lambda}{2} r^2 \right] {}_1F_1 \left(\frac{1}{2} \left(\frac{5}{2} + a \right), \frac{5}{2}; \lambda r^2 \right), \quad (11)$$

corresponding to a tower of P -wave energy states

$$\frac{W}{\sqrt{\lambda}} = 2(4n_r + 5)^{1/2}. \quad (12)$$

Our next step is to consider *massive* constituents. It is

easy to see that the solutions of Eq. (5) can then be expressed in terms of *parabolic cylinder functions*⁹ $D_\nu(z)$:

$$u(r) = D_\nu(z), \quad (13)$$

with $z = (2\lambda)^{1/2}(r + m/\lambda)$ and $\nu = W^2/8\lambda - \frac{1}{2}$.

Thus, in view of (7), the energy eigenvalues can be obtained, for $m > 0$, as solutions of the equation

$$D_\nu(m\sqrt{2/\lambda}) = 0. \quad (14)$$

It is remarkable that the massive regime departs from the harmonic behavior, characteristic of the massless case, being described by the *positive* zeros of the parabolic cylinder functions, Eq. (14). In other words, as far as $J = 0$ solutions are concerned, we predict that light mesons and the different families of heavy mesons correspond to different regimes, associated, respectively, to the massless case and to each one of the massive cases, as discussed above. Similar behavior is also expected for the $J \neq 0$ solutions of Eq. (1) which, however, would require a separate discussion.

We now wish to point out that the present scheme can be extended in a straightforward way by introducing in Eq. (1), as in the case of the Kemmer equation, additional two-body interaction terms⁴ I_{12} , each one being characterized by its specific Lorentz nature: $I_{12} = \sum_{k=0}^5 b_k O_k V_k(r)$. Here, b_k are constants, V_k are given functions of r , and O_k are the operators

$$O_1 = \beta_1 \beta_2, \quad O_2 = \frac{1}{2}(1 - \alpha_1 \cdot \alpha_2),$$

$$O_3 = \frac{1}{2} \beta_1 \beta_2 (\sigma_1 \cdot \sigma_2 + \alpha_1 \cdot \alpha_2),$$

$$O_4 = \frac{1}{2} (\sigma_1 \cdot \sigma_2 - \Gamma_1 \Gamma_2),$$

and

$$O_5 = \beta_1 \beta_2 \Gamma_1 \Gamma_2,$$

with $\sigma = (i/2)\boldsymbol{\alpha} \times \boldsymbol{\alpha}$, and $\Gamma = -i\alpha_x \alpha_y \alpha_z$.

In brief, our prescription here is to treat the Kemmer interaction terms at the level of first-order perturbation theory, in order to avoid spurious singularities in the radial equations which would destroy the normalizability of the solutions, as discussed by Childers¹⁰ in a similar context, involving Breit interaction terms.

Finally, we remark that a generalization of Eq. (1) for the case of constituents of unequal masses can also be performed similarly.⁸ This corresponds to having a system consisting of a Dirac particle of given type and mass m_1 and an antiparticle of *another* type, of mass m_2 , a typical instance being, of course, the kaons belonging to the 0^- pseudoscalar light-meson sector. For the particular instance of 1S_0 solutions, instead of the radial Eq. (5), we now get

$$u'' + \frac{1}{4} \left[W - \frac{(m_1 - m_2)^2}{W} \right] \times \left[W - \frac{1}{W} (m_1 + m_2 + 2\lambda r)^2 \right] u = 0. \quad (15)$$

In the so-called *one-body limit*, when $W = W_1 + m_2$

and $m_1/m_2 \ll 1$, Eq. (15) results in $u'' + W_1(W_1 - 2\lambda r) = 0$, an equation which can be readily solved in terms of the Airy function $\text{Ai}(z)$. One finds for W_1 the expression

$$\frac{W_1}{\sqrt{\lambda}} = \sqrt{2} |a_n|^{3/4}, \quad (16)$$

where a_n is the n th zero of the function $\text{Ai}(z)$, $n = 1, 2, \dots$.

In conclusion, the following remarks are in order. If it is assumed that the $\eta(550) 0^-$ state of the pseudoscalar light-meson sector is described by the 1S_0 solution Eq. (3), then its correct mass can be obtained by taking $\sqrt{\lambda} = 0.158$ GeV, in the massless regime of ordinary u and d quarks. With the same value of λ , the kaon, as described by a 1S_0 negative-parity solution, Eq. (15), acquires its correct mass for $m_1 = 0$ and a strange-quark

mass $m_2 = 0.461$ GeV. Notice that, by using Eq. (14), we can get the mass of the $\eta_c 0^-$ state of the J/ψ family again with the same λ and a charmed-quark mass of 1.35 GeV.

Finally, we remark that an adequate choice of the two-body relativistic interaction terms I_{12} is required to obtain the spin-dependent interactions responsible for a correct description of the fine splitting of the 3P_J states in quarkonia and, particularly, of the transition rates for radiative decays involving heavy-meson excitations in both J/ψ and Υ spectroscopies.¹¹ However, a discussion of such topics, in the light of the present approach, requires a detailed discussion of the $J \neq 0$ solutions of Eq. (1), which are outside of the limited scope of this work.

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