Two-body momentum correlations in a weakly interacting one-dimensional Bose gas

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We analyze the two-body momentum correlation function for a uniform, weakly interacting, one-dimensional Bose gas. We show that the strong positive correlation between opposite momenta, expected in a Bose-Einstein condensate with a true long-range order, almost vanishes in a phase-fluctuating quasicondensate where the long-range order is destroyed. Using the Luttinger liquid approach, we derive an analytic expression for the momentum correlation function in the quasicondensate regime, showing (i) the reduction and broadening of the opposite-momentum correlations (compared to the singular behavior in a true condensate) and (ii) an emergence of *anti*correlations at small momenta. We also numerically investigate the momentum correlations in the crossover between the quasicondensate and the ideal Bose-gas regimes using a classical field approach and show how the anticorrelations gradually disappear in the ideal-gas limit.

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I. INTRODUCTION

Many-body correlation functions contain valuable information about the physics of quantum many-body systems and therefore their measurement constitutes an important probe of the correlated phases of such systems. In recent years, ultracold-atom experiments have shown that atomic correlations can be accessed via many experimental techniques, including high-precision absorption [1–3] or fluorescence imaging [4–6], detection of atom transits through a high-finesse optical cavity [7], single-atom detection using multichannel plate detectors [8–11] or scanning electron microscopy techniques [12], and the measurement of rates of two-body (photoassociation) [13] or three-body loss processes [14–16]. While the loss-rate measurements depend only on local correlations, the imaging and atom detection techniques typically depend on nonlocal correlations which are embedded in the atom-number fluctuations in small detection volumes (such as image pixels) or in the coincidence counts of time- and position-resolved atom detection events.

The development of these techniques have enabled the study of a wide range of phenomena in ultracold-atomic gases, including the Hanbury Brown–Twiss effect [7–10,17–19] and higher-order coherences [20], phase fluctuations in quasicondensates [21,22], superfluid-to-Mott-insulator transition [2,5,6,23], isothermal compressibility and magnetic susceptibility of Bose and Fermi gases [3,24–28], scale invariance of two-dimensional (2D) systems [29], the phase diagram of the 1D Bose gas [30,31], entanglement and spin squeezing in two-component and double-well systems [32–36], sub-Poissonian relative atom-number statistics [35,37,38], and violation of the Cauchy-Schwarz inequality with matter waves [39].

From a broad statistical-mechanics point of view, most of these measurements have so far given access to either equilibrium position-space density correlations or nonequilibrium momentum-space density correlations. In this paper, we address the question of equilibrium momentum-space density correlations [40] by focusing on the two-body correlation function

$$\mathcal{G}(k,k') = \langle \delta\hat{n}_k \delta\hat{n}_{k'} \rangle = \langle \hat{n}_k \hat{n}_{k'} \rangle - \langle \hat{n}_k \rangle \langle \hat{n}_{k'} \rangle, \qquad (1)$$

for a weakly interacting, uniform, 1D Bose gas. Here, $\delta\hat{n}_k = \hat{n}_k - \langle \hat{n}_k \rangle$ is the fluctuation in the population $\hat{n}_k$ of the state of momentum $\hbar k$ [see Eqs. (3), (4), and (6)].

To measure $\mathcal{G}(k,k')$ experimentally, one needs to analyze atomic density fluctuations in a set of momentum distributions. Single-shot momentum distributions of a 1D Bose gas, realizable by confining the atoms to highly anisotropic trapping potentials, can be acquired as follows. First, by turning off (or strongly reducing) the transverse confinement, one ensures that atom-atom interactions no longer play any role in the system dynamics. The longitudinal momentum distribution is unaffected by the turning off since, in 1D geometry, the turning-off time (which is on the order of the period of the transverse confining potential) is much smaller than the relevant time scales of the longitudinal (axial) motion of the atoms. Second, the longitudinal momentum distribution can, in principle, be measured using an expansion along the long axis, after switching off the longitudinal confinement, or by using a recently demonstrated technique of Bose-gas focusing [41–43] (see also [44,45]).

In the presence of a true long-range order, the Bogoliubov theory correctly describes the excitations of a Bose condensed gas, predicting strong positive correlations in $\mathcal{G}(k,k')$ between opposite momenta, $k' = -k$, for small $|k|$, as shown in Ref. [46] (see also [47]). However, true long-range order is destroyed by long-wavelength fluctuations in a 1D Bose gas [48]; for a large enough system, the gas lies in the so-called quasicondensate regime [49] where, while the density fluctuations are suppressed as in a true Bose-Einstein condensate (BEC), the phase still fluctuates along the cloud. In this paper, we show that when the system size becomes much larger than the phase correlation length, the positive correlations between the opposite momenta vanish. In the thermodynamic limit of an infinite quasicondensate, we find an analytic expression for $\mathcal{G}(k,k')$ and show that it develops zones of anticorrelation on the (k,k') plane. We also analyze the crossover from the quasicondensate to the ideal Bose-gas regime, using a classical field theory, and show how the behavior of $\mathcal{G}(k,k')$ undergoes a continuous transformation between the two limiting regimes.

The paper is organized as follows. In Sec. II we outline the generalities applicable to two-body momentum correlations for the uniform 1D Bose gas with contact interactions. Section III summarizes the known results in the regime of a true condensate. In Sec. IV, we show that the correlations between opposite momenta, which exist in the case of a true BEC, disappear in the quasicondensate regime. Here we first use a simple model of a quasicondensate (Sec. IV A), followed by the Luttinger liquid approach (Sec. IV B) leading to an exact analytic result for the two-body momentum correlation function. In Sec. V we describe the momentum correlations in the crossover from the quasicondensate up to the ideal Bose-gas limit, using a classical field method. We discuss the experimentally relevant aspects in Sec. VI, and conclude with a summary in Sec. VII.

II. GENERALITIES

We consider a uniform gas of bosons interacting via a pairwise δ-function potential in a 1D box of length L with periodic boundary conditions. In the second-quantized form, the Hamiltonian density is

$$\mathcal{H} = -\frac{\hbar^2}{2m}\hat{\psi}^\dagger\frac{\partial^2}{\partial z^2}\hat{\psi} + \frac{g}{2}\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} - \mu\hat{\psi}^\dagger\hat{\psi}, \tag{2}$$

where $\hat{\psi}(z)$ and $\hat{\psi}^\dagger(z)$ are the bosonic field operators, m is the mass of the particles, g is the interaction constant, and μ is the chemical potential. In the grand-canonical formalism that we are using, the equilibrium density $\rho = \langle\psi^\dagger\psi\rangle$ is fixed by μ and the temperature T, and the total number of particles is given by $N = \rho L$. Throughout this paper, we restrict ourselves to the weakly interacting regime, which corresponds to the dimensionless interaction parameter $\gamma = mg/\hbar^2\rho \ll 1$.

The momentum distribution $\langle\hat{n}_k\rangle$ and its correlation function $\mathcal{G}(k,k')$ are related to the first- and second-order correlation functions of the bosonic fields,

$$G_1(z_1,z_2) = G_1(z_1 - z_2) = \langle\hat{\psi}^\dagger(z_1)\hat{\psi}(z_2)\rangle \tag{3}$$

and

$$G_2(z_1,z_2,z_3,z_4) = \langle\hat{\psi}^\dagger(z_1)\hat{\psi}(z_2)\hat{\psi}^\dagger(z_3)\hat{\psi}(z_4)\rangle, \tag{4}$$

via the Fourier transforms

$$\langle\hat{n}_k\rangle = \frac{1}{L}\int\int_0^L dz_1dz_2e^{-ik(z_1-z_2)}G_1(z_1,z_2) \tag{5}$$

and

$$\mathcal{G}(k,k') = \frac{1}{L^2}\int\int\int\int_0^L d^4z e^{-ik(z_1-z_2)}e^{-ik'(z_3-z_4)} \times[G_2(z_1,z_2,z_3,z_4) - G_1(z_1,z_2)G_1(z_3,z_4)], \tag{6}$$

where $d^4z \equiv dz_1dz_2dz_3dz_4$. In Eq. (3), the dependence of $G_1(z_1,z_2)$ only on the relative coordinate $z_1 - z_2$ follows from the translational invariance of the system.

Several general statements about the momentum correlation function $\mathcal{G}(k,k')$ can be made, which are valid in any regime of the gas. First, the correlation function obeys the following sum rule:

$$\sum_{k,k'}\mathcal{G}(k,k') = \langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2, \tag{7}$$

where $\hat{N} = \int_0^L dx\hat{\psi}^\dagger(x)\hat{\psi}(x)$ is the total particle-number operator. This implies that within the canonical ensemble, one has $\sum_{k,k'}\mathcal{G}(k,k') = 0$. In the grand-canonical ensemble, the fluctuation-dissipation theorem, which connects the particle-number variance with the derivative of $\langle\hat{N}\rangle$ with respect to the chemical potential μ [3], gives

$$\sum_{k,k'}\mathcal{G}(k,k') = k_BT\frac{\partial N}{\partial\mu} = k_BTL\frac{\partial\rho}{\partial\mu}. \tag{8}$$

Second, $\mathcal{G}(k,k')$ possesses several symmetries. In thermal equilibrium, the position-space correlation functions are invariant by the simultaneous refection symmetry of all coordinates $z_i \to -z_i$. This symmetry and the bosonic commutation relations between the field operators imply, for periodic boundary conditions, that $\mathcal{G}(k,k')$ is symmetric around the axis $k' = k$ and around the axis $k' = -k$.

Finally, for systems that have correlation lengths much smaller than the system size L, the two-body momentum correlation function $\mathcal{G}(k,k')$ can be split into a "singular" part and a regular function. (We use the term singular in the sense of the Kronecker δ-function, which turns into the Dirac δ-function singularity in the thermodynamic limit of $L \to \infty$.) To show this, let us first note that if we assume the existence of a finite correlation length l_ϕ for the decay of the first-order correlation function $G_1(z_1,z_2)$, then the second-order correlation function $G_2(z_1,z_2,z_3,z_4)$ must have the following two asymptotic limits:

$$G_2(z_1,z_2,z_3,z_4) \simeq G_1(z_1 - z_2)G_1(z_3 - z_4)$$
$$\text{for } |z_1 - z_3| \gg l_\phi \text{ and } |z_1 - z_2|,|z_3 - z_4| \lesssim l_\phi, \tag{9}$$

and

$$G_2(z_1,z_2,z_3,z_4) \simeq G_1(z_1 - z_4)G_1(z_2 - z_3) + G_1(z_1 - z_4)\delta(z_2 - z_3)$$
$$\text{for } |z_1 - z_2| \gg l_\phi \text{ and } |z_1 - z_4|,|z_2 - z_3| \lesssim l_\phi. \tag{10}$$

In Eq. (10), the δ-function term appears simply as a result of normal ordering of the operators in Eq. (4). By separating out the two asymptotic limits, Eqs. (9) and (10), we can write

$$\begin{aligned}G_2(z_1,z_2,z_3,z_4) = {}& G_1(z_1 - z_2)G_1(z_3 - z_4)\\ &+ G_1(z_1 - z_4)G_1(z_2 - z_3)\\ &+ G_1(z_1 - z_4)\delta(z_2 - z_3)\\ &+ \widetilde{G}_2(z_1,z_2,z_3,z_4),\end{aligned} \tag{11}$$

where $\widetilde{G}_2(z_1,z_2,z_3,z_4)$ is the remainder term.

By substituting Eq. (11) into Eq. (6), we obtain

$$\mathcal{G}(k,k') = (\langle\hat{n}_k\rangle + \langle\hat{n}_k\rangle^2)\delta_{k,k'} + \widetilde{\mathcal{G}}(k,k'), \tag{12}$$

which shows explicitly that $\mathcal{G}(k,k')$ can be written down as a sum of a singular and regular contributions. The first term in Eq. (12) is the shot noise, the second term is the bosonic "bunching" term, which describes the exchange interaction due to Bose quantum statistics, and the last, regular term $\widetilde{\mathcal{G}}(k,k')$ [the Fourier transform of $\widetilde{G}_2(z_1,z_2,z_3,z_4)$] describes the exchange of momenta between the particles during the binary elastic-scattering processes and is nonzero only for an interacting gas. For noninteracting bosons, Wick's theorem can be applied directly to the $G_2(z_1,z_2,z_3,z_4)$ function, which then leads to a vanishing $\widetilde{G}_2(z_1,z_2,z_3,z_4)$ and hence only to the singular terms in Eq. (12).

III. TRUE CONDENSATE

At $T = 0$, the first-order correlation function decays algebraically as $G_1(z_1,z_2) \simeq (\xi/|z_1 - z_2|)^{\sqrt{\gamma}/2\pi}$ for $|z_1 - z_2| \gg \xi$ [50–52], where $\xi = \hbar/\sqrt{mg\rho}$ is the healing length. As $\gamma \ll 1$ in the weakly interacting regime, the algebraic decay is very slow, leading to an exponentially large phase correlation length. Indeed, defining $l_\phi^{(0)}$ as the length for which $G_1(z_1,z_2)$ decreases by a factor of e, we find [49]

$$l_\phi^{(0)} \sim \xi e^{2\pi/\sqrt{\gamma}}. \tag{13}$$

At finite temperatures, the algebraic decay of $G_1(z_1,z_2)$ remains valid for distances $\xi \ll |z| \ll l_T$, where $l_T = \hbar^2/mk_BT\xi = (\hbar/k_BT)\sqrt{g\rho/m}$ is the phonon thermal wavelength [51,52]. For distances $|z| \gg l_T$, on the other hand, the correlation function decays exponentially [see Eq. (25) below] with the characteristic temperature-dependent phase coherence length,

$$l_\phi(T) = \hbar^2\rho/mk_BT. \tag{14}$$

Considering now a system of size $L \ll \min\{l_\phi^{(0)},l_\phi\}$, we can assume true long-range order in the system and use the Bogoliubov theory to describe the momentum correlations, as was done in Refs. [46,47]. We briefly recall the relevant results here.

The momentum correlation function $\mathcal{G}(k,k')$ is different from zero only for $k = k'$ and $k = -k'$. For equal momenta $k = k'$, one finds $\mathcal{G}(k,k) = \langle\hat{n}_k\rangle + \langle\hat{n}_k\rangle^2$, which is similar to the ideal Bose-gas behavior, except that the standard Bose occupation numbers $\langle\hat{n}_k\rangle = (e^{(E_k-\mu)/k_BT} - 1)^{-1}$ are now replaced by

$$\langle\hat{n}_k\rangle = (1 + 2\widetilde{n}_k)\frac{E_k + g\rho}{2\epsilon_k} - \frac{1}{2}. \tag{15}$$

Here $\epsilon_k = \sqrt{E_k(E_k + 2g\rho)}$ is the energy of the Bogoliubov modes, $E_k = \hbar^2k^2/2m$ is the free particle dispersion, and $\widetilde{n}_k = (e^{\epsilon_k/k_BT} - 1)^{-1}$ are the mean occupation numbers of Bogoliubov modes. For opposite momenta $k = -k'$, one has

$$\mathcal{G}(k,-k) = (1 + 2\widetilde{n}_k)^2\left(\frac{g\rho}{2\epsilon_k}\right)^2. \tag{16}$$

A convenient way to characterize the relative strength of the opposite and equal momentum correlations is via the normalized pair-correlation function

$$\mathcal{P}(k) = \frac{\mathcal{G}(k,-k)}{\mathcal{G}(k,k)} = 1 - \frac{\langle(\hat{n}_k - \hat{n}_{-k})^2\rangle}{2\langle\delta\hat{n}_k^2\rangle}. \tag{17}$$

Here, $\mathcal{P}(k) = 1$ corresponds to perfect (maximum) correlation between the opposite momenta, whereas $\mathcal{P}(k) = 0$ corresponds to the absence of any correlation.

At $T = 0$, one has $\widetilde{n}_k = 0$ and $\mathcal{G}(k,-k) = \mathcal{G}(k,k)$, and therefore the Bogoliubov theory predicts perfect correlation between the opposite momenta, $\mathcal{P}(k) = 1$. Such perfect correlation stems from the fact that the depletion of the condensate in the Bogoliubov vacuum simply corresponds to the creation of pairs of particles with equal but opposite momenta.

At finite temperatures, $\widetilde{n}_k$ is different from zero, nevertheless the normalized pair correlation is still close to its maximum (perfect correlation) value, $\mathcal{P}(k) \simeq 1$, for phonon excitations with $k \ll 1/\xi$ for any value of $\widetilde{n}_k$. This can be understood from the fact that the phonons are mainly phase fluctuations, so that they correspond to equal-weighted sidebands at momenta k and $-k$ of the excitation spectrum. On the other hand, for particlelike excitations, with $k \gg 1/\xi$, the thermal population of particles leads to a decrease of $\mathcal{P}(k)$. More precisely, for $1/\xi < k < \sqrt{mk_BT}/\hbar$, which corresponds to particlelike excitations whose occupation numbers are large, one obtains $\mathcal{P}(k) \ll 1$. Finally, at very large momenta, $k \gg \sqrt{mk_BT}/\hbar$, for which the occupation numbers are negligibly small, one again recovers the zero-temperature result $\mathcal{P}(k) \simeq 1$.

IV. QUASICONDENSATE REGIME

A. Effect of phase fluctuations

The above results obtained using the Bogoliubov theory are valid when the temperature is small enough so that the phase correlation length is much larger than the system size, $l_\phi \gg L$. While this condition is easier to satisfy in 3D or quasi-1D systems, it is generally not fulfilled for purely 1D gases.

In a large enough 1D system or at high enough temperatures, the long-range order is destroyed by long-wavelength phase fluctuations, having a characteristic temperature-dependent correlation length l_ϕ. When $l_\phi \ll L$, such a system is said to enter into the so-called quasicondensate regime [49], in which the density fluctuations are suppressed while the phase still fluctuates. As we show here, the two-body correlation between opposite momenta is expected to vanish in the quasicondensate regime.

To give a crude, yet simple estimate of the two-body momentum correlations, we can divide the system into domains of length l_ϕ and assume that (i) within each domain, the spatial variation of the phase is small, and therefore the Bogoliubov approach for a true condensate can be applied to each domain, and (ii) the relative phases between two different domains are uncorrelated. For each domain, indexed by α, the field operator $\hat{\psi}_\alpha(z)$ can be expanded according to the Bogoliubov theory,

$$\hat{\psi}_\alpha(z) = e^{i\phi_\alpha}\left(\sqrt{\rho} + \frac{1}{\sqrt{l_\phi}}\sum_{k\neq 0}\delta\hat{\psi}_{\alpha,k}\,e^{-ikz}\right), \tag{18}$$

where the first term is the mean-field component, the second term is the fluctuating component expanded in terms of plane-wave momentum modes $\delta\hat{\psi}_{\alpha,k}$, ϕ_α is the mean global phase of the domain assumed to be a random variable distributed uniformly between 0 and 2π, and the summation is over the momenta that are quantized in units of $2\pi/l_\phi$.

Using the fact that the momentum component $\hat{\psi}_k = \frac{1}{\sqrt{L}}\int_0^L dz\hat{\psi}(z)e^{ikz}$ of the full field $\hat{\psi}(z)$ can be decomposed as $\hat{\psi}_k = \sqrt{l_\phi/L}\sum_\alpha \delta\hat{\psi}_{\alpha,k}e^{i\phi_\alpha}$ for $k \neq 0$, we obtain the following expression for the momentum correlation function:

$$\langle \hat{n}_k \hat{n}_{k'} \rangle = \left(\frac{l_\phi}{L}\right)^2 \sum_{\alpha\beta\gamma\delta} \langle \delta\hat{\psi}^\dagger_{\alpha,k} \delta\hat{\psi}_{\beta,k} \delta\hat{\psi}^\dagger_{\gamma,k'} \delta\hat{\psi}_{\delta,k'} \rangle \times \overline{e^{-i(\phi_\alpha - \phi_\beta + \phi_\gamma - \phi_\delta)}} \quad (k,k' \neq 0). \tag{19}$$

Here, the overline above the exponential factor stands for averaging over the random mean phases of different domains.

Within the Bogoliubov theory, the Hamiltonian is quadratic in $\delta\hat{\psi}_{\alpha,k}$ and one can use Wick's theorem to evaluate the four-operator correlation function in Eq. (19). Only pairs of operators belonging to the same domain give a nonzero contribution since different domains are uncorrelated. Among these pairs, only terms $\langle \delta\hat{\psi}^\dagger_{\alpha,k} \delta\hat{\psi}_{\alpha,k} \rangle$, $\langle \delta\hat{\psi}_{\alpha,k} \delta\hat{\psi}_{\alpha,-k} \rangle$, and $\langle \delta\hat{\psi}^\dagger_{\alpha,k} \delta\hat{\psi}^\dagger_{\alpha,-k} \rangle$ survive.

To evaluate these terms in the most transparent way, we make use of the classical field approximation [53] (see also Sec. V A), treating the operators $\delta\hat{\psi}_{\alpha,k}$ and $\delta\hat{\psi}^\dagger_{\alpha,k}$ as c-numbers, $\delta\psi_{\alpha,k}$ and $\delta\psi^*_{\alpha,k}$, and assuming that the respective mode occupations $\langle \hat{n}_k \rangle = \langle \delta\psi^*_{\alpha,k} \delta\psi_{\alpha,k} \rangle$ are much larger than one, $\langle \hat{n}_k \rangle \gg 1$. For $k \ll 1/\xi$, the excitations in each domain are almost purely phase fluctuations so that $\delta\psi_{\alpha,-k} = -\delta\psi^*_{\alpha,k}$ and therefore $\langle \delta\psi^*_{\alpha,k} \delta\psi^*_{\alpha,-k} \rangle \simeq -\langle \hat{n}_k \rangle$ [54]. As a result, for the regular part of the momentum correlation function, we obtain

$$\widetilde{\mathcal{G}}(k,k') \simeq \delta_{k,-k'} \left(\frac{l_\phi}{L}\right)^2 \sum_{\alpha,\beta} \overline{e^{-2i(\phi_\alpha - \phi_\beta)}} \langle \hat{n}_k \rangle^2. \tag{20}$$

Averaging over the phases gives $\overline{e^{-2i(\phi_\alpha - \phi_\beta)}} = \delta_{\alpha,\beta}$, which singles out only the diagonal in α and β terms; there are L/l_ϕ such terms in the sum in Eq. (20). Accordingly, for the correlation function with opposite momenta, we find $\mathcal{G}(k,-k) \simeq (l_\phi/L)\langle n_k \rangle^2$, whereas the correlation function for equal momenta is given by $\mathcal{G}(k,k) = \langle n_k \rangle^2 + \langle n_k \rangle \simeq \langle n_k \rangle^2$, for $\langle n_k \rangle \gg 1$. Therefore, for the normalized pair correlation $\mathcal{P}(k)$, we obtain the following simple result:

$$\mathcal{P}(k) \underset{k \ll 1/\xi}{\simeq} \frac{l_\phi}{L} \ll 1, \tag{21}$$

which shows that the correlations between the opposite momenta are inversely proportional to the system size L and therefore are vanishingly small for $L \gg l_\phi$.

The above simple model is not capable of capturing features of $\widetilde{\mathcal{G}}(k,k')$ on momentum scales smaller than, or of the order of, the inverse phase correlation length, $k \lesssim 1/l_\phi$. For such momenta, the two-body correlation function is calculated below using a more rigorous Luttinger liquid approach. The results obtained within this approach confirm the simple scaling behavior obtained in Eq. (21). Moreover, the Luttinger liquid results show that the correlation function between different momenta is no longer singular on the antidiagonal $k' = -k$ and that it develops zones of *anti*correlation.

B. Two-body correlations in the Luttinger liquid approach

The condition for the quasicondensate regime [55] is

$$T \ll T_{\rm co} \equiv \sqrt{\gamma} \frac{\hbar^2 \rho^2}{2mk_B}. \tag{22}$$

In this regime, the correlation functions in Eqs. (3) and (4) are dominated by the long-wavelength (low-energy) excitations and the Hamiltonian density reduces to that of the Luttinger liquid [52,56]

$$\mathcal{H}_L = \frac{g}{2}(\delta\hat{\rho})^2 + \frac{\hbar^2 \rho}{2m}(\partial_z \hat{\phi})^2. \tag{23}$$

Here, $\delta\hat{\rho}(z)$ is the operator describing the density fluctuations, canonically conjugate to the phase operator $\hat{\phi}(z)$, with the commutator $[\delta\hat{\rho}(z),\hat{\phi}(z')] = i\delta(z-z')$.

The density fluctuations are small in the quasicondensate regime and, as long as the relative distances considered are much larger than the healing length ξ, they can be neglected when calculating the correlation functions (3) and (4) [51,52,57,58]. As the Luttinger liquid Hamiltonian given by Eq. (23) is quadratic in $\hat{\phi}$, the first-order correlation function $G_1(z_1,z_2) = \rho\langle e^{i(\hat{\phi}(z_1) - \hat{\phi}(z_2))} \rangle$ can be expressed through the mean-square fluctuations of the phase using Wick's theorem,

$$G_1(z_1,z_2) = \rho e^{-\frac{1}{2}\langle (\hat{\phi}(z_1) - \hat{\phi}(z_2))^2 \rangle}. \tag{24}$$

Neglecting the contribution of vacuum fluctuations compared to thermal ones [59], the calculation of the mean-square phase fluctuations leads to an exponentially decaying first-order correlation function [51,52,58],

$$G_1(z_1,z_2) = \rho e^{-|z_1 - z_2|/2l_\phi} \quad (|z_1 - z_2| \gg \xi). \tag{25}$$

This defines the finite-temperature phase coherence length l_ϕ, given by Eq. (14), and leads to a Lorentzian distribution for the momentum mode occupation numbers,

$$\langle \hat{n}_k \rangle = \frac{4\rho l_\phi}{1 + (2l_\phi k)^2}, \tag{26}$$

valid for $k \ll 1/\xi$.

Similarly, the two-body correlation function $G_2(z_1,z_2,z_3,z_4) = \rho^2 \langle e^{i[\hat{\phi}(z_1) - \hat{\phi}(z_2) + \hat{\phi}(z_3) - \hat{\phi}(z_4)]} \rangle$ can be represented in terms of the first-order correlation as

$$G_2(z_1,z_2,z_3,z_4) = \frac{G_1(z_1-z_2)G_1(z_3-z_4)G_1(z_1-z_4)G_1(z_2-z_3)}{G_1(z_1-z_3)G_1(z_2-z_4)}. \tag{27}$$

By substituting Eqs. (25) and (27) into Eq. (6), we find that the two-body momentum correlation function indeed has the form of Eq. (12) [60], in which the regular part $\widetilde{\mathcal{G}}(k,k')$ can be written as

$$\widetilde{\mathcal{G}}(k,k') = \frac{l_\phi}{L}(\rho l_\phi)^2 \mathcal{F}(2l_\phi k, 2l_\phi k'), \tag{28}$$

where $\mathcal{F}(q,q')$ is a dimensionless function given by

$$\mathcal{F}(q,q') = \frac{256}{(q^2+1)^2(q'^2+1)^2[(q+q')^2+16]} \times [(q^2 + 3qq' + q'^2)qq' - 2(q^2 - qq' + q'^2) - 7], \tag{29}$$

with $q \equiv 2l_\phi k$ and $q' \equiv 2l_\phi k'$. We note that the restriction of these results to $k \ll 1/\xi$ implies $q \ll l_\phi/\xi = 2T_{\rm co}/T$, and that the scaling of $\widetilde{\mathcal{G}}(k,k')$ with the inverse size of the system L coincides with the one obtained in Eq. (21).

We now wish to check the constraints on the function $\widetilde{\mathcal{G}}(k,k')$ imposed by the sum rule, given by Eq. (8). In evaluating the different terms on the left- and right-hand sides of Eq. (8), we note that (i) for the derivative term, we can use the equation of state for the quasicondensate regime, $\rho = \mu/g$, (ii) the term $\sum_k \langle \hat{n}_k \rangle$ [coming from the singular part of $\mathcal{G}(k,k')$] is given simply by $\sum_k \langle \hat{n}_k \rangle = N = \rho L$, and (iii) the term $\sum_k \langle \hat{n}_k \rangle^2$ can be evaluated using Plancherel's theorem and Eq. (25). As a result, the sum rule is reduced to [61]

$$\frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} dq dq' \mathcal{F}(q,q') \simeq -8 + \left(\frac{T}{T_{\rm co}}\right)^2, \qquad (30)$$

where the contribution of the $\sum_k \langle \hat{n}_k \rangle$ is ignored on the grounds that it is of the order of T/T_d (where $T_d = \hbar^2\rho^2/2mk_B$), which is always much smaller than unity in the entire range of temperatures $T \lesssim T_{\rm co}$.

The evaluation of the integral on the left-hand side of Eq. (30) gives the value of -8, implying that the sum rule is indeed approximately satisfied as long as $T \ll T_{\rm co}$, i.e., deep in the quasicondensate regime. On the other hand, as the temperature increases and approaches the quasicondensation crossover $T_{\rm co}$, the term $(T/T_{\rm co})^2$ on the right-hand side of Eq. (30) becomes non-negligible, implying that our result for the pair-correlation function $\widetilde{\mathcal{G}}(k,k')$, given by Eqs. (28) and (29), is no longer valid as it fails to satisfy the sum rule [62]. The physical origin of this failure lies in the fact that the density fluctuations at temperatures near $T_{\rm co}$ are no longer negligible.

The two-body correlation function $\mathcal{G}(k,k')$, given by Eq. (12), in the quasicondensate regime, of which the regular part $\widetilde{\mathcal{G}}(k,k')$ is described by the universal dimensionless function $\mathcal{F}(q,q')$, is one of the key results of this paper. The function $\mathcal{F}(q,q')$ is shown in Figs. 1(a)–1(c); as $\mathcal{F}(q,q')$ is independent of the system size L, it essentially describes the (unnormalized) two-body momentum correlations in the thermodynamic limit. As we see, the correlation function is nonzero on the entire 2D plane of momentum pairs (k,k'); this can be contrasted with the singular behavior of correlations in the true condensate where $\mathcal{G}(k,k')$ was nonzero only for $k' = \pm k$. This effective broadening of correlations is the first consequence of large phase fluctuations in the quasicondensate regime compared to the behavior in the true condensate. Next, $\widetilde{\mathcal{G}}(k,k')$ and $\mathcal{P}(k)$ both scale as l_ϕ/L and therefore are vanishingly small as $L \gg l_\phi$. For $k = -k'$, this means that the perfect opposite-momentum correlations [$\mathcal{P}(k) = 1$], which were present in the true condensate, essentially disappear in the quasicondensate regime. Finally, we find *negative* correlations (or anticorrelations) in $\widetilde{\mathcal{G}}(k,k')$; these are pronounced mostly in the regions of $k'k < 0$ [see Fig. 1(b)]. The correlations fall to zero on a typical scale of $k \sim 1/l_\phi$ ($q = 2l_\phi k \sim 1$). This is expected, as l_ϕ is the length scale governing the first-order spatial correlation function $G_1(z_1,z_2)$ in the quasicondensate regime, and the momentum correlations depend only on $G_1(z_1,z_2)$ in this regime.

To gain further insights into the strength of the two-body correlations, we consider the *normalized* regular part of the two-body correlation function,

$$\widetilde{g}^{(2)}(k,k') \equiv \frac{\widetilde{\mathcal{G}}(k,k')}{\langle \hat{n}_k \rangle \langle \hat{n}_{k'} \rangle}. \qquad (31)$$

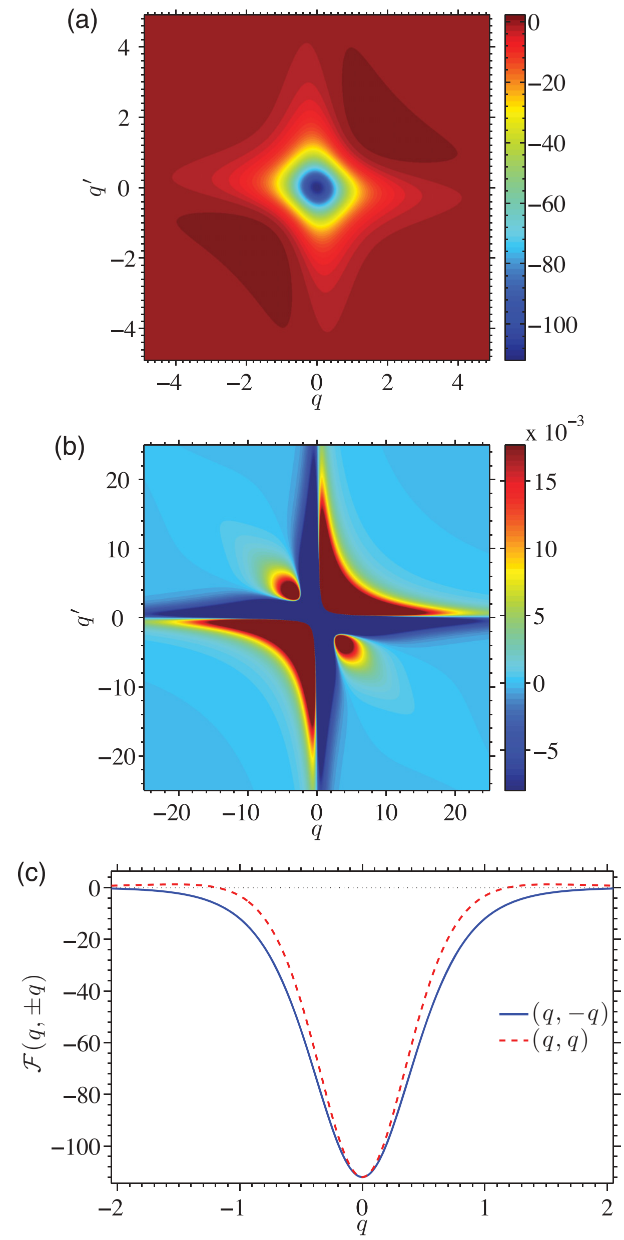

FIG. 1. (Color online) (a) Dimensionless regular part of the (unnormalized) two-body momentum correlation function, $\mathcal{F}(q,q')$, given by Eq. (29), of a uniform 1D Bose gas in the quasicondensate regime. (b) Same as in (a), but showing the details at small correlation amplitudes (see the scale on the color bar) and in a larger window of values of (q,q'). The small negative and positive amplitudes seen here get "magnified" when the function $\mathcal{F}(q,q')$ is normalized to $\langle \hat{n}_k \rangle \langle \hat{n}'_k \rangle$, as is done in Fig. 2. (c) Function $\mathcal{F}(q,q')$ along the diagonal ($q = q'$) and antidiagonal ($q = -q'$).

Using Eqs. (28) and (26), this can be rewritten as

$$\widetilde{g}^{(2)}(k,k') = \frac{l_\phi}{L} f(2l_\phi k, 2l_\phi k'), \qquad (32)$$

where

$$f(q,q') = \frac{\mathcal{F}(q,q')}{16}(1+q^2)(1+q'^2) \qquad (33)$$

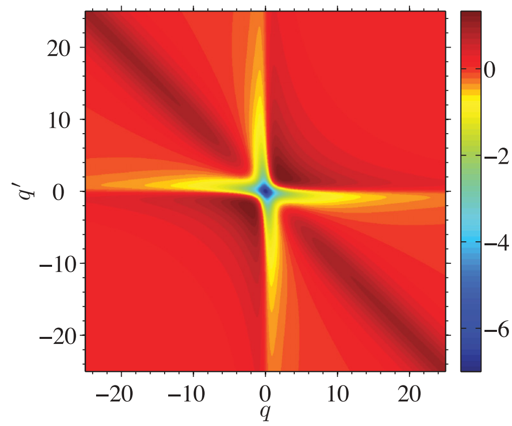

FIG. 2. (Color online) Normalized regular part of the two-body correlation function $f(q,q')$ as a function of the dimensionless momenta $q = 2l_\phi k$ and $q' = 2l_\phi k'$.

is a dimensionless universal function describing the two-body correlations of a 1D quasicondensate in the thermodynamic limit. The function $f(q,q')$ is plotted in Fig. 2. As we see, the normalization leads, at $k \gg 1/l_\phi$, to the recovery [cf. Fig. 1(b)] of positive correlations around the the antidiagonal $k' = -k$, predicted by the simple model of Sec. IV A. These correlations can be also thought of as the remnants of the nearly perfect correlations in a true condensate at $k \ll 1/\xi$.

Finally, we note that $\widetilde{g}^{(2)}(k,k')$ can be related to Glauber's normally ordered, second-order correlation function

$$g^{(2)}(k,k') = \frac{\langle \hat{\psi}_k^\dagger \hat{\psi}_{k'}^\dagger \hat{\psi}_{k'} \hat{\psi}_k \rangle}{\langle \hat{\psi}_k^\dagger \hat{\psi}_k \rangle \langle \hat{\psi}_{k'}^\dagger \hat{\psi}_{k'} \rangle}. \tag{34}$$

Indeed, by reordering the creation and annihilation operators, we can first express the $g^{(2)}(k,k')$ function in terms of the correlation function $\mathcal{G}(k,k')$, given by Eq. (1):

$$g^{(2)}(k,k') = 1 - \frac{1}{\langle \hat{n}_k \rangle}\delta_{k,k'} + \frac{\mathcal{G}(k,k')}{\langle \hat{n}_k \rangle \langle \hat{n}_{k'} \rangle}. \tag{35}$$

Using now the general structure of $\mathcal{G}(k,k')$ from Eq. (12) (valid for $L \gg l_\phi$), we obtain

$$g^{(2)}(k,k') = 1 + \delta_{k,k'} + \widetilde{g}^{(2)}(k,k'). \tag{36}$$

Here, the first term corresponds to uncorrelated atoms, the second term is the bosonic bunching term, and the last term is the normalized regular part corresponding to $\widetilde{\mathcal{G}}(k,k')$ given by Eq. (32).

According to our results, the normally ordered normalized correlation function for equal momenta is given by

$$g^{(2)}(k,k) = 2 + \widetilde{g}^{(2)}(k,k) = 2 + O(l_\phi/L), \tag{37}$$

while for opposite momenta it is given by

$$g^{(2)}(k,-k) = 1 + \widetilde{g}^{(2)}(k,-k) = 1 + O(l_\phi/L). \tag{38}$$

The small contributions $O(l_\phi/L)$ are described by Eq. (32) and are, in principle, detectable using the precision of currently available experimental techniques. Apart from the need for high precision on the signal, resolving the shape of the $\widetilde{g}^{(2)}(k,k')$ function requires experimental momentum resolution better than the separation $\Delta k = 2\pi/L$ between the individual momentum states (for resolutions that are insufficient to resolve separations of $\sim 1/L$, see Sec. VI).

As we see from Eq. (37), the amplitude of equal-momentum correlations is close to the pure thermal bunching level of $g^{(2)}(k,k) = 2$, implying large momentum-space density fluctuations. The nearly thermal level of correlations here is due to the large phase fluctuations present in a 1D quasicondensate. This makes the equal-momentum correlations analogous to those of a "speckle" pattern [18] where many sources with random phases contribute to the familiar Hanbury Brown–Twiss interference [8]. We emphasize, however, that the nearly thermal equal-momentum correlations are obtained here for a quasicondensate, which should be contrasted to the uncorrelated level of the two-point correlation function in position space [55,57], $g^{(2)}(z,z) \simeq 1$, due to the *suppressed* real-space density fluctuations. Equation (38), on the other hand, shows that the opposite-momentum correlations are close to the uncorrelated level of $g^{(2)}(k,-k) \simeq 1$, which is in contrast to the strong respective correlations [$g^{(2)}(k,-k) = 2 + 1/\langle \hat{n}_k \rangle$ at $T = 0$, and $g^{(2)}(k,-k) \simeq 2$ at finite T for $\langle \hat{n}_k \rangle \gg 1$] present in a true condensate. As we mentioned earlier, the opposite-momentum correlations are essentially destroyed by the phase fluctuations. Finally, a significant region of pairs of momenta $k' \neq k$ around the origin shows a small degree of anticorrelation, $g^{(2)}(k,k') < 1$, which was not *a priori* expected.

V. FROM THE QUASICONDENSATE TO THE IDEAL BOSE-GAS REGIME

A. Classical field approach

In the quasicondensate regime $T \ll T_{\mathrm{co}}$, higher-order correlation functions can always be expressed in terms of the first-order correlation function as in Eq. (27). Therefore, $\mathcal{G}(k,k')$ in Eq. (6) contains the same information as the momentum distribution $\langle \hat{n}_k \rangle$, given by Eq. (5). In particular, the dependence on the temperature comes about only through the phase correlation length l_ϕ. This is, however, no longer true when the temperature becomes of the order of the crossover temperature T_{co}, in which case the physics depends not only on the phase fluctuations, but also on the density fluctuations.

To compute the correlation functions at $T \gtrsim T_{\mathrm{co}}$, we resort to the classical field (or c-field) approach of Ref. [53]. In this approach, the quantum field operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ are approximated by c-number fields ψ and ψ^*, whose grand-canonical partition function (in a path-integral formulation) is given by

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\psi^* \exp\left(-\frac{1}{k_B T}\int_0^L dz\, \mathcal{H}_c\right). \tag{39}$$

Here the function $\mathcal{H}_c(\psi,\psi^*)$ is obtained from the Hamiltonian density (2) by replacing the operators with c-fields. The classical field approach is expected to be valid for high occupancy of the low-momentum modes contributing to the momentum correlation function. This condition is satisfied in a broad range of temperatures, including in the quasicondensate regime, $g\rho e^{-2\pi/\sqrt{\gamma}} < k_B T < \sqrt{\gamma}\hbar^2\rho^2/m$ [59,63], and up to the temperatures corresponding to the degenerate ideal Bose-gas regime, $\sqrt{\gamma}\hbar^2\rho^2/m < k_B T < \hbar^2\rho^2/m$ [53].

It is convenient to introduce a dimensionless field $\tilde{\psi} = \psi/\psi_0$ and a dimensionless coordinate $s = z/z_0$, with

$$\psi_0 = \left(\frac{mk_B^2T^2}{\hbar^2 g}\right)^{1/6}, \quad z_0 = \left(\frac{\hbar^4}{m^2 g k_B T}\right)^{1/3}, \tag{40}$$

and rewrite the effective "action" in Eq. (39) in the dimensionless form:

$$\frac{1}{k_B T}\int_0^L dz \mathcal{H}_c = \int_0^{L/z_0} ds \left(\frac{1}{2}|\partial_s \tilde{\psi}|^2 + \frac{1}{2}|\tilde{\psi}|^4 - \eta|\tilde{\psi}|^2\right). \tag{41}$$

This form of the action is controlled by a single dimensionless parameter

$$\eta = \left(\frac{\hbar^2}{mg^2k_B^2T^2}\right)^{1/3}\mu. \tag{42}$$

Because of the scaling relations (40), the density $\rho = \langle\psi^*\psi\rangle$ can be written as $\rho = h(\eta)(mk_B^2T^2/\hbar^2 g)^{1/3}$ using a dimensionless function $h(\eta) \equiv \langle\tilde{\psi}^*\tilde{\psi}\rangle$. Similarly, the phase correlation length l_ϕ, given by Eq. (14), can be written as $l_\phi = z_0 h(\eta)$. Thus, the length scale z_0 can be replaced by l_ϕ and therefore the one- and two-body correlation functions in Eqs. (3) and (4) scale as

$$G_1(z_1,z_2) = \rho\, h_1\left(\frac{z_1}{l_\phi},\frac{z_2}{l_\phi};\eta\right), \tag{43}$$

$$G_2(z_1,z_2,z_3,z_4) = \rho^2\, h_2\left(\frac{z_1}{l_\phi},\frac{z_2}{l_\phi},\frac{z_3}{l_\phi},\frac{z_4}{l_\phi};\eta\right), \tag{44}$$

where h_1 and h_2 are dimensionless functions.

By substituting these scaled correlation functions into Eq. (6) and using the fact that the integrand is invariant by a global translation of the coordinates, we find

$$\tilde{\mathcal{G}}(k,k') = \frac{l_\phi}{L}(\rho l_\phi)^2 \mathcal{F}(2l_\phi k, 2l_\phi k';\eta), \tag{45}$$

where $\mathcal{F}(q,q';\eta)$ is a dimensionless function parametrized by η. This relation generalizes Eq. (28) beyond the quasicondensate regime, with the departure being characterized by the value of η (see below).

To find $\mathcal{F}(q,q';\eta)$, we still need to calculate the dimensionless function h and correlations h_1 and h_2 for the rescaled fields $\tilde{\psi}$ and $\tilde{\psi}^*$, with the action given by Eq. (41). As shown in Ref. [53], this c-field problem can be mapped into the quantum-mechanical problem of a particle moving in an external potential. More precisely, expressing the action (41) in terms of the real and imaginary components of $\tilde{\psi} = x + iy$ and interpreting s as the imaginary time, the problem can be mapped to the quantum mechanics of a particle in two dimensions with the Hamiltonian

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + \tfrac{1}{2}(x^2+y^2)^2 - \eta(x^2+y^2). \tag{46}$$

Calculating the eigenvalues and matrix elements of this Hamiltonian allows one to compute the correlation function $\mathcal{F}(q,q';\eta)$. This is done in Appendix A.

B. Correlations in the crossover region

The power of the c-field approach lies in the ability to describe the momentum correlations not only in the quasicondensate regime, but also in the entire crossover region between the quasicondensate and the degenerate ideal Bose gas. As shown in Appendix B, in the quasicondensate regime where $\eta \gg 1$ (corresponding to a positive chemical potential μ), we recover the results of Sec. IV B, with Eq. (29) referring to $\mathcal{F}(q,q';+\infty) \equiv \mathcal{F}(q,q')$. The opposite limit $\eta \ll -1$ corresponds to the degenerate ideal Bose-gas regime with negative μ. In this case, the quartic term in the Hamiltonian (46) has a negligible effect on the lowest-energy eigenstates and the problem is reduced to a simple two-dimensional harmonic oscillator. As shown in Appendix C, in this limit, we obtain the ideal Bose-gas result of $\mathcal{F}(q,q';-\infty) = 0$.

In Fig. 3(a), we show how the antidiagonal correlation function $\mathcal{F}(q,-q;\eta)$ changes from its quasicondensate value of Eq. (29) to zero as η is continuously changed from $+\infty$ to $-\infty$. To quantify the width of the crossover in terms of η, we consider the peak value of the correlation function, $\mathcal{F}(0,0;\eta)$, and plot it as a function of η in Fig. 3(b). As we see, $\mathcal{F}(0,0;\eta)$ goes from its minimum value of about -112 in the quasicondensate regime ($\eta \gg 1$) to zero in the ideal Bose-gas regime ($\eta \ll -1$).

We can define the crossover region to correspond to $\eta_1 < \eta < \eta_2$, where the bounds η_1 and η_2 are chosen, respectively, at 20% and 80% of the value of $\mathcal{F}(0,0;\eta)$ in the quasicondensate regime; our numerical solutions give then $\eta_1 \simeq -1.1$ and $\eta_2 \simeq 2.0$. Recalling that the dimensionless parameter η is defined via Eq. (42), this can be converted into the crossover bounds

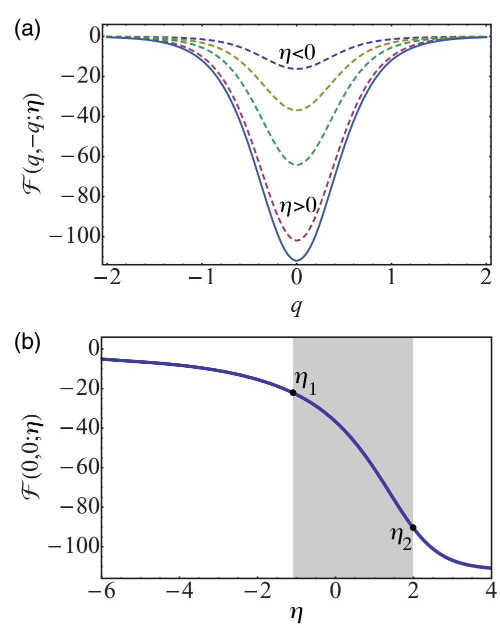

FIG. 3. (Color online) (a) The antidiagonal correlation function $F(q,-q;\eta)$ for (from top to bottom, dashed lines) $\eta = -1.87$, 0, 1.12, and 2.56. The lowest (solid) curve is for the limiting quasicondensate regime, $\mathcal{F}(q,-q) \equiv \mathcal{F}(q,-q;+\infty)$, described by Eq. (29) and shown in Fig. 1(c). (b) The minimum value of $F(q,-q;\eta)$ as a function of η. The shaded area shows the crossover region between $\eta_1 < \eta < \eta_2$ (see text).

on the chemical potential, $\eta_1 < \mu/\mu_{\rm co} < \eta_2$, where $\mu_{\rm co} \equiv k_B T (mg^2/\hbar^2 k_B T)^{1/3}$ [64]. Similarly, recalling that the density ρ was determined by the dimensionless function $h(\eta)$, via $\rho = h(\eta)(mk_B^2 T^2/\hbar^2 g)^{1/3}$, we can use the numerically found values of $h(\eta)$ to rewrite the crossover bounds in terms of the density as $0.5 < \rho/\rho_{\rm co} < 1.6$, where $\rho_{\rm co} \equiv (mk_B^2 T^2/\hbar^2 g)^{1/3}$ [64].

VI. EXPERIMENTAL CONSIDERATIONS

The results obtained so far are directly applicable to experimentally measured momentum distributions and correlation functions as long the momentum resolution is sufficient to resolve the individual momentum states separated by $\Delta k = 2\pi/L$. However, typical resolution in ultracold-atom experiments is insufficient to resolve momentum scales of the order of $1/L$. Because of this, the measured signal corresponds to integrated atom-number counts N_k in individual detection "bins" (such as camera pixels in absorption imaging) corresponding to the momentum k.

To address the situation with low momentum-resolution and relate our calculated correlation functions to the experimentally accessible quantities, we assume that the detection bin size Δ_k in momentum space fulfills $\Delta_k \gg 1/L$. In addition, we assume that $\Delta_k \ll 1/l_\phi$, so that the bulk of the momentum distribution is still well resolved. With these assumptions, the average (over many experimental runs) atom number in a bin $\langle N_k\rangle$ is related to the original average mode occupation number $\langle \hat{n}_k\rangle$ via $\langle N_k\rangle = (L\Delta_k/2\pi)\langle \hat{n}_k\rangle$, i.e., it accounts for a factor equal to the number of original momentum states contributing to the bin, $\Delta_k/\Delta k = L\Delta_k/2\pi$. Next, the average correlation between the bin population fluctuations is related to the correlation function $\mathcal{G}(k,k')$, given by Eqs. (12) and (28), via

$$\langle N_k N_{k'}\rangle - \langle N_k\rangle\langle N_{k'}\rangle = \langle N_k\rangle \delta_{k,k'} + \langle N_k\rangle\langle N_{k'}\rangle \left[\frac{2\pi}{L\Delta_k}\delta_{k,k'} + \frac{l_\phi}{L} f(2l_\phi k, 2l_\phi k')\right], \tag{47}$$

where the universal function $f(q,q')$ is given by Eq. (33). In Eq. (47), the first term is the shot noise, the second term corresponds to the bunching term in Eq. (12), and the third term is the contribution of the regular part, $\widetilde{\mathcal{G}}(k,k')$.

The shot-noise term is much smaller than the bunching term as long as highly populated momentum states are considered, i.e., $\langle \hat{n}_k\rangle \gg 1$. The latter condition is satisfied for momenta $k \lesssim 1/l_\phi$ (containing the bulk of the momentum distribution), and therefore the shot-noise term can be safely neglected for these momenta.

Comparing now the bunching term and the regular component (with the comparison being relevant only for $k = k'$), we see that they both scale inversely proportionally to the system size L (and therefore are small), but the regular component is much smaller than the bunching term as the dimensionless function $f(2l_\phi k, 2l_\phi k')$ is of the order of one and we have assumed $\Delta_k \ll 1/l_\phi$. However, the ratio of these two terms is independent of L and therefore is finite in the thermodynamic limit. As this ratio is proportional to $\Delta_k l_\phi \ll 1$, detecting the contribution of the regular component is going to depend on actual experimental parameters and the precision (signal-to-noise) with which the atom-number fluctuations can be measured. High-precision measurements of atom-number fluctuations, capable of resolving small signals like this or even below the shot-noise level, have been demonstrated in many ultracold-atom experiments [22,24–27,30,31,65].

Considering now the cross correlation between atom-number counts in different bins, $k' \neq k$, we see that the only contribution to Eq. (47) comes from the regular component. This scales as $1/L$, but again such a magnitude of the cross correlation should be accessible with state-of-the-art measurement techniques, as demonstrated, e.g., in Ref. [24].

VII. SUMMARY

To summarize, we have calculated the two-body momentum correlations for a weakly interacting, uniform, 1D Bose gas. Our results span the entire quasicondensate regime, where the correlations are derived analytically in terms of a universal dimensionless function $f(2l_\phi k, 2l_\phi k')$, as well as the crossover to the ideal Bose-gas regime, where the correlations are calculated numerically using the classical field method. A natural extension of the approaches employed here would be the calculation of these correlations for harmonically trapped gases [66], which is more appropriate for quantitative comparisons with experiments beyond the widely used local-density approximation. Calculating and understanding the momentum correlations in the strongly interacting regimes would require the development of alternative theoretical approaches and remains an open problem. The knowledge of such correlations is important in the studies of nonequilibrium dynamics from a known initial state and the subsequent thermalization in isolated quantum systems [67–70].

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APPENDIX A: CLASSICAL FIELD APPROACH: DIAGONALIZING THE EFFECTIVE HAMILTONIAN

In this appendix, we outline how the classical field correlation functions can be computed using the the equivalent quantum-mechanical problem of a particle in an external potential. We recall that the classical-to-quantum mapping is done by expressing the c-field $\psi = x + iy$ via its real and imaginary parts which, in turn, are treated as coordinates of a quantum-mechanical particle in imaginary time with the Hamiltonian given by Eq. (46). Here we imply the scaling of Eq. (40) but omit the tilde on top of the coordinates and c-fields for the sake of notational simplicity.

Using the notations of effective quantum mechanics, the first- and second-order correlation functions of the c-fields,

$$G_1(s_1,s_2) = \langle \psi^*(s_1)\psi(s_2)\rangle \tag{A1}$$

and

$$G_2(s_1,s_2,s_3,s_4) = \langle \psi^*(s_1)\psi(s_2)\psi^*(s_3)\psi(s_4)\rangle, \tag{A2}$$

are given by

$$G_1 = \frac{\mathrm{Tr}\big[U_{L-s_1'}\Psi_1 U_{s_1'-s_2'}\Psi_2 U_{s_2'}\big]}{\mathrm{Tr}[U_L]} \tag{A3}$$

and

$$G_2 = \frac{\mathrm{Tr}\big[U_{L-s_1'}\Psi_1 U_{s_1'-s_2'}\Psi_2 U_{s_2'-s_3'}\Psi_3 U_{s_3'-s_4'}\Psi_4 U_{s_4'}\big]}{\mathrm{Tr}[U_L]}, \tag{A4}$$

where we have omitted the arguments of G_1 and G_2 for notational brevity. Here $U_s = e^{-sH}$ is the imaginary-time evolution operator generated by the Hamiltonian (46). In Eq. (A4), we take into account the automatic time ordering implied by the path integral by introducing $s_1' \geqslant \cdots \geqslant s_4'$, which is the ordered permutation of $s_1,\ldots,s_4$. The operator Ψ_k stands for $\psi = x + iy$ if s_k' equals s_2 or s_4, or for $\psi^* = x - iy$ if s_k' equals s_1 or s_3.

In the limit $L \to \infty$, the ground state $|0\rangle$ gives the dominant contribution to both the numerator and denominator in Eq. (A4), and hence the correlation functions reduce to

$$G_1 = \langle 0|\Psi_1 U_{s_1'-s_2'}\Psi_2|0\rangle \tag{A5}$$

and

$$G_2 = \langle 0|\Psi_1 U_{s_1'-s_2'}\Psi_2 U_{s_2'-s_3'}\Psi_3 U_{s_3'-s_4'}\Psi_4|0\rangle, \tag{A6}$$

where we have set the ground-state energy to $\epsilon_0 = 0$.

The expectation values on the right-hand sides of Eqs. (A5) and (A6) are best evaluated in the eigenbasis of the Hamiltonian H. Let $|\alpha\rangle$ be the set of eigenstates of H with energy eigenvalues ϵ_α,

$$H|\alpha\rangle = \epsilon_\alpha|\alpha\rangle. \tag{A7}$$

The eigenstates $|\alpha\rangle = |n,m\rangle$ are classified by the principal (n) and angular momentum (m) quantum numbers such that in polar coordinates, the eigenfunctions

$$\langle r,\theta|\alpha\rangle = \frac{1}{\sqrt{2\pi}}\phi_n^m(r)e^{im\theta} \tag{A8}$$

obey the following eigenvalue equation:

$$\left[-\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{m^2}{2r^2} + \frac{r^4}{2} - \eta r^2\right]\phi_n^m(r) = \epsilon_n^m\phi_n^m(r)\,. \tag{A9}$$

In terms of the matrix elements

$$A_{\alpha\beta} = \langle\alpha|\psi|\beta\rangle = \langle\alpha|x+iy|\beta\rangle, \tag{A10}$$

the correlation functions are

$$G_1(s_1,s_2) = \sum_\alpha e^{-|s_1-s_2|(\epsilon_\alpha-\epsilon_0)}|A_{\alpha 0}|^2 \tag{A11}$$

and

$$G_2(s_1,s_2,s_3,s_4) = \sum_{\alpha\beta\gamma} e^{-K}A_{\alpha 0}^* A_{\alpha\beta}A_{\gamma\beta}^* A_{\gamma 0}, \tag{A12}$$

for the case $s_1 > s_2 > s_3 > s_4$, where we have defined

$$\begin{aligned} K = {} & \epsilon_0(s_4 - s_1) + \epsilon_\gamma(s_3 - s_4) \\ & + \epsilon_\beta(s_2 - s_3) + \epsilon_\alpha(s_1 - s_2). \end{aligned} \tag{A13}$$

For different orderings of s_1,s_2,s_3, and s_4, similar expressions can be obtained. Although there are, in general, $4! = 24$ cases to consider, by noticing that the expectation value for $G_2(s_1,s_2,s_3,s_4)$ remains invariant under the exchange of $s_1 \rightleftarrows s_3$, or $s_2 \rightleftarrows s_4$, and also under the simultaneous exchange of $s_1 \rightleftarrows s_2$ and $s_3 \rightleftarrows s_4$, we realize that it is sufficient to compute $G_2(s_1,s_2,s_3,s_4)$ in just three cases: $s_1 > s_2 > s_3 > s_4$, $s_1 > s_2 > s_4 > s_3$, and $s_1 > s_3 > s_2 > s_4$. The remaining 21 expressions can be obtained from these using symmetry considerations.

Solving the Schrödinger equation (A9) numerically and evaluating the matrix elements given by Eq. (A10) yields the correlation functions in Eqs. (A5) and (A6). It should be noted that the sums in Eqs. (A11) and (A12) contain only a finite number of terms because of the selection rule, $A_{\alpha\beta} \propto \delta_{m_\alpha,m_\beta+1}$, and the fact that the bra-ket states with a very large difference in the respective values of n give negligible matrix elements due to very different nodal structure.

APPENDIX B: QUASICONDENSATE LIMIT

In this appendix, we show that the classical field approximation correctly predicts the correlation functions in Eqs. (25) and (27), in the limit $\eta \gg 1$. In this limit, the wave functions of the lowest-lying states differ from zero significantly only for $r \simeq r_0 = \sqrt{\eta}$, so that the Hamiltonian becomes separable into the azimuthal and radial degrees of freedom. This has two consequences on the classical field calculations.

First, the wave functions $\phi_n^m(r)$ are approximately independent of m, while the azimuthal kinetic energy is reduced to

$$H \simeq -\frac{1}{2r_0^2}\frac{\partial^2}{\partial\theta^2} = \frac{m^2}{2r_0^2}. \tag{B1}$$

Accordingly, for the matrix elements (A10), we obtain

$$\begin{aligned} A_{\alpha\alpha'} &= \delta_{m,m'+1}\big\langle\phi_n^m\big|r\big|\phi_{n'}^{m'}\big\rangle \approx r_0\delta_{m,m'+1}\big\langle\phi_n^m\big|\phi_{n'}^{m'}\big\rangle \\ &= r_0\delta_{m,m'+1}\delta_{n,n'}. \end{aligned} \tag{B2}$$

This allows one to restrict summations in Eqs. (A11) and (A12) to just the leading term with $n_\alpha = n_\beta = n_\gamma = 0$.

Second, the fact that the energy eigenvalues ϵ_n^m separate into m-independent and angular parts,

$$\epsilon_n^m = \epsilon_n + \frac{m^2}{2r_0^2}, \tag{B3}$$

allows one to calculate the exponentially decaying terms for $G_1(s_1,s_2)$ in Eq. (A11). More precisely, the only relevant energy differences are

$$\epsilon_0^1 - \epsilon_0^0 = \frac{1}{2r_0^2} = \frac{1}{2\eta} \tag{B4}$$

and

$$\epsilon_0^2 - \epsilon_0^1 = \frac{3}{2r_0^2} = \frac{3}{2\eta} = 3\big(\epsilon_0^1 - \epsilon_0^0\big). \tag{B5}$$

Using Eqs. (B2)–(B5), we then find that Eq. (A11) reduces to $G_1(s_1,s_2) = \eta e^{-|s_1-s_2|/2\eta}$. Going back to natural units, using Eqs. (40) and (42), this gives the quasicondensate equation of state $\rho \simeq \mu/g$, and we recover Eq. (25) of the main text.

Considering now Eq. (A12), together with the other required cases for time ordering, a similar albeit more lengthy calculation shows that Eq. (A12) reduces to Eq. (27) of the main text.

APPENDIX C: IDEAL BOSE-GAS LIMIT

The limit $\eta \ll -1$ corresponds to the highly degenerate, ideal Bose-gas regime. In this case, the classical field problem can be mapped onto a two-dimensional quantum harmonic oscillator. Here we show how the classical field approximation recovers the correlation functions expected for the ideal Bose gas.

For $\eta \ll -1$, the Hamiltonian (46) becomes quadratic,

$$H \simeq \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + |\eta|(x^2 + y^2), \tag{C1}$$

and its matrix elements can be obtained from the standard results for the quantum harmonic oscillator with frequency $\omega = \sqrt{2|\eta|}$. We thus have

$$\langle\alpha|x|0\rangle = \langle\alpha|y|0\rangle = (2\omega)^{-1/2}, \tag{C2}$$

where $n_\alpha = 0$ and $m_\alpha = 1$ corresponds to the first excited state with energy $\epsilon_\alpha - \epsilon_0 = \omega = \sqrt{2\eta}$. Then, Eq. (A11) becomes

$$\begin{aligned} G_1(s_1,s_2) &= e^{-|s_1-s_2|(\epsilon_\alpha-\epsilon_0)}|\langle\alpha|x+iy|0\rangle|^2 \\ &= \frac{1}{\sqrt{2|\eta|}} e^{-\sqrt{2|\eta|}|s_1-s_2|}. \end{aligned} \tag{C3}$$

Going back to natural units, using Eqs. (40) and (42), we have $\rho = \psi_0^2/\sqrt{2|\eta|} = \sqrt{mk_B^2T^2/2\hbar^2|\mu|}$, and therefore

$$G_1(z_1,z_2) = \rho e^{-|z_1-z_2|mk_BT/\hbar^2\rho} = \rho e^{-|z_1-z_2|/l_\phi}, \tag{C4}$$

which is the result for a highly degenerate, ideal Bose gas.

The calculation of the G_2 function is more elaborate as there are different terms on the right-hand side of Eq. (A12) to compute, for different orderings of s_1,s_2,s_3, and s_4. However, the analogy with a simple harmonic oscillator makes the calculation possible, leading to the recovery of the Wick's theorem (valid for quadratic Hamiltonians), and therefore

$$\begin{aligned} G_2(s_1,s_2,s_3,s_4) &= G_1(s_1-s_2)G_1(s_3-s_4) \\ &\quad + G_1(s_1-s_4)G_1(s_2-s_3). \end{aligned} \tag{C5}$$

This immediately leads to the first two terms on the right-hand side of Eq. (11). The last (regular) term in Eq. (11) is identically zero in a noninteracting gas, whereas the third (δ-function) term, which comes from the commutator $[\psi(s_2),\psi^*(s_3)] = \delta(s_2-s_3)$, has a negligible contribution in the highly degenerate, ideal Bose-gas regime considered here.

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[54] The results that we obtain here, such as Eqs. (20) and (21), can be derived directly within the Bogoliubov approach for each domain (i.e., without invoking the additional classical field approximation), assuming $\langle \hat{n}_k \rangle \gg 1$.

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[59] Vacuum fluctuations have a non-negligible effect on the correlation function $G_1(z_1,z_2)$ if the characteristic distance l_T, over which $G_1(z_1,z_2)$ decays algebraically, is much larger than $l_\phi^{(0)}$, i.e., if $l_T \gg l_\phi^{(0)}$. This condition can be rewritten as $k_BT \ll g\rho \exp(-2\pi/\sqrt{\gamma})$, implying that the vacuum fluctuations are important only at temperatures that are exponentially small in the weakly interacting regime, $\gamma \ll 1$. Such temperatures are beyond the reach of current ultracold-atom experiments.

[60] The shot-noise term $\langle \hat{n}_k \rangle$ cannot be obtained in this approach, but it is negligible compared to $\langle \hat{n}_k \rangle^2$ as long as $k \ll \max\{k_BT/\hbar\sqrt{\rho g/m}, \sqrt{mk_BT}/\hbar\}$. The latter condition is fulfilled for $k \lesssim 1/l_\phi$.

[61] In obtaining this equation, we have approximated the discrete sums over $q = 2l_\phi k$ and $q' = 2l_\phi k'$ by continuous integrals, and extended the integration limits from $(-q_{\max}, q_{\max})$ to $(-\infty,\infty)$. This can be done because the maximum momentum cutoff $k_{\max} \sim 1/\xi$ in the Luttinger liquid theory [52] leads to $q_{\max} \sim l_\phi/\xi$, which is much larger than unity according to Eq. (22), and because the function $\mathcal{F}(q,q')$ decays to zero sufficiently fast.

[62] For a more accurate and consistent estimate of the breakdown of the sum rule as T increases and approaches T_{co}, one would need to evaluate the contribution of $\sum_k \langle \hat{n}_k \rangle^2$ to the right-hand side of Eq. (30) to leading order in T/T_{co}, which is, however, beyond the scope of this paper.

[63] The condition $g\rho e^{-2\pi/\sqrt{\gamma}} < k_BT < \sqrt{\gamma}\hbar^2\rho^2/m$ (see also [59]) follows from the fact that at these temperatures, the $G_1(z_1,z_2)$ function decays exponentially and as long as the (small-momentum) bulk of the momentum distribution is concerned, the mode occupancies here are indeed high. If, on the other hand, one is concerned with position-space correlations [57,58], then the (low) occupancies of large-momentum modes become relevant too, in which case the vacuum fluctuations become relevant as soon as $g\rho > k_BT$. Accordingly, for position-space correlations, the range of validity of the classical field theory in the quasicondensate regime is $g\rho < k_BT < \sqrt{\gamma}\hbar^2\rho^2/m$.

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