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TWO CELLS WITH N POINTS OF LOCAL NONCONVEXITY

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ABSTRACT. A subset S of the plane is a two cell provided S is homeomorphic to $\{x \mid \|x\| \leqq 1\}$.

THEOREM. *Let S be a two cell with exactly n points of local nonconvexity. Then S is expressible as a union of $n+1$ compact convex sets with mutually disjoint interiors.*

I. **Introduction.** We will prove that a two cell S in R^2 which has exactly n points of local nonconvexity is expressible as a union of $n+1$ compact convex sets with mutually disjoint interiors. It follows immediately that S is $n+1$ polygonally connected, i.e., an L_{n+1} set and thus we have as a corollary a special case of a result of Valentine [1].

Throughout, the symbols $\cup$, $\cap$ and $\sim$ denote set union, set intersection and set difference respectively. The interior, closure, and boundary of a set $S \subset R^2$ are denoted by int S, cl S and bd S, respectively. The convex hull of a set $S \subset R^2$ is denoted by $H(S)$. If $x, y \in S$ then $[xy]$ and (xy) denote the closed and open line segments joining x to y, respectively. The Euclidean norm is given by $\| \ \|$ and d denotes the Hausdorff metric on compact subsets of the plane as given in Valentine [2]. If S is a set, $|S|$ denotes its cardinality. We define the distance between a point x and a set S as $\inf\{\|x-y\| \mid y \in S\}$; we denote this distance by $p(x, S)$. By the ϵ ball about a set S we mean $\{x \mid p(x, S) < \epsilon\}$.

DEFINITION 1. A point $x \in S$ is called a point of *local convexity* of S if there exists a neighborhood N of x such that $N \cap S$ is convex. If such a neighborhood does not exist, x is called a point of *local nonconvexity*.

DEFINITION 2. A set S is said to be *starshaped* relative to a point p if for each $x \in S$, $[xp] \subset S$.

DEFINITION 3. A segment $[xy]$ is said to be a *crosscut* of a set S provided $x, y \in$ bd S and $(xy) \subset$ int S.

DEFINITION 4. A set $S \subset R^2$ is a *two cell* provided S is homeomorphic to $\{x \mid \|x\| \leqq 1\}$.

Given a two cell S and a crosscut $[xy]$ of S, $[xy]$ induces a natural decomposition of S into two new two cells D^1xy and D^2xy such that

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$D^1xy \cap D^2xy = [xy]$. For a proof of this, see Newman [1]. Specifically, the points x and y divide bd S into two disjoint relatively open connected subsets of bd S, say C^1xy and C^2xy, and D^1xy is the set whose boundary is given by $[xy] \cup C^1xy$ and D^2xy is the set whose boundary is given by $[xy] \cup C^2xy$. If S is a set, $C(S)$ and $L(S)$ will denote its points of local convexity and nonconvexity, respectively, and $\varnothing$ will denote the empty set. The following two theorems constitute the main results of this paper.

THEOREM 1. *Let $S \subset R^2$ be a two cell such that $|L(S)| = n$, $n \geqq 2$. Then there exists a crosscut $[xy]$ of S such that $x, y \in C(S)$ and $|L(D^ixy)| \leqq n-1$ for $i = 1, 2$.*

THEOREM 2. *Let $S \subset R^2$ be a two cell such that $|L(S)| = n$. Then $S = \bigcup_{i=1}^{n+1} C_i$ where C_i, $1 \leqq i \leqq n+1$, are compact, convex and have mutually disjoint interiors.*

Theorem 1 is of some independent interest, since it makes induction arguments readily accessible. Also, note that if $n = 0$ in Theorem 2, then the latter reduces to a special case of the important theorem of Tietze [1], which we state for later reference.

THEOREM 3. *Let S be a closed connected set in R^d, all of whose points are points of local convexity. Then S is convex.*

We also shall utilize the following result of Valentine [1].

THEOREM 4. *Let S be a closed connected set in R^d such that $L(S)$ is not empty. Then given $x \in S$, there exists $y \in L(S)$ such that $[xy] \subset S$.*

II. Preliminary results and proof of Theorems 1 and 2.

THEOREM 5. *Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of compact sets in R^2 converging to a set A in the Hausdorff metric d such that for each i, $A_i = \bigcup_{j=1}^{n+1} C_j^i$ where each C_j^i is convex and compact for $1 \leqq j \leqq n+1$ and* int $C_j^i \cap$ int $C_k^i = \varnothing$, *for $k \neq j$ and $1 \leqq k$, $j \leqq n+1$. Then $A = \bigcup_{j=1}^{n+1} C_j$ where C_j is compact and convex for $1 \leqq j \leqq n+1$ and* int $C_j \cap$ int $C_k = \varnothing$ *for $j \neq k$, $1 \leqq j, k \leqq n+1$.*

PROOF. By the theorem of Blaschke we may assume without loss of generality that for each j the sequence $\{C_j^i\}_{i=1}^{\infty}$ converges to a set C_j. Now C_j is compact and convex, since C_j is the limit of compact convex sets. Since for each i and $j \neq k$ int $C_j^i \cap$ int $C_k^i = \varnothing$, we have int C_j $\cap$ int $C_k = \varnothing$, and this completes the proof.

THEOREM 6. *Let $S \subset R^2$ be a two cell such that $|L(S)| = 1$. Then $S = C_1 \cup C_2$ where C_1 and C_2 are compact, convex and* int $C_1 \cap$ int $C_2 = \varnothing$.

PROOF. Throughout the proof let $L(S)=\{x\}$.

Case 1. Suppose there exist z, $q\in$ bd S, such that $z\neq x$, $q\neq x$, $[zx]\cup[xq]\subset$ bd S and $[zx]\cap[xq]=x$. Then clearly we may choose a crosscut $[xy]$ where $y\in C(S)$ such that $x\in C(D^1xy)\cap C(D^2xy)$. Since $y\in C(S)$, we have $y\in C(D^1xy)\cap C(D^2xy)$. Then D^1xy and D^2xy are convex by Tietze's Theorem and int $D^1xy\cap$int $D^2xy=\emptyset$ by definition. Thus D^1xy and D^2xy are the required sets.

Case 2. Suppose for each $y\in$ bd S, $y\neq x$, that $[xy]\not\subset$ bd S. For this case, we need a lemma.

LEMMA 1. *Let S be a two cell satisfying Case* 2. *Then if* $t\in$ bd $S, t\neq x$, $(xt)\subset$ int S.

PROOF. Suppose the lemma is false. By Theorem 4, S is starshaped relative to x, so $[xt]\subset S$. By hypothesis, $(xt)\not\subset$ bd S, so let $z\in(xt)\cap$int S. Since S is starshaped relative to x, for each $m\in(xt)\cap$int S we have $(xm]\subset$int S. Then, since we are denying the lemma, we may not find a sequence $\{z_i\}_{i=1}^{\infty}$ such that $z_i\in(xt)\cap$int S such that $\{z_i\}_{i=1}^{\infty}$ converges to t, for otherwise $(xt)=\bigcup_{i=1}^{\infty}(xz_i]$ and $\bigcup_{i=1}^{\infty}(xz_i]\subset$int S by the last sentence, so $(xt)\subset$int S, a contradiction. Thus let $q\neq t$ be the closest point of bd $S\cap[zt]$ to x. Then $[tq]\subset$bd S and $(qx)\subset$int S. Let $q_1\in(tq)$. Since q_1 is a point of local convexity, and since a two cell is the closure of its interior, we may choose $\epsilon>0$, so that $B(q_1,\epsilon)\cap S$ is convex and $D=\text{int}(B(q_1,\epsilon)\cap S)\neq\emptyset$, where $B(q_1,\epsilon)=\{p\mid \|p-q_1\|<\epsilon\}$. Let $\mathcal{C}=H(D\cup\{x\})\sim\{x\}$. Then $\mathcal{C}$ is an open set and $\mathcal{C}\subset$int S since x is a seeing point. Now let $\{r_i\}_{i=1}^{\infty}$ be a sequence of boundary points converging to q, such that $r_i\neq q$, $r_i\notin(tq)$ for each i. Then $\{r_i\}_{i=1}^{\infty}$ must approach q from the open half space generated by the line containing $[xt]$, opposite from the open half space containing $\mathcal{C}$. Choose $\delta>0$, so that $E=B(q,\delta)\cap S$ is convex. Then, clearly we have $\text{int}(E\cap\mathcal{C})\neq\emptyset$. Since $\{r_i\}_{i=1}^{\infty}$ converges to q, there exists an integer k, such that if $i\geq k$, we have $r_i\in E$. Let $x_1\in\text{int}(E\cap\mathcal{C})$ be such that $q\in(r_kx_1)\subset$int S. This contradicts that q is a boundary point.

Returning to Case 2, let f be the homeomorphism mapping the closed unit disc onto S. Then bd S is homeomorphic to the unit circle. Let $f(x^*)=x$. Let $y^*\neq x^*$ be on the unit circle. Then x^* and y^* naturally divide the unit circle into two nonempty, relatively open, disjoint, connected subsets, say $B^1_{x^*y^*}$ and $B^2_{x^*y^*}$. Let $\{x_i^1\}_{i=1}^{\infty}$ and $\{x_i^2\}_{i=1}^{\infty}$ be sequences of points in bd S such that the inverse images of x_i^1 and x_i^2 under f are in $B^1_{x^*y^*}$ and $B^2_{y^*x^*}$, respectively, for each i, and both sequences coverge to x. Now define the set S_i for each i to be that subset of S whose boundary is given by $[x_i^1x]\cup[xx_i^2]\cup f(B_i)$ where B_i is $B^1_{x_i^1x_i^2}$ if $x\notin B^1_{x_i^1x_i^2}$ and B_i is $B^2_{x_i^1x_i^2}$ if $x\notin B^2_{x_i^1x_i^2}$. Now since S is simply connected, $S_i \subset S$, and S_i for each i is well defined by Lemma 1. Then $\{S_i\}_{i=1}^{\infty}$ converges to S in the Hausdorff metric. Since for each i, $x_i^1, x_i^2 \in C(S)$, we have $x_i^1, x_i^2 \in C(S_i)$. Then, the only possible point of local nonconvexity of S_i is x. For some integer k, we must have for each $i \geqq k$, that $x \in L(S_i)$ for otherwise S would be convex. Now by Case 1 each S_i, $i \geqq k$, is a union of two compact convex sets with mutually disjoint interiors. Thus, by Theorem 5, S is a union of two compact convex sets with mutually disjoint interiors.

Case 3. Suppose there exists $z \in \operatorname{bd} S$ such that $z \neq x$, $[xz] \subset \operatorname{bd} S$ and there does not exist $z_1 \in \operatorname{bd} S$, $z_1 \neq z$, such that $[xz] \subset [xz_1] \subset \operatorname{bd} S$. Further, suppose for each $y \in \operatorname{bd} S \sim [xz]$ that $[xy] \not\subset \operatorname{bd} S$. Then, following a similar proof as in Lemma 1 we show that if $y \in \operatorname{bd} S \sim [xz]$, then $(xy) \subset \operatorname{int} S$. Finally, as in Case 2, we construct a sequence of compact sets $\{S_i\}_{i=1}^{\infty}$ such that $|L(S_i)| = 1$ for each i beyond some k and $\{S_i\}_{i=1}^{\infty}$ converges to S in the Hausdorff metric, and then apply Theorem 5.

PROOF OF THEOREM 1. We begin with a lemma.

LEMMA 2. *Let S be a two cell such that $|L(S)|$ is finite. Let $[xy]$ be a crosscut of S such that $x, y \in C(S)$. Then there exists a convex subset Axy of S such that $(xy) \subset \operatorname{int} Axy$.*

PROOF. Choose $\epsilon > 0$ so that $B(x, \epsilon) \cap S$ is convex, $B(y, \epsilon) \cap S$ is convex, $\operatorname{cl} B(x, \epsilon) \cap L(S) = \emptyset$ and $\operatorname{cl} B(y, \epsilon) \cap L(S) = \emptyset$. For each $z \in (xy)$, choose $\epsilon_z > 0$ so that $\operatorname{cl} B(z, \epsilon_z) \subset \operatorname{int} S$. Since $[xy]$ is compact select a finite subcover $\{B(x, \epsilon/2), B(y, \epsilon/2), B(z_1, \epsilon z_1/2), \cdots, B(z_n, \epsilon z_n/2)\}$ of the cover $\{B(x, \epsilon/2), B(y, \epsilon/2), B(z, \epsilon z/2) \mid z \in (xy)\}$. Choose $\delta > 0$ so that $\delta < \min\{\epsilon/2, \epsilon z_1/2, \cdots, \epsilon z_n/2\}$. Let B be the closure of the $\delta/\sqrt{2}$ ball about the set $[xy]$. Then $B \cap S$ is clearly connected, closed and $B \cap S \cap L(S) = \emptyset$. Then $B \cap S$ is convex by Tietze's Theorem and this is the required set Axy.

Now suppose Theorem 1 is false. Let $r, s \in L(S)$ such that $C_{rs}^1 \cap L(S) = \emptyset$ or $C_{rs}^2 \cap L(S) = \emptyset$, which is possible since $|L(S)|$ is finite. Let us suppose that $C_{rs}^1 \cap L(S) = \emptyset$. Let $\mathcal{C} = \{[xy] \mid [xy]$ is a crosscut and $x, y \in C(S)\}$. Since $|L(S)|$ is finite, every interior point of S is contained in an element of $\mathcal{C}$, so $\operatorname{int} S \subset \bigcup \mathcal{C}$. Since we are assuming that Theorem 1 is false, if $[xy] \in \mathcal{C}$, then $x, y \in C_{rs}^1$ or $x, y \in C_{rs}^2$.

Consider any two crosscuts in $\mathcal{C}$, say $[x_1y_1]$ and $[x_2y_2]$ such that $x_1, y_1 \in C_{rs}^1$ and $x_2, y_2 \in C_{rs}^2$. These crosscuts can not intersect. To see this suppose $z \in (x_1y_1) \cap (x_2y_2)$. Then the set R whose boundary is given by $[x_1y_1]$ and the portion B of bd S in C_{xy}^1 between x_1 and y_1, is convex by Tietze's Theorem. Since $B \cap [x_2y_2] = \emptyset$, this forces x_2 or y_2

to be an interior point of R, and hence an interior point of S, a contradiction.

To continue, the interior of S is connected since it is the homeomorphic image of a connected set. Let $[xy]\in\mathcal{C}$. Let Axy be as in Lemma 2. Then letting $\mathcal{A}_1=\{\text{int } Axy \mid [xy]\in\mathcal{C},\ x,\ y\in C_{rs}^1\}$ and $\mathcal{A}_2=\{\text{int } Axy \mid [xy]\in\mathcal{C},\ x,\ y\in C_{rs}^2\}$, we have int $S\subset\bigcup\mathcal{A}_1\cup\bigcup\mathcal{A}_2$ since $\bigcup\mathcal{C}=\bigcup\mathcal{A}_1\cup\bigcup\mathcal{A}_2$. Since int S is connected, we must have $\bigcup\mathcal{A}_1\cap\bigcup\mathcal{A}_2\neq\emptyset$, which says that there exists $[x_1y_1]$ and $[x_2y_2]$ in $\mathcal{C}$ with x_1, $y_1\in C_{rs}^1$, x_2, $y_2\in C_{rs}^2$ and int $Ax_1y_1\cap$int $Ax_2y_2\neq\emptyset$. Let $z\in$int Ax_1y_1 $\cap$int Ax_2y_2. Then z lies in a crosscut having endpoints in C_{rs}^1 and in a crosscut having endpoints in C_{rs}^2, which says these crosscuts intersect, which is a contradiction by the last paragraph. Thus Theorem 1 holds.

PROOF OF THEOREM 2. We know Theorem 2 holds when $n=0$, and $n=1$ by Tietze's Theorem and Theorem 6, respectively. Thus, assume the theorem holds for $0\leqq k\leqq n-1$ and we will show the result holds for n. We begin by considering cases. Let $x\in L(S)$.

Case 1. Suppose there exist z, $q\in$bd$S$ such that $z\neq x$, $q\neq x$, $[zx]\cup[xq]\subset$bd S and $[zx]\cap[xq]=\{x\}$. Then choose a crosscut $[xy]$ where $y\in C(S)$ such that $x\in C(D^1xy)\cap C(D^2xy)$. Since $y\in C(S)$, $y\in C(D^1xy)\cap C(D^2xy)$. Now suppose $d=|L(D^1xy)|$. Then $d\leqq n-1$ and by hypothesis $D^1xy=\bigcup_{i=1}^{d+1}C_i$ where for each i, $1\leqq i\leqq d+1$, C_i is compact and convex and for $1\leqq i$, $j\leqq d+1$, $i\neq j$, int $C_i\cap$int $C_j=\emptyset$. Since $|L(D_{xy}^1)|=d$ and since $x\in C(D^1xy)\cap C(D^2xy)$, we have $|L(D^2xy)|=n-d-1$. Thus, by hypothesis $D^2xy=\bigcup_{i=1}^{n-d}B_i$ where for each i, $0\leqq 1\leqq n-d$, B_i is compact, convex and for $i\neq j$, $1\leqq i$, $j\leqq n-d$, int B_i $\cap$int $B_j=\emptyset$. Thus S is a union of $(d+1)+(n-d)=n+1$ compact convex sets with mutually disjoint interiors.

Case 2. Suppose for each $y\in$bd S, $y\neq x$, that $[xy]\not\subset$bd S. We shall need the following lemma.

LEMMA 3. *Let S be a two cell such that $|L(S)|=n\geqq 1$ satisfying Case 2. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in* bd *S with $x_i\to x$, $x\in L(S)$ and $x_i\neq x$ for each i. Then there exists an integer k such that for each $i\geqq k$, $(xx_i)\subset$*int *S.*

PROOF. We proceed by induction. We know by Lemma 1 the lemma holds when $|L(S)|=1$. So suppose the lemma holds for $1\leqq k\leqq n-1$. By Theorem 1, there exists a crosscut $[rs]$ of S such that r, $s\in C(S)$ and $|L(D_{rs}^i)|\leqq n-1$ for $i=1$, 2. Suppose $x\in L(D_{rs}^1)$. Then since there exists an $\epsilon>0$ such that $B(x,\epsilon)\cap S\subset D_{rs}^1$, the result holds. The same argument applies if $x\in D_{rs}^2$.

To complete Case 2, we argue the same way as in Case 2 of Theorem 6. We construct a sequence of sets $\{S_i\}_{i=1}^{\infty}$ such that $\{S_i\}_{i=1}^{\infty}$ converges to S in the Hausdorff metric and using Case 1 of this theorem we show each S_i is representable as $n+1$ compact convex sets with mutually disjoint interiors and then apply Theorem 5. This completes Case 2.

Case 3. Suppose there exists $x \in L(S)$ and $z \in \text{bd } S$, $z \neq x$ such that $[zx] \subset \text{bd } S$ and there does not exist $z_1 \in \text{bd } S$, $z_1 \neq z$ such that $[xz] \subset [xz_1] \subset \text{bd } S$. Further suppose for each $y \in \text{bd } S \sim [xz]$, that $[x, y] \not\subset \text{bd } S$. Then using a similar proof as in Lemma 3, we prove that if $\{x_i\}_{i=1}^{\infty}$ is a sequence in bd $S \sim [xz]$ converging to x, then there exists an integer k, such that for each $i \geqq k$, $(x_i x) \subset \text{int } S$ and use the same construction as in Case 2 to get the theorem. This completes the proof of Theorem 2.

III. **Consequences of Theorem 2.** As immediate corollaries of Theorem 2 we have

COROLLARY 1. *Let S be a two cell such that $|L(S)| = n$. Then S is an L_{n+1} set.*

COROLLARY 2. *Let S be a two cell such that $|L(S)| = n$. Then for every line L, $L \cap S$ is a union of at most $n+1$ closed line segments.*

Corollary 2 is a generalization of the familiar fact that the intersection of a line and a compact convex set is a point or a closed line segment.

Examples are easily constructable to show the number $n+1$ is best in Theorem 2.

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