Two-coloring random hypergraphs

Dimitris Achlioptas[†] Jeong Han Kim[†]

Michael Krivelevich[§] Prasad Tetali[¶]

December 17, 1999

Technical Report
MSR-TR-99-99

Microsoft Research
Microsoft Corporation
One Microsoft Way
Redmond, WA 98052

[†]Microsoft Research, One Microsoft Way, Redmond, WA 98052, U.S.A. Email: {optas,jehkim}@microsoft.com

[‡]Department of Mathematics, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 60078, Israel. Research performed in part while visiting Microsoft Research. Email: krivelev@math.tau.ac.il

[§]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A. Research performed in part while visiting Microsoft Research, and in part supported by NSF Grant no. 9800351. Email: tetali@math.gatech.edu

Two-coloring random hypergraphs

Dimitris Achlioptas[†] Jeong Han Kim[†] Michael Krivelevich[§] Prasad Tetali[¶]

December 17, 1999

Abstract

A 2-coloring of a hypergraph is a mapping from its vertex set to a set of two colors such that no edge is monochromatic. Let $H = H(k, n, p)$ be a random k-uniform hypergraph on a vertex set V of cardinality n, where each k-subset of V is an edge of H with probability p, independently of all other k-subsets. Let $m = p\binom{n}{k}$ denote the expected number of edges in H. Let us say that a sequence of events $\mathcal{E}_n$ holds *with high probability* (w.h.p.) if $\lim_{n\to\infty} \Pr[\mathcal{E}_n] = 1$.

It is easy to show that if $m = c2^k n$ then w.h.p. H is not 2-colorable for $c > \frac{\ln 2}{2}$. We prove that there exists a constant $c > 0$ such that if $m = (c2^k/k)n$, then w.h.p. H is 2-colorable.

1 Introduction

For an integer $k \geq 2$, a *k-uniform hypergraph* H is an ordered pair $H = (V, E)$, where V is a finite non-empty set, called the set of *vertices* of H, and E is a family of distinct k-subsets of V, called the *edges* of H. For general hypergraph terminology and background we refer the reader to [3]. A 2-*coloring* of a hypergraph $H = (V, E)$ is a partition of its vertex set V into two (color) classes, R and B (for Red and Blue, say), so that no edge in E is monochromatic. A hypergraph is *2-colorable* if it admits a 2-coloring.

Hypergraph 2-colorability, sometimes also called “Property B”, has been studied for about forty years (see, e.g. [7, 8, 2, 14]). For $k = 2$, i.e. for graphs, the problem is well understood, since graph 2-colorability is equivalent to the graph having no odd cycle. For $k \geq 3$, though, much less is known and deciding the 2-colorability of k-uniform hypergraphs is NP-complete for every fixed $k \geq 3$ [12].

In this paper we discuss 2-colorability of random k-uniform hypergraphs for $k \geq 3$. (For the evolution of odd cycles in random graphs see [9]). Let $H(k, n, p)$ be a random k-uniform hypergraph on n labeled vertices $V = \{1, \ldots, n\}$, where each k-subset of V is chosen to be an edge of H independently and with probability $p = p(n)$. We will study asymptotic properties of $H(k, n, p)$, i.e. we will consider $k \geq 3$ to be arbitrary but fixed, while n, the number of vertices, tends to infinity. We will say that a hypergraph property A holds *with high probability* (w.h.p.) in $H(k, n, p)$ if $\lim_{n\to\infty} \Pr[H(k, n, p)$ has $A] = 1$. The main question in this setting is:

As p is increased, when does $H(k, n, p)$ stop being w.h.p. 2-colorable?

As it will be convenient to discuss the answer to this question in terms of the expected number of edges in $H(k, n, p)$, we denote $m = p\binom{n}{k}$.

[†]Microsoft Research, One Microsoft Way, Redmond, WA 98052, U.S.A. Email: {optas,jehkim}@microsoft.com

[‡]Department of Mathematics, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 60078, Israel. Research performed in part while visiting Microsoft Research. Email: krivelev@math.tau.ac.il

[§]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A. Research performed in part while visiting Microsoft Research. Email: tetali@math.gatech.edu

Alon and Spencer considered the above question in [1]. They noted that by considering the expected number of 2-colorings of $H(k,n,p)$ it is easy to show that if $m = c2^k n$, where $c > \frac{\ln 2}{2}$, then w.h.p. $H(k,n,p)$ is not 2-colorable. Their main contribution was providing a lower bound on the expected number of edges necessary for $H(k,n,p)$ not to be 2-colorable w.h.p. In particular, by applying the Lovász Local Lemma, they were able to show that if $m = (c2^k/k^2)n$ then w.h.p. $H(k,n,p)$ is 2-colorable, for some small constant $c > 0$. Thus, the gap between the upper and the lower bounds of [1] is of order k^2.

It is interesting to compare the 2-colorability of random k-uniform hypergraphs with the satisfiability problem for random k-SAT formulas. For a set of n Boolean variables, let C_k denote the set of all $2^k\binom{n}{k}$ possible disjunctions of k distinct, non-complementary literals (k-clauses) on those variables. A random k-SAT formula, $F(k,n,m)$, with m clauses over n variables is formed by selecting uniformly at random m clauses from C_k and taking their conjunction. The question now is "as m is increased when does $F(n,k,m)$ stop being satisfiable w.h.p. ?" Again, by considering the expected number of solutions (here, satisfying assignments), it is easy to show if $m = c2^k n$, where $c > \ln 2$, then w.h.p. $F(k,n,m)$ is unsatisfiable. In the opposite direction, Chao and Franco [5] proved that, for $k \geq 4$, a random k-SAT formula with $m = c(2^k/k)n$ clauses is w.h.p. satisfiable, if $c < 1/4$. Chvátal and Reed [6] extended the result of [5] to all $k \geq 2$ (and simplified it), while Frieze and Suen [11], inter alia, improved the constant to $c < 1$.

The similarity between the two problems is quite apparent, though probably cannot be translated into a formal statement. This similarity stimulated Alon and Spencer [1] to try and derive a result for random hypergraph 2-colorability analogous to the random k-SAT result of [6]. While their result [1], as mentioned above, falls short of that goal, the authors proposed a randomized 2-coloring algorithm, similar to the one used by Chvátal and Reed [6], and conjectured that w.h.p. it 2-colors $H(k,n,p)$, as long as $m = c(2^k/k)n$, for some absolute constant $c > 0$. If true, that would reduce the gap between the upper and lower bounds for random hypergraph 2-colorability to a factor of k (from k^2).

In this paper we introduce a deterministic algorithm which is similar to the one proposed by Alon and Spencer, except for one crucial difference that simplifies the analysis greatly. We prove that our algorithm w.h.p. finds a proper 2-coloring of $H(k,n,p)$ if $m = c(2^k/k)n$, for an absolute constant $c > 0$.

Theorem 1 *There exists a deterministic linear time algorithm which w.h.p. 2-colors $H(k,n,p)$ if the edge probability $p = p(n)$ satisfies*

$$p\binom{n}{k} = c\frac{2^k}{k}n \ ,$$

where $c \leq 1/50$. For $k \geq 40$, we can replace $1/50$ with $1/10$.

Let us note that a recent result of Friedgut [10] can be used to show that for each $k \geq 3$, there exists a function $r_k(n)$ such that if $m = (r_k(n) - \epsilon)n$ then w.h.p. $H(k,n,p)$ is 2-colorable, whilst if $m = (r_k(n) + \epsilon)n$ then w.h.p. $H(k,n,p)$ is not 2-colorable. Naturally, $c\frac{2^k}{k} < r_k(n) < c'2^k$, for some absolute constants $c, c' > 0$. It is widely believed that one can replace $r_k(n)$ by a constant r_k. Closing the asymptotic gap in the order of $r_k(n)$ is a challenging open problem in that direction.

The rest of the paper is organized as follows. In Section 2 we present our algorithm, analyze its performance on $H(k,n,p)$ and prove Theorem 1. Section 3 is devoted to a concluding discussion. As noted before, throughout the paper we assume n to be large enough whenever needed, while keeping in mind that k is fixed. Also, for the sake of clarity of presentation we routinely omit floor and ceiling signs.

2 Proof of the main result

In this section we first present a deterministic algorithm A for 2-coloring k-uniform hypergraphs when $k \geq 6$. We show that for such k, algorithm A 2-colors $H(k,n,p)$ w.h.p. for $p(n)$ as in Theorem 1. We treat the (easy) case $3 \leq k \leq 5$ separately at the end of this section.

2.1 Algorithm description

As stated above, we assume that $k \geq 6$. An input of the algorithm is a k-uniform hypergraph $H = (V, E)$ with $V = \{1, \ldots, n\}$. To describe the algorithm it will be convenient to fix in advance an ordering on the vertices of V, say, the natural ordering $1, \ldots, n$. This ordering induces the corresponding lexicographic order on the subsets of V. Thus, for example, $\{1, 2\} < \{1, 2, 3\} < \{3\}$. Also, for the sake of presentation, we assume, for now, that the number of vertices n is even.

The algorithm proceeds in rounds, $t = 0, 1, \ldots$, coloring *two* vertices in each round. Given a partial coloring of the vertices of V, we say that a k-subset of V is *i-monochromatic* if precisely i of its vertices have been colored and all i of them have received the same color.

ALGORITHM A

(1) If there are $(k-3)$- or $(k-2)$-monochromatic edges
    then
        let $x < y$ be the smallest uncolored vertices in the smallest such edge,
    else
        let $x < y$ be the smallest uncolored vertices.

(2) Color x Red; color y Blue.

Before proceeding to the analysis of the performance of A on $H(k,n,p)$, we wish to briefly compare it with algorithms for random k-SAT suggested in [5, 6, 11]. All these algorithms set the value of one variable at a time, giving priority to variables that appear in clauses that are yet unsatisfied and have few remaining unset variables. While the algorithms differ in the exact rule for choosing which variable to set among those appearing in "short" clauses, their asymptotic performance is within a constant factor. Similarly, we will also give priority to vertices participating in short edges, where short now means an edge many of whose vertices have already been colored, all in the same color. Those edges are clearly the most dangerous, and it is thus quite natural to try to eliminate them first. However, in contrast with the algorithms for k-SAT, our algorithm A colors two vertices at each round. There are two advantages to this strategy. First, an edge containing both vertices x, y, colored at the current round, is not dangerous anymore, as x and y get different colors. Secondly, we are able to keep track of the number of vertices colored Red and Blue so far – after $2t$ vertices have been colored, exactly t of them are Red and the remaining t are blue. Put differently, *in order to successfully color a short edge we do not need to know the color of its already colored vertices.* This feature will be especially helpful in the analysis of the behavior of A on random hypergraphs.

2.2 Proof of Theorem 1

We will prove the statement in Theorem 1 for even values of n, with a slightly larger c' (namely, $c' = 1.01/50$). For odd n, the result then follows by considering $H(k, n+1, p)$, where $p\binom{n}{k} = c(2^k/k)n$.

We need to show that w.h.p. in $H(k,n,p)$, the 2-coloring resulting from applying algorithm A contains no monochromatic edge. We will prove a slightly stronger claim.

Claim 1 *With probability $1 - O(n^{-1/2})$ no edge becomes $(k-1)$-monochromatic during the algorithm's execution.*

To see why Claim 1 implies Theorem 1, note that if an edge has two of its vertices colored in the same round then it cannot be monochromatic. Thus, for an edge to become monochromatic it must at some point be $(k-1)$-monochromatic (and, in general, to become i-monochromatic an edge must first be $(i-1)$-monochromatic).

Proof of Claim 1. As often happens in the analysis of deterministic algorithms on random structures, it will be convenient to assume that the choice of the random hypergraph is made *in parallel* with its coloring, rather than assuming that a member of $H(k,n,p)$ is chosen before the execution of A begins.

For a family E_0 of distinct k-subsets of V, *exposing the edges from* E_0 amounts to deciding for each k-tuple $e \in E_0$, whether $e \in E(H(k,n,p))$ independently and with probability $p(n)$. Thus, if the family of all k-subsets of V is represented as a union of pairwise disjoint families $\{E_i\}$, exposing the families E_i in some order generates a random hypergraph $H(k,n,p)$.

Now let us describe how the edges of $H(k,n,p)$ are exposed as Algorithm A proceeds.

EXPOSURE PROCEDURE

1. For $0 \le t \le n/2 - 1$ repeat:

   Suppose we are in round t of Algorithm A, and are about to color vertex x in Red and vertex y in Blue. Let R_{t-1} be the set of vertices colored Red and B_{t-1} be the set of vertices colored Blue, in rounds $0, \ldots, t-1$.

   Expose all edges, having $k-4$ vertices in R_{t-1}, containing x and having three vertices outside $R_{t-1} \cup B_{t-1} \cup \{x,y\}$.

   Expose all edges, having $k-4$ vertices in B_{t-1}, containing y and having three vertices outside $R_{t-1} \cup B_{t-1} \cup \{x,y\}$.

2. After all vertices from V have been colored, expose any yet unexposed edges.

It is easy to see that each k-subset of V is exposed exactly once during the above exposure procedure. Hence its output is distributed according to $H(k,n,p)$. It is important to observe that in part 2 of the above exposure procedure all exposed edges have at most $k-4$ vertices of one color. In order for an edge of H to become $(k-1)$-monochromatic, it should first become $(k-3)$-monochromatic, and therefore it could only have been exposed during part 1 of the above exposure procedure. Therefore, to prove Claim 1, it is enough to restrict our attention to this first part, performed along with the execution of the algorithm.

For each edge $e \in E(H)$, exposed in round t of the algorithm, let e' be the triple of its uncolored vertices at the end of round t. We denote

$$F^{(t)} = \{e' \subset e : e \in E(H) \text{ and } e \text{ is exposed in round } t\} \ .$$

($F^{(t)}$ is a set, not a multiset, i.e. we treat multiple copies of a triple as one.) We will refer to triples from $F^{(t)}$ as t-triples and it will be notationally convenient to define $F^{(t)} = \emptyset$ for $t \ge n/2$.

For the purpose of the analysis, we group rounds into *phases*. We use t_i and $\hat{t_i}$ to denote the first and the last round of the ith phase, respectively, and the phase itself is defined as follows. The ith phase consists of the sequence of rounds $t_i, t_i + 1, \ldots, \hat{t_i}$, if the number of $(k-3)$- and

$(k-2)$- monochromatic edges is zero at the beginning of round t_i and round $\hat{t_i}+1$, but remains positive during the rounds t_i+1 through $\hat{t_i}$. In particular, at the beginning of a new phase there are no $(k-3)$- or $(k-2)$-monochromatic edges, and during a phase there is at least one such monochromatic edge. It will be notationally convenient to consider round $n/2-1$ as the beginning of a last, trivial phase. Notice that precisely $2t_i$ vertices are colored just before phase i starts and $2(\hat{t_i}+1)$ vertices are colored after phase i ends. For phase i, we denote

$$F_i = \bigcup_{r=t_i}^{\hat{t_i}} F^{(r)} \quad .$$

Since there are no $(k-3)$-monochromatic edges of H right before phase i starts, we observe that if an edge e becomes $(k-1)$-monochromatic during phase i, then e must (a) have been exposed during phase i, and (b) if w, z are its two vertices colored during phase i, F_i must contain two distinct triples corresponding to edges e_w, e_z (different from e) containing w, z, respectively. As there at most $n/2$ phases, to prove Claim 1 it suffices to prove the following lemma.

Lemma 1

$$\Pr[\ F_i \text{ contains distinct triples } e_1, e_2, e_3 \text{ so that } e_1 \cap e_2 \neq \emptyset,\ e_1 \cap e_3 \neq \emptyset] = O(n^{-3/2}) \quad . \qquad (1)$$

Proof. We first note that conditional on R_t, B_t, each triple from $V \setminus (R_t \cup B_t)$ appears in $F^{(t)}$ independently of all other such triples and with probability

$$q(t) := 1-(1-p)^{2\binom{t}{k-4}} \quad .$$

In order to eliminate dependencies between the appearance of distinct triples in F_i, we will condition on R_{t_i} and B_{t_i}. As our argument will work for any given R_{t_i} and B_{t_i}, Lemma 1 will follow.

For $t_i \le t \le \hat{t_i}$, it will be convenient to define a superset $F_i^{(t)}$ of $F^{(t)}$ in which every triple in $V \setminus (R_{t_i} \cup B_{t_i})$ appears independently and with probability $q(t)$. To form $F_i^{(t)}$ we will add to $F^{(t)}$ each triple in $V \setminus (R_{t_i} \cup B_{t_i})$ but not in $V \setminus (R_t \cup B_t)$ with probability $q(t)$, independently of all other such triples. Also, we will introduce an auxiliary set

$$F_i' = \bigcup_{r=t_i}^{t_i+\log^2 n} F_i^{(r)} \quad .$$

Note now that each triple from $V \setminus (R_{t_i} \cup B_{t_i})$ appears in F_i' independently and with probability

$$1 - \prod_{t=t_i}^{t_i+\log^2 n} (1-q(t)) \le 1-(1-q(t_i+\log^2 n))^{\log^2 n} \quad ,$$

as $q(t)$ is increasing in t. Moreover, F_i is a subset of F_i' unless $\hat{t_i} > t_i + \log^2 n$.

Letting

$$q_i := 1-(1-q(t_i+\log^2 n))^{\log^2 n}$$

F_i' is a random set of triples from $V \setminus (R_{t_i} \cup B_{t_i})$, with each triple appearing independently and with probability *at most* q_i. Thus, recalling that F_i is a subset of F_i' unless $\hat{t_i} > t_i + \log^2 n$, we see that the probability in (1) is bounded by the sum of

$$\Pr[\ F_i' \text{ contains three distinct edges } e_1, e_2, e_3 \text{ so that } e_1 \cap e_2 \neq \emptyset,\ e_1 \cap e_3 \neq \emptyset\ ] \qquad (2)$$

and

$$\Pr[\ \hat{t}_i > t_i + \log^2 n\ ]\ . \tag{3}$$

Since $(1-a)^b \geq 1 - ab$ for all $a > 0$ and any nonnegative integer b, we have

$$\begin{aligned} q_i &\leq 2p\binom{t_i + \log^2 n}{k-4}\log^2 n \\ &\leq 2\,\frac{c2^k n}{k}\,\frac{(n-k)!k!}{n!}\,\frac{(t_i+\log^2 n)^{k-4}}{(k-4)!}\log^2 n \\ &\leq \frac{33ck^3\log^2 n}{n^3}\left(\frac{2(t_i+\log^2 n)}{n}\right)^{k-4}\ . \end{aligned} \tag{4}$$

Now, as $k \geq 4$ is fixed, the probability in (2) is readily bounded from above by

$$\binom{n-2t_i}{3}\left(3\binom{n-2t_i-1}{2}\right)^2 q_i^3 = O((n-2t_i)^7 q_i^3) = O(k^2(\log n)^6/n^2)) = n^{-2+o(1)}\ . \tag{5}$$

(We used that for any $a, b > 0$ and any nonnegative integers i, j, $(a-b)^i b^j \leq a^{i+j}/\binom{i+j}{i}$.)

For the probability in (3) we observe that in any round t with $t_i + 1 \leq t \leq \hat{t}_i$, at least one $(k-3)$- or $(k-2)$-monochromatic edge is 2-colored. Thus the number of such edges after round t is at most

$$\left|\bigcup_{r=t_i}^{t} F^{(r)}\right| - (t - t_i)\ ,$$

which must be positive for $t \leq \hat{t}_i$. In particular, if $\hat{t}_i > t_i + \log^2 n$, then $|F_i'| \geq \log^2 n$. Note now that $|F_i'|$ is dominated by a binomial random variable $\mathrm{Bin}\left(\binom{n-2t_i}{3}, q_i\right)$. Letting $2t_i/n = \alpha$, we get

$$\begin{aligned} \binom{n-2t_i}{3} q_i &\leq \frac{(n-2t_i)^3}{6}\,\frac{33ck^3\log^2 n}{n^3}\left(\frac{2(t_i+\log^2 n)}{n}\right)^{k-4} \\ &\leq \frac{17ck^3}{3}(1-\alpha)^3\alpha^{k-4}\log^2 n \\ &\leq \frac{17ck^3}{3}\left(1-\frac{k-4}{k-1}\right)^3\left(\frac{k-4}{k-1}\right)^{k-4}\log^2 n \\ &\leq 0.9\log^2 n\ , \end{aligned}$$

for $c \leq 1.01/50$ and $k \geq 6$, and for $c \geq 1.01/10$ and $k \geq 40$. Thus, by considering the Chernoff bound for the tail of the Binomial random variable, we see that the probability in (3) is at most $1/n^2$, concluding the proof of the lemma. $\square$

To conclude the proof of Theorem 1 we will present a deterministic algorithm which w.h.p. 2-colors $H(k, n, p)$ if $m = cn$, for $c < 1/6$. Postponing that proof for a moment, we observe that for $3 \leq k \leq 5$, $\frac{101}{5000}\frac{2^k}{k} < \frac{1}{6}$, and thus along with bound for $k \geq 6$ above we get Theorem 1.

Let a component of a hypergraph be "bad" if it contains more than one cycle, or more than two edges sharing more than one vertex. Recall now that for all $k \geq 2$, if $c < \frac{1}{k(k-1)}$ then w.h.p. there are no bad components in $H(k, n, p)$ [4]. Our deterministic algorithm for $3 \leq k \leq 5$ is as follows: if H contains a bad component then exit, reporting failure. Otherwise, to color a component, repeatedly remove edges containing vertices of degree 1. Let $e_1, \ldots, e_q$ be the removed edges, in order of removal. Since $k \geq 3$ and the component is not bad, it is not hard to see that this procedure removes all the edges. Now, we add the edges back to H in reverse order, coloring vertices as follows: while adding back an edge $e_i \in E(H)$, if e_i contains two uncolored vertices, color them using distinct colors. Otherwise, take a vertex of current degree 1 in e_i and use it to make the edge bichromatic. By the ordering of the edges, one of the above two cases always happens. Finally, all uncolored vertices, which are exactly the isolated vertices of H, are colored arbitrarily.

3 Concluding remarks

- It is natural to wonder if Algorithm A in fact performs significantly better than what we have demonstrated. However, one can show that for larger values of c (e.g., $c = 1$), there exists a round t^* such that w.h.p. the number of $(k-2)$-monochromatic edges at the beginning of round t^* is greater than $(n - 2t^*)$. As a result, w.h.p. the graph induced by the uncolored vertices of these edges contains an odd cycle and hence the algorithm fails. Thus, our analysis of Algorithm A is tight up to the value of the constant c.

- Our algorithm A suggests the following algorithm for r-coloring $H(k, n, p)$ for any fixed $r \geq 2$ and $k \geq 3$.

  ALGORITHM A_r

  (1) If there are $(k-r-1)$- or $(k-r)$-monochromatic edges
  then
  let $x_1 < x_2 < \cdots < x_r$ be the smallest uncolored vertices in the smallest such edge,
  else
  let $x_1 < x_2 < \cdots < x_r$ be the smallest uncolored vertices.

  (2) Color x_i with color i, for $i = 1, \ldots, r$.

  Thus, Algorithm A is A_2. Again, the key property is that at the end of every round an equal number of vertices have been assigned each color. An analysis similar to that of A, shows that the above algorithm w.h.p. r-colors a random k-uniform hypergraph $H(k, n, p)$ if the edge probability $p = p(n)$ satisfies

$$p\binom{n}{k} = c\frac{r^k}{k}n \ ,$$

  where $c < c^* = c^*(r)$. (For example, taking $c^*(r) = (r+1)!/(r+1)^{2(r+1)}$ suffices).

- Finally, we remark that using a non-rigorous technique of statistical physics, namely the replica method, in [13] it is suggested that the threshold for the satisfiability of random k-SAT formulas is at the number of clauses $m = c2^k n$ (in fact with $c = \ln 2$). We feel that improving asymptotically the easy upper bound or the existing lower bound for either the satisfiability problem or the 2-colorability problem would represent significant progress on this topic.

Acknowledgment. We would like to thank Van H. Vu for useful discussion.

References

[1] N. Alon and J. Spencer, *A note on coloring random k-sets*, unpublished manuscript.

[2] J. Beck, *On 3-chromatic hypergraphs*, Discrete Math. 24 (1978), 127–137.

[3] C. Berge, **Hypergraphs: combinatorics of finite sets**, North Holland, Amsterdam, 1989.

[4] B. Bollobás, **Random graphs**, Academic Press, London–New York, 1985.

[5] M.-T. Chao and J. Franco, *Probabilistic analysis of a generalization of the unit-clause literal selection heuristics for the k-satisfiability problem*, Inform. Sci. 51 (1990), 289–314.

[6] V. Chvátal and B. Reed, *Mick gets some (the odds are on his side)*, Proceedings of the 33^{rd} IEEE FOCS, IEEE, 1992, 620–627.

[7] P. Erdős, *On a combinatorial problem I*, Nordisk Mat. Tidskrift 11 (1963), 5–10.

[8] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in *Infinite and finite sets*, (A. Hajnal et al., eds.), North Holland, Amsterdam, 1975, 609–628.

[9] P. Flajolet, D. Knuth and B. Pittel, *The first cycles in an evolving graph*, Discrete Math. 75 (1989), 167–215.

[10] E. Friedgut, *Necessary and sufficient conditions for sharp thresholds of graph properties, and the k-SAT problem*, J. Amer. Math. Soc. 12 (1999), 1017–1054.

[11] A. Frieze, S. Suen, *Analysis of two simple heuristics on a random instance of k-SAT*, J. Algorithms 20 (1996), 312–355.

[12] L. Lovász , *Coverings and colorings of hypergraphs*, in: Proc. 4^{th} S. E. Conf. on Combinatorics, Graph Theory and Computing, 1973, Utilitas Math., 3–12.

[13] R. Monasson and R. Zecchina, *Entropy of the K-satisfiability problem*, Phys. Rev. Lett. 76 (1996), 3881–3885.

[14] J. Radhakrishnan and A. Srinivasan, *Improved bounds and algorithms for hypergraph two-coloring*, Proceedings of the 39^{th} IEEE FOCS, IEEE (1998), 684–693.